K-Theory Correspondences and the Fourier-Mukai Transform

by

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B.Sc., University of Victoria, 2016

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ABSTRACT

The goal of this thesis is to give an introduction to the geometric picture of bivariant K-theory developed by Emerson and Meyer in [11] building on the ideas Connes and Skandalis, and then to apply this machinery to give a geometric proof of a result of Emerson stated in a recent article [8]. We begin by giving an overview of topological K-theory, necessary for developing bivariant K-theory. Then we discuss Kasparov's analytic bivariant K-theory, and from there develop topological bivariant K-theory. In the final chapter we state and prove the result of Emerson.

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Finally, I want to thank all of my friends at UVic. You collectively have made my Master's a wonderfully enjoyable experience being there whenever I needed a break, or people to "talk shop" with.

Dedication

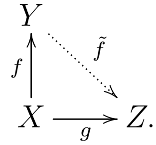
To my parents, Frank and Theresa.

How it feels to stare
at a couple of years
organized into 130 pages
the ups and downs
and weeks of frustration
and work that took ages
to take an axe
and cut the rough,
the bad, and also the pointless
seems a shame
to those who'll read
and feel some dissapointment
that in the paper
it's all so easy
to me, the weathered author
when in fact
there were many trials
that they were never there for
it seems to me
that to document the struggle
would have somewhat more worth
but i guess how it goes
when sharing knowledge
is showing only what works

Chapter 1

Introduction

Algebraic topology is an area of mathematics which attempts to classify topological spaces by assigning to them certain algebraic objects, such as groups, in order to distinguish them. More specifically, the basic method of algebraic topology is to define a family of functors $\{F_i\}_{i\in\mathbb{N}}$, either covariant or contravariant, from a subcategory of the category of topological spaces to the category of abelian groups or vector spaces. Such a family is called a *homology* or *cohomology* theory, depending on whether the F_i are covariant or contravariant respectively. One usually requires that $F_i(X) \cong F_i(Y)$ whenever X and Y are equivalent in some sense, for instance if they are of the same homotopy type. Thus, if $F_i(X) \ncong F_i(Y)$ for some i then X and Y are not equivalent. Another useful feature of this idea is that functoriality gives us information about maps between topological spaces. For instance, suppose you have spaces X, Y, and Z, with maps $f : X \to Y$ and $g : X \to Z$, and suppose you'd like to know if there was a map $\tilde{f} : Y \to Z$ making the following diagram commute



If such a map existed, then functoriality tells us that for every $i \in \mathbb{N}$ there exists a map $F_i(\tilde{f})$ making the following diagram commute

$$\begin{array}{ccc} F_i(Y) & & \\ {\scriptstyle F_i(f)}\uparrow & \overset{F_i(\tilde{f})}{\searrow} & \\ F_i(X) & \xrightarrow[F_i(g)]{} & F_i(Z). \end{array}$$

Therefore, if you could show that such a map does not exist for any specific i, then you can conclude that no such $\tilde{f}$ exists. The motivation for this general method is that equivalence or existence problems in algebra are typically easier than their counterparts in topology.

There are several ways of defining such a family of functors, but the one we will focus on in this thesis is K-theory. Given a topological space X, the K-theory of X is defined by considering how one can assign a vector space to each point $x \in X$ in a "continuous" way. Such an assignment is called a *vector bundle*. In Chapter 2 we give a more or less self contained introduction to topological K-theory, starting with the definition of a vector bundle. We conclude the chapter by summarizing the Atiyah-Singer index theorem, one of the crowning achievements of modern mathematics.

K-theory is a *contravariant* functor, meaning that a given map $f : X \to Y$ induces a map $f^* : \mathrm{K}^*(Y) \to \mathrm{K}^*(X)$. However, the index map of Atiyah and Singer (discussed at the end of Chapter 2) gives an example of how certain maps $f : X \to Y$ induce a "wrong-way map" $f_! : \mathrm{K}^*(X) \to \mathrm{K}^*(Y)$. The goal of Chapter 3 is to develop the appropriate condition, called "K-orientability", for a function to induce a wrong-way map. We conclude Chapter 3 by constructing wrong-way maps for K-oriented maps.

The main usefulness of the wrong-way construction is that it gives us more ways to tell K-groups, and elements in the K-groups, apart. In order for a cohomology theory (or homology theory) to be successful, one must be able to tell the associated algebraic structures apart. Therefore, having many ways to construct maps between the resulting algebraic objects is crucial for the success of a cohomology theory. In his paper “Equivariant KK-theory and the Novikov Conjecture” ([16]), G.G. Kasparov defined his bivariant K-groups in such a way that any class $\eta \in \mathrm{KK}(X,Y)$ induces a map from the K-theory of X to the K-theory of Y. Moreover, there is a way to compose classes in the bivariant K-groups in a way not dissimilar to how one composes functions. Thus, one can think of KK-theory as being a generalization of group homomorphism between K-groups. In fact, one can show that, most of the time, *every* map between K-groups arises from a class in KK-theory.

Kasparov’s theory is defined at the level of C*-algebras and has the disadvantage that the composition of two classes can be hard to compute. Building on the ideas of Baum, and Connes and Skandalis, Emerson and Meyer have shown how one can define bivariant K-theory for smooth manifolds geometrically using *correspondences* and in this setting the composition of classes can be computed using geometric considerations. In Chapter 4, we build the theory of topological correspondences, define geometric bivariant K-theory, and then define a natural transformation from the geometric theory of Emerson and Meyer to the analytic theory of Kasparov.

The purpose of these three chapters is to give a self-contained introduction to the subject of geometric bivariant K-theory, which could be understood by a graduate student in a (hopefully) relatively short amount of time. Although all of this material is known, it has not been well documented. For instance, certain authors use different definitions of K-orientations which are not obviously equivalent and results such as the 2 out of 3 lemma for K-orientations were proved using non-constructive methods.

In Chapter 5, we use the tools developed in the previous chapters to prove a theorem of Emerson stated in [8]. Specifically, Emerson considers the class $[D_{\mathbb{R}^d}] \in \mathrm{KK}^{\mathbb{Z}^d}_{-d}(C_0(\mathbb{R}^d),\mathbb{C})$ of the Dirac operator on $\mathbb{R}^d$. Using a process called descent, one can map this to a class $j_{\mathbb{Z}^d}([D_{\mathbb{R}^d}]) = \mathrm{KK}_{-d}(C(\mathbb{T}^d), C(\widehat{\mathbb{Z}}^d))$ which can be represented geometrically as the *Fourier-Mukai Correspondence*

$$[\mathcal{F}_d] = [\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d) \xrightarrow{\mathrm{pr}_2} \widehat{\mathbb{Z}}^d],$$

where $\mathcal{P}_d$ is a certain line bundle over $\mathbb{T}^d \times \widehat{\mathbb{Z}}^d$. This correspondence defines a map $\mathrm{K}_*(\widehat{\mathbb{Z}}^d) \to \mathrm{K}_*(\mathbb{T}^d)$ and one of the main results of Chapter 5 computes the action of this map on certain canonical classes in $\mathrm{K}_*(\widehat{\mathbb{Z}}^d)$. We define the *dual Fourier-Mukai correspondence*

$$[\overline{\mathcal{F}}_d] = [\widehat{\mathbb{Z}}^d \xleftarrow{\mathrm{pr}_2} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \overline{\mathcal{P}}_d) \xrightarrow{\mathrm{pr}_1} \mathbb{T}^d]],$$

and the main theorem of Chapter 5 is that $[\mathcal{F}_d]$ and $[\overline{\mathcal{F}}_d]$ are inverses of one another. That is,

Theorem 1.0.1. *For any d, we have*

$$[\mathcal{F}_d] \otimes [\overline{\mathcal{F}}_d] = \mathrm{id}_{\widehat{\mathrm{KK}}_*(\mathbb{T}^d, \mathbb{T}^d)} \quad \text{and} \quad [\overline{\mathcal{F}}_d] \otimes [\mathcal{F}_d] = \mathrm{id}_{\widehat{\mathrm{KK}}_*(\widehat{\mathbb{Z}}^d, \widehat{\mathbb{Z}}^d)}.$$

The action of the Fourier-Mukai transform bears striking resemblance to the Fourier-Mukai transform from algebraic geometry, and this is the reason it bears its name.

Chapter 2

K-Theory

K-theory was created by Alexander Grothendieck using locally free sheaves over an algebraic variety in order to state the Grothendieck-Riemann-Roch Theorem. Michael Atiyah and Friedrich Hirzebruch modified this idea and defined the K-theory of a space X by using vector bundles over X. The point of this chapter is to develop topological K-theory, as defined by Atiyah and Hirzebruch.

In Section 2.1 we define vector bundles, which are the basic cocycles of K-theory, exhibit some of their properties, and discuss some constructions one can do with vector bundles. In Section 2.2 we define K-theory, first for compact Hausdorff spaces and then for locally compact Hausdorff spaces. We then discuss higher K-theory and the long exact sequence, which makes K-theory a cohomology theory. In Section 2.3 we show how K-theory can be described in terms of complexes of vector bundles over a space, and use this description to define the Thom homomorphism. Following this, in Section 2.4, we compute the K-theory groups of some basic topological spaces. Finally in Section 2.5 we discuss one of the crowning achievements of K-theory, the Atiyah-Singer Index theorem.

2.1 Vector Bundles

2.1.1 Definition of a Vector Bundle

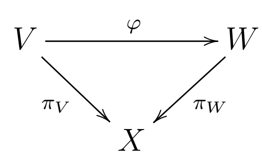
Definition 2.1.1. Let X be a topological space. A real (resp. complex) family of vector spaces over X is a topological space V together with a continuous surjection $\pi_V : V \to X$ such that, for all $x \in X$, each fibre $V_x := \pi^{-1}(\{x\})$ is a finite dimensional vector space over $\mathbb{R}$ (resp. $\mathbb{C}$) whose structure is compatible with the topology on V_x inherited as a subspace of V. We call the map π_V the projection.

Example 2.1.1. If X is a topological space, then $X \times \mathbb{C}^n$ and $X \times \mathbb{R}^n$ are families of vector spaces over X. These families are called the product bundles of rank n.

Morphisms of families combine linear maps and continuous maps.

Definition 2.1.2. If $\pi_V : V \to X$ and $\pi_W : W \to X$ are families of vector spaces over a topological space X, we say that a continuous map $\varphi : V \to W$ is a homomorphism of families (or just homomorphism, for short) if

1. The following diagram commutes:



2. for all $x \in X$, the map $\varphi_x := \varphi|_{V_x} : V_x \to W_x$ is a linear map of vector spaces.

If φ is a homeomorphism, then we call φ an isomorphism, and we write $V \cong W$.

Note that if φ is an isomorphism of families, then each φ_x is a linear isomorphism of the fibres. For simplicity, we mostly take our families to be complex, although he discussion works for both real and complex families.

In general, families are too wild for one to reasonably study. To mend this, we include an additional local assumption, which leads us to the definition of a vector bundle.

Definition 2.1.3. If a family $\pi_V : V \to X$ is isomorphic to $X \times \mathbb{C}^n$ for some n, the we call V trivial. We call V locally trivial if, for all $x \in X$, there exists an open neighbourhood $U \subset X$ containing x such that $V|_U := \pi^{-1}(U) \cong U \times \mathbb{C}^n$ for some n (which depends on U). A locally trivial family $\pi_V : V \to X$ is called a vector bundle.

The additional assumption of local triviality immediately yields that if V is a vector bundle over X, then the rank of V is constant over the connected components of X. This is because local triviality implies that the map $x \mapsto \dim(V_x)$ is locally constant. If it is globally constant (for instance, if X is connected) then we define the rank of V to be the common dimension of the fibres.

A common way to show that a family of vector spaces is a vector bundle is by using certain functions, called sections, from the base space X to the family V.

Definition 2.1.4. A section of a family $\pi_V : V \to X$ is a continuous map $s : X \to V$ such that $\pi_V \circ s = \mathrm{id}_X$. The collection of all sections of a family V is denoted $\Gamma(V)$.

Thus, a section is a continuous choice of a vector in each vector space V_x.

Example 2.1.2. There is always at least one section, namely the zero section. That is the function $s : X \to V$ defined by $s(x) = 0 \in V_x$ for all $x \in X$.

Using sections, we have an equivalent condition for a family of vector spaces to be a vector bundle.

Proposition 2.1.1. *A family $\pi_V : V \to X$ is a vector bundle if and only if for each $x \in X$ there exists an open neighbourhood U containing x and sections $\{s_1, \dots, s_n\}$ on U such that $\{s_1(y), \dots, s_n(y)\}$ is a basis of V_y for each $y \in U$. In particular, a vector bundle V is trivial if and only if we can find such sections defined globally.*

Proof. Suppose that $\varphi : U \times \mathbb{C}^n \to V|_U$ is an isomorphism and let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Then we can define sections $\{s_1, \dots, s_n\}$ on U by $s_i(x) = \varphi(x, e_i)$, and these clearly have the desired properties.

Conversely, if we have such a collection of sections $\{s_1, \dots, s_n\}$ on U with the desired properties, then we can define an isomorphism $\phi : U \times \mathbb{C}^n \to E|U$ by

$$s_i : (u, (v_1, \dots, v_n)) \mapsto \sum_{i=1}^{n} v_i s_i(u).$$ □

Swan's theorem [21] says that vector bundles are classified up to isomorphism by the finitely generated projective module comprised of their sections. That is, $V \cong W$ if and only if $\Gamma(V) \cong \Gamma(W)$ as $C(X)$-modules.

It is important to see some examples of vector bundles to make them seem not so unfamiliar. Thus, we give several examples of real and complex vector bundles.

Example 2.1.3. The tangent bundle to a smooth manifold is always a vector bundle.

Example 2.1.4. The tangent space to the circle, TS^1, is a trivial bundle. Indeed, it has a global non-vanishing section given by

$$x \mapsto \left(x, \frac{\partial}{\partial\theta}\Big|_x \right).$$

The tangent bundle to the 2-sphere is a non-trivial vector bundle, since, by the Hairy Ball theorem, there are no non-vanishing global sections.

Example 2.1.5. Consider the strip $[0,1] \times \mathbb{R}$ and consider the relation $(0,t) \sim (1,-t)$. Then $\ddot{\mathrm{M}} := [0,1] \times \mathbb{R}/\sim$ is a vector bundle over the circle $[0,1]/\{0\} \sim \{1\}$. Indeed, to show this we just need to show local triviality. On $(0,1)$ we can take $s(x) = (x,1)$

and on $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ we take

$$s(x) = \begin{cases} (x, 1) & \text{if } x \in [0, \frac{1}{2}) \\ (x, -1) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

This bundle is called the Möbius bundle. It is homeomorphic to the open Möbius band. By the intermediate value theorem, it is another example of a non-trivial vector bundle.

Example 2.1.6. We will define a canonical line bundle on $\mathbb{C}\mathrm{P}^n$, the complex projective space. Define

$$H := \{(L, z) \in \mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1} : z \text{ is a point in the line } L\}.$$

We give H the topology inherited as a subspace of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}^{n+1}$ and define the projection $\pi : H \to \mathbb{C}\mathrm{P}^n$ to be projection onto the first coordinate. Then the inverse image of each point $L \in \mathbb{C}\mathrm{P}^n$ is that line as a subspace of $\mathbb{C}^{n+1}$, which carries an obvious vector space structure. In particular, the fibers are one dimensional. Thus, to see that H is locally trivial we have to find a section which is locally non-vanishing. Consider the sets

$$U_i := \{[z_0, \dots, z_n] \in \mathbb{C}\mathrm{P}^n : z_i \neq 0\};$$

here we are using homogeneous coordinates: $[z_1, \dots, z_n] = [w_1, \dots, w_n]$ if and only if $(z_1, \dots, z_n) = \lambda(w_1, \dots, w_n)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. The U_i cover $\mathbb{C}\mathrm{P}^n$, and on each U_i we define the map

$$s([z_0, \dots, z_n]) = \left([z_0, \dots, z_n], \left(\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}\right)\right) \in H_{[z_0, \dots, z_n]}.$$

Since the i-th coordinate of $\left(\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}\right)$ is 1, we have that s is non-vanishing on U_i, hence gives a local trivialization of H. Thus, H is a vector bundle.

Example 2.1.7. More generally, if X is compact Hausdorff and $P : X \to M_n(\mathbb{C})$ is a continuous projection valued map, then

$$\mathrm{Im}(P) := \{(x, v) \in X \times \mathbb{C}^n : v \in P(x)\mathbb{C}^n\}$$

is a vector bundle. This is proved as a part of Swan's Theorem.

Example 2.1.8. The point of this example is to show that every *complex* vector bundle over the circle and unit interval is trivial.

Lemma 2.1.1. *Let $p, q \in [0,1]$ and let $\{v_i\}$ and $\{w_i\}$ be bases for $\mathbb{C}^n$. Then there exists everywhere linearly independent sections $s_i : [p,q] \to [p,q] \times \mathbb{C}^n$ such that $s_i(p) = v_i$ and $s_i(q) = w_i$, $i = 1, \ldots, n$.*

Proof. Let $A \in GL_n(\mathbb{C})$ be such that $Av_i = w_i$. Since $GL_n(\mathbb{C})$ is path connected, there exists a path $A(t) : [p,q] \to GL_n(\mathbb{C})$ such that $A(p) = \mathrm{Id}$ and $A(q) = A$. Then $s_i(t) := (t, A(t)v_i)$ are the required sections. □

This implies the following fact.

Proposition 2.1.2. *Every complex vector bundle over the closed interval $[0,1]$ is trivial.*

This, in turn, implies the following

Proposition 2.1.3. *Every (complex) vector bundle over S^1 is trivial.*

Proof. Let E be a complex vector bundle over the circle. By the previous proposition, E is trivial over two open sets U and V, homeomorphic to the unit interval. Let s_i^U and s_i^V be trivializing sections. Connecting the sections via a path in $GL_n(\mathbb{C})$ over the overlaps using Lemma 2.1.1 yields globally defined everywhere linearly independent sections s_i, so the E is trivial. □

Note that this is in stark contrast with the real vector bundles over the circle: the Möbius bundle is not trivial. Of course, this is a consequence of $GL_1(\mathbb{R}) = \mathbb{R}^*$ failing to be path connected.

Example 2.1.9. Suppose that G is a Lie group, H is a closed subgroup (hence itself a Lie group, see [18]), and M is a finite dimensional representation of H. From this data we can form a vector bundle over the coset space G/H, called the bundle associated to the representation M.

We define $G \times_H M$ to be the quotient of the product space $G \times M$ modulo the action $(g, m) \sim (gh, h^{-1}m)$. The projection is given by the map $\pi : (g, m) \mapsto gH$. To prove that it is locally trivial we must prove the following lemma:

Lemma 2.1.2. *Suppose that $f : M \to N$ is a submersion at $x \in M$. Then there is an open subset $U \subseteq N$ containing $f(x)$ and a smooth function $g : U \to M$ such that $f \circ g = \mathrm{id}_U$.*

Proof. By the Implicit Function Theorem (see [15]) there exist charts (φ, V) and (ϕ, U) centered at x and $f(x)$ respectively such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{R}^{n+k} & \longrightarrow & \mathbb{R}^n, \end{array}$$

where the bottom map is deletion of the last k coordinates. Thus, we take g to be the map given by $\varphi^{-1} \circ i_n \circ \phi$, where $i_n(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)$. □

Since the quotient map $\pi : G \to G/H$ is a submersion, we conclude that there exists an open set U and a section $s : U \subseteq G/H \to \pi^{-1}(U) \subseteq G$. Thus, we have that

$U \times H \cong \pi^{-1}(U)$ by the map $(u,h) \mapsto s(u)h$, which has inverse

$$\begin{aligned}\pi^{-1}(U) &\to U \times H \\ u &\mapsto (\pi(u), u \cdot (s(\pi(k)))^{-1}).\end{aligned}$$

Since $U \times M \cong (U \times H) \times_H M$ by the map $(u,m) \mapsto ((u,e),m)$, the result follows.

Remark. This construction is important in representation theory, as it defines *induction*. Indeed, if H is a closed subgroup of a Lie group G and M is a representation of H, then the sections of the bundle $G \times_H M$ form a representation of G, and we denote

$$\mathrm{Ind}_H^G(M) = \Gamma(G \times_H M).$$

See [4] for more details.

Example 2.1.10. The following example plays a major role in Chapter 5. Let $\widehat{\mathbb{Z}}^d$ denote the group of continuous group homomorphisms[1] $\mathbb{Z}^d \to \mathbb{T}$. The *Poincaré bundle* over $\mathbb{T}^d \times \widehat{\mathbb{Z}}^d$ is the bundle $\mathcal{P}_d$ defined to be the trivial bundle $(\mathbb{R}^d \times \widehat{\mathbb{Z}}^d) \times \mathbb{C}$ modulo the relation

$$(x, \chi, \lambda) \sim (x + n, \chi, \chi(n)\lambda) \quad \text{for } n \in \mathbb{Z}^d.$$

To see that this is actually a vector bundle, let $z \in \mathbb{T}^d$ and let $\exp : \mathbb{R}^d \to \mathbb{T}^d$ denote the exponential map. Choose a neighbourhood $U \subseteq \mathbb{T}^d$ containing z such that $\exp^{-1}(U)$ is a countable disjoint union of open sets, each containing a point in the integer lattice.

[1] In general, for a group G the group $\widehat{G} := \mathrm{Hom}(G, \mathbb{T})$ of continuous group homomorphisms is called the *Pontryagin dual* of G.

Let U_k denote the open set which contains $k \in \mathbb{Z}^d$. Define the map

$$\sigma : \bigsqcup_{k \in \mathbb{Z}^d} (U_k \times \widehat{\mathbb{Z}}^d) \to \mathcal{P}_d|_U$$
$$(x, \chi) \mapsto [(x, \chi, \chi(k))] \quad \text{for } x \in U_k;$$

this is continuous since the U_k are disjoint. Now, observe that if $x \in U_k$ and $n \in \mathbb{Z}^d$ then $x + n \in U_{k+n}$ so that

$$\begin{aligned}\sigma(x+n, \chi) &= [(x+n, \chi, \chi(k+n))] = [(x+n, \chi, \chi(n)\chi(k))] \\ &= [(x, \chi, \chi(k))] = \sigma(x, \chi),\end{aligned}$$

for any $\chi \in \widehat{\mathbb{Z}}^d$ because of the relation put on $\mathcal{P}_d$. Thus, σ defines a non-vanishing section $\sigma : U \times \widehat{\mathbb{Z}}^d \to \mathcal{P}_d|_{U \times \widehat{\mathbb{Z}}^d}$, which shows that $\mathcal{P}_d|_{U \times \widehat{\mathbb{Z}}^d}$ is trivial and therefore that $\mathcal{P}_d$ is a vector bundle.

2.1.2 Operations on Vector Bundles

Several of the constructions from linear algebra carry over to the setting of vector bundles, and we outline a couple of such constructions here.

Direct Sum. If V and W are vector bundles over X then we define their direct sum, $V \oplus W$, as follows. The total space (that is, the topological space without the vector bundle structure) is the set

$$\{(\xi, \eta) \in V \times W : \pi_V(\xi) = \pi_W(\eta)\} \subseteq V \times W,$$

given the subspace topology. The projection map and fibrewise vector space structures are clear. We thus have that $(V \oplus W)_x = V_x \oplus W_x$ and $V \oplus W$ locally is $U \times (\mathbb{C}^n \oplus \mathbb{C}^m)$.

Tensor Product. As one might expect, we can also take the tensor product of vector bundles over a common space. As a set, we define

$$V \otimes W := \bigsqcup_{x \in X} V_x \otimes W_x.$$

To topologize $V \otimes W$, take $U \subset X$ such that $V|U$ and $W|U$ are trivial. This yields a set bijection

$$\varphi_U : \bigsqcup_{x \in U} V_x \otimes W_x \to U \times [\mathbb{C}^n \otimes \mathbb{C}^m]$$

which is a linear isomorphism on each fiber. We topologize $V \otimes W$ so that each φ_U is a homeomorphism. Considering all such $U \subset X$ determines a basis for a topology on $V \otimes W$, making it a vector bundle.

Other Constructions from Linear Algebra. Following this trend, we can also form the following bundles:

1. $\mathrm{Hom}(E, F)$, the bundle whose fibre at $x \in X$ is $\mathrm{Hom}(E_x, F_x)$,
2. $E^* := \mathrm{Hom}(E, \mathbb{C})$, the dual bundle of E,
3. $\Lambda^*(E)$, the exterior algebra of E. This is actually a bundle of algebras,
4. $\mathbb{C}l(E)$, the complex Clifford bundle of a real vector bundle E; see Section 3.1.

For more information on a general set up for these constructions, see Section 1.2 of [1]. We note the difference between $\mathrm{Hom}(E, F)$ and $\mathrm{HOM}(E, F)$: $\mathrm{HOM}(E, F)$ is, by definition, the collection of vector bundle homomorphisms $E \to F$. The two are related by $\mathrm{HOM}(E, F) = \Gamma(\mathrm{Hom}(E, F))$.

Metrics on Vector Bundles. A metric on a vector bundle is just a continuous choice of inner product on each fibre. More specifically,

Definition 2.1.5. For a real vector bundle V over X, a (Euclidean) metric on V is a vector bundle homomorphism $V \otimes V \to X \times \mathbb{R}$ whose restriction to each fibre is symmetric and positive definite.

For a complex vector bundle E over X, a (Hermitian) metric on E is a vector homomorphism $\overline{E} \otimes E \to X \times \mathbb{C}$ whose restriction to each fibre is symmetric and positive definite; here we are denoting by $\overline{E}$ the conjugate bundle of E, that is the bundle whose total space is the same as E, but whose complex scalar multiplication is given by

$$(x + iy).\xi := (x - iy)\xi.$$

Morally, a metric on a vector bundle is a continuous choice of inner product on each fibre.

If the base space X is paracompact[2] Hausdorff, then using a partition of unity one can show that *every* vector bundle over X can be given a metric. Thus, we have, for instance,

Proposition 2.1.4. *Every exact sequence of vector bundles over a paracompact base space splits.*

Proposition 2.1.5. *If V is a vector bundle over a compact space X, then there is a vector bundle $V^\perp$ such that $V \oplus V^\perp \cong X \times \mathbb{C}^n$ for some n.*

Quotient Bundles. Suppose that V is a sub-bundle of the vector bundle W, that is a subspace of W that is itself a vector bundle; for instance, the Hopf bundle is a sub-bundle of the trivial bundle $\mathbb{C}P^1 \times \mathbb{C}^2$.

Definition 2.1.6. The quotient bundle W/V is the union of all the vector space W_x/V_x given the quotient topology.

[2]A space is called *paracompact* if there is a countable basis for the topology. Every compact space is paracompact, and every smooth manifold is paracompact, by definition. Paracompactness is a technical assumption which, together with Hausdorffness, ensures the existence of a partition of unity.

The only difficult part of showing that this is actually a vector bundle is the question of local triviality. To do this we show that any local frame for V can be (locally) extended to a frame for W. This follows from proceeding lemma about isomorphisms on vector bundles. For a proof, see Lemma 1.4.2 on p. 16 of [1].

Lemma 2.1.3. *Let Y be a closed subspace of a compact Hausdorff space X, and let V, W, be vector bundles over X. If $\varphi : V|_Y \to W|_Y$ is an isomorphism, then there exists an open set U containing Y and an extention $\tilde{\varphi} : V|_U \to W|_U$ which is an isomorphism.*

Let $\{s_1, \dots, s_k\}$ be a local frame for V over $U \subseteq X$. Then, for $x_0 \in U$, $\{s_1(x_0), \dots, s_k(x_0)\}$ is a linearly independent set in W_{x_0}, which can thus be extended to a basis

$$\{s_1(x_0), \dots s_k(x_0), t_{k+1}(x_0), \dots t_n(x_0)\} \quad \text{for} \quad W_x.$$

By Lemma 2.1.3 this frame can be extended to an open set $\tilde{U}$ containing x_0, so that $t_i : X \to W/V|_{\tilde{U}}$, $i = k+1, \dots, n$ is a local frame for W/V.

Example 2.1.11. An important consequence of this is the following special case. Suppose that $\varphi : X \to Y$ is an embedding of smooth manifolds. The *normal bundle of X in Y* is defined as the quotient bundle with fibre $T_{\varphi(x)}Y/T_xX$. This will be very important in defining the index map of Atiyah and Singer, and more generally defining wrong way functoriality for K-oriented maps. See Sections 2.5 and 3.3.

The Pullback of a Vector Bundle. For a given map $f : Y \to X$ and a vector bundle V over X, define

$$f^*V := \{(y, \xi) \in Y \times V : \xi \in V_{f(y)}\} \subseteq Y \times V.$$

Give f^*V the subspace topology, and let the projection $\pi_{f^*V} : f^*V \to Y$ be projection onto the first coordinate. Since V was a vector bundle, it is clear that each fiber has the structure of a vector space. To see that it is locally trivial, let $\{s_i\}$ be a collection of sections which trivialize V on U. Then $s' := s \circ f$ are sections of Y which trivialize f^*V on $f^{-1}(U)$. The bundle f^*V is called the pullback of V, and we record the following properties of the pullback

1. $f^*(V \oplus W) \cong f^*V \oplus f^*W$,
2. $(f \circ g)^*(V) \cong g^*f^*(V)$, and
3. $(\mathrm{id}_X)^*(V) \cong V$.

Remark. Since $(\mathrm{id}_X)^*(V) \cong V$, there is no harm in denoting points in a vector bundle V over X as pairs (x, ξ) where $\xi \in V_x$. This is often a useful book-keeping tool.

Orientations of Vector Bundles. Recall that any two frames for a vector space V are related by an element of $GL(V)$. We deem two frames equivalent if they are related by an element in $GL(V)$ with positive determinant. A choice of equivalence class is an orientation for the vector space V. An automorphism $A \in \mathrm{Aut}(V)$ is called orientation preserving if it has positive determinant. Equivalently, if A maps a positively oriented frame to a positively oriented frame.

An orientation of a vector bundle is defined in a similar manner, but one has to take into account that frames are now only available locally.

Definition 2.1.7. A vector bundle V is called *orientable* if there exists a trivializing cover $\{U_\lambda, \varphi_\lambda\}$ such that each transition function

$$\begin{array}{ccc} (U_\lambda \cap U_\eta) \times \mathbb{C}^n & \dashrightarrow & (U_\lambda \cap U_\eta) \times \mathbb{C}^n \\ & {\scriptstyle \varphi_\lambda^{-1}}\searrow^{\cong} \quad \nearrow^{\cong}{\scriptstyle \varphi_\eta} & \\ & V|_{U_\lambda \cap U_\eta} & \end{array}$$

is orientation preserving for all $U_\lambda, U_\eta \in \{U_\lambda\}$. A choice of such cover (if one exists) is called an orientation for V.

Example 2.1.12. The Möbius bundle is *not* orientable. Indeed, since $GL_+(\mathbb{R}^n)$, the collection of $n \times n$ matrices with positive determinant, is path connected, a choice of orientation would yield a non-vanishing section, none of which exist.

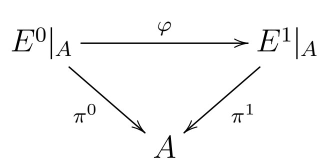
Clutching Vector Bundles. A property of vector bundles that will be useful for topological K-theory is that they can be glued over sets in which they are isomorphic. We will describe this construction now. Let $X = X^0 \cup X^1$ with X^0, X^1 open, and set $A = X^0 \cap X^1$. Suppose that E^0 is a vector bundle over X^0, E^1 is a vector bundle over X^0, and $\varphi : E^0|_A \to E^1|_A$ is an isomorphism. On $E^0 \sqcup E^1$, define $e \sim f$ if $(x, f) = (x, \varphi(x)e)$. We define

$$E^0 \cup_\varphi E^1 := E^0 \sqcup E^1 / \sim$$

equipped with the quotient topology.

Proposition 2.1.6. *As defined, $E^0 \cup_\varphi E^1$ is a vector bundle over X.*

Proof. The projection map $\pi : E^0 \cup_\varphi E^1 \to X$ comes from splicing together the two projection maps $\pi^0 : E^0 \to X$ and $\pi^1 : E^1 \to X$. It is well defined on A because the following diagram commutes



Furthermore, since φ is linear on each fiber, each fibre of $E^0 \cup_\varphi E^1$ has a well defined vector space structure.

