On chordal digraphs and semi-strict chordal digraphs

by

YingYing (Fay) Ye
B.Sc. University of Victoria 2007
M.Sc. University of Victoria 2008

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

On chordal digraphs and semi-strict chordal digraphs

by

YingYing (Fay) Ye

B.Sc. University of Victoria 2007

M.Sc. University of Victoria 2008

Supervisory Committee

Dr. Jing Huang, Supervisor

(Department of Mathematics and Statistics)

Dr. Venkatesh Srinivasan, Outside Member

(Department of Computer Science)

Supervisory Committee

Dr. Jing Huang, Supervisor

(Department of Mathematics and Statistics)

Dr. Venkatesh Srinivasan, Outside Member

(Department of Computer Science)

ABSTRACT

Chordal graphs are an important class of perfect graphs. The beautiful theory surrounding their study varies from natural applications to elegant characterizations in terms of forbidden subgraphs, subtree representations, vertex orderings, and to linear time recognition algorithms. Haskins and Rose introduced the class of chordal digraphs as a digraph analogue of chordal graphs. Chordal digraphs can be defined in terms of vertex orderings and several results about chordal graphs can be extended to chordal digraphs. However, a forbidden subdigraph characterization of chordal digraphs is not known and finding such a characterization seems to be a difficult problem. Meister and Telle studied semi-complete chordal digraphs and gave a forbidden subdigraph characterization of this class of digraphs.

In this thesis, we study chordal digraphs within the classes of quasi-transitive, extended semi-complete, and locally semi-complete digraphs. For each of these classes we obtain a forbidden subdigraph characterization of digraphs which are chordal. We also introduce in this thesis a new variant of chordal digraphs called semi-strict chordal digraphs. We obtain a forbidden subdigraph characterization of semi-strict chordal digraphs for each of the classes of semi-complete, quasi-transitive, extended semi-complete, and locally semi-complete digraphs.

Contents

List of Tables

List of Figures

ACKNOWLEDGEMENTS

I would like to thank:

my supervisor, Dr. Jing Huang, for his encouragement, patience, and mentorship. Without his guidance and persistent help this thesis would not have been possible..

my husband, Chris Zhang, and my parents, for all of the mentally suppose..

The University of Victoria, for the fellowships.

DEDICATION

Just hoping this is useful!

Chapter 1

Introduction

1.1 Background

A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H is equal to the size of a maximum clique in H [9]. One of the first classes of graphs to be recognized as being perfect was the class of chordal graphs. A graph G is *chordal* if its vertices can be linearly ordered by $\prec$ so that for any $u \prec v$ and $u \prec w$, if u is adjacent to both v and w, then v and w are adjacent. Such an ordering $\prec$ is called a *perfect elimination ordering* of G. Hajnal and Surányi [10] showed that every chordal graph is perfect.

Chordal graphs admit elegant linear time recognition algorithms and characterizations [9, 19]. It is easy to see that no chordal graph contains a cycle of length greater than three as an induced subgraph. In fact, it is proved that a graph is chordal if and only if it does not contain a cycle of length greater than three as an induced subgraph [19]. A vertex u in a graph is *simplicial* if its neighbours induce a complete subgraph. Dirac [6] proved that a graph G is chordal if and only if every induced subgraph of G contains a simplicial vertex. Rose and Tarjan [18] have obtained a very simple linear time recognition algorithm for chordal graphs.

A digraph D is called a *chordal digraph* if its vertices can be linearly ordered by $\prec$ so that for any $u \prec v$ and $u \prec w$, if $v \to u$ and $u \to w$, then $v \to w$ [12]. Such an ordering $\prec$ is called a *perfect elimination ordering* of D. Clearly, if D is chordal, then every induced subdigraph of D is chordal. It follows from the definition that every chordal digraph contains a vertex u such that every in-neighbour of u dominates every out-neighbour of u. Such a vertex u is called a *di-simplicial vertex* of D. The hereditary property of chordal digraphs implies that every induced subdigraph of a chordal digraph has a di-simplicial vertex.

Little is known about the structure of chordal digraphs. It has been an open problem to find a forbidden subdigraph characterization of chordal digraphs. In [13] a subclass of chordal digraphs was identified and studied. A digraph D is a *strict chordal digraph* if its vertices can be linearly ordered by $\prec$ so that for any $u \prec v$ and $u \prec w$, if there is an arc between u and v and an arc between u and w, then $v \leftrightarrow w$. It is easy to see that strict chordal digraphs form a subclass of chordal digraphs. It turns out that strict chordal digraphs admit a forbidden structure characterization and a polynomial time recognition algorithm [13].

In this thesis, we introduce yet another class of digraphs which we call semi-strict chordal digraphs. A digraph D is *semi-strict chordal* if its vertices can be linearly ordered by $\prec$ so that for any $u \prec v$ and $u \prec w$, if $v \to u$ and $u \to w$, then $v \leftrightarrow w$. It is easy to see that semi-strict chordal digraphs form a subclass of chordal digraphs that contains strict chordal digraphs. Call a vertex u a *semi-strict di-simplicial* vertex of a digraph D if $v \to u$ and $u \to w$ then $v \leftrightarrow w$. Every induced subdigraph of a semi-strict chordal digraph is semi-strict chordal and hence contains a semi-strict di-simplicial vertex.

In this thesis, we study both chordal digraphs and semi-strict chordal digraphs within several classes of digraphs. The main results of this thesis are forbidden subdigraph characterizations of chordal digraphs and semi-strict chordal digraphs for these classes of digraphs.

1.2 Preliminaries

We consider both graphs and digraphs, all of which are assumed to be finite and simple (i.e., containing no loops and no multiple edges or arcs).

Let D be a digraph. The vertex set of D is denoted as $V(D)$, and the arc set of D is denoted as $A(D)$. Let u, v be two vertices of D. We say that u, v are *adjacent* if there is at least one arc between u and v and we denote it by u — v; otherwise we say that they are not adjacent (and depict it as $u\ {-}{-}\ v$ in figures). If there is an arc from u to v but no arc from v to u, then we denote it by $u \mapsto v$. If $u \mapsto v$ or $v \mapsto u$ but not both, then we say u is adjacent to v by a single arc and denoted it as $u \leftharpoonup v$. If there is an arc from u to v and an arc from v to u, then we denote it by $u \leftrightarrow v$ and call the arcs between u and v *symmetric arcs*. We also use $u \rightarrow v$ to denote either $u \mapsto v$ or $u \leftrightarrow v$, i.e., u dominates v and possibly v dominates u. If $u \rightarrow v$, then u is an *in-neighbour* of v and v is an *out-neighbour* of u. The set of in-neighbours of v is denoted as $N^-(v)$ and the set of out-neighbours of v is denoted as $N^+(v)$.

The *underlying graph* of a digraph D consists of vertex set $V(D)$ and edges uv such that u, v are adjacent in D. If $V' \subseteq V(D)$, then we use $D[V']$ or $[V']$ to denote the subdigraph of D induced by V'. We use $O(D)$ to denote the subdigraph of D induced by single arcs and $S(D)$ to denote the underlying graph of the subdigraph of D induced by symmetric arcs. A digraph D is *strongly connected* if for every pair of distinct vertices u and v, there exists a directed path from u to v and a directed path from v to u.

Definition A digraph is *semi-complete* if there is an arc between any two vertices.

Definition A digraph is *quasi-transitive* if there is a complete adjacency between $N^-(v)$ and $N^+(v)$ of each vertex v [2].

Suppose that D is quasi-transitive and $\mathcal{C} = u_1, u_2, ..., u_k$ $(k \geq 3)$ is the shortest directed cycle with only single arcs (i.e., $u_i \mapsto u_{i+1}$ for each i). We claim that every non-consecutive vertices u_i and u_j $(i \neq j \pm 1)$ are pairwise adjacent by symmetric arcs. Since $u_i \in N^-(u_{i+1})$ and $u_{i+2} \in N^+(u_{i+1})$, the quasi-transitivity of D implies that u_i and u_{i+2} are adjacent, and the arc between them is symmetric, as otherwise we obtain a smaller k. That means $u_i \in N^-(u_{i+2})$. The quasi-transitivity of D also implies that u_i and u_{i+3} are adjacent because $u_{i+3} \in N^+(u_{i+2})$. Similarly, every pair of non-consecutive vertices u_i is connected by symmetric arcs.

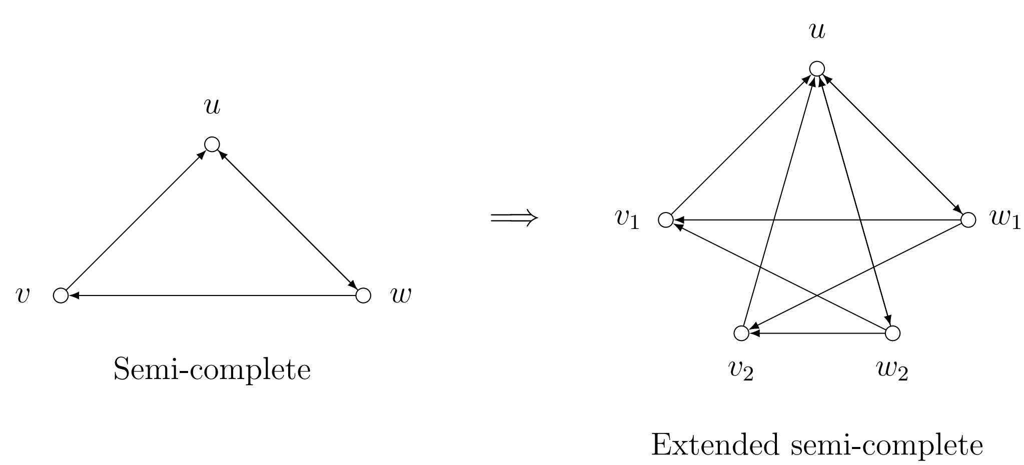


Figure 1.1: An extended semi-complete digraph blown up from a semi-complete digraph

Definition An *extended semi-complete* digraph is a digraph obtained from a semi-complete digraph by replacing each vertex with an independent set [1]. (See Figure 1.1 for an example.)

Let D be a extended semi-complete digraph. Suppose that $\mathcal{C} = u_1, u_2, ..., u_t$ is a directed cycle consisting of only single arcs and moreover there is no single arc between non-consecutive vertices of $\mathcal{C}$. We claim each pair of non-consecutive vertices is connected by symmetric arcs. Since $u_i \mapsto u_{i+1}$ and $u_{i+1} \mapsto u_{i+2}$, u_i and u_{i+2} are not from the same independent set and hence are adjacent. Our assumption thus implies u_i and u_{i+2} are connected by symmetric arcs. Now, since the arcs between u_i and u_{i+2} and between u_{i+2} and u_{i+3} are of different types, we have u_i is adjacent to u_{i+3} by symmetric arcs. Continuing this way, we can conclude that every non-consecutive vertices u_i and u_j $(i \neq j \pm 1)$ are pairwise adjacent by symmetric arcs.

Definition A digraph D is *locally semi-complete* if for every vertex v of D, both of the induced subdigraphs $D[N^-(v)]$ and $D[N^+(v)]$ are semi-complete.

Before colsing this section, we state some useful properties of chordal graphs which will be frequently used in the thesis.

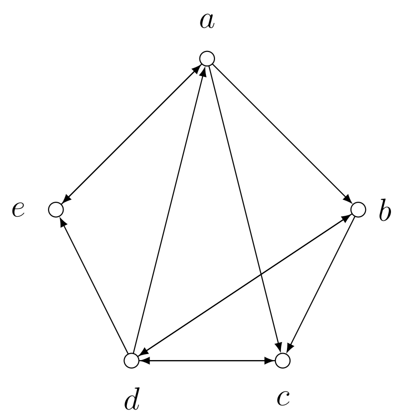
Lemma 1.1. *[16] Let G be a chordal graph. Then the following statements hold:*

1. *If u and v are a pair of adjacent simplicial vertices of G, then $N[u] = N[v]$.*
2. *If u is a vertex of G, then every connected component of $G \setminus N[u]$ contains a vertex that is simplicial in G.*

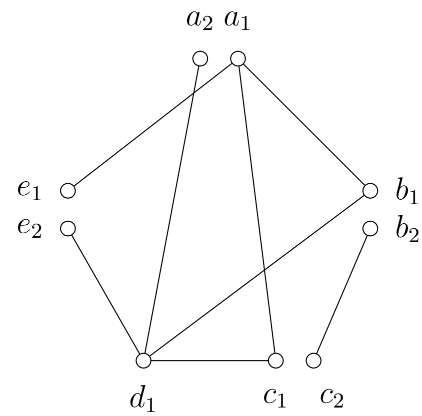
1.3 Knotting Graph Representation

Let $D = (V, A)$ be a digraph. The *knotting graph* $\bar{K}_D$ of D is defined as follows: Suppose that $v \in V(D)$. Let $\mathcal{V} = \{v_1, v_2, ...v_k\}$ $(v_i \neq \emptyset)$ where each v_i is a set of vertices of D such that $\bigcup_{i=1}^{k} v_i = N_D(v)$ and $v_i \cap v_j = \emptyset$, and call $\mathcal{V}$ a *group*. Note that each vertex of D corresponds to a group. For every pair of vertices $u, w \in N^+(v) \cup N^-(v)$, $\{u, w\} \subseteq v_i$ if either $u \in N_D^-(v), w \in N_D^+(v)$ but $u \notin N_D^-(w)$, or

$w \in N_D^-(v), u \in N_D^+(v)$ but $w \notin N_D^-(u)$. The vertices of $\bar{K}_D$ are those v_is where $v_i \in \mathcal{V}$ for every $v \in V(D)$. For every vertex pair u_i, v_j of $\bar{K}_D$, u_i is adjacent to v_j if and only if $v \in u_i$ and $u \in v_j$. Figure 1.2 and Figure 1.3 are examples of knotting graphs of digraphs.