It remains to show that $E^0 \cup_\varphi E^1$ is locally trivial. For $x \in X \setminus A$ this is clear, so suppose that $x \in A$. Since A is open we may choose an open set $U \subseteq A$ containing x for which E^0 is trivial. Since $U \subseteq A$ and $E^0 \cup_\varphi E^1|_U \cong E^0|_U$, the result follows. □

It can be shown (see [1]) that if φ^0 and φ^1 are homotopic isomorphisms, then $E^0 \cup_{\varphi^0} E^1 \cong E^0 \cup_{\varphi^1} E^1$. Thus, the isomorphism class of $E^0 \cup_\varphi E^1$ depends only on the homotopy class of the clutching function φ.

Remark. With only a bit more work, it can be shown that one can clutch vector bundles over *closed* sets as well.

2.2 Topological K-Theory

2.2.1 $\mathrm{K}^0(X)$ for Compact X

We are now able to define the K-theory of a compact Hausdorff space X. In what follows we will only consider complex K-theory, that is, all the vector bundles we will consider will be complex. In preperation, let $\mathrm{Vect}(X)$ denote isomorphism classes of complex vector bundles over X. Equipping $\mathrm{Vect}(X)$ with the direct sum operation makes it into an abelian monoid.

Definition 2.2.1. Let X be a compact Hausdorff space. The K-theory of X is defined as the Grothendieck completion of $\mathrm{Vect}(X)$. That is, $\mathrm{K}^0(X)$ is the abelian group of formal differences of isomorphism classes of (complex) vector bundles over X such that $[E^0] - [E^1] = [F^0] - [F^1]$ if and only if there is a bundle G over X such that $E^0 \oplus F^1 \oplus G \cong F^0 \oplus E^1 \oplus G$.

Remark. In general, is is not the case that the canonical map $\mathrm{Vect}(X) \to \mathrm{K}^0(X)$ is an embedding. For instance, it can be the case the non-trivial bundles are equal to trivial bundles in K-theory. Indeed, since the sphere S^2 is the boundary of the unit ball $\mathbb{D}^3$, we have that $T\mathbb{D}^3 \cong TS^2 \oplus \nu^i$, where ν^i is the inward facing normal bundle. Since $T\mathbb{D}^3$ and ν^i are trivial, we have in (real) K-Theory,

$$[TS^2] = [T\mathbb{D}^3] - [\nu^i] = [S^2 \times \mathbb{R}^2].$$

Let us begin by computing the K-theory of some basic compact spaces.

Example 2.2.1. Vector bundles over the one point space correspond to vector spaces (they are vector bundles with only one fiber). Thus, $\mathrm{Vect}(*) \cong \mathbb{N}$ so $\mathrm{K}^0(*) \cong \mathbb{Z}$.

Example 2.2.2. Since every complex bundle over S^1 and $[0,1]$ is trivial, they are in 1-1 correspondence with the positive integers, so we have that $\mathrm{K}^0(S^1) = \mathrm{K}^0([0,1]) = \mathbb{Z}$.

In order to compute the K-theory of more complicated spaces we need to note the following functorial property of K-theory. Given a map $f : Y \to X$, the pullback operation $V \mapsto f^*V$ induces a map $\mathrm{Vect}(X) \to \mathrm{Vect}(Y)$ which respects direct sums. Thus, $f : Y \to X$ induces a homomorphism $f^* : \mathrm{K}^0(X) \to \mathrm{K}^0(Y)$. This operation is functorial by the previously noted properties of the pullback.

Example 2.2.3. Let X be a locally compact space and let X^+ denote the one point compactification of X. Let $\epsilon_X : * \to X^+$ denote the map which sends a point to the point at infinity in X^+. The induced map

$$\epsilon_X^* : \mathrm{K}^0(X^+) \to \mathbb{Z}$$

is the map defined by $V \mapsto \dim(V_\infty)$. This is called the augmentation map for X.

We have to following homotopy invariance property for K-theory, which is proved in [1].

Proposition 2.2.1. *Suppose that X is a compact Hausdorff space, and suppose that $f_0, f_1 : Y \to X$ are homotopic maps. Then $f_0^* E = f_1^* E$ for all vector bundles E over X. In particular, $f_0^* = f_1^* : \mathrm{K}^0(X) \to \mathrm{K}^0(Y)$.*

Corollary 2.2.1. *If $f : Y \to X$ is a homotopy equivalence*[3] *of compact spaces, then f^* is an isomorphism.*

Example 2.2.4. Any contractible space has the same K-theory as a point, which is $\mathbb{Z}$.

Example 2.2.5. The *closed* Möbius band M is homotopy equivalent to its boundary, which is the circle. Thus,

$$\mathrm{K}^0(M) \cong \mathrm{K}^0(S^1) \cong \mathbb{Z}.$$

Although we haven't defined K-theory for locally compact spaces, it will be the case that the *open* Mobius band has different K-theory.

2.2.2 $\mathrm{K}^0(X)$ for Locally Compact X

So far we have only defined K-theory for compact Hausdorff spaces. In order to turn K-theory into a cohomology theory with a long exact sequence we need to define the higher K-groups. In order to do this we must introduce K-theory for locally compact spaces.

If X is locally compact Hausdorff recall that its *one point compactification*, denoted X^+, is the unique (up to homeomorphism) compact Hausdorff space comprised of X and a distinct point at infinity such that open sets in X are also open in X^+ and neighbourhoods of infinity are complements of compact subsets of X. K-theory for locally compact spaces is defined as the kernel of the augmentation map discussed in Example 2.2.3.

[3] A homotopy equivalence is a continuous map which is invertible up to homotopy.

Definition 2.2.2. If X is locally compact, then we define

$$\mathrm{K}^0(X) := \ker[\epsilon_X^* : \mathrm{K}^0(X^+) \to \mathbb{Z}].$$

Not that for any X we have, by definition, a short exact sequence

$$0 \longrightarrow \mathrm{K}^0(X) \longrightarrow \mathrm{K}^0(X^+) \xrightarrow{\epsilon_X^*} \mathbb{Z} \longrightarrow 0.$$

If X is compact, then $X^+ = X \sqcup \{\infty\}$, so that $\mathrm{K}^0(X^+) = \mathrm{K}^0(X) \oplus \mathrm{K}^0(*)$, which shows that we recover the original definition of K-theory for compact X.

Example 2.2.6. We have that $\mathbb{R}^+ = S^1$, so that by definition

$$\mathrm{K}^0(\mathbb{R}) := \ker[\epsilon_{\mathbb{R}}^* : \mathrm{K}^0(S^1) \to \mathbb{Z}].$$

We have seen that the augmentation map is injective (indeed, it is an isomorphism), so $\mathrm{K}^0(\mathbb{R}) = 0$.

Example 2.2.7. The one point compactification of $[0,1)$ is $[0,1]$, so as in the previous example we have $\mathrm{K}^0([0,1)) = 0$.

Remark. One reason why we don't define K-theory for locally compact spaces in terms of vector bundles over the space is for the purposes of exact sequences. For instance, later on we are going to show that we always have an exact sequence

$$0 \longrightarrow \mathrm{K}^0(X \smallsetminus A) \longrightarrow \mathrm{K}^0(X) \longrightarrow \mathrm{K}^0(A) \longrightarrow 0$$

whenever A is a closed retract of X. For the case of $X = [0,1]$, $A = \{1\}$, this gives the exact sequence

$$0 \longrightarrow \mathrm{K}^0([0,1)) \longrightarrow \mathrm{K}^0([0,1]) \longrightarrow \mathrm{K}^0(\{1\}) \longrightarrow 0.$$

Since $\mathrm{K}^0([0,1]) = \mathrm{K}^0(\{1\}) = \mathbb{Z}$, this sequence would fail to be exact if we took the standard vector bundle definition of K-theory for locally compact spaces, since in this case we would have $\mathrm{K}^0([0,1)) = \mathbb{Z}$ by homotopy invariance.

Recall that if X and Y are compact Hausdorff spaces then any map $f : Y \to X$ induces a map $f^* : \mathrm{K}^0(X) \to \mathrm{K}^0(Y)$. In order for us to get a similar property for locally compact spaces, we need to restrict ourselves to *proper* maps, that is maps $f : Y \to X$ such that $f^{-1}(K)$ is compact whenever K is. In this case, f may be extended to a continuous map $f : Y^+ \to X^+$ by sending the point at infinity to the point at infinity. In this case $f \circ \epsilon_Y = \epsilon_X$, hence, by functoriality of K^0, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{K}^0(X^+) & \xrightarrow{f^*} & \mathrm{K}^0(Y^+) \\ & {\scriptstyle \epsilon_X^*} \searrow \quad \swarrow {\scriptstyle \epsilon_Y^*} & \\ & \mathbb{Z} & \end{array}$$

which shows that f^* maps $\ker(\epsilon_X^*)$ into $\ker(\epsilon_Y^*)$. Thus, the restriction of f^* defines a map $\mathrm{K}^0(X) \to \mathrm{K}^0(Y)$.

Example 2.2.8. If A is a closed subspace of a locally compact space X, then the inclusion $A \hookrightarrow X$ induces a map $j^* : \mathrm{K}^0(X) \to \mathrm{K}^0(A)$. Indeed, the inclusion is proper if A is closed since $j^{-1}(K) = K \cap A$.

K-theory for locally compact spaces is no longer a homotopy invariant, but rather a *proper* homotopy invariant.

Proposition 2.2.2. *If $f_0, f_1 : Y \to X$ are properly homotopic, that is if there is a* *homotopy* $F : Y \times [0,1] \to X$ *between* f_0 *and* f_1 *which is a proper map, then* $f_0^* = f_1^* : \mathrm{K}^0(X) \to \mathrm{K}^0(Y)$.

Proof. We must show that a proper homotopy induces a homotopy between the extensions f_0^+ and f_1^+. This is not immediately obvious, since $(Y \times [0,1])^+$ is not equal to $Y^+ \times [0,1]$ in general. Instead, we have that $(Y \times [0,1])^+ = Y^+ \wedge [0,1]^+$, where $\wedge$ denotes the smash product[4]. This is, by definition, a quotient of $Y^+ \times [0,1]^+ = (Y^+ \times [0,1]) \sqcup Y^+$ by a certain subspace $Y^+ \vee [0,1]^+$. Define $F^\vee : Y^+ \times [0,1] \to X^+$ to be the composition

$$Y^+ \times [0,1] \hookrightarrow Y^+ \times [0,1] \sqcup Y^+ \to (Y \times [0,1])^+ \xrightarrow{F^+} X^+,$$

where we are using F^+ to denote the extension of F to the one point compactification (which exists, since F is proper). One can then check that $F^\vee$ defines a homotopy between f_0^+ and f_1^+, as required. □

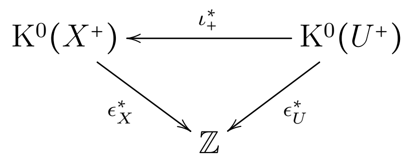
Thus, K^0 is a contravariant functor for proper maps between locally compact spaces, which is invariant with respect to proper homotopy. It turns out that K^0 is *covariant* for open inclusions, as we will now explain. Let $U \subset X$ be open and consider the inclusion $\iota : U \hookrightarrow X$. We define the map $\iota_+ : X^+ \to U^+$ by

$$\iota_+(x) = \begin{cases} x & \text{if } x \in U, \\ \infty & \text{else.} \end{cases}$$

Then ι_+ is continuous and therefore induces a map $\iota_+^* : \mathrm{K}^0(U^+) \to \mathrm{K}^0(X^+)$. We claim that $\iota_+^* : \mathrm{K}^0(U) \to \mathrm{K}^0(X)$. Indeed, this follows by noting that $\iota_+ \circ \epsilon_X = \epsilon_U$ so that, by

[4]See, for instance, [1].

functoriality, we have the commuting diagram



Definition 2.2.3. We let $\iota_!$ denote the restriction of ι_+^* to $K^0(U)$.

It is instructive to see what the map $\iota_!$ does to vector bundles. Recall that if $f : Y \to X$ is continuous and V is vector bundle over X which is trivial over U, then f^*E is trivial over $f^{-1}(U)$. Any vector bundle V over U^+ is trivial in a neighbourhood of infinity, which is the complement of a compact subset in U, say K. Since $K \subset U$, $[\iota^+]^{-1}(K) = K \subset X^+$, so that $\iota_!(V)$ will be trivial outside of K. Therefore, in a sense, $\iota_!$ extends vector bundles V over U^+ trivially to vector bundles over X^+.

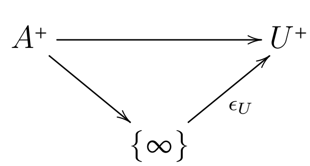
We are now in a position to get our first result towards computing K-theory of more complicated spaces.

Theorem 2.2.1. *Let X be a locally compact Hausdorff space, let $A \subseteq X$ be closed, and let $j : A \hookrightarrow X$ be the inclusion. Then the following sequence is exact*

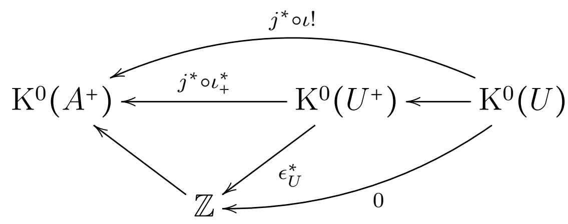
$$K^0(U) \xrightarrow{\iota_!} K^0(X) \xrightarrow{j^*} K^0(A),$$

where $U = X - A = X^+ - A^+$.

Proof. Since A is closed, the inclusion is proper so that $j^* : K^0(X) \to K^0(A)$ is well defined. The composition $\iota^+ j$ maps everything in A^+ to the point at infinity in U^+, so that we have a commutative diagram



which, by functoriality, yields the commutative diagram



This shows, in particular, that $j^* \circ \iota!$ is the zero map in K-theory. Thus, $\mathrm{Im}(\iota!) \subseteq \ker(j^*)$.

To prove the reverse inclusion, suppose that $j^*([E] - [1_n]) = 0$. This says that $[E|_{A^+}] - [1_n] = 0$, so that, by definition of the Grothendieck completion, $E|_{A^+} \oplus 1_m \cong 1_{n+m}$. In particular, $E|_{A^+} \oplus 1_m = (E \oplus 1_m)|_{A^+}$ is trivial. By Lemma 2.1.3, there exists an open subset V of X^+ which contains A^+, such that $(E \oplus 1_m)|_V$ is trivial. We observe that $X^+ \smallsetminus V$ is a compact subset of U, whence defines a neighbourhood of infinity in U^+. Using the isomorphism $E|_{X^+ \smallsetminus V} \oplus 1_m \cong 1_{n+m}$, we can clutch $(E \oplus 1_m)|_U$ with 1_{n+m} to define an element $\mathrm{K}^0(U)$, which we denote $[(E \oplus 1_m)|_U] - [1_{n+m}]$. One has that $\iota_!([E \oplus 1_m]|_U - [1_{n+m}]) = [E] - [1_n]$, whence the claim is proved. □

2.2.3 Higher K-theory and the Long Exact Sequence

We can now define the higher K-groups.

Definition 2.2.4. For X locally compact and $n \geq 1$, define

$$\mathrm{K}^{-n}(X) := \mathrm{K}^0(X \times \mathbb{R}^n).$$

Example 2.2.9. $\mathrm{K}^{-1}(*) = \mathrm{K}^0(\mathbb{R}) = 0$.

Since Theorem 2.2.1 holds for all locally compact spaces, it is readily observed that for each n we have an exact sequence

$$\mathrm{K}^{-n}(U) \xrightarrow{\iota!} \mathrm{K}^{-n}(X) \xrightarrow{j^*} \mathrm{K}^{-n}(A).$$

It is shown in [22] (and [1]) that there is a connecting morphism which gives us a long exact sequence.

Theorem 2.2.2. *There is a natural map* $\delta : \mathrm{K}^{-n}(A) \to \mathrm{K}^{-n+1}(U)$ *which fits into the following sequence, making it exact.*

$$\cdots \longrightarrow \mathrm{K}^{-n}(X) \longrightarrow \mathrm{K}^{-n}(A) \xrightarrow{\delta} \mathrm{K}^{-n+1}(U) \longrightarrow \mathrm{K}^{-n+1}(X) \longrightarrow \cdots$$

We will collect some facts which readily follow from this exact sequence.

Corollary 2.2.2 (Split Exactness of K-Theory)**.** *Let A be a closed subspace of the locally compact space X. Suppose that there is a proper map $\psi : X \to A$ such that $j \circ \psi = \mathrm{id}_A$. Then*

$$0 \longrightarrow \mathrm{K}^{-n}(X - A) \xrightarrow{\iota!} \mathrm{K}^{-n}(X) \underset{\psi*}{\overset{j*}{\rightleftarrows}} \mathrm{K}^{-n}(A) \longrightarrow 0$$

is split exact.

Proof. Since $j \circ \psi = \mathrm{id}_A$ it follows that j^* is surjective, so it remains to show that $\iota_!$ is injective. By the same argument, we have that $j^* : \mathrm{K}^{-n-1}(X) \to \mathrm{K}^{-n-1}(A)$ is also surjective, so that the connecting map $\delta : \mathrm{K}^{-n-1}(A) \to \mathrm{K}^{-n}(X \setminus A)$ is the zero map by exactness. Since the sequence is exact, $\ker(\iota_!) = \mathrm{Im}(\delta) = \{0\}$ which completes the proof. □

Corollary 2.2.3. *If X is a locally compact space, then* $\mathrm{K}^{-1}(X) = \mathrm{K}^{-1}(X^+)$.

Proof. The map $X^+ \to \{\infty\}$ is a retract so by the split exactness of K-theory we have

$$0 \longrightarrow \mathrm{K}^{-1}(X) \xrightarrow{\iota!} \mathrm{K}^{-1}(X^+) \underset{\psi *}{\overset{j*}{\rightleftarrows}} \mathrm{K}^{-1}(*) \longrightarrow 0.$$

Since $\mathrm{K}^{-1}(*) = 0$, the result follows. □

2.2.4 The Unitary Picture of K^{-1}

Before we move on we want to give an alternative description of $\mathrm{K}^{-1}(X)$ for compact spaces X that is nice for computation purposes.

Let $[X, GL_n(\mathbb{C})]$ denote the collection of homotopy classes of maps $X \to GL_n(\mathbb{C})$. Observe that we have a sequence

$$[X, GL_1(\mathbb{C})] \longrightarrow [X, GL_2(\mathbb{C})] \longrightarrow [X, GL_3(\mathbb{C})] \longrightarrow \cdots \longrightarrow [X, GL_n(\mathbb{C})] \longrightarrow \cdots,$$

induced by the maps

$$GL_n(\mathbb{C}) \to GL_{n+1}(\mathbb{C})$$
$$a \mapsto a \oplus 1,$$

so we define $[X, GL_\infty(\mathbb{C})] := \lim_{n\to\infty}[X, GL_n(\mathbb{C})]$. Then, $[X, GL_\infty(\mathbb{C})]$ has the structure of an abelian group under the operations

1. $[\mathrm{id}_n] = 0$ for any n, and

2. $[\varphi] + [\phi] = [\varphi \oplus \phi]$.

To see that this is actually an abelian group we need to record the following lemma; for a proof see [20].

Lemma 2.2.1. *For and $A, B \in GL_n(\mathbb{C})$ the matrices*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} BA & 0 \\ 0 & 1_n \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} AB & 0 \\ 0 & 1_n \end{pmatrix}$$

are connected by a path in $GL_n(\mathbb{C})$.

We have the following characterization of $\mathrm{K}^{-1}(X)$, for compact X.

Theorem 2.2.3. *For compact X,* $\mathrm{K}^{-1}(X) \cong [X, GL_\infty(\mathbb{C})]$.

The main idea of the proof we will give is due to Atiyah, see [1].

Proof. Write $[X \times \mathbb{R}]^+ = [X \times (-1,1)]^+ = C^+(X) \cup C^-(X)$, where

$$C^+(X) := [X \times (-1, 0]]^+ = X \times [-1, 0]/X \times \{-1\} \quad \text{and}$$
$$C^-(X) := [X \times [0, 1)]^+ = X \times [0, 1]/X \times \{1\}.$$

Then, $C^+(X) \cap C^-(X) = X \sqcup \{\infty\} = X^+$. Note that since X is compact $[X, GL_\infty(\mathbb{C})] = [X^+, GL_\infty(\mathbb{C})]$, so that $\varphi \in [X, GL_n(\mathbb{C})]$ may be used to clutch the product bundles $C^+(X) \times \mathbb{C}^n$ and $C^-(X) \times \mathbb{C}^n$ over X^+. Since homotopic clutching functions yield isomorphic vector bundles, this determines a well defined map $[X, GL_\infty(\mathbb{C})] \to K^{-1}(X) = K^0(X \times \mathbb{R})$ by

$$\varphi \mapsto [C^+(X) \times \mathbb{C}^n] \cup_\varphi [C^-(X) \times \mathbb{C}^n] - 1_n.$$

We will now construct an inverse to this map. Let E be a vector bundle over $[X \times \mathbb{R}]^+$. Since $C^\pm(X)$ are contractible we have that $E|_{C^\pm(X)}$ are trivial with the same rank (since $[X \times \mathbb{R}]^+$ is connected, being the union of two non-disjoint contractible subspaces). Let $\alpha^\pm : E|_{C^\pm(X)} \to C^\pm(X) \times \mathbb{C}^n$ be an isomorphism. We may think of the

$\alpha^{\pm}$ as maps $C^{\pm}(X) \to GL_n(\mathbb{C})$, so that since $C^{\pm}(X)$ are contractible and $GL_n(\mathbb{C})$ is path connected, the $\alpha^{\pm}$ are unique up to homotopy. Thus, $(\alpha^{+}|X)^{-1}(\alpha^{-}|X) : X \to GL_n(\mathbb{C})$ is unique up to homotopy, which defines a map $\mathrm{Vect}([X \times \mathbb{R}]^{+}) \to [X, GL_{\infty}(\mathbb{C})]$. By the universal property of the Grothendieck completion this induces a map on K-theory, which is readily seen to be inverse to the clutching construction.

□

Remark. It would be desirable to extend this result to the locally compact case, but unfortunately it is not true. Indeed, if the result held for non-compact spaces, then since $\mathbb{R}$ is contractible and $GL_{\infty}(\mathbb{C})$ is path connected, $0 = [\mathbb{R}, GL_{\infty}(\mathbb{C})] = \mathrm{K}^{-1}(\mathbb{R})$, which is bogus since $\mathrm{K}^{-1}(\mathbb{R}) = \mathbb{Z}$ by Bott Periodicity, as we shall see.

For the purpose of interest, we will now give an explicit description of the connecting map in certain situations. First, we need the following extension result.

Lemma 2.2.2. *Let $f_0, f_1 : A^{+} \to GL_n(\mathbb{C})$ be homotopic maps. Then there exist homotopic extensions $f_0, f_1 : X^{+} \to M_n(\mathbb{C})$ and an open subset $V \subset X^{+}$ containing A^{+} such that $f_0, f_1 : V \to GL_n(\mathbb{C})$.*

Proof. Let $f : A^{+} \times [0,1] \to GL_n(\mathbb{C}) \subset M_n(\mathbb{C})$ be a homotopy between f_0 and f_1. By the Tietze Extension Theorem we may extend f to a map $\tilde{f} : X^{+} \times [0,1] \to M_n(\mathbb{C})$. Thus, $\tilde{f}_0$ and $\tilde{f}_1$ provide the homotopic extensions.

We now show the existence of V. Since $GL_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$ and f is continuous, for each $(x,t) \in X^{+} \times [0,1]$ such that $f(x,t) \in GL_n(\mathbb{C})$ there exists a neighbourhood $U_{(x,t)}$ of (x,t) such that $f(U_{(x,t)}) \subseteq GL_n(\mathbb{C})$. Because $f(x,t) \in GL_n(\mathbb{C})$ for all $(x,t) \in A^{+} \times [0,1]$ it follows that

$$\bigcup_{(x,t) \in f^{-1}(GL_n(\mathbb{C}))} U_{(x,t)}$$

is open and contains $A^{+} \times [0,1]$. □

Now, let X be locally compact and $A \subseteq X$ be closed. Let $\varphi \in \mathrm{K}^{-1}(A) \cong K^{-1}(A^+) \cong [A^+, GL_\infty(\mathbb{C})]$. Then $\varphi : A^+ \to GL_n(\mathbb{C})$ for some n and by the previous Lemma, we may extend φ to a map $\varphi : X^+ \to M_n(\mathbb{C})$ in a way which depends only on the homotopy class of φ. We may consider φ as a vector bundle map $X^+ \times \mathbb{C}^n \to X^+ \times \mathbb{C}^n$ by $(x, v) \mapsto (x, \varphi(x)v)$. Furthermore, by the previous Lemma again, there is an open subset V such that $\varphi|_V$ is an isomorphism. Using this, we are going to define an element of $\mathrm{K}^0(U)$.

Let $K = X^+ - V \subseteq U$, let $T = U^+ - K$, and let $B = U \subseteq U^+$. Then, $\varphi|_{T\cap B} : (T \cap B) \times \mathbb{C}^n \to (T \cap B) \times \mathbb{C}^n$ is an isomorphism since $T \cap B = V \cap U \subseteq V$. Thus, we may use φ to clutch $T \times \mathbb{C}^n$ with $B \times \mathbb{C}^n$ over $T \cap B$. This yields an element of $\mathrm{K}^0(U)$ given by $(T \times \mathbb{C}^n) \cup_\varphi (B \times \mathbb{C}^n) - 1_n$.

Proposition 2.2.3. *The connecting map $\partial : \mathrm{K}^1(A) \to \mathrm{K}^0(X - A)$ from the long exact sequence agrees with the map*

$$\partial(\varphi) = (T \times \mathbb{C}^n) \cup_\varphi (B \times \mathbb{C}^n) - 1_n,$$

where T and B are as above.

For a reference, see [9].

2.3 Complexes and the Thom Isomorphism

In this section we develop another description of K-theory using *complexes* of vector bundles over a space X. This is a more natural definition for K-theory of locally compact spaces, as the definition using complexes is the same for compact and locally compact spaces. Also, complexes of vector bundles arise naturally in practice, an instance of this being seen in Section 2.5. Finally, another benefit of using complexes to define K-theory is that it allows us to define an exterior product

$$\mathrm{K}^{-n}(X) \times \mathrm{K}^{-m}(Y) \to \mathrm{K}^{-n-m}(X \times Y),$$

and therefore a graded ring structure on $\mathrm{K}^*(X)$.

For certain vector bundles there will be a distinguished class in their K-theory, and exterior multiplication by this element defines a homomorphism called the Thom Homomorphism. The fundamental theorem of K-theory says that this map is actually an isomorphism. Let's begin.

Definition 2.3.1. A *complex* (of complex vector bundles) over a space X is a sequence

$$V^\bullet : \cdots \xrightarrow{d} V^{i-1} \xrightarrow{d} V^i \xrightarrow{d} V^{i+1} \xrightarrow{d} \cdots$$

of complex vector bundles V^i over X such that

1. $d^2 = 0$, and
2. at most finitely many of the V^i are non-zero.

A *morphism* $f : V^\bullet \to W^\bullet$ between complexes is a sequence of vector bundle maps $f^i : V^i \to W^i$ which commute with the differential maps; that is, $df^{i+1} = f^i d$ for all i.

The *support* of a complex is the closure of the set of points $x \in X$ such that the sequence

$$\cdots \xrightarrow{d_x} V_x^{i-1} \xrightarrow{d_x} V_x^i \xrightarrow{d_x} V_x^{i+1} \xrightarrow{d} \cdots$$

of vector spaces is *not* exact. A complex $V^\bullet$ is called *acyclic* if its support is empty.

In the notation of Segal [22], let $L(X)$ denote compactly supported complexes over X. We may add complexes together in the obvious way, and this endows $L(X)$ with the structure of an abelian monoid. A *homotopy* between compactly supported complexes $V_0^\bullet$ and $V_1^\bullet$ over X is an compactly supported complex $V^\bullet$ over $X \times [0,1]$ such that

$$V_0^\bullet = V^\bullet|_{X\times\{0\}} \quad \text{and} \quad V_1^\bullet = V^\bullet|_{X\times\{1\}}.$$

We write $V_0^\bullet \simeq V_1^\bullet$ if there exists a homotopy between $V_0^\bullet$ and $V_1^\bullet$. We define compactly supported complexes $V_0^\bullet$ and $V_1^\bullet$ to be *equivalent* if there exist acyclic complexes $W_0^\bullet$ and $W_1^\bullet$ such that $V_0^\bullet \oplus W_0^\bullet \simeq V_1^\bullet \oplus W_1^\bullet$.

In [22], it is proved that

Theorem 2.3.1. *If X is a locally compact Hausdorff space, the quotient $L(X)/\sim$ is an abelian group naturally isomorphic to $\mathrm{K}^0(X)$. Thus, $L(X\times\mathbb{R}^n)/\sim\,\cong \mathrm{K}^{-n}(X)$ for all $n \in \mathbb{N}$.*

It is clear that the direct sum of complexes gives $L(X)/\sim$ the structure of an abelian monoid. Every acyclic complex defines the class of the identity, and the inverse of a complex $V^\bullet$ is given by the complex $TV^\bullet$ defined by $(TV)^i = V^{i+1}$.

We don't prove Theorem 2.3.1, but we will describe the map. It can be shown that it suffices to consider complexes of only two vector bundles, so we restrict our attention to this case. If X is compact, then the map is simply

$$V^\bullet \mapsto [V^0] - [V^1] \in \mathrm{K}^0(X).$$

For locally compact X, let $V^\bullet$ be a compactly supported complex over X. Find an open set U which contains $\mathrm{supp}(V^\bullet)$ and has compact closure. Since $\overline{U}$ is compact, there is a bundle $V_\perp^1$ such that $V^1 \oplus V_\perp^1$ is trivial. Since

$$0 \longrightarrow V_\perp^1 \xrightarrow{\mathrm{id}} V_\perp^1 \longrightarrow 0$$

is an acyclic, we may add it to $V^\bullet$ to get a two-term complex whose second term is trivial.

We are going to show how $V^\bullet|_U$ determines a class in $\mathrm{K}^0(U)$, and then use wrong way functoriality to push the class forward into $\mathrm{K}^0(X)$. Since $V^\bullet|_U$ is of the form

$$0 \to V^0|_U \xrightarrow{\varphi} 1_n \longrightarrow 0,$$

it follows that φ is an isomorphism outside of a compact set. Using this isomorphism we can extend $V^0|_U$ to a vector bundle over U^+ by clutching over the complement of the support. Thus defines a complex $V^\bullet|_{U^+}$ over U^+, and therefore a class in $K^0(U^+)$. It is readily observed that the class is in the kernel of the augmentation map for U. This defines a class in $\mathrm{K}^0(U)$, and we push it forward to get a class in $\mathrm{K}^0(X)$. Since the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}^0(W) & \xrightarrow{\iota_!} & \mathrm{K}^0(X) \\ \uparrow{\scriptstyle \iota_!} & \nearrow{\scriptstyle \iota_!} & \\ \mathrm{K}^0(U) & & \end{array}$$

for any $U \subseteq W \subseteq X$ open, it follows that the choice of U does not matter.

2.3.1 The Exterior Product and Bott Periodicity

As mentioned, we can use complexes to define an exterior multiplication in K-theory. Define the tensor product of complexes $E^\bullet$ and $F^\bullet$ to be the complex $E^\bullet \otimes F^\bullet$ with

$$(E^\bullet \otimes F^\bullet)^k = \bigoplus_{p+q=k} E^p \otimes F^q,$$

and differential is given by

$$d_i = \bigoplus_{p+q=k} \left(\left(d_p^E \otimes 1_{F^q} \right) + (-1)^p \left(1_{E^p} \otimes d_q^F \right) \right);$$

the $(-1)^p$ is needed so that it squares to zero.