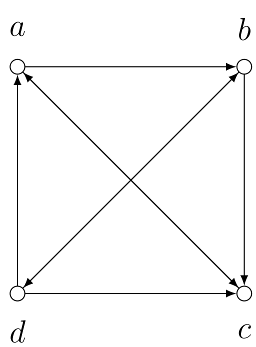
$a_1 = \{b, c, e\}$, $a_2 = \{d\}$,
$b_1 = \{a, d\}$, $b_2 = \{c\}$,
$c_1 = \{a, d\}$, $c_2 = \{b\}$,
$d_1 = \{a, b, c, e\}$,
$e_1 = \{a\}$, $e_2 = \{d\}$.



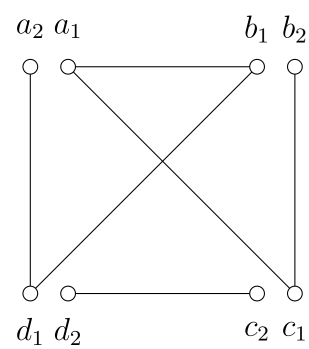
Chordal digraph D

Knotting graph $\bar{K}_D$

Figure 1.2: A chordal digraph D and its knotting graph $\bar{K}_D$



$a_1 = \{b, c\}$, $a_2 = \{d\}$,
$b_1 = \{a, d\}$, $b_2 = \{c\}$,
$c_1 = \{a, b\}$, $c_2 = \{d\}$,
$d_1 = \{a, b\}$, $d_2 = \{c\}$.



Non-chordal digraph D'

Knotting graph $\bar{K}_{D'}$

Figure 1.3: A non-chordal digraph D' and its knotting graph $\bar{K}_{D'}$

Lemma 1.2. *Let D be a digraph and $\bar{K}$ be the knotting graph of D. Then D has a di-simplicial vertex if and only if $\bar{K}$ has a group such that every vertex of that group has degree at most one.*

Proof. For the necessity, suppose that D is a chordal digraph. Then it has a di-simplicial vertex. Let v be the di-simplicial vertex of D. Then we have a group $\mathcal{V} = \{v_1, v_2, ..., v_r\}$ such that v_i is a vertex of $\bar{K}$ and $\bigcup_{i=1}^{r} v_i = N_D(v)$, $v_i \cap v_j = \emptyset$. Suppose to the contrary that $v_i \in \mathcal{V}$ has degree at least two and assume that u_j, w_k are two vertices that are adjacent to v_i. Then by the way we construct $\bar{K}$, there exist vertices $u, w \in V(D)$, such that $\{u, w\} \subseteq v_i$, $\{v\} \subseteq u_j$ and $\{v\} \subseteq w_k$, where $\{u, w\} \subseteq N_D(v)$. Then either $u \in N^-(v), w \in N^+(v)$ but $u \notin N^-(w)$, or $w \in N^-(v), u \in N^+(v)$ but $w \notin N^-(u)$. In either case, v is not a di-simplicial vertex of D, a contradiction. Suppose now that $\bar{K}$ has a group such that every vertex of that group has degree at most one and let $\mathcal{V}$ be such a group. For every vertex pair u, w such that $\{u, w\} \subseteq N_D(v)$, if $u \to v \to w$, then we have $u \to w$. Therefore, v is a di-simplicial vertex of D. □

By the hereditary property of chrodal digraph, every induced subdigraph of a chordal digraph is also a chordal digraph and hence contains a di-simplicial vertex. Thus Lemma 1.2 immediately implies the following:

Proposition 1.1. *Let D be a digraph. Then for every induced subdigraph D_s of D, the knotting graph $\bar{K}_s$ of D_s has a group such that every vertex of that group has degree at most one if and only if D is chordal.*

1.4 Outline of Thesis

No forbidden subdigraph (or substructure) characterization of chordal digraphs or of semi-strict chordal digraphs is known. In this thesis we focus on the study of them within the classes of quasi-transitive, extended semi-complete, and locally semi-complete digraphs.

Chapter 1 is an introduction of this thesis. We provide the background and the intention of this study. The notations and the basic definitions are presented in section 1.2. In section 1.3, the concept of knotting graphs of digraphs is introduced and a proposition about what the knotting graph should look like if the digraph is chordal is stated.

In Chapter 2, we obtain chordal digraph characterizations for several special digraph classes. We first provide an existing result about semi-complete chordal digraph. We then describe the feature for the vertex which cannot be a di-simplicial vertex by the concept called witness triple. The main part of this chapter is to state forbidden characterizations for quasi-transitive, extended semi-complete and locally semi-complete digraphs, and give formal proofs of them.

In Chapter 3, we turn to semi-strict chordal digraphs. We again define the witness triple for a vertex in the semi-strict chordal sense. We then restrict the study of semi-strict chordal digraphs to the classes of semi-complete, quasi-transitive, extended semi-complete and locally semi-complete digraphs. It turns out that the same list of forbidden subdigraphs applies to the first three digraph classes, while for locally semi-complete digraphs the list of forbidden subdigraphs is larger.

The last chapter summarizes and concludes the work presented in this thesis and suggests possible directions for the future research.

Chapter 2

Chordal Digraphs

2.1 Introduction

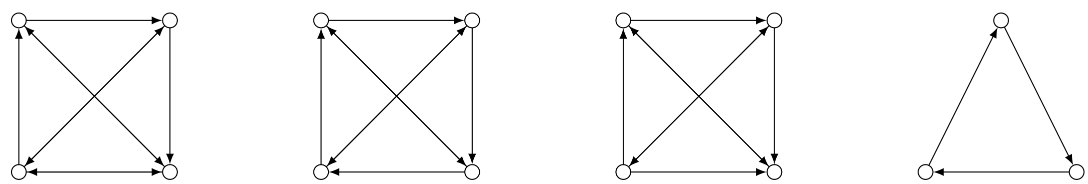
Recall that a digraph D is chordal if every induced subdigraph of D has a di-simplicial vertex. No characterization of chordal digraphs by forbidden induced subdigraphs is known. However, special subclasses of chordal digraphs have been studied [11, 15, 16]. Before we go to the special subclasses, we first look at the following lemma, which holds for general digraphs.

Lemma 2.1. *[16] Let D be a digraph. If D is chordal, then $S(D)$ is a chordal graph. Moreover, every di-simplicial vertex of D is a simplicial vertex of $S(D)$.*

Proof. Suppose that D is chordal. Let $\prec$ be the perfect elimination ordering of D. We prove that $S(D)$ contains no chordless cycle of length at least 4. Assume to the contrary that $\mathcal{C} = u_1, u_2, ..., u_k$ $(k \geq 4)$ is a chordless cycle in $S(D)$. Assume without loss of generality that $u_1 \prec u_k$ and $u_1 \prec u_2$. Since D is a chordal digraph, $u_1 \leftrightarrow u_2$ and $u_1 \leftrightarrow u_k$ imply that $u_2 \leftrightarrow u_k$, which contradicts the assumption that $\mathcal{C}$ is a chordless cycle. Suppose that u is a di-simplicial vertex of D but not a simplicial vertex of $S(D)$. Then there are vertices v and w in $S(D)$ such that $u \leftrightarrow v$ and $u \leftrightarrow w$

but v and w are not adjacent by symmetric arcs, which contradicts the fact that u is a di-simplicial vertex of D. □

Semi-complete chordal digraphs have been characterized in [16]. It is easy to verify that none of the four semi-complete digraphs in Figure 2.1 is chordal. It turns out that these digraphs are the smallest semi-complete digraphs which are not chordal.



(a) (b) (c) (d)

Figure 2.1: Forbidden induced subdigraphs for semi-complete chordal digraphs

In each digraph in Figure 2.1, one can see that none of the vertices is a di-simplicial vertex. Therefore, none of the digraphs in Figure 2.1 is a chordal digraph. Immediately, we have the following:

Lemma 2.2. *No chordal digraph contains any of the digraph in Figure 2.1 as an induced subdigraph.*

Theorem 2.1. *[16] Let D be a semi-complete digraph. Suppose that $S(D)$ is chordal. Then D is a chordal digraph if and only if D does not contain any of the digraphs in Figure 2.1 as an induced subdigraph.*

2.2 Witness Triples

In this section, we examine the cases when a vertex in a digraph fails to be a di-simplicial vertex. By Lemma 2.1, we know that if D is a chordal digraph then $S(D)$

is a chordal graph. In order for a vertex u to be di-simplicial in D, u is simplicial in $S(D)$. We assume now $S(D)$ is chordal and only work on those cases where u is a simplicial vertex in $S(D)$. Denote by X the set of simplicial vertices in $S(D)$.

Let D be a digraph and let (u, v, w) be an ordered triple of vertices of D. We call (u, v, w) a *witness triple* for u of the *first type* if $u \leftrightarrow w$ and either $u \mapsto v \mapsto w$ or $w \mapsto v \mapsto u$ (See Figure 2.2 (a)). The witness triple of the second type, the third type and the fourth type are defined similarly (see Figure 2.2 (b), (c), (d)). When there is a witness triple for u, we also say that u has a witness triple.

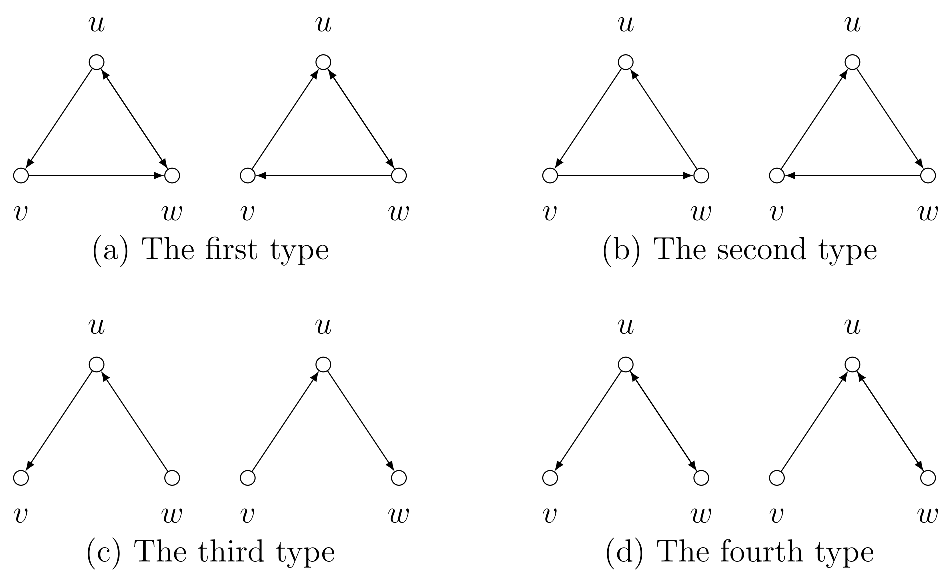


Figure 2.2: Witness triples of various types for u

Lemma 2.3. *Let D be a digraph. Suppose that $S(D)$ is chordal. Then a vertex u in X is a di-simplicial vertex of D if and only if there is no witness triple for u in D.*

Proof. Assume that D has a witness triple (u, v, w). We show that u is not di-simplicial in D with the following: Suppose that we are in the case when $u \mapsto v$. In whatever types of witness triples, since we have $w \in N^-(u)$ and $v \in N^+(u)$ but v is not an out-neighbour of w, by the definition of di-simplicial vertices, we know that u

is not a di-simplicial vertex. Suppose that $v \mapsto u$. Then in any type of witness triples, $v \in N^-(u)$ and $w \in N^+(u)$ but w is not an out-neighbour of v. Therefore, u is not a di-simplicial vertex. For the converse, suppose that u is not a di-simplicial vertex in D. This means there exist vertices $v \in N^-(u)$ and $w \in N^+(u)$ such that w is not an out-neighbour of v, or $v \in N^+(u)$ and $w \in N^-(u)$ such that v is not an out-neighbour of w. If both $u \leftrightarrow v$ and $u \leftrightarrow w$ but v is not adjacent to w by symmetric arcs, then $u \notin X$, which is not the case we need to consider with. Hence u and v or u and w are joined by a single arc. Without loss of generality, assume that u and v is connected by a single arc. If $u \leftrightarrow w$, then u has a witness triple of the first or the fourth type. If the arc between u and w is single, then u has a witness triple of the second or the third type. In each of the possible cases, u has a witness triple in D. $\square$

Recall that a digraph $D = (V, A)$ is quasi-transitive if for any three vertices x, y, z, if $x \rightarrow y$, $y \rightarrow z$, then x and z are adjacent. Suppose that D is a quasi-transitive digraph and $S(D)$ is chordal. Then for every vertex $u \in X$, if $v \in N^-(u)$ and $w \in N^+(u)$, or $v \in N^+(u)$ and $w \in N^-(u)$, then v and w are adjacent. That means u cannot have witness triples of the third or the fourth type. Consequently, if u is not di-simplicial, then u can only have witness triples of the first or the second type.