We define the exterior product in K-theory as follows. Let X and Y be locally compact spaces and let $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ denote the projections onto X and Y, respectively. Exterior multiplication is defined to be the map defined on complexes by

$$\begin{aligned} \hat{\otimes} : \mathrm{K}^{-n}(X) \times \mathrm{K}^{-m}(Y) &\to \mathrm{K}^{-n-m}(X \times Y) \\ (E^\bullet, F^\bullet) &\mapsto \pi_X^*(E^\bullet) \otimes \pi_Y^*(F^\bullet). \end{aligned}$$

We have that $\mathrm{supp}(E^\bullet \otimes F^\bullet) = \mathrm{supp}(E^\bullet) \cap \mathrm{supp}(F^\bullet)$, whence this yields a compactly supported complex over $X \times Y \times \mathbb{R}^{n+m}$.

If $\delta : X \to X \times X$ is the diagonal map, then the composition

$$\mathrm{K}^{-n}(X) \otimes \mathrm{K}^{-m}(X) \xrightarrow{\hat{\otimes}} \mathrm{K}^{-n-m}(X \times X) \xrightarrow{\delta^*} \mathrm{K}^{-n-m}(X)$$

defines a graded ring structure on K-theory. If X is compact, then it is observed that it extends the ring structure on K^0 coming from tensor products of vector bundles.

One important feature of topological K-theory is that it is periodic in the sense that for all locally compact spaces there is a natural isomorphism $\mathrm{K}^{-n}(X) \to \mathrm{K}^{-n-2}(X)$ given by exterior multiplication by a certain class $\beta \in \mathrm{K}^0(\mathbb{R}^2)$ called the *Bott class*.

Definition 2.3.2. The class of the complex

$$\beta : 0 \longrightarrow \mathbb{R}^2 \times \mathbb{C} \xrightarrow{c} \mathbb{R}^2 \times \mathbb{C} \longrightarrow 0,$$

where $c((x,y),z) = (x+iy)z$ is called the Bott class. The map given by

$$\beta : \mathrm{K}^{-n}(X) \to \mathrm{K}^{-n-2}(X)$$
$$\eta \mapsto \eta \hat{\otimes} \beta$$

is called the Bott map.

Example 2.3.1. The Bott class can be represented by the formal difference $[H]-[1] \in \mathrm{K}^0(\mathbb{R}^2)$, where H is the Hopf bundle on $\mathbb{C}P^1 = S^2 = (\mathbb{R}^2)^+$, defined in Example 2.1.6.

With this in hand, the Bott Periodicity Theorem is the following.

Theorem 2.3.2 (Bott Periodicity)**.** *The Bott homomorphism*

$$\beta : K^{-n}(X) \to K^{-n-2}(X)$$

is an isomorphism.

Example 2.3.2. Bott Periodicity shows that $\mathrm{K}^0(\mathbb{R}^2) = \mathbb{Z}$ generated by β and $\mathrm{K}^{-1}(\mathbb{R}^2) = \mathrm{K}^{-2}(\mathbb{R}) \cong \mathrm{K}^0(\mathbb{R}) = 0$. Since $S^2 = (\mathbb{R}^2)^+$, we have, by definition, the exact sequence

$$0 \longrightarrow \mathrm{K}^0(\mathbb{R}^2) \longrightarrow \mathrm{K}^0(S^2) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

whence $\mathrm{K}^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$, generated by the trivial bundle of rank 1 and the class of the Hopf Bundle $[H]$.

It can be shown that the Bott map commutes with all the maps in the long exact sequence, so that it truncates to a six term exact sequence

$$\begin{array}{ccccc} K^0(X-A) & \xrightarrow{\iota!} & K^0(X) & \xrightarrow{j^*} & K^0(A) \\ \uparrow{\scriptstyle \partial} & & & & \downarrow{\scriptstyle \partial\beta} \\ K^{-1}(A) & \xleftarrow[j^*]{} & K^{-1}(X) & \xleftarrow[\iota!]{} & K^{-1}(X-A) \end{array}$$

which is a powerful tool that allows K-theory to be computed.

In the next section we give a sort of generalization of Bott periodicity.

2.3.2 The Thom Isomorphism

We will now give another very important theorem, called the Thom isomorphism theorem.

Theorem 2.3.3 (Thom Isomorphism Theorem I). *Suppose that V is a complex vector bundle over X. Then there is a complex $\tau_V^\bullet$ over V such that exterior multiplication by $\tau_V^\bullet$ induces an isomorphism*

$$\tau_*^V : \mathrm{K}^*(X) \cong \mathrm{K}^*(V).$$

One sees this to be a generalization of Bott periodicity when one considers $X \times \mathbb{R}^2$ to be a complex vector bundle over X of complex rank 1.

We will sketch a proof of this result (in fact, a slightly more general result) later, but for now we will describe the complex τ_V. Let V be a complex vector bundle and let s be some section of V. Consider the complex[5]

$$0 \longrightarrow \mathbb{C} \xrightarrow{\wedge s} \Lambda^1_{\mathbb{C}} V \xrightarrow{\wedge s} \Lambda^2_{\mathbb{C}} V \xrightarrow{\wedge s} \cdots,$$

where $\xi \wedge s := \xi \wedge s(x)$ for $\xi \in \Lambda^i_{\mathbb{C}} V_x$. The support of this complex is precisely the zero set of s. If $\pi_V : V \to X$ denotes the bundle projection, then the diagonal map $\delta_V : V \to \pi_V^* V$ is a section whose set of zeros is precisely the zero section of V. We let $\tau_V^\bullet$ denote the complex formed above with $\pi_V^* V$ and δ_V. Its support is isomorphic to X, so only defines a class in $\mathrm{K}^0(V)$ when X is compact. However, even when X is

[5] for a vector bundle V, $\Lambda^*_{\mathbb{C}} V$ denotes the bundle of exterior algebras of the fibres of V.

not compact, τ_V may be exterior multiplied with classes in $\mathrm{K}^*(X)$ to give compactly supported complexes in $\mathrm{K}^*(V)$.

2.4 Computation of Some K-Groups

Example 2.4.1 ($\mathbb{R}^n$)**.** All of the groups $\mathrm{K}^0(\mathbb{R}^n)$ can be computed by Bott periodicity and depend only on the parity of n. We have

$$\mathrm{K}^0(\mathbb{R}^n) = \mathrm{K}^{-n}(*) = \begin{cases} \mathbb{Z} & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1. \end{cases}$$

and

$$\mathrm{K}^{-1}(\mathbb{R}^n) = \mathrm{K}^{-n-1}(*) = \begin{cases} \mathbb{Z} & \text{if } n = 2k+1 \\ 0 & \text{if } n = 2k. \end{cases}$$

Example 2.4.2 (S^n)**.** Since $S^n = (\mathbb{R}^n)^+$ we have, by definition, an exact sequence

$$0 \longrightarrow \mathrm{K}^0(\mathbb{R}^n) \longrightarrow \mathrm{K}^0(S^n) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Thus we have $\mathrm{K}^0(S^n) = \mathrm{K}^0(\mathbb{R}^n) \oplus \mathbb{Z}$. Furthermore, since $\mathrm{K}^{-1}(X) = \mathrm{K}^{-1}(X^+)$, we have $\mathrm{K}^{-1}(S^n) = \mathrm{K}^0(\mathbb{R}^{n+1})$.

Example 2.4.3 ($\mathbb{T}^n$)**.** Let X be compact Hausdorff and consider the space $X \times S^1$. Let $j : X \times \{1\} \hookrightarrow X \times S^1$ be the inclusion. Then the map $\mathrm{pr}_1 : X \times S^1 \to X \times \{1\}$ induces a splitting

$$0 \longrightarrow \mathrm{K}^0(X \times \mathbb{R}) \xrightarrow{\iota!} \mathrm{K}^0(X \times S^1) \underset{\psi*}{\overset{j*}{\rightleftarrows}} \mathrm{K}^0(X) \longrightarrow 0$$

by the split exactness of K^0. Hence $\mathrm{K}^0(X \times S^1) = \mathrm{K}^0(X \times \mathbb{R}) \oplus \mathrm{K}^0(X) = K^{-1}(X) \oplus$

$\mathrm{K}^0(X)$. Replacing X with $X \times \mathbb{R}$ we obtain

$$\mathrm{K}^{-1}(X \times S^1) = \mathrm{K}^0(X \times \mathbb{R} \times S^1) = \mathrm{K}^0(X \times \mathbb{R}^2) \oplus \mathrm{K}^0(X \times \mathbb{R}) = \mathrm{K}^0(X) \oplus \mathrm{K}^{-1}(X).$$

Putting this together, we see that $\mathrm{K}^*(X \times S^1) = \mathrm{K}^0(X) \oplus \mathrm{K}^0(X) \oplus \mathrm{K}^{-1}(X) \oplus \mathrm{K}^{-1}(X)$.

As a consequence of this, we see by induction that $\mathrm{K}^*(\mathbb{T}^n) = \mathbb{Z}^{2^n}$, where $\mathbb{T}^n$ is the n-torus.

In what follows, we use the notation $[V, W, \varphi]$ to denote the class of the complex

$$0 \longrightarrow V \xrightarrow{\varphi} W \longrightarrow 0$$

in K-theory.

Example 2.4.4 ($\mathbb{R}P^2$)**.** Think of $\mathbb{R}P^2$ as the closed disk in $\mathbb{R}^2$ with antipodal points on the boundary identified. In particular, we may think of $\mathbb{R}P^1$ as a subspace of $\mathbb{R}P^2$ whose complement in $\mathbb{R}P^2$ is the open disk $\mathbb{D}$. The six term exact sequence is then

$$\begin{array}{ccccc} \mathrm{K}^0(\mathbb{D}) & \xrightarrow{\iota!} & \mathrm{K}^0(\mathbb{R}P^2) & \xrightarrow{j^*} & \mathrm{K}^0(\mathbb{R}P^1) \\ \uparrow{\scriptstyle\partial} & & & & \downarrow{\scriptstyle\partial\beta} \\ \mathrm{K}^{-1}(\mathbb{R}P^1) & \xleftarrow[j^*]{} & \mathrm{K}^{-1}(\mathbb{R}P^2) & \xleftarrow[\iota!]{} & \mathrm{K}^{-1}(\mathbb{D}) \end{array}$$

We begin by making some preliminary calculations.

1. First, $\mathbb{D} \cong \mathbb{R}^2$, so $\mathrm{K}^{-1}(\mathbb{D}) = 0$ and $\mathrm{K}^0(\mathbb{D})$ is generated by the triple $[1, 1, z]$, as this represents the Bott class for $\mathbb{R}^2$.

2. The map $[z] \mapsto z^2$ provides a homeomorphism between $\mathbb{R}P^1$ and S^1 hence induces an isomorphism $\mathrm{K}^0(S^1) \to \mathrm{K}^0(\mathbb{R}P^1)$ and $[S^1, GL_\infty(\mathbb{C})] = \mathrm{K}^{-1}(S^1) \to \mathrm{K}^{-1}(\mathbb{R}P^1) = [\mathbb{R}P^1, GL_\infty(\mathbb{C})]$. In particular, since $[z]$ generates $[S^1, GL_\infty(\mathbb{C})]$, $[z^2]$ generates $[\mathbb{R}P^1, GL_\infty(\mathbb{C})]$.

With this is mind, the six term exact sequence reduces to

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\iota!} & \mathrm{K}^0(\mathbb{R}P^2) & \xrightarrow{j^*} & \mathbb{Z} \\ {\scriptstyle \partial}\uparrow & & & & \downarrow{\scriptstyle \partial\beta} \\ \mathbb{Z} & \xleftarrow[j^*]{} & \mathrm{K}^{-1}(\mathbb{R}P^2) & \xleftarrow[\iota!]{} & 0 \end{array}$$

We must compute the map $\partial : \mathbb{Z} \to \mathbb{Z}$, which we can do by seeing where it sends $[z^2]$. We compute

$$\begin{aligned} \partial([z^2]) &= [1,1,z^2] = [1,1,z^2] + [1,1,\mathrm{id}] \quad \text{since } [1,1,\mathrm{id}] \text{ is acyclic,} \\ &= [1_2,1_2,z^2 \oplus \mathrm{id}] = [1_2,1_2,z \oplus z] \quad \text{homotopy invariance,} \\ &= [1,1,z] + [1,1,z]. \end{aligned}$$

Thus, ∂ maps a generator of $\mathrm{K}^{-1}(\mathbb{R}P^1)$ to twice a generator of $\mathrm{K}^0(\mathbb{D})$ so, at the level of the integers, ∂ is multiplication by 2 and is in particular injective. Thus, $\mathrm{K}^{-1}(\mathbb{R}P^1) = 0$. Since $\ker(\iota) = 2\mathbb{Z}$ we get the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{K}^0(\mathbb{R}P^2) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

whence $K^0(\mathbb{R}P^2) = \mathbb{Z} \oplus \mathbb{Z}_2$.

Example 2.4.5. Declare two points $z, w \in \partial\mathbb{D} = S^1$ equivalent is there exists α such that $\alpha^n + 1 = 0$ and $z = \alpha w$. Then by an analogous calculation as above we see that $K^0(\mathbb{D}/\sim) = \mathbb{Z} \oplus \mathbb{Z}_n$ and $K^1(\mathbb{D}/\sim) = 0$.

Example 2.4.6. The real projective plane $\mathbb{R}P^2$ is the one point compactification of the open Möbius band $\ddot{M}$. Thus, $\mathrm{K}^0(\ddot{M}) = \mathbb{Z}_2$ and $\mathrm{K}^{-1}(\ddot{M})$=0.

Example 2.4.7 (The Klein Bottle)**.** Let $\mathcal{K}$ denote the Klein Bottle. Topologically, $\mathcal{K}$ is the union of a circle and an open Möbius band, so we obtain the six term exact sequence

$$\begin{array}{ccccc} \mathbb{Z}_2 & \longrightarrow & \mathrm{K}^0(\mathcal{K}) & \longrightarrow & \mathbb{Z} \\ \partial \big\uparrow & & & & \big\downarrow \partial\beta \\ \mathbb{Z} & \longleftarrow & \mathrm{K}^{-1}(\mathcal{K}) & \longleftarrow & 0 \end{array}$$

We claim that the boundary map on the left hand side is the zero map. Indeed, the canonical map $M \to \mathcal{K}$ from the *closed* Möbius band M to the Klein bottle induces a commuting diagram

$$\begin{array}{ccccccccc} \overbrace{\mathrm{K}^{-1}(\ddot{M})}^{=0} & \xrightarrow{0} & \overbrace{\mathrm{K}^{-1}(M)}^{=\mathbb{Z}} & \xrightarrow{\times 2} & \overbrace{\mathrm{K}^{-1}(S^1)}^{=\mathbb{Z}} & \xrightarrow{1\mapsto[1]} & \overbrace{\mathrm{K}^0(\ddot{M})}^{=\mathbb{Z}_2} & \xrightarrow{0} & \overbrace{\mathrm{K}^0(M)}^{=\mathbb{Z}} \\ \Big\| & & \Big\uparrow & & \times 2 \Big\uparrow & & \Big\| & & \Big\uparrow \\ \mathrm{K}^{-1}(\ddot{M}) & \longrightarrow & \mathrm{K}^{-1}(\mathcal{K}) & \longrightarrow & \mathrm{K}^{-1}(S^1) & \xrightarrow[\partial]{} & \mathrm{K}^0(\ddot{M}) & \xrightarrow{0} & \mathrm{K}^0(\mathcal{K}), \end{array}$$

from which it is clear that $\partial = 0$. Thus, from the six term exact sequence

$$\mathrm{K}^0(\mathcal{K}) = \mathbb{Z} \oplus \mathbb{Z}_2 \quad \text{and} \quad \mathrm{K}^{-1}(\mathcal{K}) = \mathbb{Z}.$$

2.5 The Atiyah-Singer Index Theorem

One of the great triumphs of topological K-theory is the Atiyah-Singer Index theorem, which asserts that the index of an elliptic differential operator on a compact manifold can be computed purely from K-theory considerations. In this section, we spell out what this theorem says, omitting many details.

2.5.1 Differential Operators

Definition 2.5.1. A *differential operator* of order m on $\mathbb{R}^n$ is a linear operator $L : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ which acts on functions as

$$(Lf)(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha f(x).$$

Here $\alpha \in \mathbb{N}^n$ is a multi-index, with $|\alpha| = \sum \alpha_i$, and

$$D^\alpha = (-i)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Closely related to a differential operator L is its symbol. To motivate the definition, recall the Fourier inversion formula: for $f \in C_c^\infty(\mathbb{R}^n)$ we have

$$f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi.$$

Recall as well that $\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$. With this, we have

$$\begin{aligned}(Lf)(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha f(x) &= \sum_{|\alpha|\leq m} a_\alpha(x) \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{D^\alpha f}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\sum_{|\alpha|\leq m} a_\alpha(x) \xi^\alpha \right) \hat{f}(\xi) d\xi\end{aligned}$$

Definition 2.5.2. Let $L = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be a differential operator on $\mathbb{R}^n$. Its *symbol* is the polynomial function on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$\sigma_L(x, \xi) = \sum_{|\alpha|\leq m} a_\alpha(x) \xi^\alpha;$$

the *principal symbol* is the top order terms of the symbol, that is,

$$\sigma_L^p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

A differential operator L is called *elliptic* if the principal symbol is non-zero for all (x,ξ) with $\xi \neq 0$.

We interpret the principal symbol as a map $\sigma_L^p : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{End}(\mathbb{C})$. Ellipticity says that the symbol lands in the invertible linear maps for all (x,ξ) with $\xi \neq 0$.

More generally, if V is any finite dimensional vector space, then the partial differential operators ∂_j are defined using the usual limit definition. If V_0 and V_1 are finite dimensional complex vector spaces then, as before, we call a linear operator $L : C_c^\infty(\mathbb{R}^n, V_0) \to C_c^\infty(\mathbb{R}^n, V_1)$ a differential operator of order m if it has the form

$$(Lf)(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha f(x),$$

where the $a_\alpha : \mathbb{R}^n \to \mathrm{Hom}(V_0, V_1)$ are smooth. The principal symbol, as before, is defined to be function

$$\sigma_L^p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha \in \mathrm{Hom}(V_0, V_1).$$

A differential operator $L : C_c^\infty(\mathbb{R}^n, V_0) \to C_c^\infty(\mathbb{R}^n, V_1)$ is called elliptic if $\sigma_L^p(x,\xi)$ is invertible for all (x,ξ) with $\xi \neq 0$.

We now want to extend this discussion to the case the $\mathbb{R}^n$ is replaced with an arbitrary smooth manifold M and the vector spaces V_0, V_1 are replaced with vector bundles S^0 and S^1 over M. In this setting, the principal symbol of a differential operator will be a section $\sigma^p : T^*M \to \mathrm{Hom}(\pi^* S^0, \pi^* S^1)$, where $\pi : T^*M \to M$ is the vector bundle projection. In what follows, if S is a vector bundle over M, we let $\Gamma_c^\infty(S)$ denote the compactly supported smooth sections of S, that is, the smooth sections which are zero outside of a compact set.

Definition 2.5.3. A linear operator

$$L : \Gamma_c^\infty(S^0) \to \Gamma_c^\infty(S^1),$$

where S^0 and S^1 are smooth vector bundles over a smooth manifold M, is called a *differential operator* of order m if

i. L is a *local* operator: $L : \Gamma_c^\infty(S^0|_U) \to \Gamma_c^\infty(S^1|_U)$ for all open $U \subseteq M$, and

ii. for every chart U with $S^0|_U$ and $S^1|_U$ trivial, there is a differential operator $\tilde{L} : C_c^\infty(U, \mathbb{C}^n) \to C_c^\infty(U, \mathbb{C}^k)$ of order m such that the following diagram commutes

$$\begin{array}{ccc} C_c^\infty(U,\mathbb{C}^n) & \xrightarrow{\tilde{L}} & C_c^\infty(U,\mathbb{C}^k) \\ \cong\uparrow & & \uparrow\cong \\ \Gamma_c^\infty(S^0|_U) & \xrightarrow[L]{} & \Gamma_c^\infty(S^1|_U), \end{array}$$

where the vertical arrows are induced by the trivializations for $S^{0,1}|_U$.

In order to define the principal symbol of a differential operator, we need to recall some facts about the cotangent bundle. We say that a germ of functions f at a point $p \in M$ vanishes of order k if it can be written as the product of k functions which vanish at p. Letting J_p^k denote the vector space of germs of smooth functions which vanish to order k at p. Then, by definition, $T_p^*M = J_p^1/J_p^2$, and for $f \in C^\infty(M)$, df denotes the equivalence class of $f - f(p)$ in T_p^*M. Let $\pi : T^*M \to M$ denote the projection.

Definition 2.5.4. Let $L : \Gamma_c^\infty(S^0) \to \Gamma_c^\infty(S^1)$ be a differential operator. For $(x, \xi) =$

$(x, df) \in T_x^*(M)$ and $e = s(x) \in S_x^0$ for $s \in \Gamma_c^\infty(S^0)$, we define

$$\sigma_L^p(x,\xi)(e) := \frac{i^m}{m!} L((f - f(x))^m s)(x) \in S_x^1.$$

The *principal symbol* of L is defined as the section

$$\sigma_L^p : T^*M \to \mathrm{Hom}(\pi^*(S^0), \pi^*(S^1))$$

given by $(x,\xi) \mapsto \sigma_L^p(x,\xi)$. We say that L is *elliptic* if $\sigma_L^p(x,\xi)$ is invertible outside of the zero section of T^*M.

2.5.2 The Index Map and the Index Theorem

Given an elliptic differential operator $L \in \Gamma^\infty(S^0) \to \Gamma^\infty(S^1)$ between vector bundles over a compact manifold M, we want to associate an integer called that *topological index* of L. We will do this by assigning to L a class $[\sigma_L^p] \in \mathrm{K}^0(T^*M)$. We will then define a map $\mathrm{ind}_t : K^0(T^*M) \to \mathbb{Z}$ called the topological index map, and the topological index of L is defined as $\mathrm{ind}_t([\sigma_L^p]) \in \mathbb{Z}$.

First, we define the class $[\sigma_L^p] \in \mathrm{K}^0(T^*M)$. The symbol defines a complex

$$[\sigma_L^p] : 0 \longrightarrow \pi^* S^0 \xrightarrow{\sigma_L^p} \pi^* S^1 \longrightarrow 0,$$

where $\sigma_L^p(x,\xi,e) = \sigma_L^p(x,\xi)(e)$. If L is elliptic, then by definition the support of this complex is contained in the zero section of T^*M. Since this is homeomorphic to M and M is compact, we conclude that $[\sigma_L^p] \in \mathrm{K}^0(T^*M)$.

We will now define the topological index map. Let $\varphi : X \to \mathbb{R}^n$ be an embedding, for some n sufficiently large. We may extend this to an embedding $(v, d\varphi) : TX \to T\mathbb{R}^n$. If N is the normal bundle of X in $\mathbb{R}^n$, then we note that $\pi^*N \oplus \pi^*N \cong \pi^*N \otimes \mathbb{C}$

is the normal bundle of TX in $T\mathbb{R}^n$. In particular, it is complex, whence we have the Thom isomorphism

$$\mathrm{K}^*(TX) \cong \mathrm{K}^*(\pi^*N \otimes \mathbb{C}).$$

Identifying $\pi^*N \otimes \mathbb{C}$ with a tubular neighbourhood of TX in $T\mathbb{R}^n$ gives the wrong way map $\mathrm{K}^*(\pi^*N \otimes \mathbb{C}) \to \mathrm{K}^*(T\mathbb{R}^n)$. Since $T\mathbb{R}^n \cong \mathbb{R}^{2n}$, Bott periodicity identifies $\mathrm{K}^*(T\mathbb{R}^n) \cong \mathrm{K}^*(*) = \mathbb{Z}$.

Definition 2.5.5. The *topological index map* is defined to be the composition

$$\mathrm{ind}_t : \mathrm{K}^*(T^*X) \cong \mathrm{K}^*(\pi^*N \otimes \mathbb{C}) \to \mathrm{K}^*(T\mathbb{R}^n) \cong \mathbb{Z}.$$

It can be shown that this map is independent of choice of embedding and of tubular neighbourhood. We are finally ready to state the Atiyah-Singer Index Theorem.

Theorem 2.5.1 (Atiyah-Singer Index Theorem)**.** *Let $L : \Gamma^\infty(S^0) \to \Gamma^\infty(S^1)$ be an elliptic differential operator between smooth vector bundles S^0, S^1, over a compact, smooth manifold M. Then*

i. both the kernel and cokernel of L are finite dimensional and,

ii. $\mathrm{ind}_t([\sigma_L^p]) = \dim(\ker(L)) - \dim(\mathrm{coker}(L))$.

Chapter 3

Wrong Way Functoriality

Motivated by the topological index construction of Atiyah and Singer, we are going to define “wrong way maps" for certain smooth maps between smooth manifolds. The correct condition for a map to define a wrong way map in K-theory is the requirement that it admit a K-orientation. A K-orientation for a map $f : X \to Y$ is defined as a K-orientation of the tangent bundle of f, that is the bundle $TX \oplus f^*TY$.

A K-orientation of a vector bundle is a condition involving its Clifford bundle and the representation theory of the fibres of this bundle. In Section 3.1 we define the Clifford algebra of a Euclidean vector space, and talk about its representation theory. We then discuss K-orientations of vector bundles, and their relationship with orientations, in Section 3.2. Armed with this background, we then construct wrong way maps for K-oriented maps in Section 3.3

3.1 Clifford Algebras

Clifford algebras and their representation theory play a major role in topological K-theory and bivariant K-theory. In this section we will define the Clifford algebras and discuss their representation theory. This section follows the wonderful book [17].

Definition 3.1.1. Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean[1] vector space; the (real) *Clifford Algebra* of V is defined as

$$Cl(V) \coloneqq T(V)/I(V),$$

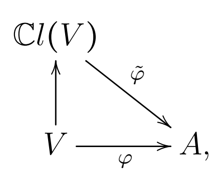
where $T(V) = \sum_{n\geq 0} V^{\otimes n}$ is the tensor algebra[2] of V and $I(V)$ is the (two-sided) ideal generated by $v \otimes v + \|v\|^2$, for $v \in V$. The *complex Clifford algebra* of V is defined as

$$\mathbb{C}l(V) \coloneqq Cl(V) \otimes_{\mathbb{R}} \mathbb{C}.$$

Example 3.1.1. The Clifford algebra $Cl(\mathbb{R})$ is the algebra generated by 1 and e, where $e^2 = -1$. Thus, $Cl(\mathbb{R}) \cong \mathbb{C}$. Since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$, it follows that $\mathbb{C}l(\mathbb{R}) \cong \mathbb{C} \oplus \mathbb{C}$.

Clifford algebras are the universal associative, unital, complex algebra associated to V satisfying $v \cdot v = -\|v\|^2 1$ in the following sense.

Proposition 3.1.1. *Let A be an associative, unital, complex algebra and $\varphi : V \to A$ be a real linear map such that $\varphi(v)^2 = -\|v\|^2 1$ for all $v \in V$. There is a unique $\mathbb{C}$-algebra homomorphism $\tilde{\varphi} : \mathbb{C}l(V) \to A$ such that the following diagram commutes*



where the map $V \to \mathbb{C}l(V)$ is the canonical embedding.

This universal property implies a $\mathbb{Z}_2$-grading $\mathbb{C}l(V) = \mathbb{C}l^0(V) \oplus \mathbb{C}l^1(V)$ coming from the $+1$ and -1 eigenspaces of the automorphism induced by $v \mapsto -v$. This grading is seen to be fundamental by the following proposition

[1] a finite dimensional, real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$
[2] by convention, we take $V^{\otimes 0} \coloneqq \mathbb{R}$

Proposition 3.1.2. *There is a natural isomorphism*

$$\mathbb{C}l(V \oplus W) \cong \mathbb{C}l(V) \hat{\otimes} \mathbb{C}l(W),$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded algebras.

Proof. The isomorphism is induced by the map

$$V \oplus W \to \mathbb{C}l(V) \hat{\otimes} \mathbb{C}l(W)$$
$$(v, w) \mapsto v \hat{\otimes} 1 + 1 \hat{\otimes} w;$$

this satisfies the universal property since

$$(v\hat{\otimes}1 + 1\hat{\otimes}w)^2 = (v\hat{\otimes}1)^2 + (v\hat{\otimes}1)(1\hat{\otimes}w) + (1\hat{\otimes}w)(v\hat{\otimes}1) + (1\hat{\otimes}w)^2$$
$$= v^1\hat{\otimes}1 + v\hat{\otimes}w - v\hat{\otimes}w + 1\hat{\otimes}w^2 = -\|(v,w)\|^2(1\hat{\otimes}1). \qquad \square$$

Just as an orientation of a vector space V determines a volume form in the exterior algebra of V, and orientation of V defines a volume element in the complex Clifford algebra $\mathbb{C}l(V)$. This element will play an important role in studying the representation theory of $\mathbb{C}l(V)$, so we take some time now to record some facts.

Definition 3.1.2. Let V be an oriented Euclidean vector space of dimension n. The (oriented) *volume element* for $\mathbb{C}l(V)$ is defined as

$$\omega_V := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n,$$

where $\{e_1, \ldots, e_n\}$ is any positively oriented[3] orthonormal basis for V and $\lfloor \cdot \rfloor$ denotes

[3] Any two orthonormal bases of a vector space V are related by an orthogonal matrix. Two bases are in the same orientation class if they are related by a special orthogonal matrix. An orientation of a vector space is a choice of orientation class, and a frame is called positively oriented if it is in this chosen orientation class.

the floor function.

Proposition 3.1.3. *The volume element $\omega_V \in \mathbb{C}l(V)$ is well defined by a choice of orientation for V and satisfies the following properties*

i. $\omega_V^2 = 1$,

ii. $\omega_V \xi = -\xi \omega_V$ *for all* $\xi \in V$ *if* $\dim V$ *is even,*

iii. $\omega_V \xi = \xi \omega_V$ *for all* $\xi \in V$ *if* $\dim V$ *is odd, and*

iv. under the isomorphism $\mathbb{C}l(V) \to \mathbb{C}l^0(V \oplus \mathbb{R})$ *induced by the map*

$$\begin{aligned} V &\to \mathbb{C}l^0(V \oplus \mathbb{R}) \\ \xi &\mapsto \xi e, \end{aligned}$$

where e *is the canonical basis element for* $\mathbb{R}$*, the volume element for* V *maps to the volume element for* $V \oplus \mathbb{R}$ *if* V *is odd dimensional, and* $-i$ *times the volume element for* $V \oplus \mathbb{R}$ *if* V *is even dimensional. Here we are giving* $V \oplus \mathbb{R}$ *the product orientation.*

Proof. See [17]. □

3.1.1 Representation Theory of $\mathbb{C}l(\mathbb{R}^n)$

The representation theory of $\mathbb{C}l(\mathbb{R}^n)$ plays a major role in K-theory in the sense that it gives us Thom Isomorphisms. Indeed, suppose that V is a real vector bundle over a space X, and associated to V is a complex vector bundle S over X such that for all $x \in X$, the fibre S_x is an irreducible representation of $\mathbb{C}l(V_x)$. Then the Thom Isomorphism Theorem says that there is a natural isomorphism $\lambda_*^V : \mathrm{K}^*(X) \to \mathrm{K}^{*-\dim V}(V)$ induced by the action of V on S. We therefore devote this section to understanding the irreducible representations of $\mathbb{C}l(\mathbb{R}^n)$. They turn out to be quite simple, because of the following classification theorem.

Theorem 3.1.1. *For all* $n \in \mathbb{N}$ *there is an isomorphism* $\mathbb{C}l(\mathbb{R}^{n+2}) \to \mathbb{C}l(\mathbb{R}^n) \otimes \mathbb{C}l(\mathbb{R}^2)$. *Thus, one has the following table computing* $\mathbb{C}l(\mathbb{R}^n)$ *up to isomorphism:*

	$n=1$	$n=2$	$\cdots$	$n=2k$	$n=2k+1$
$\mathbb{C}l(\mathbb{R}^n)$	$\mathbb{C}\oplus\mathbb{C}$	$M_2(\mathbb{C})$	$\cdots$	$M_{2^k}(\mathbb{C})$	$M_{2^k}(\mathbb{C})\oplus M_{2^k}(\mathbb{C})$

In particular, if v_n *denotes the number of inequivalent irreducible representations of* $\mathbb{C}l(\mathbb{R}^n)$, *and* d_n *denotes the complex dimension, then we have*

$$v_{2n} = 1, \quad d_{2n} = 2^n,$$
$$v_{2n+1} = 2, \quad d_{2n+1} = 2^n.$$

Proof. See [17]. □

It will be useful to have a bit of a handle on what these irreducible representations are, so we will take some time to understand them. Representations of $\mathbb{C}l(\mathbb{R}^n)$ always split into the direct sum of irreducible representations, so if we can find a representation of minimal dimension, it must be irreducible.