Extended semi-complete digraph is a digraph obtained from a semi-complete digraph D by substituting each vertex of D by an independent vertex set. Following from the definition, vertices of an extended semi-complete digraph in the same independent set have the same in- and out- neighbours. Let v and w be vertices from the same independent set and u be a vertex from another independent set. Then u is adjacent to both v and w. Suppose that $u \in X$. Then the arcs between u and v and between u and w are both single. Moreover, $u \mapsto v$ if and only if $u \mapsto w$, and $v \mapsto u$ if and only if $w \mapsto u$. That means u cannot have witness triples of the third or the fourth type. Consequently, u can only have witness triples of the first or the second type.

A digraph D is locally semi-complete if both of $D[N^-(u)]$ and $D[N^+(u)]$ are semi-complete for every vertex u in D. In other words, given $u \in X$, if $\{v, w\} \subseteq N^-(u)$ or $\{v, w\} \subseteq N^+(u)$, then v and w are adjacent. Hence u cannot have witness triples of the fourth type. That means if u is not di-simplicial, then u has witness triples of the first, the second or the third type.

	Witness triple of the			
	first type	second type	third type	fourth type
Semi-complete digraphs	✓	✓		
Quasi-transitive digraphs	✓	✓		
Extended semi-complete digraphs	✓	✓		
Locally semi-complete digraphs	✓	✓	✓	
General digraphs	✓	✓	✓	✓

Table 2.1: Witness triples for a vertex in the different type of the digraphs

The following proposition is a summary of the above discussion.

Proposition 2.1. *Let D be a digraph and u be a simplicial vertex of $S(D)$. Suppose that u is not a di-simplicial vertex. Then the following statements hold:*

1. *if D is quasi-transitive or extended semi-complete, then u has witness triples of the first or the second type;*

2. *if D is locally semi-complete, then u has witness triples of the first, the second or the third type (Table 2.1).*

2.3 Quasi-transitive Digraphs

In this section, we give a forbidden subdigraph characterization of quasi-transitive chordal digraphs. The following lemma will be useful for the proof of the main result.

Lemma 2.4. *Suppose that D is quasi-transitive and $S(D)$ is chordal. If D does not contain any of the digraphs in Figure 2.1 as an induced subdigraph, then $O(D)$ does not contain a directed cycle.*

Proof. Suppose to the contracy that $O(D)$ contains a directed cycle. Let $\mathcal{C} = u_1, u_2, ..., u_l$ be such a cycle of the shortest length l. Since $\mathcal{C}$ is the shortest directed cycle in $O(D)$, we know from Chapter 1 that every non-consecutive vertices u_i and u_j $(i \neq j \pm 1)$ in $\mathcal{C}$ are joined by symmetric arcs. Since D does not contain Figure 2.1 (b) or (d) as an induced subdigraph, l cannot be 3 or 4. If $l = 5$, then $\mathcal{C}' = u_1, u_3, u_5, u_2, u_4$ is a chordless cycle of length 5 in $S(D)$, which is a contradiction to the assumption that $S(D)$ is chordal. If $l \geq 6$, then $\mathcal{C}'' = u_1, u_4, u_2, u_5$ is a chordless cycle of length 4 in $S(D)$, a contradiction. $\square$

Now, we are ready to prove the main result of this section.

Theorem 2.2. *Let D be a quasi-transitive digraph. Suppose that $S(D)$ is chordal. Then D is a chordal digraph if and only if it does not contain any of the digraphs in Figure 2.1 as an induced subdigraph.*

Proof. The necessity follows from Lemma 2.2. For the sufficiency, suppose that D does not contain any of the digraphs in Figure 2.1 as an induced subdigraph. It suffices to show that D has a di-simplicial vertex. Suppose that D does not have di-simplicial vertices and let u be a vertex in X. By Proposition 2.1, u has witness triple of the first or the second type. If u has a witness triple of the second type, then D contains a copy of the digraph in Figure 2.1 (d). Thus, assume that u has a witness triple of the first type.

Let (u, v, w) be a witness triple of the first type. We show that there exists a vertex $v' \in X$ such that (u, v', w) is also a witness triple of the first type. Since (u, v, w) is a witness triple of the first type, we have either $w \mapsto v \mapsto u$ or $u \mapsto v \mapsto w$.

Without loss of generality, assume that $w \mapsto v \mapsto u$. (the other case can be proved in a similar way.) By Lemma 1.1, there exists a vertex v' in the component of v in $S(D) \setminus N[w]$ such that $v' \in X$ (it is possible to have $v = v'$). Since v and v' are in the same component of $S(D) \setminus N[w]$, there is a path joining them in $S(D) \setminus N[w]$. Let $\mathcal{P}_k : v = v_1, v_2, ..., v_k = v'$ be such a path. We prove by induction on k that (u, v_k, w) is also a witness triple of the first type. It is true when $k = 1$. So assume that $k \geq 2$. Let (u, v_{k-1}, w) be a witness triple of the first type such that $w \mapsto v_{k-1} \mapsto u$. Since $\{u, v_k\} \subseteq N^+(v_{k-1})$ and $\{w, v_k\} \subseteq N^-(v_{k-1})$, v_k is adjacent to both u and w. Moreover, w and v_k are connected by a single arc because v_k is not adjacent to w in $S(D)$. Since $u \in X$, u and v_k are connected by a single arc, as otherwise we have $w \leftrightarrow v_k$, which is not the case. If $v_k \mapsto w$, then the digraph induced by $\{u, v_k, w, v_{k-1}\}$ is a copy of Figure 2.1 (b) or (c), a contradiction to the assumption. Thus, $w \mapsto v_k$. If $u \mapsto v_k$, then $\{u, v_k, w, v_{k-1}\}$ induces a digraph as Figure 2.1 (c), a contradiction. Hence $v_k \mapsto u$. Therefore, (u, v_k, w) is a witness triple of the first type, where $v_k = v' \in X$.

We have proved that for each vertex $u \in X$, there exists a vertex $u' \in X$ such that (u, u', w) is a witness triple of the first type. Since u' is in X, there exists a vertex $u'' \in X$ such that (u', u'', w') is a witness triple of the first type. Continuing this way, we obtain a sequence of vertices $u, u', u'', ... \in X$ along with witness triples $(u, u', w), (u', u'', w')$,... of the first type. Since D is finite, the sequence will have a repeated vertex. Then either we find a di-simplicial vertex, which contradicts the assumption, or obtain a sequence of vertices $u_1, u_2, ..., u_l \in X$ such that $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_l, u_1, w_l)$ are witness triples of the first type. Assume that such a sequence is chosen so that l is the least. Since (u_i, u_{i+1}, w_i) is a witness triple of the first type, we have $u_i \leftrightarrow w_i$, and the arcs between u_i and u_{i+1} and between w_i and u_{i+1} are single arcs. By Lemma 2.4, the cycle: $\mathcal{C} = u_1, u_2, ..., u_l$

is not a directed cycle in $O(D)$. Therefore, without loss of generality, assume that $u_1 \mapsto u_2$ and $u_1 \mapsto u_l$. We prove by contradiction that the sequence does not exist by considering the following cases:

- **Case 1:** $l = 2$. Then (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the first type, where $u_i \in X$. Assume that $u_1 \mapsto u_2$. (The proof for the case when $u_2 \mapsto u_1$ is the same.) Then we have $u_2 \mapsto w_1$ and $w_2 \mapsto u_1$. Because $w_1 \in N^+(u_1)$ and $w_2 \in N^-(u_1)$, and since D is quasi-transitive, we have w_1 and w_2 are adjacent. If $w_1 \leftrightarrow w_2$, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 2.1 (a). If w_1 and w_2 are connected by a single arc, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 2.1 (b) or (c).

- **Case 2:** $l = 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2)$ and (u_3, u_1, w_3) are witness triples of the first type, where w_1, w_2 and w_3 are distinct. From the assumption, $u_1 \mapsto u_2$ and $u_1 \mapsto u_3$. By the definition of witness triples of the first type, we have $u_2 \mapsto w_1$, $u_1 \leftrightarrow w_1$, $w_3 \mapsto u_1$ and $u_3 \leftrightarrow w_3$. Moreover, $u_2 \leftrightarrow w_2$ and either $u_2 \mapsto u_3 \mapsto w_2$ or $w_2 \mapsto u_3 \mapsto u_2$. Since the subdigraph induced by $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ is strongly connected, it is a semi-complete subdigraph. Therefore, the vertices $u_1, u_2, u_3, w_1, w_2, w_3$ are pairwise adjacent. There are two cases for the witness triple (u_2, u_3, w_2): either $u_2 \mapsto u_3 \mapsto w_2$ (Figure 2.3 (a)) or $w_2 \mapsto u_3 \mapsto u_2$ (Figure 2.3 (b)). Suppose that $u_2 \mapsto u_3 \mapsto w_2$. If $w_3 \mapsto u_2$, then (u_3, u_2, w_3) becomes a witness triple of the first type, which yields a smaller value of l, a contradiction to the choice of l. If $w_3 \leftrightarrow u_2$, then (u_2, u_1, w_3) is a witness triple of the first type, which again contradicts the choice of l. Hence $u_2 \mapsto w_3$. But then the subdigraph induced by $\{u_1, u_2, w_3\}$ is a copy of Figure 2.1 (d). Suppose that $w_2 \mapsto u_3 \mapsto u_2$ (Figure 2.3 (b)). If $u_3 \mapsto w_1$, then (u_1, u_3, w_1) is a witness triple of the first type and we obtain a sequence u_1, u_3 for a shorter length, a contradiction. If $u_3 \leftrightarrow w_1$, then (u_3, u_2, w_1) is a witness triple of the first type, which contradicts the choice of l. Hence the only possible case left is $w_1 \mapsto u_3$. But then the subdigraph induced by $\{u_2, u_3, w_1\}$ is a copy of Figure 2.1 (d).