The unique irreducible representation of the Clifford algebras of even dimensional spaces is quite easy to describe, so we start there. Suppose that $n = 2k$, and let $\Lambda^*_{\mathbb{C}}\mathbb{C}^k$ denote the complex exterior algebra of $\mathbb{C}^k$. There is a canonical map $c : \mathbb{R}^{2k} \to \mathrm{End}(\Lambda^*_{\mathbb{C}}\mathbb{C}^k)$ given by

$$\xi \mapsto c(\xi) := \lambda_\xi - \lambda^*_\xi,$$

where

$$\lambda_\xi(w) = \xi \wedge w \text{ and } \lambda^*_\xi(w_1 \wedge \cdots \wedge w_m) = \sum_{j=1}^{m} (-1)^{j+1} \langle w_j, \xi \rangle_{\mathbb{C}} w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_m,$$

and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is the standard inner product on $\mathbb{C}^k$. We have that $\lambda_\xi^2 = 0$, $(\lambda_\xi^*)^2 = 0$, and

$$\lambda_\xi^*(\lambda_\xi) = \|\xi\|^2 \cdot 1 - \lambda_\xi(\lambda_\xi^*),$$

whence the map $\xi \mapsto c(\xi)$ extends to a (skew self-adjoint) representation of $\mathbb{C}l(\mathbb{R}^{2k})$. A dimension count shows that this is the irreducible representation. Furthermore, we note that $\Lambda_{\mathbb{C}}^*\mathbb{C}^k$ has a natural $\mathbb{Z}_2$-grading induced by the volume element ω_{2k}. Indeed, $\Lambda_{\mathbb{C}}^*\mathbb{C}^k$ splits as $(1+c(\omega_{2k}))\Lambda_{\mathbb{C}}^*\mathbb{C}^k \oplus (1-c(\omega_{2k}))\Lambda_{\mathbb{C}}^*\mathbb{C}^k$, i.e. into the ± 1-eigenspaces for ω_{2k}. We note as well that, for each $\xi \in \mathbb{R}^n$, the operator $c(\xi)$ is odd, since $\xi\omega_{2k} = -\omega_{2k}\xi$. Observe that switching the orientation of $\mathbb{R}^{2k}$ has the effect of reversing the $\mathbb{Z}_2$-grading.

To find the irreducible representations of $\mathbb{C}l(\mathbb{R}^{2k-1})$, let S be an irreducible representation of $\mathbb{C}l(\mathbb{R}^{2k})$. Then, as before, S splits into the direct sum of the ± 1-eigenspaces of the volume element ω_{2k}. Note that the inclusion of the first $2k-1$ coordinates $\mathbb{R}^{2k-1} \to \mathbb{R}^{2k}$ induces the map

$$\begin{aligned} \mathbb{R}^{2k-1} &\to \mathrm{End}(S), \\ v &\mapsto c(ve_{2k}), \end{aligned}$$

where e_{2k} is the standard basis vector not in $\mathbb{R}^{2k-1}$. This map satisfies the universal property, whence extends to give a representation $\mathbb{C}l(\mathbb{R}^{2k-1}) \to \mathrm{End}(S)$. Since each element of $\mathbb{R}^{2k}$ acts as an odd operator, it follows that $\mathbb{R}^{2k-1}$ acts as *even* operators, and therefore leaves each eigenspace of ω_{2k} invariant. Thus, each summand is a representation of $\mathbb{C}l(\mathbb{R}^{2k-1})$ and a dimension count shows that it is irreducible. Since the volume element for $\mathbb{C}l(\mathbb{R}^{2k-1})$ maps to the volume element of $\mathbb{C}l(\mathbb{R}^{2k})$, it follows that it acts as +1 on one summand, and -1 on the other. Thus, these two representations are inequivalent.

3.1.2 The Spin Group

We motivate this section by posing the following question. First, note that $O(n)$ acts on representations of $\mathbb{C}l(\mathbb{R}^n)$ as follows. Indeed, suppose that (S,c) is a representation of $\mathbb{C}l(\mathbb{R}^n)$ and $A \in O(n)$. Then A induces a new representation of $\mathbb{C}l(\mathbb{R}^n)$ via

$$\begin{aligned} \mathbb{R}^n &\to \mathrm{End}(S) \\ \xi &\mapsto A.c(\xi) := c(A\xi); \end{aligned}$$

since $A \in O(n)$ this map extends by the universal property. We let $A.S$ denote the new representation. When are S and $A.S$ equivalent?

The answer lies in understanding a certain subgroup of the invertible elements of $\mathbb{C}l(V)$, called the spin group. First, we make the following observation.

Proposition 3.1.4. *For any non-zero vector $v \in V$ and any $w \in V$, we have that*

$$-\mathrm{Ad}_v(w) = -vwv^{-1} = w - 2\frac{\langle v, w\rangle}{\|v\|^2}v.$$

In other words, $-\mathrm{Ad}_v$ reflects w across $v^\perp$.

We note that

$$\|\mathrm{Ad}_v(w)\|^2 = \left\langle w - 2\frac{\langle v, w\rangle}{\|v\|^2}v, w - 2\frac{\langle v, w\rangle}{\|v\|^2}v\right\rangle = \|w\|^2. \tag{3.1}$$

At this point we are interested in a certain subgroup of $\mathbb{C}l(V)$ called the spin group. This group is (apparently) important in physics, but for our purposes it will be a useful tool in determining when two representations of $\mathbb{C}l(V)$ are equivalent.

Definition 3.1.3. Let V be a Euclidean vector space, and let $S(V)$ denote the unit sphere in V. We define the group $\mathrm{Spin}(V)$ as

$$\mathrm{Spin}(V) := \{v_1 \cdots v_{2k} \in \mathbb{C}l(V) : v_1, \ldots, v_{2k} \in S(V)\}.$$

Proposition 3.1.4 and Equation (3.1) say that the adjoint representation $\mathrm{Ad} : \mathrm{Spin}(V) \to O(V)$. In fact, since $-\mathrm{Ad}_v$ is a reflection across $v^\perp$, and each such reflection has determinant -1, we see that Ad yields a representation $\mathrm{Spin}(V) \to SO(V)$, being the product of an even number of such reflections. It is shown in [17] that the adjoint representation is surjective, and moreover that we have the following exact sequence.

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\mathrm{Ad}} SO(V) \longrightarrow 0. \tag{3.2}$$

This says that $\mathrm{Spin}(V)$ is the double cover of $SO(V)$. The important point for us, though, is the surjection $\mathrm{Spin}(V) \longrightarrow SO(V)$, as it provides us with an answer to our original question.

Proposition 3.1.5. *If $A \in SO(\mathbb{R}^n)$, and S is any representation of $\mathbb{C}l(\mathbb{R}^n)$, then S and $A.S$ are equivalent. If $A \in O(n) \setminus SO(n)$, then $A.S.$ is equivalent to the opposite representation of S.*

By opposite representation we mean the following: If n is even, then it is the same representation but with the $\mathbb{Z}_2$-grading reversed, and if n is odd then it is the representation induced by

$$\mathbb{R}^n \to \mathrm{End}(S)$$
$$\xi \mapsto -c(\xi),$$

where c is Clifford multiplication.

Proof. Observe that by the exact sequence (3.2), it follows that A lifts to some element

$w \in \mathrm{Spin}(n)$. It follows that Clifford multiplication by w yields the needed equivalence, since, by definition, for all $v \in V$, $Av = wvw^{-1}$ so that, for all $s \in S$,

$$c(A\xi)c(w) = c(w\xi w^{-1})c(w) = c(w\xi) = c(w)c(\xi),$$

i.e. the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{c(w)} & S \\ {\scriptstyle c(\xi)}\downarrow & & \downarrow{\scriptstyle a(\xi)} \\ A & \xrightarrow[c(w)]{} & S. \end{array}$$

Now, suppose that A is orientation reversing. If n is odd, then $-A$ is orientation preserving, so S and $(-A).S$ are equivalent. Thus, it suffices to show this when $A = -1$, but this gives the opposite representation, by definition.

If n is even, then there is an element $w \in \mathbb{C}l(\mathbb{R}^n)$ such that $\alpha(w)\xi = A\xi$ for all $\xi \in \mathbb{R}^n$, where α is the grading operator (see [17]). If w is even, then S and $A.S$ are equivalent and if w is odd then S and $(-A).S$ are equivalent, and since $-1 \in SO(n)$, it follows that S and $A.S$ are equivalent, *as ungraded representations* in both cases. To see that the grading is reversed, we note that if $a(\omega_{2k}) = -c(\omega_{2k})$. □

3.2 K-Orientations of Vector Bundles

A K-orientation is, as the name suggests, an "orientation class" for K-theory in the sense that it gives a Thom isomorphism. A K-orientation of a vector bundle V can be given explicitly as a complex vector bundle S, called the *spinor*, of irreducible representations of the Clifford bundle $\mathbb{C}l(V)$. More specifically, we make the following definition, which is a combination of the definitions due to Connes and Skandalis, [5], and Higson and Roe, [14].

Definition 3.2.1. Let X be a space and let V be a real vector bundle over X. A *spinor triple* for V is a triple (q, S_V, c), where

i. q is a metric on V (see Definition 2.1.5),

ii. S_V is a complex, Hermitian vector bundle over X, and

iii. $c : V \to \mathrm{End}(S)$ is an $\mathbb{R}$-linear vector bundle map such that

a. $c(\xi) = -c(\xi)^*$ and $c(\xi)^2 = -q(v,v)^2 \cdot 1$ for all $\xi \in V$,

b. $(S_V)_x$ is an irreducible $\mathbb{C}l(V_x, q_x)$-module for all $x \in X$, and

c. if V is of even rank, then S_V is $\mathbb{Z}_2$-graded and $c(\xi)$ is an odd operator for all $\xi \in V$, that is $S_V = S_V^0 \oplus S_V^1$ and

$$c(\xi)(S_V^0) \subseteq S_V^1 \text{ and } c(\xi)(S_V^1) \subseteq S_V^0 \text{ for all } \xi \in V.$$

In order to compare spinor triples we introduce the notion of *concordance*.

Definition 3.2.2. Suppose that $\mathbf{S}_0 = (q_0, S_0, c_0)$ and $\mathbf{S}_1 = (q_1, S_1, c_1)$ are spinors for a vector bundle V over X. We say that $\mathbf{S}_0$ and $\mathbf{S}_1$ are *concordant* if there is a vector bundle W over $X \times [0,1]$ equipped with a spinor $\mathbf{S}$ such that

$$W|_{X\times\{0\}} \cong V, \text{ with } \mathbf{S}|_{X\times 0} \cong \mathbf{S}_0 \quad \text{and} \quad W|_{X\times\{1\}} \cong V \text{ with } \mathbf{S}|_{X\times 1} \cong \mathbf{S}_1$$

Definition 3.2.3. Let V be a real vector bundle over X. A *K-orientation* for V is a choice of concordance class of spinor triple for V. A manifold M is called K-orientable if its tangent bundle admits a K-orientation.

Because of the notion of concordance, the metric plays a minor role in the definition of a K-orientation. Thus, unless we need to be specific about the metric we are using, we usually drop it from the notation.

Example 3.2.1. An almost complex structure on a real vector bundle V is an endomorphism $J : V \to V$ such that $J^2 = -1$; using J we can define complex scalar multiplication on the fibres of V by

$$(x + iy).\xi := x\xi + yJ(\xi) \quad \text{for } x, y \in \mathbb{R} \text{ and } \xi \in V.$$

Any almost complex Euclidean vector bundle V over a space X, in particular the real vector bundle $TX \oplus TX$, is canonically K-oriented, by taking $S_V = \Lambda^*_{\mathbb{C}} V$, where we are viewing V as a complex vector bundle using the complex structure, and $c(\xi) = \lambda_\xi - \lambda^*_\xi$, as above. We give S_V a $\mathbb{Z}_2$-grading as in the previous section. We note that if X is compact, then the complexes

$$0 \longrightarrow \mathbb{C} \longrightarrow \Lambda^1_{\mathbb{C}} V \xrightarrow{d} \Lambda^2_{\mathbb{C}} V \xrightarrow{d} \cdots \quad \text{and} \quad 0 \longrightarrow \sum_k \Lambda^{2k}_{\mathbb{C}} V \xrightarrow{c} \sum_k \Lambda^{2k+1}_{\mathbb{C}} V \longrightarrow 0$$

determine the same class in $\mathrm{K}^*(V)$. More generally, if X is any space then these complexes define equivalent classes in $\mathrm{RK}^0_X(V)$, the *representable K-theory of V with X-compact support*, defined in Section 4.2. It is called the Thom class.

Example 3.2.2. Every trivial bundle is K-orientable, and any choice of global frame yields a K-orientation by deeming that frame to be orthonormal. If two frames are related by an $SO(n)$-valued function A, and if A lifts continuously to a $\mathrm{Spin}(n)$-valued function, then the two frames yield equivalent K-orientations as seen by Proposition 3.1.5. We remark that although there are only (up to equivalence) two irreducible $\mathbb{Z}_2$-graded modules for $\mathbb{C}l(\mathbb{R}^n)$, the irreducible $\mathbb{Z}_2$-graded modules for $\mathbb{C}l(X \times \mathbb{R}^n)$ are parametrized by the line bundles over X. Indeed, if S is a spinor for $X \times \mathbb{R}^n$ and L is a line bundle over X, then we may form another spinor for $X \times \mathbb{R}^n$ using $S \otimes L$; see [14] for more details.

Example 3.2.3. The trivial bundle $X \times \mathbb{R}$ has two canonical K-orientations, given by multiplication by x and $-x$, respectively. These are called the *positive* and *negative* K-orientations of $X \times \mathbb{R}$ respectively.

Example 3.2.4. If $f : X \to Y$ is a map and V is a K-oriented vector bundle over Y, then f^*V obtains a K-orientation in an obvious way. If X and Y are smooth and $f : X \to Y$ is a diffeomorphism, then since $f^*(TY) \cong TX$, we see that K-orientability is a diffeomorphism invariant.

Example 3.2.5. Every Lie group is K-orientable since the tangent bundle of a Lie group is trivial. In fact, an orientation of a Lie group induces a K-orienation of the Lie group; see Lemma 3.2.2.

3.2.1 K-Orientations and Orientations

As the name might suggest, a choice of K-orientation for a vector bundle V determines an orientation for V. Indeed, one can think of a K-orientation as an orientation "with a slight bit of extra structure"[4]. With this in mind, we devote this section to comparing the two concepts.

Lemma 3.2.1. *There is a 1-1 correspondence between K-orientations of V and K-orientations of $V \oplus \mathbb{R}$.*

Proof. First assume that V is of odd rank. If V is K-oriented with spinor (S_V, c_V) then we K-orient $V \oplus \mathbb{R}$ using the bundle $S_{V\oplus\mathbb{R}} := S_V \oplus S_V$ and the action given by the extension of

$$c_{V\oplus\mathbb{R}}(\xi, x) = \begin{pmatrix} 0 & c_V(\xi) - x \\ c_V(\xi) + x & 0 \end{pmatrix}.$$

The bundle $S_{V\oplus\mathbb{R}}$ is obviously $\mathbb{Z}_2$-graded, and each $(\xi, x) \in V \oplus \mathbb{R}$ acts as an odd operator.

[4]This remark is due to Paul Baum and Ronald Douglas, see [3]

On the other hand, suppose that $V \oplus \mathbb{R}$ is K-oriented with spinor $S = S^0 \oplus S^1$. If e denotes the canonical basis element of $\mathbb{R}$, then V acts on S^0 via the extension of the map

$$\xi \mapsto c_{V\oplus\mathbb{R}}(\xi e).$$

Let us verify that these two constructions are inverse. Starting with a K-orientation for V, inducing one on $V \oplus \mathbb{R}$, and then inducing back on V we see that the vector spaces agree, so we must verify that the actions are the same. We have

$$c_{V\oplus\mathbb{R}}(\xi e) = \begin{pmatrix} 0 & c_V(\xi) \\ c_V(\xi) & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_V(\xi) & 0 \\ 0 & -c_V(\xi) \end{pmatrix},$$

so, after projection onto the first coordinate, that these actions agree. Conversely, to show the two K-orientations on $V \oplus \mathbb{R}$ agree we must show that the following diagram commutes

$$\begin{array}{ccc} S^0 \oplus S^1 & \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & -c(e)\end{pmatrix}} & S^0 \oplus S^0 \\ {\scriptstyle c(\xi,x)}\Big\downarrow & & \Big\downarrow{\scriptstyle \begin{pmatrix}0 & c(\xi e)-x\\ c(\xi e)+x & 0\end{pmatrix}} \\ S^0 \oplus S^1 & \xrightarrow[\begin{pmatrix}1 & 0\\ 0 & -c(e)\end{pmatrix}]{} & S^0 \oplus S^0. \end{array} \tag{3.3}$$

If $s_0 \in S^0$ then traversing down and right we have

$$s_0 \mapsto -c(e)c(\xi, x)s_0 = -c(e)(c(\xi) + xc(e))s_0 = (c(\xi e) + x)s_0,$$

which is the same as if one traverses right and then down. Similarly, if $s_1 \in S^1$ then traversing right then down gives

$$s_1 \mapsto -(c(\xi e) - x)c(e)s_1 = (c(\xi) + xc(e))s_1 = c(\xi, x)s_1,$$

which is what one gets if they transverse down and then right. Thus, (3.3) commutes.

Now suppose that V is of even rank. Then, if V is K-oriented with spinor S we define the action of $V \oplus \mathbb{R}$ on S to be the extension of

$$c_{V\oplus\mathbb{R}}(\xi, x) = c_V(\xi) - ixc_V(\omega_V). \tag{3.4}$$

If $V \oplus \mathbb{R}$ is K-oriented with spinor S, then V acts on S via $\xi \mapsto c_{V\oplus\mathbb{R}}(\xi)$. We $\mathbb{Z}_2$-grade S as $S = (1 + c_{V\oplus\mathbb{R}}(\omega_V))S \oplus (1 - c_{V\oplus\mathbb{R}}(\omega_V))S$. To show that these constructions are inverse, we must show that

$$c_{V\oplus\mathbb{R}}(\xi, x) = c_{V\oplus\mathbb{R}}(\xi) - ixc_{V\oplus\mathbb{R}}(\omega_V).$$

Since S is an irreducible $\mathbb{C}l(V \oplus \mathbb{R})$-module, the properties of $\omega_{V\oplus\mathbb{R}}$ imply that it acts as +1 or -1, and by choosing the appropriate orientation on V, we may assume that it acts as +1. Next, we observe that, by definition, $e\omega_V = \omega_V e = -i\omega_{V\oplus\mathbb{R}}$. Thus,

$$c_{V\oplus\mathbb{R}}(\xi, x) = c_{V\oplus\mathbb{R}}(\xi) + c_{V\oplus\mathbb{R}}(x) = c_{V\oplus\mathbb{R}}(\xi) + xc_{V\oplus\mathbb{R}}(e\omega_{V\oplus\mathbb{R}}) = c_{V\oplus\mathbb{R}}(\xi) - ixc_{V\oplus\mathbb{R}}(\omega_V),$$

as needed. An irreducibility arguement shows that if $S = S^0 \oplus S^1$ is a $\mathbb{Z}_2$ grading for a spinor of V, then we may orient V so that S^0 is the +1-eigenspace of ω_V and S^1 is the −1-eigenspace of ω_V, which completes the proof. □

Remark. There are two seemingly distinct ways one can K-orient $V \oplus \mathbb{R}$ given a K-orientation for V. We compare the two in Appendix B.

Proposition 3.2.1. *A K-orientation on a vector bundle V induces an orientation of V.*

Proof. First suppose that $n = 2k$, so that $S_V = S_V^0 \oplus S_V^1$. If $\mathbf{e} = \{e_1, \dots e_n\}$ is an orthonormal frame for V, then irreducibility and local considerations show that $\omega_{\mathbf{e}} :=$

$i^k e_1 \cdots e_n$ acts as +1 or -1 on S_v^0. If it acts as +1, we deem this frame to be positively oriented, and if it acts as -1, we deem it to be negatively oriented.

If V is of rank $n = 2k - 1$, then irreducibility implies that $\omega_{\mathbf{e}}$ acts as +1 or -1. As before, we deem a frame $\mathbf{e}$ positive if $\omega_{\mathbf{e}}$ acts as +1, and negatively oriented if it acts as -1. □

Although a converse to this theorem is in general false, see Example 3.2.7, we do have the following partial converse.

Lemma 3.2.2. *Suppose that M is a parallelizable manifold, i.e. its tangent bundle is trivial. Then, an orientation of M induces a K-orientation of M. Moreover, if M and N are both parallelizable and $f : M \to N$ is orientation preserving, then f is K-orientation preserving when M and N are K-oriented as above. In particular, when M and N are K-oriented as above, any diffeomorphism is either K-orientation preserving or reversing, and this is detectable by computing the determinant of the Jacobian at any point of $m \in M$.*

Proof. Suppose that $\{e_1, \ldots, e_n\}$ is a positively oriented global frame for TM. We declare this frame to be orthonormal, and therefore may identify TM isometrically with $M \times \mathbb{R}^n$, which gives a K-orientation for M. Different positively oriented frames yield concordant spinor triples, so this is well defined.

The second statement is clear, since df maps a postively oriented frame $\{e_1 \ldots, e_n\}$ for TM to a positively oriented frame $\{f_1, \ldots, f_n\}$ for TN via the following commuting diagram

$$\begin{array}{ccc} TN & \xleftarrow{df} & TM \\ {\scriptstyle f_i}\uparrow & & \uparrow{\scriptstyle e_i} \\ N & \xrightarrow[f^{-1}]{} & M. \end{array}$$

□

Example 3.2.6. The Möbius bundle is *not* K-orientable, since it is not orientable.

Example 3.2.7. The manifold $SU(3)/SO(3)$ is orientable but not K-orientable, see [17], p. 393. Thus, K-orientability is strictly finer than orientability.

Thus, one can think of a K-orientation as an orientation "with a slight bit of extra structure[5]". Using this train of thought we define the "opposite K-orientation" as well as state and prove a 2-out-of-3 lemma for K-orientations.

Definition 3.2.4. Suppose that V is a K-oriented vector bundle with K-orientation (S_V, c). We define the *opposite K-orientation* in the following way:

i. if V is of even rank, then $(S_V^0 \oplus S_V^1, c)^{op} := (S_V^1 \oplus S_V^0, c)$ and

ii. if V is of odd rank, then $(S_V, c)^{op} := (S_V, c^-)$, where c^- is the map induced by

$$\xi \mapsto -c(\xi).$$

Remark. Observe that reversing the K-orientation reverses the corresponding orientation, and vice versa.

The fundamental lemma for constructing K-orientations from given ones is the 2-out-of-3 lemma, which we now state and prove.

Theorem 3.2.1 (The 2-Out-Of-3 Lemma for K-Orientations)**.** *Suppose that*

$$0 \longrightarrow V \longrightarrow U \longrightarrow W \longrightarrow 0$$

is a short exact sequence of vector bundles. If any two of them are K-oriented, then the third receives a canonical K-orientation.

[5]This remark is due to Paul Baum and Ronald Douglas, see [3]

If a splitting $W \to U$ (resp. $U \to V$) is given, then the K-orientation that W (resp. V) receives induces the original K-orientation on U when paired with the given K-orientation on V (resp. W). Furthermore, this K-orientation is unique.

Proof. Since any short exact sequence of vector bundles splits, we may assume that $U = V \oplus W$; we will show at the end that the choice of splitting makes no difference, up to concordance. Also, by Lemma 3.2.1 we may assume that all bundles are of even rank, hence the spinors are $\mathbb{Z}_2$-graded.

If V and W are K-oriented via (S_V, c_V) and (S_W, c_W) respectively, then V obtains a K-orientation using $S_V \hat{\otimes} S_W$, where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product, and the action induced by

$$(\xi, \eta) \mapsto c_V(\xi)\hat{\otimes}1 + 1\hat{\otimes}c_W(\xi).$$

Conversely, if $V \oplus W$ and W are K-oriented, let $\mathrm{Hom}^g_{\mathbb{C}l(W)}(S_W, S_{V\oplus W})$ denote the bundle of linear maps which graded commute with the action of $\mathbb{C}l(W)$. The bundle $\mathbb{C}l(W)$ acts on $\mathrm{Hom}^g_{\mathbb{C}l(W)}(S_W, S_{V\oplus W})$ by composition, and since the evaluation map

$$\mathrm{Hom}^g_{\mathbb{C}l(W)}(S_W, S_{V\oplus W})\hat{\otimes}S_W \to S_{V\oplus W}$$

is an equivalence as S_W is irreducible, it follows that $S_V := \mathrm{Hom}^g_{\mathbb{C}l(W)}(S_W, S_{V\oplus W})$ is a spinor for V satisfying the conclusion of the theorem.

We now show that any two choices of splitting yield the same K-orientation. If ϕ and ϕ' are two different splittings $W \to U$, then they induce an automorphism of $V \oplus W$ given by the matrix

$$\begin{pmatrix} 1 & \phi + \phi' \\ 0 & 1 \end{pmatrix}.$$

This matrix is connected by a path (namely the path obtained by multiplying $\phi + \phi'$ by t) in $SO(n)$ to the identity, which shows that the corresponding K-orientations are concordant.

To prove uniqueness, suppose that S_V is another K-orientation such that $S_V\hat{\otimes}S_W$ and $S_{V\oplus W}$ are equivalent. Consider the map

$$\begin{aligned} S_V &\to \mathrm{Hom}_{\mathbb{C}l(W)}(S_W, S_V\hat{\otimes}S_W) = \mathrm{Hom}_{\mathbb{C}l(W)}(S_W, S_{V\oplus W}) \\ x &\mapsto (T_x : y \mapsto x\hat{\otimes}y). \end{aligned}$$

First, observe that for each $x \in S_V$ the map T_x graded commutes with the action of $\mathbb{C}l(W)$. Indeed, note that $\partial(T_x) = \partial x$ and for $\xi \in W$ and $y \in S_W$ we have

$$T_x(c_W(\xi)y) = x\hat{\otimes}(c_W(\xi)y) = (-1)^{\partial x}(1\hat{\otimes}c_W(\xi))(x\hat{\otimes}y) = (-1)^{\partial x}c_{V\oplus W}(\xi)T_x(y).$$

The map $x \mapsto T_x$ is an injection, hence an isomorphism by dimension consideration, so it remains to show that it intertwines the Clifford multiplications, that is show that the following diagram commutes

$$\begin{array}{ccc} S_V & \longrightarrow & \mathrm{Hom}^g_{\mathbb{C}l(W)}(S_W, S_V\hat{\otimes}S_W) \\ {\scriptstyle c_V(\eta)}\downarrow & & \downarrow{\scriptstyle \tilde{c}_V(\eta)} \\ S_V & \longrightarrow & \mathrm{Hom}^g_{\mathbb{C}l(W)}(S_W, S_V\hat{\otimes}S_W), \end{array}$$

for $\eta \in V$. For $x \in S_V$ this amounts to showing that

$$\tilde{c}_V(\eta)T_x = T_{c_V(\eta)x},$$

which we do pointwise. For $y \in S_W$ we compute

$$(\tilde{c}_V(\eta)T_x)(y) = (c_V(\eta) \otimes 1)(x\hat{\otimes}y) = (c_V(\eta)x)\hat{\otimes}y = T_{c_V(\eta)x}(y),$$

as needed. This completes the proof. □

Example 3.2.8. Suppose that V and W are complex vector bundles. Then $V \oplus W$ is complex, and the K-orientation that W obtains from $V \oplus W$ and V from the 2 out of 3 lemma is equivalent to the one coming from its complex structure.

Example 3.2.9. If V is an odd dimensional K-oriented vector bundle with K-orientation (S_V, c_V), then the K-orientation that $V \oplus V$ receives is given by

$$\left((S_V \otimes S_V) \oplus (S_V \otimes S_V), \begin{pmatrix} 0 & c_V \otimes 1 + i \otimes c_V \\ c_V \otimes 1 + i \otimes c_V & 0 \end{pmatrix} \right).$$

3.3 Wrong Way Functoriality for K-Oriented Maps

The main tool in construction wrong way maps is the Thom isomorphism, which we are now in a position to discuss in full generality.

3.3.1 The Thom Isomorphism Revisited

Now that we have developed the theory of K-orientations, we can now explain the relevance to topological K-theory. Recall that an *orientation* of a vector bundle E over a manifold M is a class in the cohomology of E which induces an isomorphism $H^*(M) \to H^*(E)$. A K-orientation of a vector bundle V turns out to be an orientation class for K-theory.

We discuss the class. Let $\pi : V \to X$ be a K-oriented vector bundle. If V is of rank d, then the direct sum $V \oplus \mathbb{R}^d$ is given a canonical K-orientation coming from V by the 2-out-of-3 lemma and, furthermore, is of even rank. Let $S_{V\oplus\mathbb{R}^d}$ denote the spinor bundle, and let $S_{V\oplus\mathbb{R}^d} = S^0_{V\oplus\mathbb{R}^d} \oplus S^1_{V\oplus\mathbb{R}^d}$, which is $\mathbb{Z}_2$-graded. We thus have a complex over $V \oplus \mathbb{R}^d$ given by

$$\lambda^V : 0 \longrightarrow \pi^* S^0_{V\oplus\mathbb{R}^d} \xrightarrow{c_V} \pi^* S^1_{V\oplus\mathbb{R}^d} \longrightarrow 0,$$

where $c_V(\xi,\eta) = c_V(\xi)(\eta)$ denotes Clifford multiplication. Since $c_V(\xi)^2 = -\|\xi\|^2$, it follows that the support of this complex is the zero section of V, which is homeomorphic to X. Thus, λ^V defines a class in $\mathrm{K}^0(V \oplus \mathbb{R}^d) = \mathrm{K}^{-\dim V}(V)$ only when X is compact. In this case, it is called the *Thom class*. Nevertheless, observe that if $\eta \in \mathrm{K}^0(X)$ is represented by a compactly supported complex over X, then $\pi^*(\eta) \otimes \lambda^V$ is compactly supported. Thus, $\pi^*(\eta) \otimes \lambda^V \in \mathrm{K}^{-\dim V}(V)$. Replacing X and V with $X \times \mathbb{R}^n$ and $V \times \mathbb{R}^n$ respectively, we have defined a map

$$\lambda^V_* : \mathrm{K}^*(X) \to \mathrm{K}^{*-\dim V}(V).$$

Definition 3.3.1. Let V be a K-oriented vector bundle over a space X. The *Thom homomorphism* corresponding to the K-orientation on V is the map

$$\lambda^V_* : \mathrm{K}^*(X) \to \mathrm{K}^{*-\dim V}(X)$$

The main theorem in K-theory is the following

Theorem 3.3.1 (Thom Isomorphism Theorem II)**.** *For any K-oriented vector bundle V over a locally compact space X, the Thom homomorphism $\lambda^*_V : \mathrm{K}^*(X) \to \mathrm{K}^{*-\dim V}(V)$ is an isomorphism.*

This can be proved, for smooth manifolds X and assuming Bott Periodicity, as follows. Following [22], it is sufficient, using the continuity of K-theory, to prove that the Thom homomorphism $K^0(X) \to \mathrm{K}^{-\dim V}(V)$ is an isomorphism for X compact. Once this reduction is made, one can use induction on the number of trivializations for V is needed to cover X. If X is a smooth manifold, then the base case is Bott periodicity, and the inductive step is achieved by the long exact sequence, or the Mayer-Vietoris sequence.

We record the following useful fact about the Thom isomorphism. Recall that if E and F are K-oriented vector bundles then their direct sum $E \oplus F$ obtains a canonical K-orientation by the 2 out of 3 lemma and we have the following.

Proposition 3.3.1 (Transitivity of the Thom Isomorphism)**.** *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{K}^*(X) & \xrightarrow{\lambda_*^{E\oplus F}} & \mathrm{K}^{*-\dim(E\oplus F)}(E \oplus F) \\ & {\scriptstyle \lambda_*^{E}} \searrow \quad \nearrow {\scriptstyle \lambda_*^{\pi_E^*(F)}} & \\ & \mathrm{K}^{*-\dim E}(E) & \end{array}$$

3.3.2 K-Orientations of Maps

We showed in Chapter 2 that K-theory is contravariant for continuous maps and covariant for open embeddings. Open embeddings are examples of a family of maps with the property that their tangent bundle is K-oriented, and K-theory is covariant for such maps. In this section we discuss K-oriented maps and show how they induce wrong-way maps in K-theory.

Definition 3.3.2. A K-orientation on a smooth map $f : X \to Y$ is a K-orientation of the real vector bundle $TX \oplus f^*TY$. We call the bundle $TX \oplus f^*TY$ the tangent bundle of f.

Example 3.3.1. Any smooth map between K-oriented manifolds is canonically K-oriented using the 2 out of 3 lemma.

Example 3.3.2. An obvious, but important, example is the identity map $\mathrm{id}_X : X \to X$, where X is any smooth manifold. This map is canonically K-oriented since $TX \oplus$

i^*TX has a canonical complex structure. More generally, if $i : U \hookrightarrow X$ is an open embedding of U into X, then i is K-oriented.

Example 3.3.3. The vector bundle projection $\pi : V \to X$ is K-orientable if and only if V is. Indeed we have that $TV = \pi^*V \oplus \pi^*TX$; this follows from the exact sequence

$$0 \longrightarrow \pi^*(V) \xrightarrow{c} TV \xrightarrow{d\pi} \pi^*TM \longrightarrow 0,$$

where $c(\eta, \xi) = c'_{\eta,\xi}(0)$ is the derivative of the curve $c_{\eta,\xi}(t) = \eta + t\xi$ evaluated at zero. We therefore have that $TV \oplus \pi^*TX = \pi^*V \oplus \pi^*(TX \oplus TX)$, which shows that π is K-orientable if V is by the 2-out-of-3 lemma.

Conversely, if π is K-orientable, then so is π^*V by the 2-out-of-3 lemma. Consequently, so is $V = (\pi \circ \zeta_0)^*V = \zeta_0^*\pi^*V$, where $\zeta_0 : X \to V$ is the zero section.

Example 3.3.4. If V is a K-orientable vector bundle, then the zero section ζ_0 is K-orientable. Indeed, we have

$$TX \oplus \zeta_0^*TV \cong TX \oplus TX \oplus V.$$

Example 3.3.5. If $f : X \to Y$ and $g : Y \to Z$ are K-oriented maps, then so is $(g \circ f) : X \to Z$. Indeed, we observe that

$$TX \oplus (g \circ f)^*TZ \oplus f^*TY \oplus f^*TY = TX \oplus f^*TY \oplus f^*(TY \oplus g^*TZ),$$

is K-oriented since f and g are, hence so is $TX \oplus (g \circ f)^*TZ$ by the 2-out-of-3 lemma.

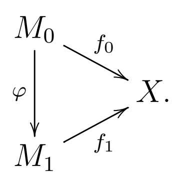
Example 3.3.6. Suppose that $\varphi : X \to Y$ is an immersion. Then φ is K-orientable if and only if the normal bundle of X in Y is K-orientable. Indeed, let $N_\varphi X$ denote the normal bundle of X in Y. We have the equality

$$TX \oplus \varphi^* TY = TX \oplus TX \oplus N_\varphi X,$$

from which the result follows by the 2-out-of-3 lemma.

Example 3.3.7. If X is a K-orientable manifold and $f : X \to Y$ is a smooth map, then K-orientations of f are in 1-1 correspondence with K-orientations of Y. Indeed, this follows from the 2 out of 3 lemma. In particular, the diagonal map $\delta : X \to X \times X$ is K-oriented if and only if X is K-oriented, where $X \times X$ is K-oriented using the complex structure on $T(X \times X)$.

Example 3.3.8. Suppose that $f_0 : M_0 \to X$, $f_1 : M_1 \to X$, and there is a diffeomorphism $\varphi : M_0 \to M_1$ making the following diagram commute:



If f_1 is K-oriented, then f_0 obtains a canonical K-orientation coming from f_1 and φ. Indeed, the result follows by the isomorphism $\varphi^*(TM_1 \oplus f_1^* TX) = TM_0 \oplus f_0^* TX$.

3.3.3 The Wrong-Way Construction

Using the Thom isomorphism, we will now explain how K-theory is *covariant* for K-oriented maps using a construction much akin to the topological index map construction of Atiyah and Signer. This construction is studied in detail in [10].

First, we recall how to construct wrong way maps for open embeddings. Suppose that $\varphi : X \to Y$ is an open embedding, that is $\varphi(X)$ is an open subset of Y

diffeomorphic to X. There is a canonical map $\varphi_+ : Y^+ \to X^+$ given by

$$\varphi_+(y) = \begin{cases} x & \text{if } \varphi(x) = y, \\ \infty & \text{else.} \end{cases}$$

We define $\varphi_! : \mathrm{K}^*(X) \to \mathrm{K}^*(Y)$ to be the restriction of the map $\varphi_+^* : \mathrm{K}^*(X^+) \to \mathrm{K}^*(Y^+)$ to the kernel of the augmentation map.

Example 3.3.9. If $\varphi : X \to Y$ is a homeomorphism, then we observe that $\varphi_! = (\varphi^*)^{-1}$. Indeed, this follows directly from the definition and the fact that $(\varphi^*)^{-1} = (\varphi^{-1})^*$ by functoriality.

Now, suppose that $\varphi : X \to Y$ is any embedding, and assume further that φ is K-oriented. This, by Example 3.3.6, is equivalent to a K-orientation on the normal bundle $N_\varphi X$ of X in Y. Let $\tau : V \to Y$ be a tubular neighbourhood of X in Y. If $\dim(X) = n$ and $\dim(Y) = m$, then the K-orientation assumption implies that there is a Thom isomorphism $\lambda_*^V : \mathrm{K}^*(X) \to \mathrm{K}^{*-(m-n)}(V)$, and we define $\varphi_!$ to be the composition

$$\begin{array}{ccc} \mathrm{K}^{*-(m-n)}(V) & \xrightarrow{\tau_!} & \mathrm{K}^{*-(m-n)}(Y) \\ {\scriptstyle \lambda_*^V} \uparrow & \overset{\varphi_!}{\nearrow} & \\ \mathrm{K}^*(X) & & \end{array}$$

In order to show that this definition is independent of choice of tubular neighbourhood, we need the following lemma which we cite from [15].

Lemma 3.3.1. *Tubular neighbourhoods of embedded submanifolds are unique in the following sense. If $\tau_0, \tau_1 : N_\varphi X \to Y$ are tubular neighbourhoods of X in Y, then the is a one parameter family $\{\tau_t : N_\varphi X \to Y\}$ of tubular neighbourhoods from τ_0 to τ_1. Such a family is called an isotopy.*

If τ_0 and τ_1 are tubular neighbourhoods of X in Y, then we want to use this theorem to construct a homotopy between the two maps at the level of the one point compactifications of X and Y. Let τ_t be an isotopy between τ_0 and τ_1, and let $\iota_t : \tau_t(N_\varphi X) \hookrightarrow Y$ denote the inclusion. The maps $(\tau_t)_+$ then factor as

$$Y^+ \xrightarrow{(\iota_t)_+} (N_\varphi X)^+ \xrightarrow{(\tau_t^{-1})_+} X^+,$$

which shows that $(\tau_0)_+$ and $(\tau_1)_+$ are homotopic. Thus, in particular, $(\tau_0)_! = (\tau_1)_!$, which shows that the definition of $\varphi_!$ is independent of choice of tubular neighbourhood.

Proposition 3.3.2. *We have the following functorial properties:*

1. $(\mathrm{id}_X)_! = \mathrm{id}_{\mathrm{K}^*(X)}$ *and*

2. *if f and g are K-oriented embeddings, then* $(f \circ g)_! = f_! \circ g_!$.

To prove this, we first need a lemma about vector spaces.

Lemma 3.3.2. *If $U \subset V \subset W$ is an increasing sequence of finite dimensional Euclidean vector spaces, then W/U is naturally isomorphic to $W/V \oplus V/U$.*

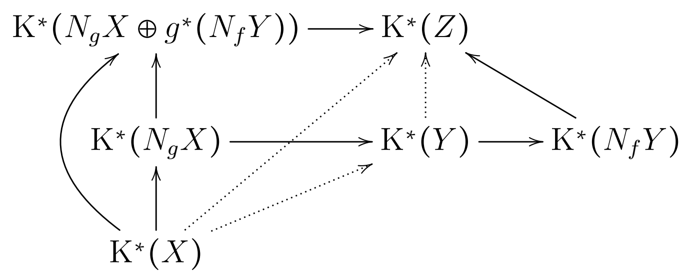
Proof. There is an exact sequence

$$0 \longrightarrow V/U \longrightarrow W/U \longrightarrow W/V \longrightarrow 0,$$

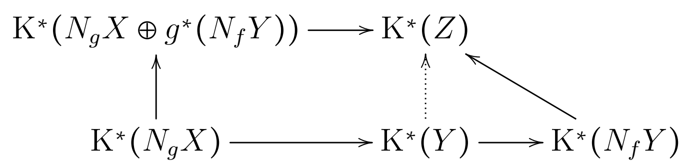
which has a natural splitting using the Euclidean structure. □

Proof of Proposition 3.3.2. The first statement is clear. To prove the second, we first note that if $N_g X$ is the normal bundle for $g : X \to Y$ and $N_f Y$ is the normal bundle for $f : Y \to Z$, then the normal bundle for $f \circ g$ is given by $N_g X \oplus g^*(N_f Y)$ by

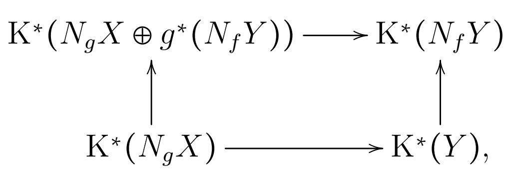
Lemma 3.3.2. Thus, our goal is too show that the following diagram commutes

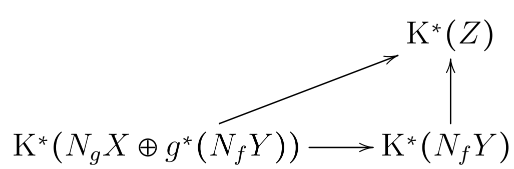


The vertical arrows on the left commute because of the transitivity of the Thom isomorphism, and the bottom triangle commutes by definition. We therefore want to show that the following diagram commutes



Using tubular neighbourhoods, we may think of N_gX as an open subspace of Y, and $N_gX \oplus g^*(N_fY)$ as an open subspace of N_fY. With this point of view, we can "factor" the above diagram into the following two diagrams:





The diagram on the left commutes because of the naturality of the Thom isomorphism with respect to open inclusions, and the diagram on the right commutes because of functoriality with respect to open inclusions. □

Let us record the following homotopy invariance property of wrong way functoriality for K-oriented embeddings.

Lemma 3.3.3. *Suppose that f_0 and f_1 are K-oriented embeddings from X to Y such that there is a K-oriented homotopy $F : X \times [0,1] \to Y$ between f_0 and f_1 which is an embedding for each fixed $t \in [0,1]$. Then $(f_0)_! = (f_1)_!$.*

Such a homotopy is called a K-oriented regular homotopy.

Proof. Let V_0 and V_1 denote the normal bundles of X in Y using f_0 and f_1 respectively. The definition of wrong-way maps gives rise to the following diagram, which we wish to show commutes:

$$\begin{array}{ccccc} & & \mathrm{K}^*(V_0) & & \\ & \overset{\lambda^0_*}{\nearrow} & & \overset{\tau^0_!}{\searrow} & \\ \mathrm{K}^*(X) & & & & \mathrm{K}^*(Y), \\ & \underset{\lambda^1_*}{\searrow} & & \underset{\tau^1_!}{\nearrow} & \\ & & \mathrm{K}^*(V_1) & & \end{array}$$

where λ^i_* are the Thom isomorphisms, and τ_i are tubular neighbourhoods. We will do this by defining a map $\mathrm{K}^*(V_0) \to \mathrm{K}^0(V_1)$ which makes each of the triangles commute.

Since F is a regular homotopy, it follows from homotopy invariance of vector bundles that there exists an isomorphism $\varphi : V_1 \to V_0$. We claim that $\varphi^* : \mathrm{K}^*(V^0) \to \mathrm{K}^*(V^1)$ is the map we want, so we set about showing this.

Let $\tau^0 : V_0 \to Y$ be a tubular neighbourhood. Since the choice of tubular neighbourhood does not matter, we may choose τ^1 to be the composition

$$\tau^1 : V^1 \xrightarrow{\varphi^{-1}} V^0 \xrightarrow{\tau^0} Y.$$

Since $\varphi_! = (\varphi^*)^{-1}$, we have that $(\tau^0 \circ \varphi^{-1})_! = \tau^0_! \circ \varphi^{-1}_! = \tau^0_! \circ \varphi^*$. It therefore remains to show that $\varphi^* \lambda^0_* = \lambda^1_*$.

Recall that the Thom isomorphism is given by multiplication by the Thom class for a given K-orientation. Using φ we obtain two different K-orientations for V_0, and we must show that the are equivalent. Since F is a K-oriented regular homotopy, the two K-orientations are concordant whence equivalent. This completes the proof. $\square$

Finally, let $f : X \to Y$ be any K-oriented map. Choose an embedding $\varphi : X \to \mathbb{R}^k$ for sufficiently large k. Then (f, φ) is a K-oriented embedding and we define $f_! : \mathrm{K}^*(X) \to \mathrm{K}^{*-(m-n)}(Y)$ to be the composition

$$\begin{array}{ccc} \mathrm{K}^{*-(m-n)-k}(Y \times \mathbb{R}^k) & \xrightarrow{\beta_k^{-1}} & \mathrm{K}^{*-(m-n)}(Y) \\ \uparrow{\scriptstyle (f,\varphi)_!} & \nearrow{\scriptstyle f_!} & \\ \mathrm{K}^*(X) & & \end{array}$$

where β_k^{-1} denotes the inverse to the Bott map. To see that this is independent of embedding, let $\varphi_0 : X \to \mathbb{R}^n$ and $\varphi_1 : X \to \mathbb{R}^m$ be embeddings. Then $(f, \varphi_0, \varphi) : X \to Y \times \mathbb{R}^n \times \mathbb{R}^m$ is an embedding, and we will show that using $(f, \varphi_0, \varphi_1)$ to construct $f_!$ gives the same map as if we used (f, φ_0) or (f, φ_1).

Note that the map

$$\begin{aligned} F : X \times [0,1] &\to Y \times \mathbb{R}^n \times \mathbb{R}^m \\ (x,t) &\mapsto (f(x), \varphi_0(x), t\varphi_1(x)) \end{aligned}$$

is a K-oriented regular homotopy from $(f, \varphi_0, \varphi_1)$ to $(f, \varphi_0, 0)$. Thus $(f, \varphi_0, \varphi_1)_! = (f, \varphi_0, 0)_!$ by Lemma 3.3.3.

Since $(f, \varphi_0, 0) = \zeta_0 \circ (f, \varphi_0)$, where $\zeta_0 : Y \times \mathbb{R}^n \to Y \times \mathbb{R}^n \times \mathbb{R}^m$ is the zero section, the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}^*(Y \times \mathbb{R}^n) & \xrightarrow{(\zeta_0)_!} & \mathrm{K}^*(Y \times \mathbb{R}^n \times \mathbb{R}^m) \\ \uparrow{\scriptstyle (f,\varphi_0)_!} & \nearrow{\scriptstyle (f,\varphi_0,0)_!} & \\ \mathrm{K}^*(X) & & \end{array}$$

We note that $(\zeta_0)_!$ is equal to the Bott map $\beta_m : \mathrm{K}^*(Y \times \mathbb{R}^n) \to \mathrm{K}^*(Y \times \mathbb{R}^n \times \mathbb{R}^m)$, so that the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}^*(X) & \xrightarrow{(f,\varphi_0,0)_!} & \mathrm{K}^*(Y \times \mathbb{R}^n \times \mathbb{R}^m) \\ {\scriptstyle (f,\varphi_0)_!}\big\downarrow & \nearrow{\scriptstyle \beta_m} & \big\downarrow{\scriptstyle \beta_{n+m}^{-1}} \\ \mathrm{K}^*(Y \times \mathbb{R}^n) & \xrightarrow[\beta_n^{-1}]{} & \mathrm{K}^*(Y). \end{array}$$

This shows that using (f,φ_0) or (f,φ_0,φ_1) to construct $f_!$ gives the same map, which shows, by symmetry, that the choice of embedding φ is irrelevant. Thus, we have proved the following theorem.

Theorem 3.3.2. *Suppose that $f : X \to Y$ is a smooth, K-oriented map between smooth manifolds X and Y of dimensions n and m, respectively. Then there is a well-defined wrong way map*

$$f_! : \mathrm{K}^*(X) \to \mathrm{K}^{*-(m-n)}(Y).$$

We now show that if f was already an embedding, then these two constructions give the same map. For notational clarity, let $f_{em!}$ denote the wrong way construction for embeddings.

Proposition 3.3.3. *If $f : X \to Y$ is a K-oriented embedding, then $f_! = f_{em!}$.*

Proof. Let $\varphi : X \to \mathbb{R}^n$ be an embedding. Since f is an embedding, it follows that the map

$$\begin{aligned} F : X \times [0,1] &\to Y \times \mathbb{R}^n \\ (x,t) &\mapsto (f(x), t\varphi(x)) \end{aligned}$$

is a K-oriented regular homotopy between the maps (f,φ) and $(f,0)$. Thus, by

Lemma 3.3.3 we have that $(f, \varphi)_! = (f, 0)_!$. It follows from our previous calculations that $\beta^{-1}(f, 0)_! = f_!$, which completes the proof. $\square$

Chapter 4

Bivariant K-Theory

Bivariant K-theory is a bifunctor defined on the category of C*-algebras by G.G. Kasparov for his work on the Novikov Conjecture in [16]. In Section 4.1 we give a brief survey of C*-algebras and Kasparov's bivariant K-theory. In Section 4.2 we define a group related to KK-theory that will be used in the following section.

The key feature of Kasparov's bivariant K-theory is the existence of the intersection product. Because of this, we can define a new category whose objects are C*-algebras and whose morphisms are given by classes in the bivariant K-group. One can hope that when one restricts the objects to commutative C*-algebras, the groups can be defined using a more geometric construction and the intersection product can be computed using geometric considerations. In Section 4.3 we define geometric bivariant K-theory using topological correspondences, and define a geometric intersection product. Finally, we show the two categories agree in Section 4.4. All of the work in this Chapter is based heavily on the work of Emerson and Meyer in [11], building on the ideas of Connes and Skandalis [5] and Baum [3].

4.1 C*-Algebras and Analytic KK-Theory

C*-algebras are a natural generalization of locally compact Hausdorff spaces, and K-theory extends rather easily to the category of C*-algebras. Building on the ideas of Atiyah, G.G. Kasparov extended K-theory to a bifunctor on C*-algebras in [16]. The point of this section is to give a brief introduction to C*-algebras and Kasparov's KK-theory, in order to motivate the rest of this thesis. We begin with the definition of a C*-algebra.

Definition 4.1.1. A *C*-algebra* is a Banach algebra A equipped with a conjugate linear anti-involution $a \mapsto a^*$ which satisfies the following identity:

$$\|a^* a\| = \|a\|^2. \tag{4.1}$$

This involution map is called the *adjoint*, and Equation (4.1) is called the *C* identity*.

There are several good resources that one can use to learn about C*-algebras, including [9], [12], [14], [20], and [23]. The definition of a C*-algebra is very robust in the sense that given some data, it is very often the case that one can associate to it a C*-algebra[1]. The most important example for the purposes of this thesis is Example 4.1.3. The two other basic examples are as follows.

Example 4.1.1. The complex numbers $\mathbb{C}$ are a C*-algebra, with the standard norm and adjoint given by complex conjugation. More generally, the matrix algebras $M_n(\mathbb{C})$ form a C*-algebra, with the adjoint given by the conjugate transpose.

Example 4.1.2. Let H be a Hilbert space and let $\mathbb{B}(H)$ denote the bounded operators on H. By the Riesz representation theorem, given any $T \in \mathbb{B}(H)$, there is a

[1] This is an intentionally vague statement.

unique operator $T^* \in \mathbb{B}(H)$ such that $\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle$; T^* is called the *adjoint* of T and the mapping $T \mapsto T^*$, together with the operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$ makes $\mathbb{B}(H)$ into a C*-algebra. More generally, if A is any $*$-closed (that is, $a^* \in A$ whenever $a \in A$), norm closed subalgebra of $\mathbb{B}(H)$, then A is a C*-algebra. A remarkable fact (due the Gelfand-Naimark-Segal) is that every C*-algebra can be realized in this way.

The main example for the purposes of non-commutative geometry is the following.

Example 4.1.3. Suppose that X is a locally compact Hausdorff space. Then $C_0(X)$, the collection of functions which vanish at infinity, is a (commutative) C*-algebra when equipped with the sup norm $\|f\|_\infty = \sup_X |f(x)|$. This C*-algebra is unital if and only if X is compact.

We are now going to describe how every unital *commutative* C*-algebra is isomorphic (as a C*-algebra) to $C(X)$ for some space compact Hausdorff space X.

Definition 4.1.2. Let A be a unital, commutative C*-algebra. We define the *spectrum* of A to be $\widehat{A} := \mathrm{Hom}(A, \mathbb{C})$, endowed with the topology of pointwise convergence. The *Gelfand transform* is the map $A \to C(\widehat{A})$ given by $a \mapsto \hat{a}$, where

$$\hat{a}(\chi) := \chi(a).$$

Gelfand's theorem is the following.

Theorem 4.1.1 (Gelfand)**.** *If A is a commutative, unital C*-algebra, then $\widehat{A}$ is a compact Hausdorff space and the Gelfand transformation*

$$\begin{aligned} A &\to C(\widehat{A}) \\ a &\mapsto \hat{a} \end{aligned}$$

is an isomorphism of C-algebras.*

This result suggests that C*-algebras "extend" the notion of spaces, and is why C*-algebras are sometimes referred to a "non-commutative spaces". One can therefore ask whether or not K-theory can be extended to the category of C*-algebras in such a way that $\mathrm{K}_0(C(X)) \cong \mathrm{K}^0(X)$ when X is compact Hausdorff[2]. The answer is yes, and the proper definition is motivated by Swan's Theorem.

Theorem 4.1.2 (Swan)**.** *Suppose that X is a compact Hausdorff space. There is a 1-1 correspondence between vector bundles over X and finitely generated, projective modules over $C(X)$ given by*

$$V \mapsto \Gamma(V).$$

Replacing $C(X)$ with an arbitrary unital C*-algebra A motivates the following definition.

Definition 4.1.3. For a unital C*-algebra A, we define $\mathrm{K}_0(A)$ to be the Grothendieck completion of the semi-group of finitely generated projective modules over A. If A is non-unital, then we define

$$\mathrm{K}_0(A) := \ker\{\epsilon_+^* : \mathrm{K}_0(A^+) \to \mathbb{Z}\},$$

where A^+ is the unitization of A, and $\epsilon_+ : A^+ \to A^+/A \cong \mathbb{C}$ is the quotient map.

If $A = C_0(X)$, then $A^+ \cong C(X^+)$, so this definition coincides with our previous one.

Remark. Definition 4.1.3 is the definition of K-theory for general rings. A remarkable feature of C*-algebras is that their K-theory satisfies Bott periodicity, whereas the algebraic K-theory of rings in general does not. This is further evidence that C*-algebras are a generalization of spaces.

[2]Since $X \mapsto C(X)$ is contravariant and $X \mapsto \mathrm{K}^0(X)$ is contravariant, it follows that $X \mapsto \mathrm{K}(C(X))$ would be covariant, and this is why the dimension appears at the foot of the K

In Atiyah's paper [2], it became apparent that elliptic operators on a smooth manifold X could be used to give maps $\mathrm{K}^*(X) \to \mathbb{Z}$. Thus, Atiyah postulated that a dual theory to K-theory, called K-homology, should be built from "abstract elliptic operators". Inspired by this, Kasparov defined K-homology groups using *Fredholm modules* modulo the appropriate equivalence relation. For a wonderful exposition on K-homology, see [7].

By only a slight generalization of the definitions used to build K-homology, Kasparov then defined his bivariant K-functor called KK-theory. Just as K-homology classes give rise to maps $K_*(A) \to \mathbb{Z}$, a class in $\mathrm{KK}_*(A, B)$ gives rise to a map $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$. The rest of this section we work towards defining $\mathrm{KK}_*(A, B)$ for (separable[3]) C*-algebras A and B. For a nice introduction to KK-theory, we recommend [13].

Definition 4.1.4. Let A be a C*-algebra. A *Hilbert A-module* is a right A-module $\mathcal{E}$ equipped with an A-valued form $\langle -, - \rangle : \mathcal{E} \times \mathcal{E} \to A$ such that

i. $\langle \eta, a\xi_1 + b\xi_2 \rangle = a\langle \eta, \xi_1 \rangle + b\langle \eta, \xi_2 \rangle$ for $a, b \in \mathbb{C}$,

ii. $\langle \eta, \xi a \rangle = \langle \eta, \xi \rangle a$,

iii. $\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle$,

iv. $\mathrm{spec}(\langle \eta, \eta \rangle) \subseteq [0, \infty)$ where $\mathrm{spec}(a)$ denotes the spectrum of $a \in A$[4],

v. $\langle \eta, \eta \rangle = 0 \Leftrightarrow \eta = 0$, and

vi. $\mathcal{E}$ is complete with respect to the norm $\|\eta\| = \|\langle \eta, \eta \rangle\|^{1/2}$.

Example 4.1.4. A Hilbert $\mathbb{C}$-module is precisely a Hilbert space.

[3]The separability assumption is needed for some technical reasons.

[4]This is the notion of *positivity* in a C*-algebra.

Example 4.1.5. Any C*-algebra A is a Hilbert A-module with inner product given by $\langle a, b\rangle = a^*b$. For simplicity assume that A is unital; the non-unital case follows because A is closed in its unitization. Completeness follows because the norm given by the A-module structure is the same as the C*-norm:

$$\|\eta\|_A = \|\eta^*\eta\|^{1/2} = (\|\eta\|^2)^{1/2} = \|\eta\|.$$

Hilbert A-Module maps differ from linear maps between Hilbert spaces in that it is not automatic that an adjoint exists. For this, we make the following definition.

Definition 4.1.5. A module map $T : \mathcal{E}_0 \to \mathcal{E}_1$ between Hilbert A-modules $\mathcal{E}_0$ and $\mathcal{E}_1$ is called an *operator* if there exists a map $T^* : \mathcal{E}_1 \to \mathcal{E}_0$ such that

$$\langle T^*\eta, \xi\rangle = \langle \eta, T\xi\rangle.$$

The collection of all operators between Hilbert A-modules $\mathcal{E}_0$ and $\mathcal{E}_1$ is denoted $\mathbb{B}(\mathcal{E}_0, \mathcal{E}_1)$.

We now need the appropriate notion of compact and Fredholm operators. Note that $\mathbb{B}(\mathcal{E})$ is a C*-algebra for any Hilbert A-module $\mathcal{E}$.

Definition 4.1.6. An operator $T \in \mathbb{B}(\mathcal{E}_0, \mathcal{E}_1)$ is called *finite rank* if it is of the form

$$T(\eta) = \sum_{k=1}^{n} \xi_k \langle \eta_k, \eta\rangle$$

for some vectors $\eta_1, \ldots, \eta_k \in \mathcal{E}_0$ and $\xi_1, \ldots, \xi_n \in \mathcal{E}_1$. An operator $T \in \mathbb{B}(\mathcal{E}_0, \mathcal{E}_1)$ is called *compact* if it is in the norm closure of the finite rank operators. The compact operators are denoted $\mathbb{K}(\mathcal{E}_0, \mathcal{E}_1)$.

The operator T is called *Fredholm* if there exists an operator $S \in \mathbb{B}(\mathcal{E}_1, \mathcal{E}_0)$ such that $1 - ST$ and $1 - TS$ are compact. The operator S is called a *separatrix* for T.

With these definitions, we are finally able to define the bicycles which make up Kasparov's KK-groups.

Definition 4.1.7. Let A and B be (separable) C*-algebras. A *Fredholm* (A, B)-*bimodule* is a triple $(\mathcal{E}, \pi, F)$ where

1. $\mathcal{E}$ is a Hilbert B-module,
2. $\pi : A \to \mathbb{B}(\mathcal{E})$ is a *-homomorphism (that is, a representation of A by operators on the Hilbert B-module $\mathcal{E}$),
3. $F \in \mathbb{B}(\mathcal{E})$ is a Fredholm operator such that

$$\pi(a)(F - F^*), \quad \pi(a)(F^2 - 1), \quad \text{and } [\pi(a), F] \tag{4.2}$$

are compact for all $a \in A$.

The bimodule $(\mathcal{E}, \pi, F)$ is called *even* if $\mathcal{E}$ splits as the direct sum $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ or orthogonal submodules on which the elements of A act as even operators and F acts as an odd operator; we also say that $(\mathcal{E}, \pi, F)$ is $\mathbb{Z}_2$-graded. If $(\mathcal{E}, \pi, F)$ has no such $\mathbb{Z}_2$-grading, we say it is *odd*. A Fredholm (A, B)-module is called *degenerate* if all the operators in (4.2) are equal to 0.

Example 4.1.6. Any *-homomorphism $\varphi : A \to B$ yields a Fredholm (A, B)-module. Indeed, we let B be our Hilbert B-module with left action of A defined as

$$\pi_\varphi(a)b := \varphi(a)b.$$

By considering an approximate unit in B, we see that A acts as compact operators whence $[\varphi] = [B, \pi_\varphi, 0]$ is a Fredholm (A, B)-module, as needed.

Kasparov's bivariant K-theory is defined as *homotopy classes* of Fredholm (A, B)-modules, which we now define.

Definition 4.1.8. A homotopy of Fredholm (A, B)-bimodules $(\mathcal{E}_0, \pi_0, F_0)$ and $(\mathcal{E}_1, \pi_1, F_1)$ is a $(A, B \otimes C([0,1]))$-bimodule which restricts to $(\mathcal{E}_i, \pi_i, F_i)$ for $i = 0, 1$.

Definition 4.1.9. We let $\mathrm{KK}_0(A, B)$ denote homotopy classes of even Fredholm (A, B)-bimodules, and $\mathrm{KK}_1(A, B)$ denote homotopy classes of odd Fredholm (A, B)-bimodules. We define

$$\mathrm{KK}_*(A, B) := \mathrm{KK}_0(A, B) \oplus \mathrm{KK}_1(A, B).$$

Classes in $\mathrm{KK}_*(A, B)$ are called *Kasparov morphisms.*

Example 4.1.7. Any K-oriented map $f : X \to Y$ between smooth manifolds X and Y determines a *wrong way* class $[f_!] \in \mathrm{KK}_*(C_0(X), C_0(Y))$ via a process which we now describe.