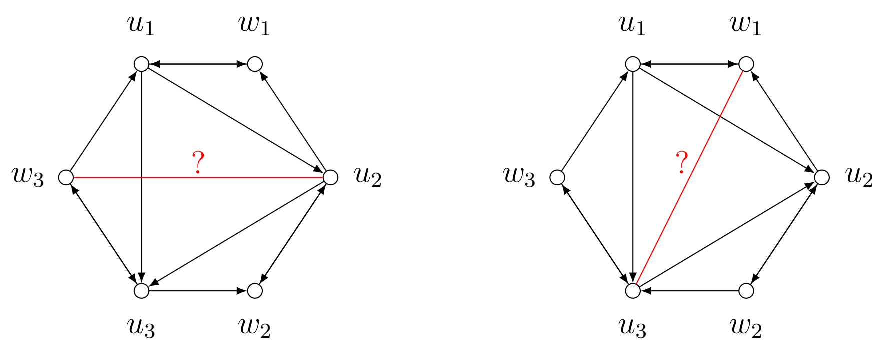


Figure 2.3: Two cases when sequence size is three

- **Case 3:** $l > 3$. The witness triples $(u_1, u_2, w_1), (u_2, u_3, w_2)$, ..., (u_l, u_1, w_l) are of the first type and $u_i \in X$. For the sequence $u_1, u_2, ..., u_l$, we have $u_1 \mapsto u_2$ and $u_1 \mapsto u_l$. By the definition of witness triples of the first type, we have $u_2 \mapsto w_1$, $u_1 \leftrightarrow w_1$, $w_l \mapsto u_1$ and $w_l \leftrightarrow u_l$. Thus, u_2 is adjacent to w_l because $w_l \in N^-(u_1)$ and $u_2 \in N^+(u_1)$. If $u_2 \leftrightarrow w_l$, then (u_2, u_1, w_l) is a witness triple of the first type and we obtain a shorter sequence u_1, u_2, a contradiction to the assumption that $l > 3$. If $u_2 \mapsto w_l$, then $\{u_1, u_2, w_l\}$ induces a copy of Figure 2.1 (d). Hence $w_l \mapsto u_2$. Since $u_l \in N^-(w_l)$ and $u_2 \in N^+(w_l)$, we know that u_2 is adjacent to u_l. Since $u_l \in X$, there cannot be symmetric arcs between u_2 and u_l. If $u_2 \mapsto u_l$, then (u_l, u_2, w_l) is a witness triple for u_l of the first type and we obtain a shorter sequence $u_2, u_3, ..., u_l$, a contradiction. The only possible left is $u_l \mapsto u_2$. Since $u_l \in N^-(u_2)$ and $w_1 \in N^+(u_2)$, u_l is adjacent to w_1. If $w_1 \leftrightarrow u_l$, then (u_l, u_2, w_1) is a witness triple of the first type and there is a shorter sequence $u_2, u_3, ..., u_l$, a contradiction. Similarly, if $u_l \mapsto w_1$, then (u_1, u_l, w_1) is also a witness triple of the first type, and there is a shorter sequence u_1, u_l. Hence $w_1 \mapsto u_l$. But then the subdigraph induced by $\{u_2, u_l, w_1\}$ is a copy of Figure 2.1 (d), a contradiction.

Therefore, D has a di-simplicial vertex. $\square$

2.4 Extended Semi-complete Digraphs

We have characterized the forbidden subdigraphs for quasi-transitive digraphs. It turns out that they are consisting with the forbidden subdigraphs for semi-complete chordal digraphs. In fact, there is one more digraph class which has the same forbidden subdigraphs. Such digraph class is the class of extended semi-complete digraphs. The following proposition can be proved in a similar way as for Lemma 2.4

Proposition 2.2. *Suppose that D is an extended semi-complete digraph and $S(D)$ is chordal. If D does not contain any of the digraphs in Figure 2.1 as an induced subdigraph, then $O(D)$ does not contain a directed cycle.*

Theorem 2.3. *Let D be an extended semi-complete digraph. Suppose that $S(D)$ is chordal. Then D is a chordal digraph if and only if it does not contain any of the digraphs in Figure 2.1 as an induced subdigraph.*

Proof. The necessity follows from Lemma 2.2. For the sufficiency suppose that D does not contain any of the digraphs in Figure 2.1 as an induced subdigraph. We prove that D has a di-simplicial vertex. Suppose that none of the vertex in D is di-simplicial and let $u \in X$. Proposition 2.1 shows that u has a witness triple of the first or the second type. Clearly, u cannot have witness triples of the second type because the second type itself is a forbidden subdigraph of D (Figure 2.1 (d)). Hence u has a witness triple of the first type.

Let (u, v, w) be a witness triple of the first type. Then we can show that there must exist a vertex $v' \in X$ such that (u, v', w) is a witness triple of the first type.

By the definition of witness triples, we have either $w \mapsto v \mapsto u$ or $u \mapsto v \mapsto w$, and $u \leftrightarrow w$. Without loss of generality, we assume that $w \mapsto v \mapsto u$. By Lemma 1.1, the component of $S(D) \setminus N[w]$ that contains v always contains a simplicial vertex v' (it is possible to have $v = v'$). Moreover, there is a path joining v and v' in $S(D) \setminus N[w]$. Let $\mathcal{P}_k : v = v_1, v_2, ..., v_k = v'$ be such a path. We use mathematical induction on k to prove that (u, v_k, w) is a witness triple of the first type. From the assumption, the statement is true when $k = 1$. Assume that for $k \geq 2$, (u, v_{k-1}, w) is a witness triple of the first type such that $w \mapsto v_{k-1} \mapsto u$. Since $v_k \in S(D) \setminus N[w]$ which is in $\mathcal{P}_k$, we have $v_{k-1} \leftrightarrow v_k$. Moreover, since $w \mapsto v_{k-1}$, we know that w and v_k are not in the same independent set and hence are adjacent. The arc between them must be a single arc because $v_k \in S(D) \setminus N[w]$. Because $v_{k-1} \mapsto u$ and $v_{k-1} \leftrightarrow v_k$, we know that u and v_k are adjacent. In addition to that, there is a single arc between u and v_k, as otherwise, $u \notin X$. If $v_k \mapsto w$, then $\{u, v_k, w, v_{k-1}\}$ induces a copy of Figure 2.1 (b) or (c), which contradicts the assumption. Thus, we have $w \mapsto v_k$. If $u \mapsto v_k$, then the digraph induced by $\{u, v_{k-1}, w, v_k\}$ is a copy of Figure 2.1 (c), a contradiction. Hence, we have $v_k \mapsto u$, which means (u, v_k, w) is a witness triple of the first type with $v_k = v' \in X$.

We have showed that for each vertex $u \in X$, there exists a vertex $u' \in X$, such that (u, u', w) is a witness triple of the first type. Since u' is in X, there exists a vertex $u'' \in X$ such that (u', u'', w') is a witness triple of the first type. Since D is finite, we will eventually either find a di-simplicial vertex and obtain a contradiction, or yield a sequence of vertices $u_1, u_2, ..., u_l \in X$ such that $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_l, u_1, w_l)$ are witness triples of the first type. We prove such a sequence does not exist by assuming that it is chosen with a shortest length l. Since (u_i, u_{i+1}, w_i) is a witness triple of the first type, we know that the arcs between u_i and u_{i+1} and between w_i and u_{i+1} are single arcs, and $u_i \leftrightarrow w_i$. Thus, $u_1, u_2, ..., u_l$ form a cycle in $O(D)$.

By Proposition 3.1, this cycle cannot be a directed cycle. Hence, without loss of generality, we assume that $u_1 \mapsto u_2$ and $u_1 \mapsto u_l$. The following are the cases for the values of l:

- **Case 1:** $l = 2$. We have (u_1, u_2, w_1) and (u_2, u_1, w_2) as witness triples of the first type, where $u_i \in X$. We know that either $u_1 \mapsto u_2$ or $u_2 \mapsto u_1$. Without loss of generality, assume that $u_1 \mapsto u_2$. Then $u_2 \mapsto w_1$ and $w_2 \mapsto u_1$. Since u_1 and w_1 are joined by symmetric arcs but u_1 and w_2 are joined by a single arc, w_1 and w_2 are not coming from the same independent set and hence they are adjacent. If they are connected by symmetric arcs, then $\{u_1, u_2, w_1, w_2\}$ induces a copy Figure 2.1 (a). Otherwise, $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 2.1 (b) or (c).

- **Case 2:** $l = 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2)$ and (u_3, u_1, w_3) are witness triples of the first type, where w_1, w_2 and w_3 are distinct. By the assumption, $u_1 \mapsto u_2$ and $u_1 \mapsto u_3$. Then we have $u_2 \mapsto w_1$, $u_1 \leftrightarrow w_1$, $w_3 \mapsto u_1$ and $u_3 \leftrightarrow w_3$. Moreover, $u_2 \leftrightarrow w_2$ and either $u_2 \mapsto u_3 \mapsto w_2$ or $w_2 \mapsto u_3 \mapsto u_2$. Suppose that $u_2 \mapsto u_3 \mapsto w_2$. Since $w_3 \mapsto u_1$ and $u_1 \mapsto u_2$, they are not coming from the same independent set and hence u_2 and w_3 are adjacent. If $w_3 \mapsto u_2$, then there is a smaller sequence u_2, u_3 because(u_3, u_2, w_3) is a witness triple of the first type, which is a contradiction to the assumption of the value of l. If $u_2 \leftrightarrow w_3$, then (u_2, u_1, w_3) is a witness triple of the first type, which contradicts the choice of l. If $u_2 \mapsto w_3$, then the subdigraph induced by $\{u_1, u_2, w_3\}$ is a copy of Figure 2.1 (d). Suppose that $w_2 \mapsto u_3 \mapsto u_2$. Since $u_3 \mapsto u_2$ and $u_2 \mapsto w_1$, we know that u_3 is adjacent to w_1. If $u_3 \mapsto w_1$, then (u_1, u_3, w_1) is a witness triple of the first type and we obtain a sequence u_1, u_3 for a shorter length, a contradiction. If $u_3 \leftrightarrow w_1$, then there is a smaller sequence u_2, u_3 because (u_3, u_2, w_1) is a witness triple of the first type, a contradiction. Hence we have $w_1 \mapsto u_3$, but then $\{u_2, u_3, w_1\}$ induces a copy of Figure 2.1 (d), a contradiction.

- **Case 3:** $l > 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2)$, ..., (u_l, u_1, w_l) are witness triples of the first type and $u_i \in X$. For the sequence $u_1, u_2, ..., u_l$, we have $u_1 \mapsto u_2$ and $u_1 \mapsto u_l$. Thus, $u_2 \mapsto w_1$, $u_1 \leftrightarrow w_1$, $w_l \mapsto u_1$ and $w_l \leftrightarrow u_l$. Since $u_1 \mapsto u_2$ and $w_l \mapsto u_1$, we know that u_2 and w_l are not in the same independent set and hence they are adjacent. If $u_2 \leftrightarrow w_l$, then we get a shorter sequence u_1, u_2 because (u_2, u_1, w_l) is a witness triple of the first type, a contradiction. If $u_2 \mapsto w_l$, then the digraph induced by $\{u_1, u_2, w_l\}$ is a copy of Figure 2.1 (d), a contradiction. Thus, $w_l \mapsto u_2$. Since $u_l \leftrightarrow w_l$ and $w_l \mapsto u_2$, they are not coming from the same independent set and hence they are adjacent. Note that $u_l \in X$, so if $u_2 \leftrightarrow u_l$, then $u_2 \leftrightarrow w_l$, which contradicts the fact that $w_l \mapsto u_2$. If $u_2 \mapsto u_l$, then (u_l, u_2, w_l) is a witness triple of the frist type and we have a shorter sequence $u_2, u_3, ..., u_l$, which contradicts the assumption. Thus, $u_l \mapsto u_2$. Since $u_2 \mapsto w_1$, we know that u_l and w_1 are not in the same indpendent set and they are adjacent. If $u_l \leftrightarrow w_1$, then (u_l, u_2, w_1) is a witness triple of the first type and there is a shorter sequence $u_2, u_3, ..., u_l$, a contradiction. If $u_l \mapsto w_1$, then (u_1, u_l, w_1) is a witness of the first type and we again obtain a shorter sequence u_1, u_l, a contradiction. If $w_1 \mapsto u_l$, then $\{u_2, u_l, w_1\}$ is a copy of Figure 2.1 (d), a contradiction. Hence, no such sequence existes.

Therefore, D has a di-simplicial vertex. □

2.5 Locally Semi-complete Digraphs

In the previous two sections, we have studied the forbidden subdigraphs for both quasi-transitive and extended semi-complete chordal digraphs. It turns out that these two digraph classes share the same forbidden subdigraphs class. In this section, we study locally semi-complete chordal digraphs.

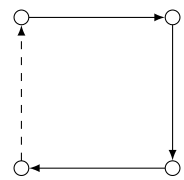


Figure 2.4: Induced directed cycles of length $l \geq 4$ with only single arcs

Theorem 2.4. *Let D be a locally semi-complete digraph. Suppose that $S(D)$ is chordal. Then D is a chordal digraph if and only if it does not contain any of the digraphs in Figure 2.1 or Figure 2.4 as an induced subdigraph.*

Proof. For the necessity, we known from Lemma 2.2 that if a digraph contains any of the subdigraphs in Figure 2.1, then it is not a chordal digraph. We can also show that each digraph in Figure 2.4 does not contain a di-simplicial vertex. Thus no chordal digraph contains a digraph in Figure 2.4 as an induced subdigraph. It remains to prove the suffciency. We prove it by contrapositive.