First, assume that f is an embedding. Let N_f denote the normal bundle of X in Y using f, and let $\tau : N_f \to Y$ be a tubular neighbourhood. The tubular neighbourhood identifies $C_0(N_f)$ as an ideal in $C_0(Y)$, thus gives a class $[\tau] \in \mathrm{KK}_*(C_0(N_f), C_0(Y))$, by Example 4.1.6. Next, the K-orientation on f determines a K-orientation on N_f by the 2-out-of-3 lemma. Let $\pi : N_f \to X$ be the vector bundle projection, and let (S, c) be the spinor for N_f. Define $F \in \mathbb{B}(\Gamma(\pi^* S))$ and $\rho_\pi : C_0(X) \to \mathbb{B}(\Gamma(\pi^* S))$ by

$$F(\xi, \eta) = \frac{c(\xi)\eta}{(1 + |\xi|^2)^{1/2}} \quad \text{and} \quad \rho_\pi(f)(s) = m_{f \circ \pi} s,$$

respectively, where $m_{f\circ\pi}$ denotes multiplication by $f \circ \pi$. This determines a class $[\Gamma(\pi^* S), \rho_\pi, F] \in \mathrm{KK}_*(C_0(X), C_0(N_f))$. We define

$$[f_!] = [\Gamma(\pi^* S), \rho_\pi, F] \otimes_{C_0(N_f)} [\tau] \in \mathrm{KK}_*(C_0(X), C_0(Y)).$$

If f is not an embedding, then choose an embedding $\varphi : X \to \mathbb{R}^n$. We define $[f_!]$ the be the composition of $[(f,\varphi)_!] \in \mathrm{KK}_*(C_0(X), C_0(Y \times \mathbb{R}^n))$ with the Thom isomorphism $\mathrm{KK}_*(C_0(X), C_0(Y \times \mathbb{R}^n)) \cong \mathrm{KK}_*(C_0(X), C_0(Y))$.

$\mathrm{KK}_0(A,B)$ turns out to be an abelian group via the direct sum, with inverses given by reversing the $\mathbb{Z}_2$-grading. The most important feature of Kasparov theory is the category structure. That is, Kasparov morphisms can be composed.

Theorem 4.1.3. *For any separable C*-algebras A, B, and C, there is a bilinear pairing*

$$\mathrm{KK}_*(A,B) \times \mathrm{KK}_*(B,C) \to \mathrm{KK}_*(A,C),$$

called the intersection product*, which maps Kasparov morphisms $f \in \mathrm{KK}_i(A,B)$ and $g \in \mathrm{KK}_j(B,C)$ to a morphism $f \hat{\otimes}_B g \in \mathrm{KK}_{i+j}(A,C)$.*

The existence of the Kasparov product allows us to define a new category.

Definition 4.1.10. The category KK_* is defined to be the category whose objects are separable C*-algebras and whose morphisms are given by Kasparov morphisms.

It can be shown that for any C*-algebra A, $\mathrm{KK}_*(\mathbb{C}, A) \cong \mathrm{K}_*(A)$. Thus, bivariant K-theory truely extends K-theory. Furthermore, because of the intersection product any class $\eta \in \mathrm{KK}(A,B)$ determines a groups homomorphism $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$ as follows:

$$\mathrm{K}_*(A) \cong \mathrm{KK}_*(\mathbb{C}, A) \xrightarrow{\hat{\otimes}\eta} \mathrm{KK}_*(\mathbb{C}, B) \cong \mathrm{K}_*(B).$$

In fact, for a large class of C*-algebras, *every* map between K-groups arises in this way (see [13], for instance).

The existence of the intersection product is a technical result due to Kasparov, and it is not constructive. In the case when A and B are continuous functions on smooth manifolds, one could hope to be able to compute the intersection product using geometric considerations. This turns out to be the case, and the remainder of this chapter is devoted to developing a topological description of bivariant K-theory. Topological bivariant K-theory is defined using *topological correspondences*, an idea originally due to Connes and Skandalis in [5]. The theory of correspondences was then fully developed by Emerson and Meyer in [11] and [10], and this is the description that we follow.

4.2 Representable K-Theory

One of the ingredients in a correspondence will be a class in the *representable K-theory* of a space, defined in [10].

Definition 4.2.1. Suppose that $b : M \to X$ is a smooth map. The *representable K-theory* of M with X-compact support, denoted by $\mathrm{RK}_X^{-n}(M)$, is defined as equivalence classes of Fredholm $(C_0(X \times \mathbb{R}^n), C_0(M \times \mathbb{R}^n))$-modules where $C_0(X \times \mathbb{R}^n)$ acts by first pulling back to $C_b(M \times \mathbb{R}^n)$ (bounded continuous functions on $M \times \mathbb{R}^n$) using b. We define $\mathrm{RK}_X^*(M) = \bigoplus_{n \geq 0} \mathrm{RK}_X^{-n}(M)$. If $\xi \in \mathrm{RK}_X^*(M)$, then $\deg \xi$ is defined as the number n such that $\xi \in \mathrm{RK}_X^{-n}(M)$, if such an n exists.

The data coming from $\mathrm{RK}_X^*(M)$ will act as a multiplier for K-theory and therefore, motivated by the Thom homomorphism, it would be nice to describe classes in $\mathrm{RK}_X^*(M)$ as certain complexes over M. The rest of this section is devoted to showing that such complexes (to be defined) always give classes in $\mathrm{RK}_X^*(M)$.

Definition 4.2.2. Let M and X be manifolds and let $b : M \to X$ be a map. We say a set $N \subseteq M$ is X*-compact* if $b^{-1}(Y) \cap N$ is a compact subset of M for all compact subsets $Y \subseteq X$. A complex of vector bundles $V^\bullet$ over M is X-compactly supported if its support

$$\mathrm{supp}(V^\bullet) = \{m \in M : V_m^\bullet \text{ is } \textit{not} \text{ exact}\}$$

is X-compact.

Example 4.2.1. If $b : M \to X$ is proper then every closed set is X-compact. Since the support of any complex of vector bundles is automatically closed (see [22]), every complex is therefore X-compactly supported.

Example 4.2.2. If X is compact, then the only X-compact sets are the compact sets. Indeed, if X is compact then $N \subseteq M$ is X-compact only if $b^{-1}(X) \cap N = M \cap N = N$ is compact. Thus, the only X-compactly supported complexes are the compactly supported complexes.

Example 4.2.3 (The Thom Class)**.** Let V be an even rank vector bundle over X with spinor $S_V = S_V^0 \oplus S_V^1$. Then the complex

$$0 \longrightarrow \pi_V^*(S_V^0) \xrightarrow{c} \pi_V^*(S_V^1) \longrightarrow 0,$$

$c(v, \xi) = c(v)\xi$, is X-compactly supported. Indeed, its support is the zero section.

An X-compactly supported complex $V^\bullet$ is said to be *acyclic* if its support is empty. A *homotopy* between X-compactly supported complexes $V_0^\bullet$ and $V_1^\bullet$ over M is an X-compactly supported complex $V^\bullet$ over $M \times [0,1]$ such that

$$V_0^\bullet = V^\bullet|_{X\times\{0\}} \quad \text{and} \quad V_1^\bullet = V^\bullet|_{X\times\{1\}}.$$

We write $V_0^\bullet \simeq V_1^\bullet$ if there exists a homotopy between $V_0^\bullet$ and $V_1^\bullet$. We define X-compactly supported complexes $V_0^\bullet$ and $V_1^\bullet$ to be *equivalent* if there exist acyclic complexes $W_0^\bullet$ and $W_1^\bullet$ such that $V_0^\bullet \oplus W_0^\bullet \simeq V_1^\bullet \oplus W_1^\bullet$.

Definition 4.2.3. We define $\mathrm{VK}_X^0(M)$ to be the collection of equivalence classes of X-compactly supported complexes over M. It is an abelian group with respect to the direct sum. We extend $\mathrm{VK}^0_X(M)$ to a graded theory by defining

$$\mathrm{VK}^{-n}_X(M) = \mathrm{VK}^0_X(M \times \mathbb{R}^n).$$

Example 4.2.4. If X is compact, then $\mathrm{VK}^*_X(M) = \mathrm{K}^*(M)$, since the only X-compact sets are compact sets.

We will now show that every X-compactly supported complex over M defines a class in $\mathrm{RK}^0_X(M)$.

Proposition 4.2.1. *There is a natural map*

$$\mathrm{VK}^0_X(M) \to \mathrm{RK}^0_X(M).$$

Proof. For a given class $\xi \in \mathrm{VK}^0_X(M)$, we are going to construct a class $[(b,\xi)^*] \in \mathrm{RK}^0_X(M)$; we use this notation to denote the dependence on b. For simplicity, we assume that ξ is given by a complex

$$\xi : 0 \longrightarrow \xi^0 \xrightarrow{u} \xi^1 \longrightarrow 0$$

with u a unitary outside an X-compact subset of M. We let $L^2(M, \xi^0 \oplus \xi^1)$ be the required Hilbert $C_0(M)$-module, with the action being multiplication. Let

$$F := \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix},$$

and define

$$[(b,\xi)^*] := [L^2(M, \xi^0 \oplus \xi^1), \pi_b, F],$$

where $\pi_b(f)$ denotes multiplication by $f \circ b$. We claim that $[(b,\xi)^*]$ is a Fredholm

$(C_0(X), C_0(M))$-module. Indeed, since $\pi_b(f)(F - F^*) = [\pi_b(f), F] = 0$ for all $f \in C_0(X)$ we need only show that $\pi_b(f)(F^2 - 1)$ is compact. If $f \in C_c(X)$, then observe that the support of $\pi_b(f)(F^2 - 1)$ is a compact subset of M, since its support is given by $b^{-1}(\mathrm{supp}(f)) \cap \mathrm{supp}(\xi)$, which is compact since ξ is X-compactly supported. If f is not compactly supported, then we can find a sequence $\{f_n\} \subseteq C_0(X)$ converging to f, and therefore, since the compact operators are closed,

$$\pi_b(f)(F^2 - 1) = \lim_{n \to \infty} \pi_b(f_n)(F^2 - 1) \in \mathbb{K}(L^2(M, \xi^0 \oplus \xi^1)).$$

Thus, $[(b, \xi)^*] \in \mathrm{RK}^0_X(M)$.

To see that this map is well-defined we note that an acyclic complex maps to a degenerate Kasparov module, and that a homotopy of complexed maps to a homotopy of Kasparov modules. □

Example 4.2.5. If $b : M \to X$ is proper, then any vector bundle over M determines a class on $\mathrm{RK}^0_X(M)$

4.2.1 The Thom Homomorphism

Let $\pi_V : V \to X$ be a K-oriented vector bundle. Since the K-orientation uniquely determines a K-orientation for $V \oplus \mathbb{R}$ we assume, for simplicity, that V is even rank. In this case the spinor S_V is $\mathbb{Z}_2$-graded and we have the Thom class

$$\lambda_V = [0 \longrightarrow \pi_V^*(S_V^0) \xrightarrow{c} \pi_V^*(S_V^1) \longrightarrow 0] \in \mathrm{RK}^0_X(V).$$

We want to show that λ_V induces a map $\lambda_V^* : \mathrm{RK}^0_X(X) \to \mathrm{RK}^0_X(V)$ by $\eta \mapsto \lambda_V \otimes \pi_V^*(\eta)$. This, of course, amounts to showing that it has X-compact support. For any $\eta \in$

$\mathrm{RK}_X^*(X)$ we have that

$$\operatorname{supp}(\lambda_V \otimes \pi_V^*(\eta)) = \operatorname{supp}(\lambda_V) \cap \pi_V^{-1}(\operatorname{supp}(\eta)),$$

which is a closed subset of the zero section in V. It follows that $\operatorname{supp}(\lambda_V \otimes \pi_V^*(\eta))$ is X-compact, whence the map $\lambda_V^* : \mathrm{RK}_X^0(V) \to \mathrm{RK}_X^0(V)$ is well defined. This map is called the *Thom homomorphism.* By replacing X and V with $X \times \mathbb{R}^n$ and $V \times \mathbb{R}^n$ respectively, we get a Thom homomorphism

$$\lambda_V^* : \mathrm{RK}_X^{-n}(X) \to \mathrm{RK}_X^{-n}(V)$$

for all $n \in \mathbb{N}$.

Example 4.2.6. For compact X, this is the familiar Thom isomorphism.

4.3 K-Theory Correspondences

4.3.1 Definition and Examples of Correspondences

Definition 4.3.1. A *(smooth) correspondence* between smooth manifolds X and Y is a quadruple

$$\Phi = X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y,$$

where

- M is a smooth manifold,
- $\xi \in \mathrm{RK}_X^*(M)$,
- $b : M \to X$ is a smooth map, and
- $f : M \to Y$ is smooth and K-oriented.

We sometimes refer to f as the *forward map* and b the *backward map* of Φ. The *sum* of two correspondences is given by their disjoint union:

$$[X \xleftarrow{b_0} (M_0, \xi_0) \xrightarrow{f_0} Y] + [X \xleftarrow{b_1} (M_1, \xi_1) \xrightarrow{f_1} Y] := [X \xleftarrow{b_0 \sqcup b_1} (M_0 \sqcup M_1, \xi_0 \sqcup \xi_1) \xrightarrow{f_0 \sqcup f_1} Y],$$

where $f_0 \sqcup f_1$ is K-oriented in the obvious way. The *degree* of a correspondence is defined to be $\dim Y + \deg \xi - \dim M$, if this is defined.

In practice, it is often the case that ξ is represented by an element in $\mathrm{VK}^0_X(M)$.

Example 4.3.1. For any smooth manifold X, we have the "identity correspondence" given by

$$[\mathrm{id}_X] = [X \xleftarrow{\mathrm{id}} (X, \mathbf{1}) \xrightarrow{\mathrm{id}} X].$$

Example 4.3.2. If Y is compact, then any map $b : Y \to X$ induces a correspondence

$$[b^*] := [X \xleftarrow{b} (Y, \mathbf{1}) \xrightarrow{\mathrm{id}_Y} Y].$$

Example 4.3.3. If $f : X \to Y$ is K-oriented, then f induces a correspondence

$$[f_!] := [X \xleftarrow{\mathrm{id}_X} (X, \mathbf{1}) \xrightarrow{f} Y].$$

In particular, since $\mathbb{R}^2$ is a K-oriented vector bundle over a point, we have a correspondence

$$[* \longleftarrow (*, \mathbf{1}) \longrightarrow \mathbb{R}^2],$$

which will turn out to represent the Bott element in $\mathrm{K}^0(\mathbb{R}^2)$, see Theorem 4.4.2.

Example 4.3.4. If X is a compact, K-orientable manifold then we have two correspondences

$$[\sigma_X] = [* \longleftarrow (X, \mathbf{1}) \xrightarrow{\Delta} X \times X]$$
$$[\rho_X] = [X \times X \xleftarrow{\Delta} (X, \mathbf{1}) \longrightarrow *],$$

where Δ is the diagonal map, K-oriented in the obvious way. These will be the *unit* and *counit* in Poincaré duality, discussed in Section 5.2.

More generally, if X is compact then we have correspondences

$$[\tilde{\sigma}_X] = [* \longleftarrow (X, \mathbf{1}) \xrightarrow{(\zeta, \mathrm{id})} TX \times X]$$
$$[\tilde{\rho}_X] = [X \times TX \xleftarrow{(\pi, \mathrm{id})} (TX, \mathbf{1}) \longrightarrow *],$$

where $\zeta : X \to TX$ is the zero section and $\pi : TX \to X$ is the vector bundle projection. These give a more general statement of Poincaré duality, see for instance [5].

Example 4.3.5. Let $\widehat{\mathbb{Z}}^d$ denote the Pontryagin dual of $\mathbb{Z}^d$. Recall the *Poincaré bundle* $\mathcal{P}_d$ defined in Example 2.1.10. It is the line bundle $\mathcal{P}_d = [\mathbb{R}^d \times \widehat{\mathbb{Z}}^d] \times_{\mathbb{Z}^d} \mathbb{C}$ over $\mathbb{T}^d \times \widehat{\mathbb{Z}}^d$ defined by the relation

$$(x, \chi, \lambda) \sim (x + n, \chi, \chi(n)\lambda), \quad \text{for } n \in \mathbb{Z}^d.$$

The *Fourier-Mukai correspondence* is defined as

$$\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d) \xrightarrow{\mathrm{pr}_2} \widehat{\mathbb{Z}}^d,$$

where $\mathbb{T}^d$ is K-oriented as a product of circles, and $\widehat{\mathbb{Z}}^d$ is K-oriented using the canonical diffeomorphism $\widehat{\mathbb{Z}}^d \to \mathbb{T}^d$. This correspondence plays a major role in the latter part of this article.

Definition 4.3.2. If $\Psi = (M, \xi, b, f)$ is a correspondence between X and Y then we define $-\Psi := (M, \xi, b, -f)$ where $-f$ means the same map as f but with the opposite K-orientation.

4.3.2 Equivalence of Correspondences

Equivalence of correspondences is generated by three steps:

- Isomorphism,
- Thom modification, and
- Bordism.

The notion of isomorphism is rather clear, but because of its importance in Chapter 5 we give an explicit definition.

Definition 4.3.3. We deem two correspondences

$$\begin{array}{ccccc} & \overset{b_0}{\swarrow} & (M_0, \xi_0) & \overset{f_0}{\searrow} & \\ X & & & & Y \\ & \underset{b_1}{\nwarrow} & (M_1, \xi_1) & \underset{f_1}{\nearrow} & \end{array} \tag{4.3}$$

to be *isomorphic* if there is a diffeomorphism $\varphi : M_0 \to M_1$ which fits into (4.3) making it commute such that

- $\xi_0 = \varphi^*(\xi_1)$ and
- the K-orientations on $\varphi^*(TM_1 \oplus f_1^*TY) \cong TM_0 \oplus f_0^*TY$ agree.

Next, we need to take into account Thom isomorphisms.

Definition 4.3.4. If V is a K-oriented vector bundle over M, and $\lambda_*^V : \mathrm{RK}_X^*(M) \to \mathrm{RK}_X^*(V)$ is the Thom Isomorphism, then we define the *Thom modification of*

$$\Phi = [X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y]$$

along V to be the correspondence

$$\Phi^V = [X \xleftarrow{b\circ\pi_v} (V, \lambda_*^V(\xi)) \xrightarrow{f\circ\pi_v} Y].$$

Here we are K-orienting $f \circ \pi_v$ as the composition of K-oriented maps (see Examples 3.3.3 and 3.3.5). If Φ is a Thom modification of Ψ along some vector bundle, then we write $\Psi \sim_{Tm} \Phi$.

Finally, we take bordism into account. A *correspondence with boundary*, or *∂-correspondence*, is a correspondence

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y,$$

where M is a manifold with boundary. Suppose that $\partial M = \partial_0 M \sqcup \partial_1 M$. Give the inward facing normal bundle at $\partial_0 M$ the positive K-orientation and the inward facing normal bundle at $\partial_1 M$ the negative K-orientation. Such a ∂-correspondence Φ induces a correspondence on its boundary as

$$\partial_i \Phi := [X \xleftarrow{b|_{\partial_i M}} (\partial_i M, \xi_0 \partial_i M) \xrightarrow{f|_{\partial_i M}} Y].$$

Here we give $f|_{\partial_i M}$ the K-orientation coming from the 2-out-of-3 Lemma.

Definition 4.3.5. Two smooth correspondences Φ_0 and Φ_1 are called *bordant* if there is a ∂-correspondence Φ such that

$$(-1)^i \Phi_i = \partial_i \Phi \quad \text{for } i = 0, 1.$$

If Φ and Ψ are bordant correspondences we write $\Phi \sim_b \Psi$.

Putting isomorphism, Thom modification, and bordism together yields equivalence of correspondences. We are finally in a position to define $\widehat{\mathrm{KK}}_*(X,Y)$ for smooth manifolds X and Y.

Definition 4.3.6. For smooth manifolds X and Y, we define $\widehat{\mathrm{KK}}_*(X,Y)$ to be the set of equivalence classes of correspondences from X to Y, where equivalence is generated by isomorphism, Thom modification, and bordism.

4.3.3 Examples of Equivalence

In this section we will give some examples of equivalent correspondences which we shall use throughout this article.

Example 4.3.6. For any correspondence $\Phi = (M,\xi,b,f)$ from X to Y, $\Phi \sqcup -\Phi$ bordant to the zero correspondence. Indeed, to show this we have to cook up a ∂-correspondence is $\Phi \sqcup -\Phi$. This is achieved by the correspondence

$$\Phi^I = [X \xleftarrow{b\circ \mathrm{pr}_1} (M\times I, \mathrm{pr}_1^*\xi) \xrightarrow{f\circ\mathrm{pr}_1} Y],$$

where $I = [0,1]$ is the unit interval K-oriented using the positive K-orientation, and $f\circ\mathrm{pr}_1$ is the composition of K-oriented maps.

Example 4.3.7. Let $\Phi = (M,\xi,b,f)$ be a correspondence from X to Y. Let $N\subseteq M$ be an open subset and suppose there is some $\eta \in \mathrm{RK}_X^*(N)$ which maps to ξ under the canonical mapping. Then

$$[X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y] \sim_b [X \xleftarrow{b|_N} (N,\eta) \xrightarrow{f|_N} Y].$$

Indeed, for our manifold with boundary we take

$$\widetilde{M} = N\times\{0\} \cup M\times(0,1] \subseteq M\times[0,1];$$

this is an open subset of $M \times [0,1]$ whence is itself a smooth manifold. The forward and backward maps, $\tilde{f}$ and $\tilde{b}$ respectively, are defined via the commuting diagram

$$\begin{array}{ccccc} & & M \times [0,1] & & \\ & \swarrow^{b \circ \mathrm{pr}_1} & \uparrow & \searrow^{f \circ \mathrm{pr}_1} & \\ X & \xleftarrow[\tilde{b}]{} & \widetilde{M} & \xrightarrow[\tilde{f}]{} & Y \end{array}$$

Since the inclusion $\widetilde{M} \to M \times [0,1]$ is a local diffeomorphism, it is canonically K-oriented, which K-orients $\tilde{f}$. It remains to specify the K-theory data. The map $\mathrm{pr}_1 : N \times [0,1] \to N$ is proper, hence $\mathrm{pr}_1^*(\eta) \in \mathrm{RK}_X^*(N \times [0,1])$. Since $N \times [0,1]$ is open in $\widetilde{M}$, it extends to a class $\tilde{\xi} \in \mathrm{RK}_X^*(\widetilde{M})$ with the desired properties. Thus

$$[X \xleftarrow{\tilde{b}} (\widetilde{M}, \tilde{\xi}) \xrightarrow{\tilde{f}} Y]$$

is the required ∂-correspondence.

Example 4.3.8. If $\Phi = (M, \xi, b, f)$ and $\Psi = (M, \eta, b, f)$ are two correspondences from X to Y which differ only in their K-theory data, then

$$[X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y] + [X \xleftarrow{b} (M,\eta) \xrightarrow{f} Y] = [X \xleftarrow{b} (M,\xi+\eta) \xrightarrow{f} Y]$$

in $\widehat{\mathrm{KK}}_*(X,Y)$. Proving this requires some cohomological properties of $\mathrm{RK}_X^*(M)$, so we cite [11]. Note that, in particular, this implies that any equivalence class or correspondence is the sum of correspondences with well defined degree. Equivalence preserves the degree and sum of correspondence, so $\widehat{\mathrm{KK}}_*(X,Y)$ is a graded monoid. By Example 4.3.6, we see that it is, in fact, a graded Abelian group.

Example 4.3.9. If $\Phi = (M, \xi, b, f)$ is any correspondence from Y to Y, then Φ is equivalent to a correspondence $\tilde{\Phi}$ whose forward map is a submersion.

Indeed, choose an embedding $\varphi : M \to \mathbb{R}^n$ for some n and let V be the normal bundle to the embedding $(f,\varphi) : M \to Y \times \mathbb{R}^n$. Since f is K-oriented, the map (f,φ) is K-oriented by the 2-out-of-3 lemma, whence V is too. Let $\tau : V \to Y \times \mathbb{R}^n$ be a tubular neighbourhood. We will show that we can take

$$\tilde{\Psi} = [X \xleftarrow{b\circ\pi_v} (V, \lambda_*^V(\xi)) \xrightarrow{\mathrm{pr}_1\circ\tau} Y],$$

where $\pi_V : V \to M$ is the vector bundle projection.

First, by a Thom modification along V we have that

$$[X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y] \sim_{Tm} [X \xleftarrow{b\circ\pi_v} (V, \lambda_*^V(\xi)) \xrightarrow{f\circ\pi_V} Y].$$

Now, consider the homotopy $F : ((x,\xi),t) \mapsto \mathrm{pr}_1 \circ \tau(x, t\xi)$ between $f \circ \pi_V$ and $\mathrm{pr}_1 \circ \tau$. We claim that there is a K-orientation on F so that

$$[X \longleftarrow (V \times [0,1], \lambda_*^V(\xi)) \xrightarrow{F} Y]$$

provides a bordism between $(V, \tau_*^V(\xi), b \circ \pi_V, f \circ \pi_V)$ and $(V, \tau_*^V(\xi), b \circ \pi_V, \mathrm{pr}_Y \circ \tau)$. Indeed, first note that F factors as

$$V \times [0,1] \xrightarrow{m} V \xrightarrow{\mathrm{pr}_Y\circ\tau} Y,$$

where $m((x,\xi),t) = (x,t\xi)$, so that it is sufficient to give a K-orientation to m. Since $\pi_V \circ m = \pi_V \circ \mathrm{pr}_V$ we have

$$T(V \times [0,1]) \oplus m^*(TV) = T(V \times [0,1]) \oplus \mathrm{pr}_V^*(TV),$$

which is canonically K-oriented. We conclude that

$$[X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y] = [X \xleftarrow{b\circ\pi_v} (V,\lambda_*^V(\xi)) \xrightarrow{\mathrm{pr}_Y\circ\tau} Y]$$

in $\widehat{\mathrm{KK}}_*(X,Y)$.

4.3.4 Composition of Correspondences

We will now discuss how to compose correspondences, assuming first a transversality condition.

Definition 4.3.7. Correspondences $\Phi = (M,\xi,b_M,f_M)$ and $\Psi = (N,\eta,b_N,f_N)$ from X to Y and from Y to Z, respectively, are *transverse* if the map

$$df_M - db_N : T_mM \oplus T_nN \to T_{f_M(m)}Y$$

is surjective for all $(m,n) \in M \times_Y N := \{(x,y) \in M \times N : f_M(x) = b_N(y)\}$.

Transversality ensures that the fibered product $M \times_Y N$ is a smooth manifold. Let $\mathrm{pr}_M : M \times_Y N \to M$ (resp. pr_N) is the projection onto M (resp. N).

Definition 4.3.8. If $\Phi = (M,\xi,b_M,f_M)$ and $\Psi = (N,\eta,b_N,f_N)$ are transverse correspondences from X to Y and from Y to Z respectively, then their *intersection product* is the correspondence

$$\Phi \otimes \Psi := [X \xleftarrow{b_M\circ\mathrm{pr}_M} (M \times_Y N, \mathrm{pr}_M^*(\xi)\hat{\otimes}\mathrm{pr}_N^*(\eta)) \xrightarrow{f_N\circ\mathrm{pr}_N} Z] \in \widehat{\mathrm{KK}}_*(X,Z),$$

where $f_N \circ \mathrm{pr}_N$ is given the K-orientation discussed below.

In order for this to make sense, we must give $f_N \circ \mathrm{pr}_N$ a K-orientation. Since f_N is K-oriented, it is sufficient to K-orient pr_N as the composition of K-oriented maps is K-oriented. To do this, first observe that transversality implies that we have an exact sequence

$$0 \longrightarrow T(M \times_Y N) \longrightarrow T(M \times N)|_{M \times_Y N} \xrightarrow{df_M - db_N} (f_M \circ \mathrm{pr}_M)^* TY \longrightarrow 0.$$

In particular, we have an isomorphism $T(M \times_Y N) \oplus (f_M \circ \mathrm{pr}_M)^* TY \cong \mathrm{pr}_M^* TM \oplus \mathrm{pr}_N^* TN$. This implies the equality

$$\begin{aligned} T(M \times_Y N) \oplus \mathrm{pr}_N^*(TN) \oplus (f_M \circ \mathrm{pr}_M)^*(TY \oplus TY) \\ \cong \mathrm{pr}_M^*(TM \oplus f_M^* TY) \oplus \mathrm{pr}_N^*(TN \oplus TN) \end{aligned} \tag{4.4}$$

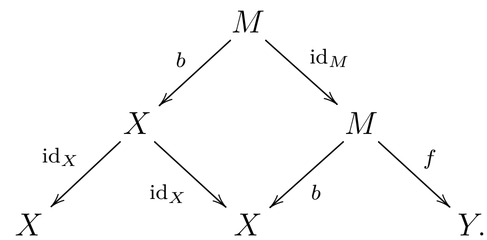
which, since f_M is K-oriented, gives a K-orientation to pr_N by the 2 out of 3 lemma. Two different choices of splitting will yield isomorphisms which are connected by a path, whence give the same K-orientation.

Now, given any two correspondences $\Phi = (M, \xi, b_M, f_M)$ and $\Psi = (N, \eta, b_N, f_N)$, it need not be the case that they are transverse. However, Example 4.3.9 shows that Φ is equivalent to a correspondence $\tilde{\Phi}$ whose forward map is a submersion. Since any map is transverse to a submersion, we may compose these correspondences. It is shown in [11] that this is well defined on equivalence classes of correspondences whence the composition product is defined for all $(\Phi, \Psi) \in \widehat{\mathrm{KK}}_*(X, Y) \times \widehat{\mathrm{KK}}_*(Y, Z)$.

Proposition 4.3.1. *If Φ is any correspondence for X to Y, then*

$$[\mathrm{id}_X] \otimes \Phi = \Phi \otimes [\mathrm{id}_Y] = \Phi.$$

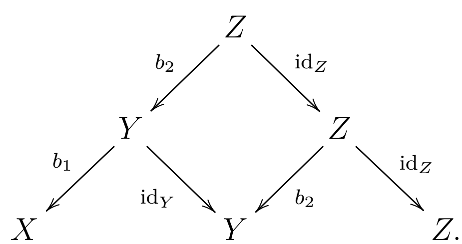
Proof. Let $\Phi = [X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y]$. Since id_X is a submersion, it is transverse to b so we can compose these correspondences using the composition product. The fibred product of X and M is M, and the composition diagram is



The K-orientation that id_M receives is the standard one, whence $f \circ \mathrm{id}_M$ is K-oriented in the same was as f. Thus, $[\mathrm{id}_X] \otimes \Phi = \Phi$, as claimed. The fact that $\Phi \otimes [\mathrm{id}_Y] = \Phi$ follows by the same argument. □

Proposition 4.3.2. *If $b_1 : Y \to X$ and $b_2 : Z \to Y$ are proper maps, then $[b_1^*] \otimes [b_2^*] = [(b_1 \circ b_2)^*]$.*

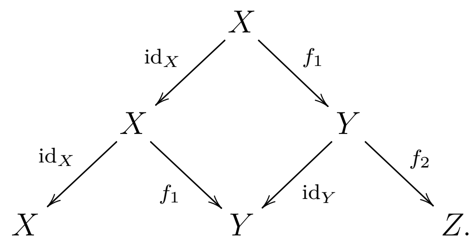
Proof. The fibred product of Y and Z is Z, and the composition diagram is given by



The K-orientation the id_Z receives is the standard one, whence the result follows. □

Proposition 4.3.3. *If $f_1 : X \to Y$ and $f_2 : Y \to Z$ are K-oriented maps then $[(f_1)_!] \otimes [(f_2)_!] = [(f_2 \circ f_1)_!]$.*

Proof. The fibred product of X and Y is X and the composition diagram



By checking the spinors, it follows from the uniqueness property of the 2 out of 3 lemma that $f_2 \circ f_1$ is K-oriented as in Example 3.3.5, from which the result follows. □

These results show that composition of correspondences is a generalization of function composition.

There is another product, the exterior product, which we now define.