Suppose that D does not contain any of the digraphs in Figure 2.1 or Figure 2.4 as an induced subdigraph. If D is a chordal digraph and has a di-simplicial vertex, then we are done. Hence, suppose that D does not have any di-simplicial vertex. Let u be a vertex that is in X. By Proposition 2.1, u has witness triples of the first, the second or the third type. Since the witness triple of the second type itself is a forbidden digraph (Figure 2.1 (d)). Hence u has witness triples of the first or the third type.

Assume that (u, v, w) is a witness triple of the first type. We claim that there exists a vetex $v' \in X$ such that (u, v', w) is also a witness triple of the first type. Without loss of generality, assume that $w \mapsto v \mapsto u$. (The case when $u \mapsto v \mapsto w$ can be proved in a similar way.) By Lemma 1.1, there exists a vetex $v' \in X$ in the component of v in $S(D) \setminus N[w]$. Since v and v' are in the same component, there exists a path which is joining them. Let $\mathcal{P}_l : v = v_1, v_2, ..., v_l = v'$ be such a path. We prove by induction on l that (u, v_l, w) is a witness triple of the first type. This is clearly true when $l = 1$. So assume that $l \geq 2$ and (u, v_{l-1}, w) is a witness triple of the first type and $w \mapsto v_{l-1} \mapsto u$. We also have $v_{l-1} \leftrightarrow v_l$. Since both u and v_l are out-neighbours of v_{l-1}, they are adjacent. Similarly, since both w and v_l are in-neighbours of v_{l-1}, they are adjacent. Moreover, w and v_l are joined by a single arc because $v_l \in S(D) \setminus N[w]$. If $v_l \mapsto w$, then $\{u, v_l, w, v_{l-1}\}$ induces a copy of Figure 2.1 (a), (b) or (c), which contradicts the assumption. Thus, we have $w \mapsto v_l$. But then if $u \leftrightarrow v_l$, we have $u \notin X$, which is a contradiction. If $u \mapsto v_l$, then $\{u, v_l, w, v_{l-1}\}$ induces a copy of Figure 2.1 (c), also a contradiction. Hence, we have $v_l \mapsto u$, which gives a witness triple (u, v_l, w) of the first type.

Assume now that (u, v, w) is a witness triple of the third type. We show that there exist vertices $v', w' \in X$ so that (u, v', w') is a witness triple of the third type. By the definition of witness triples of the third type, we have either $v \mapsto u \mapsto w$ or $w \mapsto u \mapsto v$. Without loss of generality, assume that $v \mapsto u \mapsto w$. By Lemma 1.1 there exists a vertex $v' \in X$ and it is in the component of v in $S(D) \setminus N[w]$. Since v and v' are in the same component in $S(D) \setminus N[w]$, there exists a path $\mathcal{P}' : v = v_1, v_2, ..., v_k = v'$ in $S(D) \setminus N[w]$. We show by induction on k that (u, v', w) is a witness triple of the third type: If $k = 1$, then (u, v_1, w) is a witness triple of the third type by our assumption. Assume that $k \geq 2$ and (u, v_{k-1}, w) is a witness triple of the third type such that $v_{k-1} \mapsto u \mapsto w$. Suppose that w is adjacent to v_k. Then v_{k-1} and w are adjacent because $v_{k-1} \leftrightarrow v_k$, against our assumption. Hence w and v_k are not adjacent. Since $(u, v_k) \subseteq N^+(v_{k-1})$, u and v_k are adjacent. Since v_k and w are not adjacent, we must have $v_k \mapsto u$. Therefore, (u, v_k, w) is a witness triple of the third type, where $v_k = v' \in X$. By the same argument as above, we can also show that (u, v', w') is a witness triple of the third type for $w' \in X$.

We know from above that if u has a witness triple of the third type, then there always exist vertices $v, w \in X$ such that (u, v, w) is a witness triple of the third type. Now, one may ask what happens if u only has witness triples of the third type? We claim that each of them can only have witness triples of the third type. Again, assume that $v \mapsto u \mapsto w$. (The proof for the case when $w \mapsto u \mapsto v$ is similar.) Since v and w are both in X and neither of them is di-simplicial, they have witness triples of the first, the second or the third type. If they have witness triples of the second type, then we obtain a copy of Figure 2.1 (d), a contradiction. Therefore, they have witness triples of the first or the third type. If both v and w only have witness triples of the third type, then we are done. Otherwise, suppose that v has a witness triple of the first type. Then there exist vertices v_1, v_2 such that $v_1 \in X$ and (v, v_1, v_2) is a witness triple of the first type. By the definition of witness triples of the first type, we have $v \leftrightarrow v_2$. Since $\{u, v_2\} \subseteq N^+(v)$, u and v_2 are adjacent. Moreover, w is not adjacent to v implies that w is not adjacent to v_2. Since u is an in-neighbour of w but v_2 is not adjacent to w, we know that there is a single arc from v_2 to u. Hence (v, u, v_2) is not a witness triple of any type and so that u and v_1 are distinct. Suppose that $v \mapsto v_1 \mapsto v_2$. If $w \to v_1$, then v and w must be adjacent, a contradiction. If $v_1 \to w$, then v_2 and w must be adjacent, a contradiction. Hence w and v_1 are not adjacent. Because $\{u, v_1\} \subseteq N^+(v)$, u and v_1 are adjacent. Moreover, since v_1 is not adjacent to w, and w is an out-neighbour of u, we have $v_1 \mapsto u$. At this point, we have (u, v_1, w) as a witness triple of the third type (see Figure 2.5), and both v_1, w are in X. Now, consider the witness triple (u, v_1, w). Assume that v_1 is not di-simplicial. Clearly, v_1 does not have witness triples of the second type. If v_1 only has witness triples of the third type, then we have (u, v, w) as a witness triple of the third type, and v_1 only has witness triples of the third type. Suppose v_1 has a witness triple of the first type. Let (v_1, v_3, v_4) be such a witness triple of the first type with $v_3 \in X$. Clearly, $v_4 \notin \{u, v, w, v_1, v_2\}$ because v_1 is adjacent to v_4 by symmetric arcs but not adjacent to the other vertices by symmetric arcs. We claim that $v_3 \notin \{u, v, w, v_1, v_2\}$. Since D is locally semi-complete, v_4 is adjacent to each of u, v and v_2 but not to w. This immediately implies that $v_4 \mapsto u$. Hence (v_1, u, v_4) is not a witness triple of any type and in particular $v_3 \neq u$. If $v_3 = v$. Then $v_4 \mapsto v$ and $\{v, v_1, v_2, v_4\}$ induces a copy of Figure 2.1 (a), (b) or (c), a contradiction. So $v_3 \neq v$. Similarly, we can conclude that $v_3 \neq v_2$. Thus, v_3 is distinguish with u, v, w, v_1 and v_2. Now, we can use a similar discussion as for (v, v_1, v_2) to show that (u, v_3, w) is also a witness triple of the third type (see Figure 2.5). Continuing this way, we will either run out of vertices, or end up with a vertex v_n, so that (u, v_n, w) is a witness triple of the third type and v_n only has witness triples of the third type. Due to the symmetry of the vertices v and w, we can use a similar discussion as above to show that there exists a vertex w_m such that (u, v_n, w_m) is a witness triple of the third type and w_m only has witness triples of the third type.

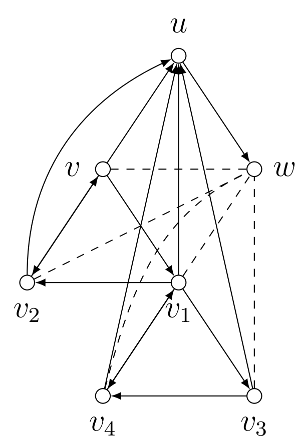


Figure 2.5: Vertex u only has witness triples of the third type

We have just proved that for each vertex $u \in X$, if u only has witness triples of the third type, then there exist vertices v and w such that each of them only has witness triples of the third type, Furthermore, (u, v, w) is a witness triple of the third type and $v_1 \mapsto u \mapsto w$. We continue to find vertices v_1 and w_1 where each of them only has witness triples of the third type. Moreover, (v, v_1, v') and (w, w', w_1) are witness triples of the third type such that $v_1 \mapsto v$ and $w \mapsto w_1$. Following this process, we either end up with a di-simplicial vertex and have a contradiction, or obtain a directed cycle $\mathcal{C}_t = u_1, u_2, ..., u_t$ $(t \geq 3)$ such that u_i only has witness triples of the third type, and $u_i \mapsto u_{i+1}$. In the following, we prove that such a cycle $\mathcal{C}_t$ does not exist: Assume that the cycle is chosen so that t is the least. Clearly, $t \neq 3$, as otherwise $\mathcal{C}_3$ is a copy of Figure 2.1 (d). Suppose that $t \geq 4$. If $\mathcal{C}_t$ is chordless, then $\mathcal{C}_t$ itself is a copy of Figure 2.4, which contradicts the assumption. Hence $\mathcal{C}_t$ should contain chords. If any chord of $\mathcal{C}_t$ is a single chord, then we obtain a smaller t, which is a contradiction to the choice of t. Thus, suppose that the chords of $\mathcal{C}_t$ are all symmetric arcs. Without loss of generality, assume that $u_i \leftrightarrow u_j$ $(j > i, j \neq i + 1)$. If $j = i + 2$, then (u_i, u_{i+1}, u_j) is a witness triple of the first type, which is a contradiction to the fact that u_i only has witness triples of the third type. If $j \neq i + 2$, then since both u_i and u_{j-1} are the in-neighbours of u_j, we know that u_i is adjacent to u_{j-1} by symmetric arcs. Then we know that u_i and u_{j-2} are connected by symmetric arcs. Finally, u_i and u_{i+2} must be adjacent by symmetric arcs. so that (u_i, u_{i+1}, u_{i+2}) is a witness triple of the first type, a contradiction. Hence, not such cycle $\mathcal{C}_t$ exist. Therefore, for every $u \in X$, u always has a witness triple of the first type.

At the begining of the proof, we showed that for every vertex $u \in X$, if u has a witness triple of the first type, then there exists $u' \in X$ such that (u, u', w) is a witness triple of the first type. There also exists a vertex $u'' \in X$ such that (u', u'', w') is a witness triple of the first type. Eventually, we will obtain a sequence of vertices $u, u', u'', ... \in X$ such that $(u, u', w), (u', u'', w')$,... are witness triples of the first type. Since the sequence will have a repeated vertex, we either find a di-simplicial vertex, or have a sequence of vertices $u_1, u_2, ..., u_r \in X$ such that $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_r, u_1, w_r)$ are witness triples of the first type. Assume that such a sequence is chosen with the shortest length r. By the definition of witness triples of the first type, we have $u_i \leftrightarrow w_i$, and either $u_i \mapsto u_{i+1} \mapsto w_i$ or $w_i \mapsto u_{i+1} \mapsto u_i$. Without loss of generality, assume that $u_1 \mapsto u_2 \mapsto w_1$. We prove by contradiction that such sequence does not exist with the following cases on the length of r:

- **Case 1:** $r = 2$. Then (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the first type where $u_i \in X$. Because $u_1 \mapsto u_2 \mapsto w_1$, we have $w_2 \mapsto u_1$. Since both w_1 and w_2 are the in-neighbours of u_1, this implies that w_1 and w_2 are adjacent. If w_1 and w_2 are connected by a single arc, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 2.1 (b) or (c). If it is symmetric, then the digraph induced by $\{u_1, u_2, w_1, w_2\}$ is a copy of Figure 2.1 (a).

- **Case 2:** $r = 3$. The witness triples $(u_1, u_2, w_1), (u_2, u_3, w_2)$ and (u_3, u_1, w_3) are of the first type with $u_i \in X$. It is not hard to see that w_1, w_2 and w_3 are distinct. Clearly, $\{u_1, u_2, u_3\}$ cannot induce a C_3 because otherwise it is a copy of Figure 2.1 (d). Hence, we may assume that $u_1 \mapsto u_2$ and $u_1 \mapsto u_3$. Immediately, we have $w_3 \mapsto u_1$. Suppose that $u_2 \mapsto u_3 \mapsto w_2$. Then u_2 and w_3 are adjacent because both of them are the in-neighbours of u_3. If they are connected by symmetric arcs, then (u_2, u_1, w_3) is a witness triple of the first type and there is a shorter sequence u_1, u_2, a contradiction. If $w_3 \mapsto u_2$, then (u_3, u_2, w_3) is a witness triple of the first type and we have a shorter sequence u_2, u_3, a contradiction. If $u_2 \mapsto w_3$, then the subdigraph induced by $\{u_1, u_2, w_3\}$ is a copy of Figure 2.1 (d), a contradiction to the assumption. Suppose that $w_2 \mapsto u_3 \mapsto u_2$. Then u_3 is adjacent to w_1 because they are the out-neighbours of u_1. If $u_3 \leftrightarrow w_1$, then (u_3, u_2, w_1) is a witness triple of the first type and we obtain a shorter sequence u_2, u_3, a contradiction. If $u_3 \mapsto w_1$, then (u_1, u_3, w_1) is a witness triple of the first type and we again have a shorter sequence u_1, u_3, a contradiction. If $w_1 \mapsto u_3$, then $\{u_2, u_3, w_1\}$ is a copy of Figure 2.1 (d), which is also a contradiction.

- **Case 3:** $r > 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_r, u_1, w_r)$ are witness triples of the first type with $u_i \in X$. Suppose that r is the least. Clearly, u_i and u_{i+1} are connected by single arcs. Let $\mathcal{C}_r = u_1u_2...u_r$ be a cycle of single arcs. Suppose that $\mathcal{C}_r$ is a directed cycle. Then $\mathcal{C}_r$ is not chordless, because otherwise it is a copy of Figure 2.4. If the chords of $\mathcal{C}_r$ are all single arcs, then there always contains a smaller directed cycle, which is a forbidden digraph. Hence $\mathcal{C}_r$ has symmetric arcs as its chords. Without loss of generality, assume that the arcs between u_1 and u_i are the rightmost symmetric arcs (rightmost symmetric arcs means there is no other symmetric arcs between u_j and u_k for all $1 \leq j, k \leq i$). If $i = 3$, then (u_3, u_2, u_1) is a witness triple of the first type and there is a shorter sequence u_2, u_3, a contradiction to the choice of r. Hence $i > 3$. Since $\{u_1, u_{i-1}\} \subseteq N^-(u_i)$, u_1 and u_{i-1} are adjacent. Since u_1 and u_i is connected by the rightmost symmetric arcs, u_1 and u_{i-1} can only be connected by a single arc. If $u_1 \mapsto u_{i-1}$, then (u_1, u_{i-1}, u_i) is a witness triple of the first type and we obtain a shorter sequence $u_1, u_{i-1}, ..., u_r$, a contradiction. If $u_{i-1} \mapsto u_1$, then there exists a copy of Figure 2.4, which contradicts the assumption. Hence $\mathcal{C}_r$ is not a directed cycle. We assume without loss of generality that $u_1 \mapsto u_r$. Immediately, we have $w_r \mapsto u_1$. Since $\{u_2, u_r, w_1\} \subseteq N^+(u_1)$, they are pairwise adjacent. If $u_r \mapsto w_1$, then (u_1, u_r, w_1) is a witness triple of the first type, and, u_1, u_r form a sequence of the length 2, a contradiction. Suppose that $w_1 \mapsto u_r$. Then if u_2 and u_r are connected by symmetric arcs, the digraph induced by $\{u_1, u_2, u_r, w_1\}$ is a copy of Figure 2.1 (c). If $u_r \mapsto u_2$, then $\{u_2, u_r, w_1\}$ induces a copy of Figure 2.1 (d). If $u_2 \mapsto u_r$, then u_2 and w_r are adjacent because they are both in-neighbours of u_r. Clearly, $u_2 \not\mapsto w_r$, as otherwise the digraph induced by $\{u_1, u_2, w_r\}$ is a copy of Figure 2.1 (d). Moreover, $w_r \not\mapsto u_2$, as otherwise (u_r, u_2, w_r) becomes a witness triple of the first type, which yields a shorter sequence $u_2, u_3, ..., u_r$. If $u_2 \leftrightarrow w_r$, then the digraph induced by $\{u_1, u_2, w_2, w_r\}$ is Figure 2.1 (a), (b) or (c). Hence, it remains to consider the case when the arcs between u_r and w_1 are symmetric. Then we have $w_1 \leftrightarrow w_r$ because $u_r \in X$. Since both u_2 and w_r are the in-neighbours of w_1, they are adjacent. If they are joined by symmetric arcs, then $\{u_1, u_2, w_1, w_r\}$ induces a copy of Figure 2.1 (a), a contradiction. If $u_2 \mapsto w_r$, then the digraph induced by $\{u_1, u_2, w_r\}$ is Figure 2.1 (d). Hence, suppose that $w_r \mapsto u_2$. Since $\{u_2, u_r\} \subseteq N^+(w_r)$, u_2 is adjacent to u_r. If u_2 and u_r are connected by a single arc, then either (u_r, u_2, w_1) or (u_r, u_2, w_r) is a witness triple of the first type and there is a shorter sequence $u_2, u_3, ..., u_r$, a contradiction. If $u_2 \leftrightarrow u_r$, then since $\{u_2, w_r\} \subseteq N(u_r)$ in $S(D)$ but u_2 and w_r are not adjacent in $S(D)$, which is a contradiction to the fact that $u_r \in X$. Hence, no such sequence exists.

Therefore, D has a di-simplicial vertex. $\square$

Chapter 3

Semi-Strict Chordal Digraphs

3.1 Introduction

Recall from Chapter 1 that a digraph D is semi-strict chordal if its vertices can be linearly ordered by $\prec$ so that for any $u \prec v$ and $u \prec w$, if $v \rightarrow u$ and $u \rightarrow w$, then $v \leftrightarrow w$. Semi-strict chordal digraphs form a subclass of chordal digraphs and a superclass of strict chordal digraphs. Unlike strict chordal digraphs whose underlying graphs are chordal, the underlying graphs of semi-strict chordal digraphs are not necessarily chordal (see Figure 3.1). Semi-strict chordal digraphs share some properties with chordal digraphs. Recall that a vertex u in a digraph is semi-strict di-simplicial if there are symmetric arcs between every in-neighbour and every out-neighbour of u.

Lemma 3.1. *Suppose that D is a semi-strict chordal digraph. Then $S(D)$ is a chordal graph. Moreover, every semi-strict di-simplicial vertex of D is a simplicial vertex of $S(D)$.*

Proof. Suppose that D is semi-strict chordal and that $\prec$ is a perfect elimination ordering of D. If $S(D)$ is not a chordal graph, then it contains a chordless cycle $\mathcal{C} = u_1, u_2, ..., u_k$ $(k \geq 4)$. Assume that such cycle is chosen so that k is the least.

Without loss of generality, assume that $u_1 \prec u_2$ and $u_1 \prec u_k$. We have $u_1 \leftrightarrow u_2$ and $u_1 \leftrightarrow u_k$ in D. Since D is semi-strict chordal, we have $u_2 \leftrightarrow u_k$, a contradiction to the assumption that C is chordless. Therefore, $S(D)$ is a chordal graph. Suppose that u is a semi-strict di-simplicial vertex of D. If u is not a simplicial vertex of $S(D)$, then there exist vertices v and w such that $v \leftrightarrow u$ and $w \leftrightarrow u$ but $v \not\leftrightarrow w$ in D, a contradiction. $\square$

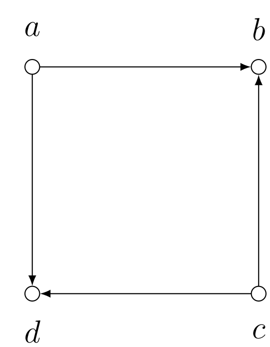


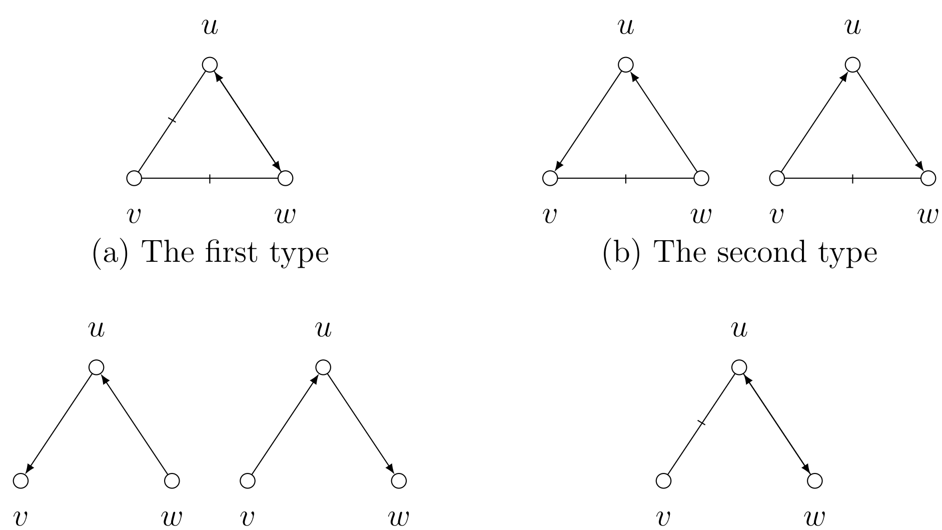
Figure 3.1: A semi-strict chordal digraph with no chordal underlying graph

In the rest of this chapter, we study some subclasses of semi-strict chordal digraphs. In section two, we give the definitions of witness triples of vertices in the sense of semi-strict chordal digraphs. In sections three to six, we give forbidden subdigraph characterizations for semi-complete semi-strict chordal, quasi-transitive semi-strict chordal, extended semi-complete semi-strict chordal and locally semi-complete semi-strict chordal digraphs. Characterizations for general semi-strict chordal digraphs remain open.

3.2 Witness Triples

Lemma 3.1 asserts that if D is semi-strict chordal then $S(D)$ is a chordal graph. That means a vertex u is a semi-strict di-simplicial vertex only if it is a simplicial vertex in $S(D)$. In this chapter, we only deal with the cases when u is a simplicial vertex in $S(D)$. Moreover, we use X to denote the set of simplicial vertices of $S(D)$.

Let D be a digraph and (u, v, w) be an ordered triple of the vertices of D. We call (u, v, w) a *witness triple* (in the semi-strict sense) for u of the *first type* if u and w are joined by symmetric arcs, and the arcs between u and v and between v and w are single arcs (See Figure 3.2 (a)). We define the witness triples of the second, the third and the fourth type in the similar way (Figure 3.2 (b), (c) and (d)). If there is a witness triple for u, we also say that u has a witness triple. It is not hard to explore that the witness triple in semi-strict chordal digraphs is a super class of the witness triple in chordal digraphs.



(a) The first type (b) The second type

(c) The third type (d) The fourth type

Figure 3.2: Witness triples (in the semi-strict sense) for u

Lemma 3.2. *Let D be a digraph. Suppose that $S(D)$ is a chordal graph. Then $u \in X$ is a semi-strict di-simplicial vertex of D if and only if there is no witness triples for u in D.*

The proof idea for Lemma 3.2 is the same as the proof for Lemma 2.3. In the rest of this section, we work on the witness triple in some special digraph classes.

We know that the underlying graph of a semi-complete digraph is complete. Given a vertex $u \in X$, the witness triple of it cannot be the third or the fourth type in a semi-complete digraph. Hence, the witness triple for any vertex from a semi-complete digarph can only be the first or the second type.

Given an extended semi-complete digraph D', we know that it is obtained from a semi-complete digraph D. Hence, the vertex $u' \in V(D')$ inherits the properties from the vertex $u \in V(D)$. Then we know that any vertex $u \in X_{D'}$ from an extended semi-complete digraph has witness triples of the first or the second type. Moreover, by the definition of extended semi-complete digraphs, vertices that are not adjacent are in the same independent set and have the same in-neighbours and out-neighbours. For every pair of vertices v' and w' that are coming from the same independent set and vertex u' from another independent set, $u' \mapsto v'$ if and only if $u' \mapsto w'$, $v' \mapsto u'$ if and only if $w' \mapsto u'$. That means u' cannot have witness triples of the third or the fourth type.

For a quasi-transitive digraph D, and for every vertex $u \in X$, if $v \rightarrow u \rightarrow w$ or $w \rightarrow u \rightarrow v$, then v and w are adjacent. That means u cannot have witness triples of the third or the fourth type. Consequently, u has witness triples of the first or the second type.

Locally semi-complete digraph has the property that for every vertex u, if $\{v, w\} \subseteq N^-(u)$ or $\{v, w\} \subseteq N^+(u)$, then v and w are adjacent. Hence u cannot have witness triples of the fourth type. That means u has witness triples of the first, the second or the third type.

We can easily check that the witness triple in certain digraph classes in the semi-strict chordal digraph point of view is matching to the regular chordal digraph. Table 3.1 illustrates the witness triples of a vertex for each digraph class. The following proposition summarizes the discussion above.