Definition 4.3.9. Let $\Phi = (M, \xi, b_M, f_M) \in \widehat{\mathrm{KK}}_*(X_1, Y_1)$ and $\Psi = (N, \eta, b_N, f_N) \in \widehat{\mathrm{KK}}_*(X_2, Y_2)$. The *exterior product* of Φ and Ψ is defined as

$$\Phi \times \Psi := [X_1 \times X_2 \overset{b_M \times b_N}{\longleftarrow} (M \times N, \mathrm{pr}_M^*(\xi) \hat{\otimes} \mathrm{pr}_N^*(\eta)) \overset{f_M \times f_N}{\longrightarrow} Y_1 \times Y_2],$$

where $f_M \times f_N$ is given the K-orientation coming from the equality

$$T(M \times N) \oplus (f_M \times f_N)^* T(Y_1 \times Y_2) = \mathrm{pr}_M^*(TM \oplus f_M^* TY_1) \oplus \mathrm{pr}_N^*(TN \oplus f_N^* TY_2).$$

Combining the intersection product with the exterior product we may compose two correspondences "over" an auxiliary space U as follows.

$$\begin{aligned} \hat{\otimes}_U : \widehat{\mathrm{KK}}_*(X_1, Y_1 \times U) \times \widehat{\mathrm{KK}}_*(U \times X_2, Y_2) &\to \widehat{\mathrm{KK}}_*(X_1 \times X_2, Y_1 \times Y_2) \\ (\Phi, \Psi) &\mapsto (\Phi \times \mathrm{id}_{X_2}) \otimes (\mathrm{id}_{Y_1} \times \Psi). \end{aligned} \tag{4.5}$$

This product will be used in Section 5.2.1 to define Poincaré duality.

4.4 Comparison with Kasparov's KK-Theory

We define $\widehat{\mathrm{KK}}_*$ to be the category whose objects are smooth manifolds and whose morphisms are given by $\widehat{\mathrm{KK}}_*(X,Y)$, with composition given by the intersection product. The main theorem to do with the category $\widehat{\mathrm{KK}}_*$ is the following.

Theorem 4.4.1. *There is a natural transformation*

$$\widehat{\mathrm{KK}}_* \to \mathrm{KK}_*.$$

Proof. The map is given on objects of $\widehat{\mathrm{KK}}_*$ by $X \mapsto C_0(X)$ and on morphisms by

$$\begin{gathered}\widehat{\mathrm{KK}}_*(X,Y) \to \mathrm{KK}_*(C_0(X), C_0(Y)) \\ [X \xleftarrow{b} (M,\xi) \xrightarrow{f} Y] \mapsto [(b,\xi)^*] \otimes_{C_0(M)} [f_!].\end{gathered}$$

It is well defined by Theorem 4.2, combined with Corollary 2.38 and Example 2.12 of [11]. See [10] for how a K-oriented map $f : X \to Y$ gives a normally non-singular map. □

It is not a concern for us whether this transformation is an equivalence, or even injective. The main idea behind $\widehat{\mathrm{KK}}_*$ is that it is simple to understand and that the intersection product, which corresponds to the Kasparov product in the analytic picture, is easily computable. Moreover, our definitions are robust enough to capture K-theory, as the following theorem shows.

Theorem 4.4.2. *The canonical map*

$$\begin{aligned}\mathrm{K}^*(X) &\to \widehat{\mathrm{KK}}_*(*, X) \\ \xi &\mapsto [(X,\xi) \xrightarrow{\mathrm{id}} X]\end{aligned} \tag{4.6}$$

is an isomorphism for all smooth manifolds X.

Proof. We will show that the map

$$\widehat{\mathrm{KK}}_*(*, X) \to \mathrm{K}^*(X)$$
$$[(M,\xi) \xrightarrow{f} X] \to f_!(\xi)$$

is the inverse to (4.6); it is shown in [5] that this map is well defined.

Since $(\mathrm{id}_X)_! = \mathrm{id}_{\mathrm{K}^*(X)}$ by Corollary 3.3.2, it follows that the composition $\mathrm{K}^*(X) \to \widehat{\mathrm{KK}}_*(*, X) \to \mathrm{K}^*(X)$ is the identity.

We now show that the composition $\widehat{\mathrm{KK}}_*(*, X) \to \mathrm{K}^*(X) \to \widehat{\mathrm{KK}}_*(*, X)$ is the identity. This amounts to showing that $[(M,\xi) \xrightarrow{f} X] = [(X, f_!(\xi)) \longrightarrow X]$ in $\widehat{\mathrm{KK}}_*(*, X)$.

Recall that $f_!$ is defined to fit into the commuting diagram

$$\begin{array}{ccc} \mathrm{K}^*(V) & \xrightarrow{\tau_!} & \mathrm{K}^*(X \times \mathbb{R}^n) \\ \uparrow \varphi_*^V & & \downarrow \beta_n^{-1} \\ \mathrm{K}^*(M) & \xrightarrow[f_!]{} & \mathrm{K}^*(X) \end{array}$$

Thus, a Thom modification of $[(X, f_!(\xi)) \xrightarrow{\mathrm{id}} X]$ along $X \times \mathbb{R}^n$ gives

$$[(X, f_!(\xi)) \xrightarrow{\mathrm{id}} X] = [(X \times \mathbb{R}^n, \tau_! \varphi_*^V(\xi)) \xrightarrow{\mathrm{pr}_1} X].$$

Using the tubular neighbourhood to view V as an open subset of $X \times \mathbb{R}^n$, we have that

$$[(X \times \mathbb{R}^n, \tau_! \varphi_*^V(\xi)) \xrightarrow{\mathrm{pr}_1} X] = [(V, \varphi_*^V(\xi)) \xrightarrow{\mathrm{pr}_1 \circ \tau} X].$$

By Example 4.3.9, we have that

$$[(V, \varphi_*^V(\xi)) \xrightarrow{\mathrm{pr}_1 \circ \tau} X] = [(V, \varphi_*^V(\xi)) \xrightarrow{f \circ \pi_v} X]$$

Since $(V, \varphi_*^V(\xi), f \circ \pi_v)$ is the Thom modification of (M, ξ, f) along V we have

$$[(V, \varphi_*^V(\xi)) \xrightarrow{f \circ \pi_V} X] = [(M, \xi) \xrightarrow{f} X],$$

which completes the proof. □

Definition 4.4.1. We define the *K-Homology* of a smooth manifold X to be the group

$$\mathrm{K}_*(X) \coloneqq \widehat{\mathrm{KK}}_*(X, *).$$

By the theorems in this section we see that K-homology acts on K-theory as maps to the integers, hence the name. It is the *dual* theory to K-theory. In Section 5.2, we shall see that K-theory coincides with K-homology for K-orientable manifolds.

Chapter 5

Fourier-Mukai Duality

5.1 The Baum-Connes-Dirac Equivalence

Bivariant K-theory can be extended to an equivariant theory which takes into account the action of a group. The Baum-Connes conjecture is a statement in Kasparov's equivariant bivariant K-theory which helps compute the K-theory of certain crossed products. Certain positive results have been shown, including the following (see [8]).

Theorem 5.1.1. *Let A be a $\mathbb{Z}^d$-C^*-algebra. Then the class*

$$[D_{\mathbb{R}^d}] \in \mathrm{KK}^{\mathbb{Z}^d}_{-d}(C_0(\mathbb{R}^d), \mathbb{C})$$

of the Dirac operator on $\mathbb{R}^d$ induces an isomorphism

$$\mathrm{K}_{*-d}(C_0(\mathbb{R}^d, A) \rtimes \mathbb{Z}^d) \cong \mathrm{K}_*(A \rtimes \mathbb{Z}^d) \tag{5.1}$$

More precisely, the class $[D_{\mathbb{R}^d}]$ induces an invertible morphism in the *non-equivariant* bivariant K-group $\mathrm{KK}_*(C_0(\mathbb{R}^d, A) \rtimes \mathbb{Z}^d, A \rtimes \mathbb{Z}^d)$, which acts as the isomorphism between the respective K-groups, shifting degree by d. This is the *Baum-Connes-Dirac*

Equivalence.

Our goal is to investigate this equivalence in the basic case that $A = \mathbb{C}$ with the trivial action of $\mathbb{Z}^d$. There is a canonical map

$$j_{\mathbb{Z}^d} : \mathrm{KK}^{\mathbb{Z}^d}_{-d}(C_0(\mathbb{R}^d), \mathbb{C}) \to \mathrm{KK}_{-d}(C_0(\mathbb{R}^d) \rtimes \mathbb{Z}^d, \mathbb{C} \rtimes \mathbb{Z}^d),$$

called *descent*, and $j_{\mathbb{Z}^d}([D_{\mathbb{R}^d}])$ is the invertible class in $\mathrm{KK}_{-d}(C_0(\mathbb{R}^d) \rtimes \mathbb{Z}^d, \mathbb{C} \rtimes \mathbb{Z}^d)$. We have that $C_0(\mathbb{R}^d) \rtimes \mathbb{Z}^d$ is equivalent, in some precise sense[1] to $C(\mathbb{T}^d)$, and $\mathbb{C} \rtimes \mathbb{Z}^d = C^*(\mathbb{Z}^d) \cong C(\widehat{\mathbb{Z}}^d)$, so that $j_{\mathbb{Z}^d}([D_{\mathbb{R}^d}])$ is an invertible element in $\mathrm{KK}_{-d}(C(\mathbb{T}^d), C(\widehat{\mathbb{Z}}^d))$. Since

$$\widehat{\mathrm{KK}}_{-d}(\mathbb{T}^d, \widehat{\mathbb{Z}}^d) \quad \text{maps to} \quad \mathrm{KK}_{-d}(C(\mathbb{T}^d), C(\widehat{\mathbb{Z}}^d)),$$

we can ask whether $j_{\mathbb{Z}^d}([D_{\mathbb{R}^d}])$ can be represented by a correspondence and, if so, can we understand the induced map $\mathrm{K}_*(\mathbb{T}^d) \to \mathrm{K}_*(\widehat{\mathbb{Z}}^d)$ geometrically. More precisely, suppose that $j_{\mathbb{Z}^d}([D_{\mathbb{R}^d}])$ can be represented by a correspondence, say $[\mathcal{F}_d]$; every toral subgroup $T \subseteq \mathbb{T}^d$ gives a class $[T] \in \mathrm{K}^*(\mathbb{T}^d)$, and we would like to know what the image of $[T]$ under the morphism $[\mathcal{F}_d]$ in $\mathrm{K}^*(\widehat{\mathbb{Z}}^d)$ is.

It is shown in [8] that the equivalence (5.1) is represented geometrically by the correspondence

$$j_{\mathbb{Z}^d}([D_{\mathbb{R}^d}]) = [\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d) \xrightarrow{\mathrm{pr}_2} \widehat{\mathbb{Z}}^d] =: [\mathcal{F}_d],$$

defined in Example 4.3.5, so the goal of this chapter is to compute

$$[\mathcal{F}_d] \otimes [\mathbb{T}^d \xleftarrow{\varphi} (\mathbb{T}^k, \mathbf{1})] \in \mathrm{K}_*(\widehat{\mathbb{Z}}^d),$$

[1] What this means is not relevant for the work here. The equivalence is *Strong Morita Equivalence*, and the important point is that a strong Morita equivalence induces a KK-equivalence, so $C_0(\mathbb{R}^d) \rtimes \mathbb{Z}^d$ is the same as $C(\mathbb{T}^d)$ as far as K-theory is concerned.

where $\varphi : \mathbb{T}^k \to \mathbb{T}^d$ is an embedding of Lie groups.

Remark. The action of $[\mathcal{F}_d]$ bears striking resemblance to the Fourier-Mukai Transform in algebraic geometry, a point which we will now elaborate on.

In the article [19], Mukai proved that if X is an abelian variety (for instance, a torus) and $\widehat{X}$ is the dual torus (for instance, $\widehat{\mathbb{Z}}^d$) then there is a canonical equivalence of categories of sheaves on X and $\widehat{X}$ even though X and $\widehat{X}$ are not equivalent as varieties. The equivanlence, which is the Fourier-Mukai transform is constructed as follows.

Let $\mathrm{pr}_X : X \times \widehat{X} \to X$ and $\mathrm{pr}_{\widehat{X}} : X \times \widehat{X} \to \widehat{X}$ be the coordinate projections. Using a certain sheaf, β, the Fourier-Mukai is roughly the composition

$$\mathcal{S} \mapsto p_X^*(\mathcal{S}) \mapsto (p_{\widehat{X}})_*(p_X^*(\mathcal{S}) \otimes \beta).$$

By identifying a vector bundle V with its sheaf of sections, this is exactly how the Fourier-Mukai correspondence acts on V. This was observed by Emerson in [8] and the work in this chapter is devoted to proving Theorem 3.11 in that article.

5.2 Poincaré Duality

A key ingredient in one of the arguments in the next section is an explicit computation of the Poincaré duality isomorphism. Therefore, the purpose of this section is to prove the following theorem.

Theorem 5.2.1 (Poincaré Duality)**.** *Let X be a compact, K-oriented smooth manifold of dimension n. Then for all smooth manifolds Y and Z there is a natural isomorphism*

$$\mathrm{PD} : \widehat{\mathrm{KK}}_*(X \times Y, Z) \to \widehat{\mathrm{KK}}_{*+n}(Y, X \times Z)$$

given on correspondences by

$$[X \times Y \xleftarrow{b_1 \times b_2} (M,\xi) \xrightarrow{f} Z] \mapsto [Y \xleftarrow{b_2} (M,\xi) \xrightarrow{b_1 \times f} X \times Z] \tag{5.2}$$

where $b_1 \times f$ *is K-oriented using the isomorphism*

$$TM \oplus (b_1, f)^* T(X \times Z) \cong b_1^* TX \oplus (TM \oplus f^* TZ).$$

We show that this map is well defined by showing that it is given by "composing over X" with a specific element $[\sigma_X] \in \widehat{\mathrm{KK}}_0(*, X \times X) \cong \mathrm{K}^0(X \times X)$, using the map defined in Equation (4.5) of Section 4.3.4.

Definition 5.2.1. Let X be a compact, K-oriented manifold and define the correspondences

$$[\sigma_X] := [* \longleftarrow X \xrightarrow{\Delta} X \times X] \in \mathrm{K}^0(X \times X)$$
$$[\rho_X] := [X \times X \xleftarrow{\Delta} X \longrightarrow *] \in \mathrm{K}_0(X \times X),$$

where $\Delta : X \to X \times X$ is the diagonal map, K-oriented using the isomorphism

$$TX \oplus \Delta^* T(X \times X) \cong TX \oplus (TX \oplus TX)$$

where TX is K-oriented by assumption, and $TX \oplus TX$ is K-oriented using the complex structure.

The correspondences $[\sigma_X]$ and $[\rho_X]$ will be the *unit* and *counit* for Poincaré duality (see [6] for this terminology). Before we proceed, we declare the following notation: if V is a real vector bundle over a space X, then we will use the notation $V_{\mathbb{C}} := V \oplus V = V \otimes_{\mathbb{R}} \mathbb{C}$ to denote the complexification of V.

We now show that the map (5.2) is given by composing with $[\sigma_X]$ on the right over X.

Proposition 5.2.1. *For a compact, K-oriented manifold X and any smooth manifolds Y and Z, we have*

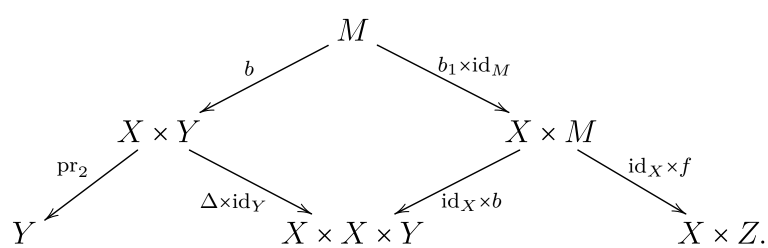
$$[\sigma_X] \otimes_X [X \times Y \xleftarrow{b_1 \times b_2} (M, \xi) \xrightarrow{f} Z] = [Y \xleftarrow{b_2} (M, \xi) \xrightarrow{b_1 \times f} X \times Z].$$

In particular, the map (5.2) *is well defined.*

Proof. This is just a computation, the main point being that the K-orientation that (b_1, f) receives from the composition product is the right one. By definition

$$\begin{aligned} &[\sigma_X] \otimes_X [X \times Y \xleftarrow{b} (M, \xi) \xrightarrow{f} Z] \\ &\quad = [Y \xleftarrow{\mathrm{pr}_2} X \times Y \xrightarrow{\Delta \times \mathrm{id}_Y} X \times X \times Y] \otimes [X \times X \times Y \xleftarrow{\mathrm{id}_X \times b} (X \times M, \xi) \xrightarrow{\mathrm{id}_X \times f} X \times Z] \end{aligned}$$

which gives rise to the composition diagram



We claim that the K-orientation that $b_1 \times \mathrm{id}_M$ receives is given by the spinor bundle $S_X \otimes \Lambda^*(TM_{\mathbb{C}})$. According to equation (4.4) the map $b_1 \times \mathrm{id}_M$ is K-oriented according to the equality

$$\begin{aligned} &[TM \oplus (b_1 \times \mathrm{id})^* T(X \times M)] \oplus ((\Delta \times \mathrm{id}_Y) \circ b)^* (T(X \times X \times Y)_{\mathbb{C}}) \\ &\qquad = b^*(T(X \times Y) \oplus (\Delta \times \mathrm{id}_Y)^* T(X \times X \times Y)) \oplus (b_1 \times \mathrm{id})^* (T(X \times M)_{\mathbb{C}}). \end{aligned}$$

The spinor bundle for the right hand side is given fibrewise by $S_X \otimes \Lambda^*_{\mathbb{C}}(T(X\times Y)_{\mathbb{C}}) \otimes \Lambda^*_{\mathbb{C}}(T(X\times M)_{\mathbb{C}})$, and the spinor bundle for $(\Delta\times\mathrm{id})$ is given by $\Lambda^*_{\mathbb{C}}(T(X\times X\times Y)_{\mathbb{C}})$. By the uniqueness property of the 2-out-of-3 lemma, it follows that $b_1 \times \mathrm{id}$ is K-oriented using $S_X \otimes \Lambda^*(TM_{\mathbb{C}})$ because of the following equality

$$S_X \otimes \Lambda^*_{\mathbb{C}}(TM_{\mathbb{C}}) \otimes \Lambda^*_{\mathbb{C}}(T(X\times X\times Y)_{\mathbb{C}}) = S_X \otimes \Lambda^*_{\mathbb{C}}(T(X\times Y)_{\mathbb{C}}) \otimes \Lambda^*_{\mathbb{C}}(T(X\times M)_{\mathbb{C}}).$$

A similar argument shows that the composition $(b_1, f) = (\mathrm{id}, f)\circ(b_1,\mathrm{id})$ receives the correct K-orientation, which completes the proof. □

By a completely analogous argument one can show that

$$\begin{aligned} \otimes_X[\rho_X] : & \widehat{\mathrm{KK}}_*(Y, X\times Z) \to \widehat{\mathrm{KK}}_*(X\times Y, Z) \\ & [Y \xleftarrow{b} (M,\xi) \xrightarrow{f} X\times Z] \mapsto [X\times Y \xrightarrow{f_1\times b} (M,\xi) \xrightarrow{f_2} Z], \end{aligned}$$

where $f_2 = \mathrm{pr}_Z \circ f : M \to Z$ is K-oriented via the equality

$$TM \oplus f^*T(X\times Z) = f_1^*TX \oplus (TM \oplus f_2^*TZ)$$

and the 2-out-of-3 lemma. This proves that 5.2 is indeed an isomorphism and we have therefore proved Theorem 5.2.1. This, in particular, implies that K-theory and K-homology agree on compact, K-oriented manifolds, up to a dimension shift.

Corollary 5.2.1. *For any compact, K-oriented manifold X of dimension n, the map*

$$\begin{aligned} \mathrm{PD} : \widehat{\mathrm{KK}}_*(*, X) &\to \widehat{\mathrm{KK}}_{*+n}(X, *) \\ [(M,\xi) \xrightarrow{f} X] &\mapsto [X \xleftarrow{f} (M,\xi)] \end{aligned}$$

is an isomorphism.

Remark. This is a special instance of a more general version of Poincaré duality shown in [5], which reads as follows:

Theorem 5.2.2. *Let X be a compact manifold and define*

$$[\tilde{\sigma}_X] := [* \longleftarrow X \xrightarrow{(\zeta,\mathrm{id})} TX \times X] \in \mathrm{K}^0(TX \times X)$$
$$[\tilde{\rho}_X] := [X \times TX \xleftarrow{(\pi,\mathrm{id})} TX \longrightarrow *] \in \mathrm{K}_0(TX \times X),$$

where $\zeta : X \to TX$ is the zero section and $\pi : TX \to X$ is the projection. Then

(i) $[\tilde{\sigma}_X] \otimes_X [\tilde{\rho}_X] = 1_{TX} \in \widehat{\mathrm{KK}}_*(TX, TX)$ and $[\tilde{\sigma}_X] \otimes_{TX} [\tilde{\rho}_X] = 1_X \in \widehat{\mathrm{KK}}_*(X, X)$ and

(ii) For any smooth manifolds A and B, $[\tilde{\sigma}_X]$ induces natural isomorphisms

$$\widehat{\mathrm{KK}}_*(X \times A, B) \to \widehat{\mathrm{KK}}_*(A, TX \times B)$$
$$\widehat{\mathrm{KK}}_*(TX \times A, B) \to \widehat{\mathrm{KK}}_*(A, X \times B).$$

The main point of Theorem 5.2.1 is that we have an explicit description of the map, which we will need in the following section.

5.3 The Action of $[\mathcal{F}_d]$ on K-Homology

In what follows we declare that, when there is a choice, we K-orient our Lie groups by specifying an orientation on the Lie algebra.

Suppose that $\hat{\varphi} : \widehat{\mathbb{Z}}^k \to \widehat{\mathbb{Z}}^d$ is an embedding of Lie groups. The purpose of this section is to compute

$$[\mathcal{F}_d] \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\hat{\varphi}} (\widehat{\mathbb{Z}}^k, \mathbf{1})] \in \mathrm{K}_*(\mathbb{T}^d).$$

This computation is the content of Theorem 5.3.1, which we now set up. By Pontryagin duality, there is a unique morphism $\varphi : \mathbb{Z}^d \to \mathbb{Z}^k$ whose Pontryagin dual is $\hat{\varphi}$. The morphism φ extends to an $\mathbb{R}$-linear map $\varphi : \mathbb{R}^d \to \mathbb{R}^k$ which preserves the integer lattice, hence descends to a map $\tilde{\varphi} : \mathbb{T}^d \to \mathbb{T}^k$. This gives rise to an exact sequence

$$0 \longrightarrow \ker(\tilde{\varphi}) \longrightarrow \mathbb{T}^d \xrightarrow{\tilde{\varphi}} \mathbb{T}^k \longrightarrow 0,$$

which lifts to an exact sequence of Lie algebras

$$0 \longrightarrow T(\ker(\tilde{\varphi})) \longrightarrow T(\mathbb{T}^d)|_{\ker(\tilde{\varphi})} \xrightarrow{d(\tilde{\varphi})} \hat{\varphi}^* T(\mathbb{T}^k) \longrightarrow 0, \tag{5.3}$$

which endows $\ker(\tilde{\varphi})$ with a K-orientation such that the map $\phi : \ker(\tilde{\varphi}) \times \mathbb{T}^k \to \mathbb{T}^d$ is K-orientation preserving. If $\tilde{\varphi}$ is injective, then it is an isomorphism of Lie groups, and we declare $\ker(\tilde{\varphi})$ to be positively K-oriented if $\tilde{\varphi}$ is K-orientation preserving and negatively K-oriented otherwise.

The goal of this section is to prove the following theorem.

Theorem 5.3.1. *Suppose that $\hat{\varphi} : \widehat{\mathbb{Z}}^k \to \widehat{\mathbb{Z}}^d$ is a Lie embedding. Then, in the notation above,*

$$[\mathcal{F}_d] \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\hat{\varphi}} (\widehat{\mathbb{Z}}^k, 1)] = (-1)^{dk + \frac{k(k-1)}{2}} [\ker(\tilde{\varphi})],$$

where $\ker(\tilde{\varphi})$ *is given the K-orientation coming from* (5.3)*.*

In order to prove this theorem we need to make several observations about the ingredients in the correspondences above. Let us begin by making a fundamental observation about the Poincaré bundle.

Let $\exp : (0,1) \to \mathbb{T}$ be the exponential map $x \mapsto e^{2\pi i x}$ and let $\widehat{\exp} : (0,1) \to \widehat{\mathbb{Z}}$

denote the map $\theta \mapsto [n \mapsto e^{2\pi i n\theta}]$. Let $e : (0,1)^2 \to \mathbb{T} \times \widehat{\mathbb{Z}}$ be the map

$$e(x,y) = (\exp(x), \widehat{\exp}(y)),$$

Lemma 5.3.1. *The map $e_! : \mathrm{K}^0((0,1)^2) \to \mathrm{K}^0(\mathbb{T} \times \widehat{\mathbb{Z}})$ induced by the embedding $e : (0,1)^2 \to \mathbb{T} \times \widehat{\mathbb{Z}}$ maps the Bott class $\beta \in \mathrm{K}^0((0,1)^2)$ to $\mathcal{P}_1 - \mathbf{1} \in \mathrm{K}^0(\mathbb{T} \times \widehat{\mathbb{Z}})$.*

Proof. Let $\mathrm{ev}_1 : \widehat{\mathbb{Z}} \to \mathbb{T}$ denote evaluation of a character at 1. We note that we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{Z}} & \xrightarrow{\mathrm{ev}_1} & \mathbb{T} \\ {\scriptstyle\widehat{\exp}}\uparrow & \nearrow{\scriptstyle\exp} & \\ (0,1) & & \end{array}$$

which yields a commutative diagram

$$\begin{array}{ccc} \mathrm{K}^{-1}(\mathbb{T}) & \xrightarrow{\mathrm{ev}_1^*} & \mathrm{K}^{-1}(\widehat{\mathbb{Z}}) \\ {\scriptstyle\exp_!}\uparrow & \nearrow{\scriptstyle\widehat{\exp}_!} & \\ \mathrm{K}^{-1}((0,1)), & & \end{array}$$

because $(\mathrm{ev}_1)_!^{-1} = \mathrm{ev}_1^*$; see Example 3.3.9. Since $\exp_!(\beta) = [z] \in \mathrm{K}^{-1}(\mathbb{T})$ and $\mathrm{ev}_1^*([z]) = [\chi] \in \mathrm{K}^{-1}(\widehat{\mathbb{Z}})$, where β is the Bott class and $[\chi]$ denotes the class of multiplication by $\chi(1)$, it follows that $\widehat{\exp}_!(\beta) = [\chi]$. Because the following diagram commutes

$$\begin{array}{ccc} (0,1) \times \widehat{\mathbb{Z}} & \xrightarrow{\exp \times \mathrm{id}_{\widehat{\mathbb{Z}}}} & \mathbb{T} \times \widehat{\mathbb{Z}} \\ {\scriptstyle\mathrm{id}_{\mathbb{T}} \times \widehat{\exp}}\uparrow & \nearrow{\scriptstyle\exp \times \widehat{\exp}} & \\ (0,1)^2, & & \end{array}$$

it is sufficient to show that $(\exp \times \mathrm{id})_!([\chi]) = \mathcal{P}_1 - \mathbf{1}$ in order to prove the claim.

We identify the 1-point compactification of $(0,1) \times \widehat{\mathbb{Z}}$ with the space $(\mathbb{T} \times \widehat{\mathbb{Z}})/(\{1\} \times \widehat{\mathbb{Z}})$. It follows then that the map $(\exp \times \mathrm{id}_{\widehat{\mathbb{Z}}})_!$ is given the map induced by the quotient map

$$q : \mathbb{T} \times \widehat{\mathbb{Z}} \to (\mathbb{T} \times \widehat{\mathbb{Z}})/(\{1\} \times \widehat{\mathbb{Z}}).$$

The class of $[\chi] \in \mathrm{K}^0((0,1) \times \widehat{\mathbb{Z}})$ is represented in $\mathrm{K}^0((\mathbb{T} \times \widehat{\mathbb{Z}})/(\{1\} \times \widehat{\mathbb{Z}}))$ by the class $[E_\chi] - [\mathbf{1}]$, where E_χ is the trivial line bundle modulo the relation

$$(0, \chi, \lambda) \sim (1, \chi, \chi(1)\lambda).$$

It follows that $q^*(E_\chi) = [[0,1] \times \widehat{\mathbb{Z}}] \times \mathbb{C}/\sim$, where $\sim$ is the relation above, so we conclude the proof by noting that the inclusion

$$[[0,1] \times \widehat{\mathbb{Z}}] \times \mathbb{C}/\sim \to \mathcal{P}_1$$

is an isomorphism. □

Lemma 5.3.2. *The class* $[\mathbb{T} \xleftarrow{pr_1} (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathcal{P}_1)]$ *is equivalent to* $[\mathbb{T} \longleftarrow (*, \mathbf{1})]$ *in* $\mathrm{K}_*(\mathbb{T})$.

Proof. Since $\mathbb{T} \times \widehat{\mathbb{Z}}$ is a boundary we have that $[\mathbb{T} \xleftarrow{\mathrm{pr}_1} (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathbf{1})] = 0$, whence

$$\begin{aligned}
[\mathbb{T} \xleftarrow{\mathrm{pr}_1} (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathcal{P}_1)] &= [\mathbb{T} \xleftarrow{\mathrm{pr}_1} (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathcal{P}_1)] - [\mathbb{T} \longleftarrow (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathbf{1})] \\
&= [\mathbb{T} \xleftarrow{\mathrm{pr}_1} (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathcal{P}_1 - \mathbf{1})],
\end{aligned}$$

by Example 4.3.8. By Lemma 5.3.1 and Example 4.3.7

$$[\mathbb{T} \xleftarrow{\mathrm{pr}_1} (\mathbb{T} \times \widehat{\mathbb{Z}}, \mathcal{P}_1 - \mathbf{1})] = [\mathbb{T} \xleftarrow{\exp \circ \mathrm{pr}_1} ((0,1)^2, \beta)] \quad \text{in } \mathrm{K}_*(\mathbb{T}).$$

Next, we note that the correspondence with boundary $[\mathbb{T} \xleftarrow{F} ((0,1)^2 \times [0,1], \beta \hat{\otimes} \mathbf{1})]$, with $F((x,y),t) = \exp(tx)$, provides a cobordism between

$$[\mathbb{T} \xleftarrow{\mathrm{pr}_1} ((0,1)^2, \beta)] \quad \text{and} \quad [\mathbb{T} \xleftarrow{1} ((0,1)^2, \beta)],$$

where 1 denotes the constant map at 1. Since $(0,1)^2$ is a tubular neighbourhood of $(\frac{1}{2},\frac{1}{2})$ in $\mathbb{R}^2$ and β is the Thom class, Thom modification along the trivial rank 2 bundle shows that

$$[\mathbb{T} \xleftarrow{1} ((0,1)^2, \beta)] \sim_{Tm} [\mathbb{T} \xleftarrow{1} (*, \mathbf{1})],$$

as claimed. □

An immediate corollary of this result is

Lemma 5.3.3. *For any d,*

$$[\mathbb{T}^d \xleftarrow{pr_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d)] = (-1)^{\frac{d(d-1)}{2}} [\mathbb{T}^d \longleftarrow (*, 1)].$$

Proof. One can prove using induction that the isomorphism

$$\theta : \mathbb{T}^d \times \widehat{\mathbb{Z}}^d \to (\mathbb{T} \times \widehat{\mathbb{Z}})^d$$
$$(z_1, \ldots, z_d, \chi_1, \ldots, \chi_d) \mapsto (z_1, \chi_1, \ldots, z_d, \chi_d)$$

has determinant $(-1)^{\frac{d(d-1)}{2}}$, so that

$$[\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d)] = (-1)^{\frac{d(d-1)}{2}} [\mathbb{T}^d \longleftarrow ((\mathbb{T} \times \widehat{\mathbb{Z}})^d, (\theta^{-1})^*(\mathcal{P}_d)].$$

Next, one observes that $(\theta^{-1})^*(\mathcal{P}_d)$ is the bundle over $(\mathbb{T} \times \widehat{\mathbb{Z}})^d$ given by the relation

$$(x_1, \chi_1, \ldots, x_d, \chi_d, \lambda) \sim \left(x_1 + n_1, \chi_1, \ldots, x_d + n_d, \chi_d, \left(\prod_{i=1}^{d} \chi_i(n_i) \right) \lambda \right),$$

which is readily seen to be equal the external product of $\mathcal{P}_1$ with itself d times. That is, $(\theta^{-1})^*(\mathcal{P}_d) = \mathcal{P}_1^{\hat{\otimes} d}$ whence

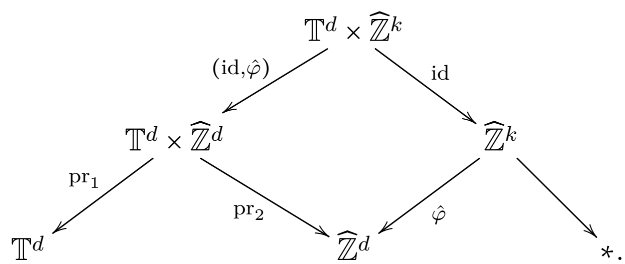
$$[\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d)] = (-1)^{\frac{d(d-1)}{2}} [\mathbb{T}^d \longleftarrow ((\mathbb{T} \times \widehat{\mathbb{Z}})^d, \mathcal{P}_1^{\hat{\otimes} d}].$$

Finally, we note that $[\mathbb{T}^d \longleftarrow ((\mathbb{T} \times \widehat{\mathbb{Z}})^d, \mathcal{P}_1^{\hat{\otimes} d})]$ is the exterior product of d copies of $[\mathbb{T} \longleftarrow (\mathbb{T} \times \mathbb{Z}, \mathcal{P}_1)]$, each of which is equivalent to a point correspondence, whence the result follows. □

We can now prove Theorem 5.3.1.