	Witness triple of the			
	first type	second type	third type	fourth type
Semi-complete digraphs	✓	✓		
Quasi-transitive digraphs	✓	✓		
Extended semi-complete digraphs	✓	✓		
Locally semi-complete digraphs	✓	✓	✓	
General digraphs	✓	✓	✓	✓

Table 3.1: Witness triples (for semi-strict chordal digraph) for a vertex

Proposition 3.1. *Let D be a digraph and u be a simplicial vertex of $S(D)$. Suppose that u is not a semi-strict di-simplicial vertex of D. Then the following statements hold:*

1. *if D is semi-complete, quasi-transitive or extended semi-complete, then u has witness triples of the first or the second type;*

2. *if D is locally semi-complete, then u has witness triples of the first, the second or the third type (Table 3.1).*

In the rest of this chapter, we work on each of the digraph classes independently.

3.3 Semi-complete Digraphs

In this section, we give a forbidden structure characterization for semi-complete semi-strict chordal digraphs.

Lemma 3.3. *Suppose that D is a semi-complete digraph and $S(D)$ is chordal. If D does not contain any of the digraphs in Figure 3.3 as an induced subdigraph, then $O(D)$ does not contain a directed cycle.*

Proof. Suppose to the contracy that $O(D)$ contains a directed cycle. Suppose that the directed cycle $\mathcal{C}_k = u_1, u_2, ..., u_k$ is chosen so that the length k is the smallest among all directed cycles in $O(D)$. Since $\mathcal{C}_k$ is the shortest directed cycle, we know that every non-consecutive vertices u_i and u_j ($i \neq j \pm 1$) in $\mathcal{C}_k$ are connected by symmetric arcs. Since $\mathcal{C}_k$ does not contain Figure 3.3 (a) or (d) as an induced subdigraph, k cannot be 3 or 4. If $k = 5$, then $\mathcal{C}' = u_1, u_3, u_5, u_2, u_4$ is a chordless cycle of length 5 in $S(D)$, which is a contradiction to the assumption that $S(D)$ is chordal. If $k \geq 6$, then $\{u_1, u_2, u_4, u_5\}$ induces a chordless cycle of length 4 in $S(D)$, a contradiction. □

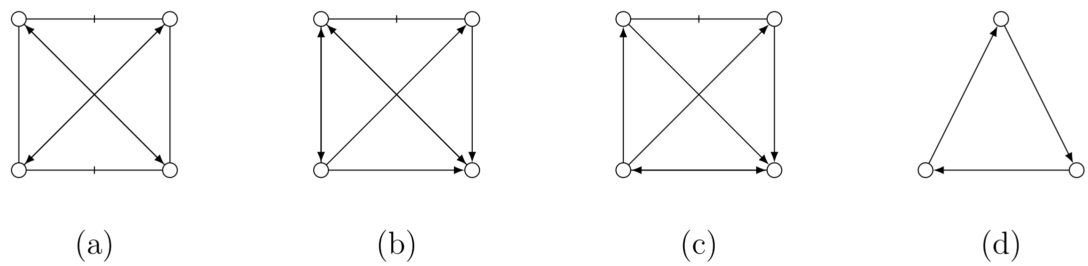


Figure 3.3: Forbidden induced subdigraphs for semi-strcit chordal digraphs

In each digraph in Figure 3.3, one can find a witness triple for every vertex. Thus, none of the digraphs in Figure 3.3 is semi-strict chordal. Immediately, we have the following result:

Lemma 3.4. *No semi-strict chordal digraph contains any of the digraph in Figure 3.3 as an induced subdigraph.*

Theorem 3.1. *Let D be a semi-complete digraph. Suppose that $S(D)$ is chordal. Then D is a semi-strict chordal digraph if and only if it does not contain any of the digraphs in Figure 3.3 as an induced subdigraph.*

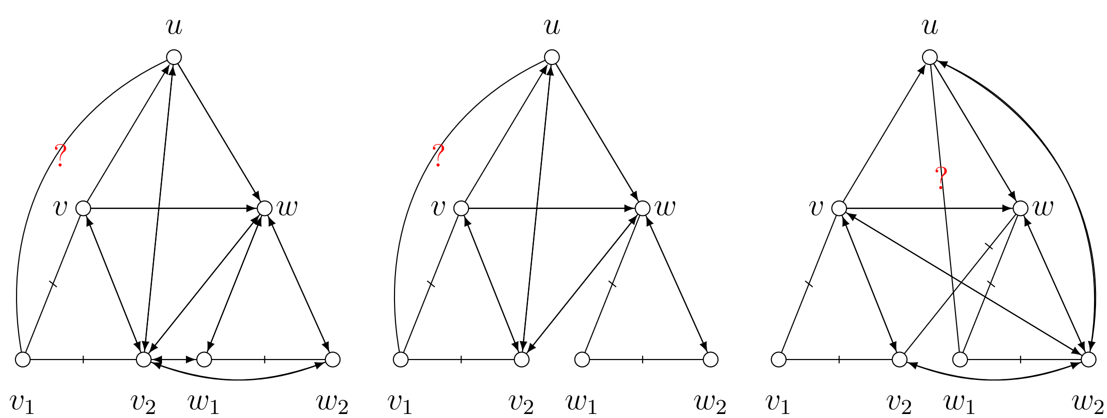
Proof. The proof for the necessity follows from Lemma 3.4. For the sufficiency, suppose that D does not contain any of the digraphs in Figure 3.3 as an induced subdigraph. we need to show that D has a semi-strict di-simplicial vertex. Assume by contradiction that none of the vertex in D is semi-strict di-simplicial. Suppose there is a vertex $u \in X$. Then by Proposition 3.1, u has a witness triple of the first or the second type.

We first consider the situation when u has a witness triple of the second type. We claim that if (u, v, w) is a witness triple of the second type, then v or w is a simplicial vertex of $S(D)$. By the definition of witness triples, we have either $w \mapsto u \mapsto v$ or $v \mapsto u \mapsto w$. Without loss of generality, assume that $v \mapsto u \mapsto w$. (Due to symmetry, the proof for $w \mapsto u \mapsto v$ is the same). Immediately, we have $v \mapsto w$, as otherwise $\{u, v, w\}$ induces a copy of Figure 3.3 (d). Suppose that neither v nor w is a simplicial vertex of $S(D)$. Then there exist vertices v_1, v_2, w_1, w_2 such that $\{v_1, v_2\} \subseteq N(v)$ and $\{w_1, w_2\} \subseteq N(w)$ in $S(D)$ and the arcs between v_1 and v_2 and between w_1 and w_2 are single arcs. If $w \leftrightarrow v_1$ or $w \leftrightarrow v_2$, then $\{v, w, v_1, v_2\}$ induces a copy of Figure 3.3 (a), contradicting to the assumption. Hence, the arcs between w and v_1 and between w and v_2 are single. By the symmetry, we know that the arcs between v and w_1 and between v and w_2 are also single. Thus, the vertices v_1, v_2, w_1 and w_2 are all distinct. Then, no matter if the arc between v_1 and w_1 is single or symmetric, the subdigraph induced by $\{v, w, v_1, w_1\}$ is a copy of Figure 3.3 (a), a contradiction. Therefore, v or w is in X.

Assume now that there is a vertex $u \in X$ such that u only has witness triples of the second type. Let (u, v, w) be such a witness triple. We claim that v or w only has witness triples of the second type. Without loss of generality, assume that $v \mapsto u \mapsto w$. Immediately, we have $v \mapsto w$ or as otherwise $\{u, v, w\}$ induces a copy of Figure 3.3 (d). We consider the following two cases:

- **Case 1:** $w \notin X$. There exist vertices w_1 and w_2 such that $w \leftrightarrow w_1$, $w \leftrightarrow w_2$ but w_1 and w_2 are connected by a single arc. From the previous claim, we know that v must be in X. If v only has witness triples of the second type, then we are done. Suppose that v has a witness triple of the first type. Assume that

(v, v_1, v_2) is such a witness triple such that $v \leftrightarrow v_2$, and the arcs between v and v_1 and between v_1 and v_2 are single. It is not hard to see that v_1, v_2, w_1, w_2 are all distinct. If v_2 and w_1 are connected by a single arc, then the subdigraph induced by $\{v, w, v_2, w_1\}$ is a copy of Figure 3.3 (a), a contradiction. Hence $v_2 \leftrightarrow w_1$. If v_2 and w_2 are connected by a single arc, then the subdigraph induced by $\{v, w, v_2, w_2\}$ is a copy of Figure 3.3 (a), a contradiction. Thus, $v_2 \leftrightarrow w_2$. And, we have $v_2 \leftrightarrow w$, or else $\{v_2, w, w_1, w_2\}$ induces a chordless cycle of length 4 in $S(D)$, which contradicts the fact that $S(D)$ is chordal. Suppose now that u and v_2 are connected by a single arc. Then the subdigraph induced by $\{u, v, w, v_2\}$ is a copy of Figure 3.3 (b), a contradiction. Hence, we have $u \leftrightarrow v_2$. If $u \leftrightarrow v_1$, then becasue $u \in X$, the v_1 and v_2 are connected by symmetric arcs, which contradicts our assumption. If u and v_1 are connected by a single arc, then (u, v_1, v_2) is a witnss triple of the first type, which is a contradiction to the fact that u only has witness triples of the second type. (see Figure 3.4 (a).)



(a) $w \notin X$ (b) $w \in X$, $w \leftrightarrow v_2$ (c) $w \in X$, $w \not\leftrightarrow v_2$

Figure 3.4: Vertex u only has witness triples of the second type

- **Case 2:** $w \in X$. If w only has witness triples of the second type, then we are done. Hence, we may assume that w has a witness triple of the first type. Let

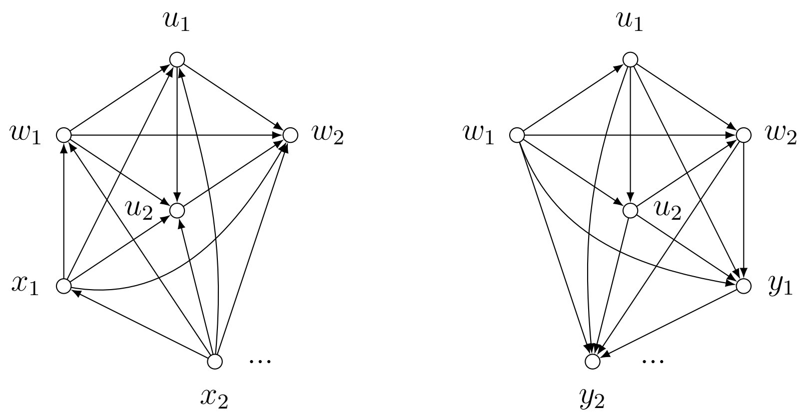
(w, w_1, w_2) be a witness triple of the first type. From Case 1, we know that v is in X. Suppose that v has a witness triple of the first type. Let (v, v_1, v_2) be such a witness triple. By the definition of witness triples of the first type, $v \leftrightarrow v_2$, $w \leftrightarrow w_2$, and arcs between v and v_1, between v_1 and v_2, between w and w_1 and between w_1 and w_2 are all single arcs. Suppose that $w \leftrightarrow v_2$. Then if u and v_2 are connected by a single arc, $\{u, v, w, v_2\}$ induces a copy of Figure 3.3 (b), a contradiction. Hence the arc between u and v_2 is symmetric. If $u \leftrightarrow v_1$, then since $u \in X$, we have $v_1 \leftrightarrow v_2$, a contradiction to our assumption. If $u \mapsto v_1$ or $v_1 \mapsto u$, then (u, v_1, v_2) is a witness triple of the first type, a contradiction. (see Figure 3.4 (b).) Suppose that w and v_2 are connected by a single arc. If v and w_2 or v_2 and w_2 are connected by a single arc, then the subdigraph induced by $\{v, w, v_2, w_2\}$ is a copy of Figrue 3.3 (a). Thus, $v \leftrightarrow w_2$ and $v_2 \leftrightarrow w_2$. If $u \mapsto w_2$ or $w_2 \mapsto u$, then $\{u, v, w, w_2\}$ induces a copy of Figure 3.3 (b). Hence u and w_2 are connected by symmetric arcs. Note that $u \in X$ and $u \leftrightarrow w_2$, so if $u \leftrightarrow w_1$, then $w_1 \leftrightarrow w_2$, a contradiction. If $u \mapsto w_1$ or $w_1 \mapsto u$, then (u, w_1, w_2) is a witness triple of the first type, a contradition.