Proof of Theorem 5.3.1. The spirit of the proof is to decompose $\mathbb{T}^d$ into the product $\ker(\tilde{\varphi}) \times \mathbb{T}^k$, and apply Lemma 5.3.3 to the $\mathbb{T}^k$ factor.

The composition diagram for $[\mathcal{F}_d] \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\hat{\varphi}} (\widehat{\mathbb{Z}}^k, \mathbf{1})]$ is



The K-orientation $\mathbb{T}^d \times \widehat{\mathbb{Z}}^k$ receives from the intersection product is given by the short exact sequence

$$0 \longrightarrow T(\mathbb{T}^d \times \widehat{\mathbb{Z}}^k) \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & d\hat{\varphi} \\ 0 & 1 \end{smallmatrix}\right]} T(\mathbb{T}^d \times \widehat{\mathbb{Z}}^d \times \widehat{\mathbb{Z}}^k) \xrightarrow{[\,0\;1\;-d\hat{\varphi}\,]} T(\widehat{\mathbb{Z}}^d) \longrightarrow 0.$$

It canonically splits, yielding an isomorphism $T(\mathbb{T}^d \times \widehat{\mathbb{Z}}^k) \times T(\widehat{\mathbb{Z}}^d) \to T(\mathbb{T}^d \times \widehat{\mathbb{Z}}^d \times \widehat{\mathbb{Z}}^k)$ given by the matrix

$$\begin{bmatrix} 1_d & 0 & 0 \\ 0 & d\hat{\varphi} & 1_d \\ 0 & 1_k & 0 \end{bmatrix}.$$

The path

$$t \mapsto \begin{bmatrix} 1_d & 0 & 0 \\ 0 & td\hat{\varphi} & 1_d \\ 0 & 1_k & 0 \end{bmatrix} \text{ connects } \begin{bmatrix} 1_d & 0 & 0 \\ 0 & d\hat{\varphi} & 1_d \\ 0 & 1_k & 0 \end{bmatrix} \text{ to } \begin{bmatrix} 1_d & 0 & 0 \\ 0 & 0 & 1_d \\ 0 & 1_k & 0 \end{bmatrix} \text{in O(dk)},$$

whence the K-orientation that $\mathbb{T}^d \times \widehat{\mathbb{Z}}^k$ receives is $(-1)^{dk}$ times the one it had originally, since these matrices have determinant $(-1)^{dk}$. Thus,

$$\mathcal{F}_d \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\hat{\varphi}} (\widehat{\mathbb{Z}}^k, 1)] = (-1)^{dk} [\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} \mathbb{T}^d \times \mathbb{Z}^k, \mathcal{P}_d|_{\hat{\varphi}})],$$

where $\mathcal{P}_d|_{\hat{\varphi}}$ is the line bundle over $\mathbb{T}^d \times \widehat{\mathbb{Z}}^k$ given by the relation

$$(x, \chi, \lambda) \sim (x + n, \chi, \hat{\varphi}(n)\lambda), \quad \text{for } x \in \mathbb{R}^d \text{ and } n \in \mathbb{Z}^d.$$

Using our orientation preserving isomorphism $\phi : \ker(\tilde{\varphi}) \times \mathbb{T}^k \to \mathbb{T}^d$, this is equivalent to

$$[\mathbb{T}^d \xleftarrow{\phi} (\ker(\tilde{\varphi}) \times \mathbb{T}^k \times \widehat{\mathbb{Z}}^k, \phi^*(\mathcal{P}_d|_{\hat{\varphi}}))],$$

and one can check that $\phi^*(\mathcal{P}_d|_{\hat{\varphi}}) = \mathbf{1} \hat{\otimes} \mathcal{P}_k$. Applying the correspondence $[(\phi^{-1})^*]$ and using Lemma 5.3.3 we have

$$\begin{aligned} [\phi^{-1}] \otimes [\mathbb{T}^d \xleftarrow{\phi} (\ker(\tilde{\varphi}) \times \mathbb{T}^k \times \widehat{\mathbb{Z}}^k, \phi^*(\mathcal{P}_d|_{\hat{\varphi}}))] &= (-1)^d [\ker(\tilde{\varphi}) \times \mathbb{T}^k \leftarrow (\ker(\tilde{\varphi}) \times \mathbb{T}^k \times \widehat{\mathbb{Z}}^k, \mathbf{1} \hat{\otimes} \mathcal{P}_d)] \\ &= (-1)^d ([\ker(\tilde{\varphi}) \leftarrow (\ker(\tilde{\varphi}), \mathbf{1})] \times [\mathbb{T}^k \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^k \times \widehat{\mathbb{Z}}^k, \mathcal{P}_k)]) \\ &= (-1)^d ([\ker(\tilde{\varphi}) \leftarrow (\ker(\tilde{\varphi}), \mathbf{1})] \times [\mathbb{T}^k \longleftarrow (*, (-1)^{\frac{k(k-1)}{2}} \mathbf{1})]) \\ &= (-1)^{d + \frac{k(k-1)}{2}} [\ker(\tilde{\varphi}) \times \mathbb{T}^k \longleftarrow (\ker(\tilde{\varphi}), \mathbf{1})]. \end{aligned}$$

Applying $[\phi]$ to $(-1)^d[\ker(\tilde{\varphi}) \times \mathbb{T}^k \longleftarrow (\ker(\tilde{\varphi}), \mathbf{1})]$ gives $[\mathbb{T}^d \longleftarrow (\ker(\tilde{\varphi}), \mathbf{1})]$, hence

$$\mathcal{F}_d \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\hat{\varphi}} (\widehat{\mathbb{Z}}^k, \mathbf{1})] = (-1)^{dk + \frac{k(k-1)}{2}} [\mathbb{T}^d \longleftarrow (\ker(\tilde{\varphi}), \mathbf{1})],$$

which was to be shown. □

Example 5.3.1. If $\hat{\varphi} : * \to \widehat{\mathbb{Z}}^d$ is the embedding of a point, then $\ker(\tilde{\varphi}) = \mathbb{T}^d$ oriented as a product of circles so that

$$[\mathcal{F}_d] \otimes [\widehat{\mathbb{Z}}^d \longleftarrow (*, \mathbf{1})] = [\mathbb{T}^d \xleftarrow{\text{id}} (\mathbb{T}^d, \mathbf{1})].$$

Example 5.3.2. If $\varphi : \widehat{\mathbb{Z}}^d \to \widehat{\mathbb{Z}}^d$ is an isomorphism, then

$$[\mathcal{F}_d] \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\hat{\varphi}} (\widehat{\mathbb{Z}}^d, \mathbf{1})] = \epsilon(\tilde{\varphi})(-1)^d [\mathbb{T}^d \longleftarrow (*, \mathbf{1})],$$

where $\epsilon(\tilde{\varphi})$ is ± 1 if $\tilde{\varphi}$ orientation preserving or not. In particular,

$$[\mathcal{F}_d] \otimes [\widehat{\mathbb{Z}}^d \xleftarrow{\text{id}} (\widehat{\mathbb{Z}}^d, \mathbf{1})] = (-1)^d [\mathbb{T}^d \longleftarrow (*, \mathbf{1})].$$

Since $\mathbb{T}^d$ and $\widehat{\mathbb{Z}}^d$ are both K-orientable manifolds, Poincaré duality provides an isomorphism $\widehat{\text{KK}}_*(\mathbb{T}^d, \widehat{\mathbb{Z}}^d) \cong \widehat{\text{KK}}_*(\widehat{\mathbb{Z}}^d, \mathbb{T}^d)$. The Poincaré dual of the Fourier-Mukai transform is

$$[\widehat{\mathbb{Z}}^d \xleftarrow{\text{pr}_2} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d) \xrightarrow{\text{pr}_1} \mathbb{T}^d] \in \widehat{\text{KK}}_*(\widehat{\mathbb{Z}}^d, \mathbb{T}^d),$$

which, abusing notation, we also denote by $[\mathcal{F}_d]$. Given a Lie embedding $\varphi : \mathbb{T}^k \to \mathbb{T}^d$, we would like to compute $[\mathcal{F}_d] \otimes [\mathbb{T}^d \xleftarrow{\varphi} (\mathbb{T}^k, \mathbf{1})]$ as in Theorem 5.3.1, which we are now going to do.

The map φ lifts to a linear injection $\varphi : \mathbb{R}^k \to \mathbb{R}^d$ which maps $\mathbb{Z}^k$ into $\mathbb{Z}^d$. Its Pontryagin dual $\hat{\varphi} : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^k$ is then a surjection which maps $\widehat{\mathbb{Z}}^d_\perp$ to $\widehat{\mathbb{Z}}^k_\perp$, where

$\widehat{\mathbb{Z}}^d_\perp$ denotes the characters which vanish on $\mathbb{Z}^d$. Thus, $\hat{\varphi}$ descends to a surjection $\hat{\varphi} : \widehat{\mathbb{Z}}^d \to \widehat{\mathbb{Z}}^k$, which gives rise to an exact sequence

$$0 \longrightarrow \ker(\hat{\varphi}) \longrightarrow \widehat{\mathbb{Z}}^d \xrightarrow{\hat{\varphi}} \widehat{\mathbb{Z}}^k \longrightarrow 0,$$

which endows $\ker(\hat{\varphi})$ with a K-orientation. With this set-up we have, as before, the following theorem.

Theorem 5.3.2. *Suppose that $\varphi : \mathbb{T}^k \to \mathbb{T}^d$ is a Lie embedding. Then*

$$[\mathcal{F}_d] \otimes [\mathbb{T}^d \xleftarrow{\varphi} (\mathbb{T}^k, \mathbf{1})] = (-1)^{dk + \frac{k(k-1)}{2}} [\ker(\hat{\varphi})].$$

In order to prove this theorem, it will be advantageous to introduce the *Pontryagin dual* of the Fourier-Mukai Transform.

Definition 5.3.1. The Pontryagin dual of the Fourier-Mukai transform is the correspondence

$$[\widehat{\mathcal{F}}_d] = [\widehat{\mathbb{Z}}^d \xleftarrow{\mathrm{pr}_2} (\widehat{\mathbb{Z}}^d \times \mathbb{T}^d, \widehat{\mathcal{P}}_d) \xrightarrow{\mathrm{pr}_2} \mathbb{T}^d],$$

where $\widehat{\mathcal{P}}_d$ is the line bundle $[\widehat{\mathbb{R}}^d \times \mathbb{T}^d] \times \mathbb{C}$ modulo the relation

$$(\chi, z, \lambda) \sim (\chi + \eta, z, \eta(z)\lambda) \quad \text{for } \eta \in \widehat{\mathbb{Z}}^d_\perp;$$

note that $\widehat{\mathbb{Z}}^d_\perp \cong \widehat{\mathbb{T}}^d$, so that $\eta(z)$ makes sense.

The proof of Theorem 5.3.1 works verbatim to show that

$$[\widehat{\mathcal{F}}_d] \otimes [\mathbb{T}^d \xleftarrow{\varphi} (\mathbb{T}^k, \mathbf{1})] = (-1)^{dk + \frac{k(k-1)}{2}} [\ker(\hat{\varphi})],$$

so in order to prove 5.3.2 it suffices to prove the the following lemma.

Lemma 5.3.4. $[\mathcal{F}_d] = [\widehat{\mathcal{F}}_d]$ *in* $\widehat{\mathrm{KK}}_*(\widehat{\mathbb{Z}}^d, \mathbb{T}^d)$.

Proof. For simplicity we take $d = 1$, the general case being completely analogous. Consider the map

$$\theta : \widehat{\mathbb{Z}} \times \mathbb{T} \to \mathbb{T} \times \widehat{\mathbb{Z}}$$
$$(\chi, z) \mapsto (\chi(1), \chi_z),$$

where $\chi_z(1) := z$. Then θ provides an isomorphism of correspondences between $[\mathcal{F}_d]$ and

$$[\widehat{\mathbb{Z}} \xleftarrow{\theta_{\mathbb{T}}} (\widehat{\mathbb{Z}} \times \mathbb{T}, \theta^*(\mathcal{P}_d)) \xrightarrow{\theta_{\widehat{\mathbb{Z}}}} \mathbb{T}] = [\widehat{\mathbb{Z}} \xleftarrow{\theta_{\mathbb{T}}} (\widehat{\mathbb{Z}} \times \mathbb{T}, \widehat{\mathcal{P}}_d) \xrightarrow{\theta_{\widehat{\mathbb{Z}}}} \mathbb{T}],$$

where $\theta_{\mathbb{T}}(\chi, z) = \chi_z$ and $\theta_{\widehat{\mathbb{Z}}}(\chi, z) = \chi(1)$.

Now, consider then embeddings $\widehat{\exp} : (0,1) \to \widehat{\mathbb{Z}}$ and $\exp : (0,1) \to \mathbb{T}$ given in the proof of Lemma 5.3.1. Using the embedding $\widehat{\exp} \times \exp : (0,1)^2 \to \widehat{\mathbb{Z}} \times \mathbb{T}$, it follows from Examples 4.3.7 and 4.3.8 that

$$\begin{aligned}[\widehat{\mathbb{Z}} \xleftarrow{\theta_{\mathbb{T}}} (\widehat{\mathbb{Z}} \times \mathbb{T}, \widehat{\mathcal{P}}_1) \xrightarrow{\theta_{\widehat{\mathbb{Z}}}} \mathbb{T}] &= [\widehat{\mathbb{Z}} \xleftarrow{\theta_{\mathbb{T}}} (\widehat{\mathbb{Z}} \times \mathbb{T}, \widehat{\mathcal{P}}_1 - \mathbf{1}) \xrightarrow{\theta_{\widehat{\mathbb{Z}}}} \mathbb{T}] \\ &= [\widehat{\mathbb{Z}} \xleftarrow{\widehat{\exp}} ((0,1)^2, \widehat{\mathcal{P}}_1 - \mathbf{1}) \xrightarrow{\exp} \mathbb{T}],\end{aligned}$$

where $\beta \in \mathrm{K}^0((0,1)^2)$ is the Bott class, as before. Define $F : (0,1)^2 \times [0,1] \to \mathbb{T}$ and $\widehat{F} : (0,1)^2 \times [0,1] \to \widehat{\mathbb{Z}}$ by

$$F((x,y),t) = \exp(ty + (1-t)x) \quad \text{and} \quad \widehat{F}((x,y),t) = \widehat{\exp}(tx + (1-t)y),$$

respectively. Then ∂-correspondence

$$[\widehat{\mathbb{Z}} \xleftarrow{\widehat{F}} ((0,1)^2 \times [0,1], \widehat{\mathcal{P}}_1 - \mathbf{1}) \xrightarrow{F} \mathbb{T}]$$

shows that

$$[\widehat{\mathbb{Z}} \xleftarrow{\widehat{\exp}} ((0,1)^2, \widehat{\mathcal{P}}_1 - \mathbf{1}) \xrightarrow{\exp} \mathbb{T}] \sim_b [\widehat{\mathbb{Z}} \xleftarrow{\widehat{\exp}} ((0,1)^2, \widehat{\mathcal{P}}_d - \mathbf{1}) \xrightarrow{\exp} \mathbb{T}].$$

Since

$$[\widehat{\mathbb{Z}} \xleftarrow{\widehat{\exp}} ((0,1)^2, \widehat{\mathcal{P}}_1 - \mathbf{1}) \xrightarrow{\exp} \mathbb{T}] = [\widehat{\mathcal{F}}_d]$$

by Examples 4.3.7 and 4.3.8 again, we are done. □

Remark. We phrase Theorem 5.3.2 in more geometric terms as follows. Suppose that $T \subseteq \mathbb{T}^d$ is a toral subgroup. Then T lifts to a linear subspace $L \subseteq \mathbb{R}^d$. Let $\hat{L}_\perp$ denote the characters of $\mathbb{R}^d$ which vanish on L. If $T^\perp \subseteq \hat{\mathbb{R}}^d/\widehat{\mathbb{Z}}^d_\perp$ is the image of $\hat{L}_\perp$ under the quotient map, then Theorem 5.3.2 says that $[\mathcal{F}_d]$ maps the class of T in $\mathrm{K}_*(\mathbb{T}^d)$ to the class of $T^\perp$ in $\mathrm{K}_*(\widehat{\mathbb{Z}}^d)$, up to a sign.

5.4 Invertibility of $[\mathcal{F}_d]$

By combining Theorems 5.3.1 and 5.3.2 with Poincaré duality, we can show that $[\mathcal{F}_d]$ is invertible, can we can give an explicit inverse. Define

$$[\overline{\mathcal{F}}_d] = [\widehat{\mathbb{Z}}^d \xleftarrow{\mathrm{pr}_1} (\widehat{\mathbb{Z}}^d \times \mathbb{T}^d, \overline{\mathcal{P}}_d) \xrightarrow{\mathrm{pr}_2} \mathbb{T}^d],$$

where $\overline{\mathcal{P}}_d$ is the line bundle over $\widehat{\mathbb{Z}}^d \times \mathbb{T}^d$ given by the relation

$$(\chi, x, \lambda) \sim (\chi, x + n, \overline{\chi}(n)\lambda) \quad \text{for } x \in \mathbb{R}^d \text{ and } n \in \mathbb{Z}^d.$$

Then we have the following.

Theorem 5.4.1. *For any d, we have*

$$[\mathcal{F}_d] \otimes [\overline{\mathcal{F}}_d] = \mathrm{id}_{\widehat{\mathrm{KK}}_*(\mathbb{T}^d, \mathbb{T}^d)} \quad \text{and} \quad [\overline{\mathcal{F}}_d] \otimes [\mathcal{F}_d] = \mathrm{id}_{\widehat{\mathrm{KK}}_*(\widehat{\mathbb{Z}}^d, \widehat{\mathbb{Z}}^d)}.$$

Thus, the class $[\mathcal{F}_d]$ is an invertible element of $\widehat{\mathrm{KK}}_*(\mathbb{T}^d, \widehat{\mathbb{Z}}^d)$

Proof. We compute directly that

$$[\mathcal{F}_d] \otimes [\overline{\mathcal{F}}_d] = [\mathbb{T}^d \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^d \times \widehat{\mathbb{Z}}^d \times \mathbb{T}^d, \mathcal{P}_d \otimes \overline{\mathcal{P}}_d) \xrightarrow{\mathrm{pr}_3} \mathbb{T}^d].$$

The Poincaré dual of $[\mathcal{F}_d] \otimes [\overline{\mathcal{F}}_d]$ is the class

$$[\mathbb{T}^{2d} \xleftarrow{\mathrm{pr}_1} (\mathbb{T}^{2d} \times \widehat{\mathbb{Z}}^d, \mathcal{P}_d|_{\overline{\delta}})] \in \mathrm{K}_*(\mathbb{T}^{2d}),$$

where $\mathcal{P}_d|_{\overline{\delta}}$ is the line bundle over $\mathbb{T}^{2d} \times \widehat{\mathbb{Z}}^d$ determined by the relation

$$(x, y, \chi, \lambda) \sim (x + n, y + m, \chi, \chi(m - n)\lambda) \quad \text{for } x, y \in \mathbb{R}^d \text{ and } n, m \in \mathbb{Z}^d.$$

One observes that this is equal to $[\mathcal{F}_{2d}] \otimes [\overline{\delta}]$, where $\overline{\delta} : \widehat{\mathbb{Z}}^d \to \widehat{\mathbb{Z}}^{2d}$ is the map which sends χ to $(\overline{\chi}, \chi)$.

We observe that the Pontryagin dual of $\overline{\delta}$ is the map

$$\begin{aligned} \overline{\Delta} : \mathbb{R}^{2d} &\to \mathbb{R}^d \\ (x, y) &\mapsto y - x. \end{aligned}$$

We identify the kernel of $\overline{\Delta}$ with $\mathbb{R}^d$ using the diagonal map $\Delta : \mathbb{R}^d \to \mathbb{R}^{2d}$. Thus, by Theorem 5.3.1, the Poincare Dual of $[\mathcal{F}_d] \otimes [\overline{\mathcal{F}}_d]$ is

$$[\mathbb{T}^{2d} \xleftarrow{\Delta} (\mathbb{T}, 1)].$$

The Poincaré dual of this element is the unit in $\widehat{\mathrm{KK}}_*(\mathbb{T}^d, \mathbb{T}^d)$, which concludes the proof. □

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Appendix A

$\mathbb{Z}_2$-Graded Algebras

Since we do so much in this thesis to do with $\mathbb{Z}_2$-graded algebras, we would like to give a brief account of what they are.

Definition A.0.1. A $\mathbb{Z}_2$-graded vector space V is a vector space equipped with a direct sum decomposition $V = V_0 \oplus V_1$.

A $\mathbb{Z}_2$-graded algebra over a field K is an algebra A over K equipped with a direct sum decomposition $A = A_0 \oplus A_1$ such that

$$A_i \cdot A_j \subseteq A_{i+j},$$

where the subscripts are read modulo 2.

As was shown in Section 3.1, the Clifford algebras are examples of $\mathbb{Z}_2$-graded algebras. We define the $\mathbb{Z}_2$-graded tensor product as follows.

Definition A.0.2. Suppose that A and B are $\mathbb{Z}_2$-graded algebras. Their $\mathbb{Z}_2$-graded tensor product, denoted $A \hat{\otimes} B$, is the algebra whose underlying vector space is $A \otimes B$ (the regular tensor product), endowed with the multiplication

$$(a_0 \otimes b_0) \cdot (a_1 \otimes b_1) = (-1)^{\partial(a_1)\partial(b_0)}(a_0 a_1 \otimes b_0 b_1).$$

Other than the Clifford algebras, the main example of a $\mathbb{Z}_2$-graded algebra is the collection of endomorphisms of a $\mathbb{Z}_2$-graded vector space. If $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space, then a map $T \in \mathrm{End}(V)$ is called *even* if $T(V_0) \subseteq V_0$ and $T(V_1) \subseteq V_1$; T is called *odd* if $T(V_0) \subseteq V_1$ and $T(V_1) \subseteq V_0$. Two maps T and S are said to *graded commute* if $TS = (-1)^{\partial(T)\partial(S)} ST$.

Appendix B

Comparison of Induced K-Orientations

Let V be a (real) vector bundle and suppose that we are given a K-orientation on the direct sum $V \oplus \mathbb{R}$. There are two seemingly distinct ways that we can give V a K-orientation, and the point of this section is to show that they are, in fact, equivalent. The way we induce a K-orientation on V depends on the rank of V, so we split the section into two subsections.

B.1 V is of Even Rank

Suppose that $\dim V = 2k$ and let $S_{V\oplus\mathbb{R}}$ be the given spinor for $V \oplus \mathbb{R}$. Let $e \in \mathbb{C}l(\mathbb{R})$ denote the generator. The two bundles on which we let $\mathbb{C}l(V)$ act are:

1. $S_{V\oplus\mathbb{R}}$ itself, with action of V given by

$$\tilde{c} : \xi \mapsto c_{V\oplus\mathbb{R}}(\xi e)$$

   and $S_{V\oplus\mathbb{R}}$ is $\mathbb{Z}_2$-graded into the $\pm i$-eigenspaces of $c_{V\oplus\mathbb{R}}(e)$, i.e.

$$S_{V\oplus\mathbb{R}} = (1 - ic_{V\oplus\mathbb{R}}(e))S_{V\oplus\mathbb{R}} \oplus (1 + ic_{V\oplus\mathbb{R}}(e))S_{V\oplus\mathbb{R}}$$

2. $\mathrm{Hom}(\mathbb{C}, S_{V\oplus\mathbb{R}})$, with the action of V given by

$$(c_V(\xi)T)(x) \coloneqq c_{V\oplus\mathbb{R}}(\xi)T(x)$$

and $\mathbb{Z}_2$-graded using the subspaces

$$\mathrm{Hom}^{\pm}_{\mathbb{C}l(\mathbb{R})}(\mathbb{C}, S_{V\oplus\mathbb{R}}) \coloneqq \{T : T(c_{\mathbb{R}}(e)x) = \pm c_{V\oplus\mathbb{R}}(e)T(x)\}.$$

Proposition B.1.1. *These two K-orientations on V are equivalent.*

Proof. The map $T \mapsto T(1)$ is a grading preserving isomorphism $\mathrm{Hom}(\mathbb{C}, S_{V\oplus\mathbb{R}}) \to S_{V\oplus\mathbb{R}}$ which induces the following action of V on $S_{V\oplus\mathbb{R}}$:

$$c_V(\xi)x \coloneqq c_{V\oplus\mathbb{R}}(\xi)x.$$

Thus we are reduced to showing that these two actions on $S_{V\oplus\mathbb{R}}$ are equivalent. This is achieved by observing that the following diagram commutes

$$\begin{CD} S_{V\oplus\mathbb{R}} @>{1-c_{V\oplus\mathbb{R}}(e)}>> S_{V\oplus\mathbb{R}} \\ @V{\tilde{c}_V(\xi)}VV @VV{c_V(\xi)}V \\ S_{V\oplus\mathbb{R}} @>>{1-c_{V\oplus\mathbb{R}}(e)}> S_{V\oplus\mathbb{R}}; \end{CD}$$

indeed, for $x \in S_{V\oplus\mathbb{R}}$ we have

$$\begin{aligned}(1 - c_{V\oplus\mathbb{R}}(e))\tilde{c}_V(\xi)x &= (1 - c_{V\oplus\mathbb{R}}(e))c_{V\oplus\mathbb{R}}(\xi e)x = c_{V\oplus\mathbb{R}}(\xi)(1 - c_{V\oplus\mathbb{R}}(e))x \\ &= c_V(\xi)(1 - c_{V\oplus\mathbb{R}}(e)).\end{aligned}$$

□

B.2 V is of Odd Rank

Let V be an odd rank vector bundle, and suppose that you are given a K-orientation on $V \oplus \mathbb{R}$. There are two canonical ways that one can give a K-orientation from this data:

1. V acts on $S^0_{V\oplus\mathbb{R}}$ (the even part of $S_{V\oplus\mathbb{R}}$) by

$$\tilde{c}_V : \xi \mapsto c_{V\oplus\mathbb{R}}(\xi e),$$

   where e is the canonical generator for $\mathbb{C}l(\mathbb{R})$, and

2. V acts on $\mathrm{Hom}_{\mathbb{C}l(\mathbb{R})}(\mathbb{C}, S_{V\oplus\mathbb{R}})$ by

$$[c_V(\xi)T](x) := i\omega c_{V\oplus\mathbb{R}}(\xi)T(x),$$

   where ω is the volume element in $\mathbb{C}l(V \oplus \mathbb{R})$.

Proposition B.2.1. *These two K-orientations are equivalent.*

For the sake of clarity we break the proof into three parts. We first define the equivalence and show that it is well defined, then show that it intertwines the actions, and finally we show that it is indeed an isomorphism.

Our equivalence is defined as follows

$$\begin{aligned} S^0_{V\oplus\mathbb{R}} &\to \mathrm{Hom}_{\mathbb{C}l(\mathbb{R})}(\mathbb{C}, S_{V\oplus\mathbb{R}}) \\ x &\mapsto [T_x : 1 \mapsto (1 - ic_{V\oplus\mathbb{R}}(e))x]. \end{aligned}$$

Claim. *This map is well defined.*

Proof. We have to show that T_x, as defined above, is in $\mathrm{Hom}_{\mathbb{C}l(\mathbb{R})}(\mathbb{C}, S_{V\oplus\mathbb{R}})$ for all $x \in S^0_{V\oplus\mathbb{R}}$. This is equivalent to showing that $(1 - ic_{V\oplus\mathbb{R}}(e))x$ is in the $+i$-eigenspace of $c_{V\oplus\mathbb{R}}(e)$ for all $x \in S^0_{V\oplus\mathbb{R}}$. We compute

$$c_{V\oplus\mathbb{R}}(e)(1 - ic_{V\oplus\mathbb{R}}(e))x = (c_{V\oplus\mathbb{R}}(e) + i)x = i(1 - ic_{V\oplus\mathbb{R}}(e))x,$$

as needed. □

Claim. *The following diagram commutes*

$$\begin{array}{ccc} S^0_{V\oplus\mathbb{R}} & \xrightarrow{(1-ic_{V\oplus\mathbb{R}}(e))} & \mathrm{Hom}_{\mathbb{C}l(\mathbb{R})}(\mathbb{C}, S_{V\oplus\mathbb{R}}) \\ {\scriptstyle \tilde{c}_V(\xi)}\downarrow & & \downarrow{\scriptstyle c_V(\xi)} \\ S^0_{V\oplus\mathbb{R}} & \xrightarrow[(1-ic_{V\oplus\mathbb{R}}(e))]{} & \mathrm{Hom}_{\mathbb{C}l(\mathbb{R})}(\mathbb{C}, S_{V\oplus\mathbb{R}}), \end{array}$$

where the horizontal arrows are the map given above.

Proof. If we first traverse right and then down we see that

$$x \mapsto (1 - ic_{V\oplus\mathbb{R}}(e))x \mapsto i\omega c_{V\oplus\mathbb{R}}(\xi)(1 - ic_{V\oplus\mathbb{R}}(e))x;$$

on the other hand if we first go down and then right we get

$$x \mapsto c_{V\oplus\mathbb{R}}(\xi e)x \mapsto (1 - ic_{V\oplus\mathbb{R}}(e))c_{V\oplus\mathbb{R}}(\xi e)x.$$

We mush show they are equal. Recall that $\omega x = x$ since $x \in S^0_{V\oplus\mathbb{R}}$, and that $c_{V\oplus\mathbb{R}}(e)(1-$

$ic_{V\oplus\mathbb{R}}(e))x = i(1 - ic_{V\oplus\mathbb{R}}(e))x$ by design. With this in mind we compute

$$\begin{aligned} i\omega c_{V\oplus\mathbb{R}}(\xi)(1 - ic_{V\oplus\mathbb{R}}(e))x &= \omega c_{V\oplus\mathbb{R}}(\xi e)(1 - ic_{V\oplus\mathbb{R}}(e))x \\ &= c_{V\oplus\mathbb{R}}(\xi e)(1 + ic_{V\oplus\mathbb{R}}(e))x \\ &= (1 - ic_{V\oplus\mathbb{R}}(e))c_{V\oplus\mathbb{R}}(\xi e)x, \end{aligned}$$

as needed. □

Claim. *The map defined above is an isomorphism.*

Proof. A dimension count shows that it is sufficient to show that it is injective. Suppose that

$$(1 - ic_{V\oplus\mathbb{R}}(e))x = 0.$$

Let ω_V be the volume element for $\mathbb{C}l(V)$; note that ω_V is an odd operator. Then we have that

$$0 = \omega_V(1 - ic_{V\oplus\mathbb{R}}(e))x = (\omega_V - i\omega)x = \omega_V x - ix,$$

since $x \in S^0_{V\oplus\mathbb{R}}$. This says that $\omega_V x = ix$, which implies that $x = 0$ since $ix \in S^0_{V\oplus\mathbb{R}}$ and $\omega_V x \in S^1_{V\oplus\mathbb{R}}$ as ω_V is odd. This completes the proof. □