Hence v or w only has witness triples of the second type. Moreover, we claim that v and w together with another vertex also form a witness triple of the second type. Without loss of generality, assume that v only has witness triples of the second type. There exists a vertex v' such that $v' \mapsto v$. If $u \mapsto v'$, then the subdigraph induced by $\{u, v, v'\}$ is a copy of Figure 3.3 (d), a contradiction. If $u \leftrightarrow v'$, then (u, v, v') is a witness triple of the first type, a contradiction. Hence, we have $v' \mapsto u$. If $w \mapsto v'$, then $\{v, v', w\}$ induces a copy of Figure 3.3 (d), a contradiction. If $w \leftrightarrow v'$, then $\{u, v, w, v'\}$ induces a copy of Figure 3.3 (c), a contradiction. Therefore, the only possibility is $v' \mapsto w$, and (v, w, v') is a witness triple of the second type.

From the above, we know that if the semi-complete digraph D contains a vertex u

such that u only has witness triples of the second type, then there exists a vertex u' also only has witness triples of the second type such that (u, u', w) or (u, w, u') is a witness triple of the second type. Due to the symmetry, we assume that (u, u', w) to be such a witness triple. Continuing this way, we obtain a sequence of vertices $u, u', u'', ...$ along with witness triples $(u, u', w), (u', u'', w')$,... of the second type. Moreover, u, u', u'', ... only have witness triples of the second type. Since D is finite, the sequence will eventually come back to a previously used vertex and form a cycle with only single arcs. Assume that such cycle $\mathcal{C}_l = u_1, u_2, ..., u_l$ is chosen so that l is the least. Suppose that $l \geq 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_l, u_1, w_l)$ are witness triples of the second type and the vertices $u_1, u_2, ..., u_l$ only have witness triples of the second type. From Lemma 3.3, we know that $O(D)$ does not contain directed cycle. Without loss of generality, assume that $u_1 \mapsto u_2$ and $u_1 \mapsto u_l$. Then we have $w_1 \mapsto u_1$, $w_1 \mapsto u_2$, $u_1 \mapsto w_l$ and $u_l \mapsto w_l$. If $u_2 \leftrightarrow u_l$, then (u_2, u_1, u_l) is a witness triple of the first type, a contradiction to the assumption. If $u_l \mapsto u_2$, then (u_l, u_2, u_1) is a witness triple of the second type and we get a shorter cycle $\mathcal{C}'' = u_2, u_3, ..., u_l$, a contradiction. If $u_2 \mapsto u_l$, then (u_2, u_1, u_l) is a witness triple of the second type, we get a shorter cycle u_1, u_2, a contradiction. Suppose that $l = 2$. Then (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the second type and both u_1, u_2 only have witness triples of the second type. Without loss of generality, assume that $u_1 \mapsto u_2$. We have $w_1 \mapsto u_1$, $w_1 \mapsto u_2$, $u_1 \mapsto w_2$ and $u_2 \mapsto w_2$. If $w_1 \leftrightarrow w_2$, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 3.3 (c), a contradiction. If $w_2 \mapsto w_1$, then $\{u_1, w_1, w_2\}$ induces a copy of Figure 3.3 (d), a contradiction. Hence $w_1 \mapsto w_2$, and (u, w_1, w_2) becomes a witness triple of the second type. Suppose that w_2 is not a simplicial vertex of $S(D)$ or has a witness triple of the first type. Then we know from above that w_1 only has witness triples of the second type. Hence, there exists a vertex x_1 such that $x_1 \mapsto w_1$. If $u_1 \leftrightarrow x_1$, then u_1 has a witness triple of the first type, a contradiction. If $u_1 \mapsto x_1$, then $\{u_1, w_1, x_1\}$ is a copy of Figure 3.3 (d), a contradiction. Therefore, we have $x_1 \mapsto u_1$. Similarly as for u_1, we can show that $x_1 \mapsto u_2$. If $w_2 \leftrightarrow x_1$, then $\{u_1, u_2, w_2, x_1\}$ induces a copy of Figure 3.3 (c), a contradiction. If $w_2 \mapsto x_1$, then the subdigraph induced by $\{u_1, w_2, x_1\}$ is a copy of Figure 3.3 (d), a contradiction. Hence $x_1 \mapsto w_2$, and we have (u_1, x_1, w_2) as a witness triple of the second type. In conclusion, $\{u_1, u_2, w_1, w_2\} \subseteq N^+(x_1)$. Since we have assume that w_2 is not in X or has a witness triple of the first type, which means x_1 can only has witness triples of the second type. There exists a vertex x_2 such that $x_2 \mapsto x_1$. Furthermore, it is easy to show that $\{u_1, u_2, w_1, w_2, x_1\} \subseteq N^+(x_2)$. Hence (u_1, x_2, w_2) is a witness triple of the second type and x_2 only has witness triples of the second type (see Figure 3.5 (a)). Continuing with the same discussion, we will either run out of vertices in X, or end up with a copy of a digraph of Figure 3.3. Hence, assume that w_2 only has witness triples of the second type. Then there exists a vertex y_1 such that $w_2 \mapsto y_1$. Clearly, u_1 and y_1 are connected by a single arc, as otherwise u_1 has a witness triple of the first type, a contradiction. If $y_1 \mapsto u_1$, then $\{u_1, w_2, y_1\}$ is a copy of Figure 3.3 (d), a contradiction. Hence $u_1 \mapsto y_1$. Similarly we have that $u_2 \mapsto y_1$. If $w_1 \leftrightarrow y_1$, then $\{u_1, w_1, w_2, y_1\}$ induces a copy of Figure 3.3 (c), a contradiction. If $y_1 \mapsto w_1$, then $\{u_1, w_1, y_1\}$ induces a copy of Figure 3.3 (d), a contradiction. Hence $w_1 \mapsto y_1$ and (u_1, w_1, y_1) is a witness triple of the second type. In conclusion, $\{u_1, u_2, w_1, w_2\} \subseteq N^-(y_1)$. We will go back to the previous cases if $y_1 \notin X$ or y_1 has a witness triple of the first type. Therefore, assume that y_1 only has witness triples of the second type. Then there exists a vertex y_2 such that $y_1 \mapsto y_2$. In a similar way, we can show that $\{u_1, u_2, w_1, w_2, y_1\} \subseteq N^-(y_2)$ and y_2 only has witness triples of the second type (see Figure 3.5 (b)). Continuing this process, we will eventually end up with a forbidden subdigraph. Therefore, no such cycle $\mathcal{C}_l$ exists.

In summary, given a vertex u in X, u always has a witness triple of the first type.



(a) (b)

Figure 3.5: To check if a sequence exists when u only has the second type

Assume that (u, v, w) is a witness tirple of the first type. We claim that v is in X. Suppose that v is not in X. Then by Lemma 1.1, there are more than one vertex in the componet of v in $S(D) \setminus N[w]$. Assume that v' is a vertex in $S(D) \setminus N[w]$ with $v \leftrightarrow v'$. Since $v' \notin S(D) \setminus N[w]$, the arc between w and v' must be a single arc. Then it is easy to see that the subdigraph induced by $\{u, v, w, v'\}$ is a copy of Figure 3.3 (a), a contradiction.

From the above claims, we know that for each vertex u in X, u has a witness triple of the first type. Moreover, if (u, v, w) is a witness triple of the first type, then $v \in X$ and also has a witness triple of the first type, say, (v, v', w') where $v' \in X$. Continuing this way, we obtain a sequence of vertices $u, v, v', ... \in X$ such that $(u, v, w), (v, v', w'), ...$ are witness triples of the first type. Because D is a finite digraph, the sequence will have a repeated vertex. We either find a semi-strict di-simplicial vertex, or end up with a sequence of vertices $u_1, u_2, ..., u_r \in X$, where $(u_1, u_2, w_1), (u_2, u_3, w_2), ...,$ (u_r, u_1, w_r) are witness triples of the first type. Assume that such sequence is the one with the shortest length r. By the definition of witness triples of the first type, we have $u_i \leftrightarrow w_i$, and the arcs between u_i and u_{i+1} and between u_{i+1} and w_i are single. If $r = 2$, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of

Figure 3.3 (a), a contradiction. Suppose that $r \geq 3$. If u_1 and w_2 is adjacent by a single arc, then the subdigraph induced by $\{u_1, u_2, w_1, w_2\}$ is a copy of Figure 3.3 (a), a contradiction. Therefore, we have $u_1 \leftrightarrow w_2$. If $u_1 \leftrightarrow u_3$, then the fact $u_1 \in X$ implies that u_3 and w_2 must be adjacent by symmetric arcs, this contradicts the assumption. If u_1 is adjacent to u_3 by a single arc, then (u_1, u_3, w_2) is a witness triple of the first type and there is a shorter sequence $u_1, u_3, ..., u_r$, a contradiction.

Therefore, D has a semi-strict di-simplicial vertex.

□

3.4 Quasi-transitive Digraphs

Form Chapter 2, we know that quasi-transitive chordal digraphs and semi-complete chordal digraphs have the same list of forbidden subdigraphs. In this section, we prove that quasi-transitive semi-strict chordal digraphs also have the same forbidden subdigraphs as semi-complete semi-strict chordal digraphs. If D is quasi-transitive and $O(D)$ contains a directed cycle $\mathcal{C}$, then $V(\mathcal{C})$ induces a semi-complete subdigraph of D. This fact along with Lemma 3.3 imply the following:

Proposition 3.2. *Suppose that D is quasi-transitive and $S(D)$ is chordal. If D does not contain any of the digraphs in Figure 3.3 as an induced subdigraph, then $O(D)$ does not contain a directed cycle.*

Theorem 3.2. *Let D be a quasi-transitive digraph. Suppose that $S(D)$ is a chordal graph. Then D is a semi-strict chordal digraph if and only if it does not contain any of the digraphs in Figure 3.3 as an induced subdigraph.*

Proof. The necessity follows from Lemma 3.4. For the sufficiency, suppose that D does not contain any of the digraphs in Figure 3.3 as an induced subdigraph. It suffices to show that D has a semi-strict di-simplicial vertex. Suppose that D does not contain any semi-strict di-simplicial vertex. Let $u \in X$, then by Proposition 3.1, u has a witness triple of the first or the second type.

In the previous section, we have showed that if (u', v', w') is a witness triple of the second type in a semi-complete digraph, then v' or w' is in X. We claim that the same property holds for quasi-transitive digraphs. Assume that (u, v, w) is a witness triple of the second type in a quasi-transitive digraph. Then we have either $w \mapsto u \mapsto v$ or $v \mapsto u \mapsto w$. Without loss of generality, assume that $v \mapsto u \mapsto w$. (The proof for the case when $w \mapsto u \mapsto v$ is the same.) If $w \mapsto v$, then $\{u, v, w\}$ induces a copy of Figure 3.3 (d), a contradiction. Hence, we have $v \mapsto w$. Suppose that neither v nor w is in X. Then there exist vertices v_1, v_2, w_1, w_2, such that $v \leftrightarrow v_1, v \leftrightarrow v_2, w \leftrightarrow w_1$ and $w \leftrightarrow w_2$, but $v_1 \not\leftrightarrow v_2$ and $w_1 \not\leftrightarrow w_2$. Since $\{u, w, v_1, v_2\} \subseteq N^+(v)$ and $\{v_1, v_2\} \subseteq N^-(v)$, there is a complete adjacency between $\{u, w, v_1, v_2\}$ and $\{v_1, v_2\}$. Furthermore, v_1 and v_2 are joined by a single arc. If $w \leftrightarrow v_1$ or $w \leftrightarrow v_2$, then the subdigraph induced by $\{v, w, v_1, v_2\}$ is a copy of Figure 3.3 (a), a contradiction. Hence w is adjacent to v_1 and v_2 by single arcs and we see that v_1, v_2, w_1, w_2 are all distinct. In addition, since $\{w_1, w_2\} \subseteq N^+(w)$ and $\{u, v, w_1, w_2\} \subseteq N^-(w)$, u is adjacent to w_1 and w_2 and v is also adjacent to w_1 and w_2. Furthermore, w_1 and w_2 are joined by a single arc. If the arcs between v and w_1 and between v and w_2 are symmetric, then $\{v, w, w_1, w_2\}$ induces a copy of Figure 3.3 (a), a contradiction. Therefore, v is adjacent to w_1 and w_2 by single arcs. Because w_1 and v is joined by a single arc and v_1 and v is joined by symmetric arcs, moreover, either $v_1 \in N^-(v)$ and $w_1 \in N^+(v)$, or $v_1 \in N^+(v)$ and $w_1 \in N^-(v)$, and v_1 is adjacent to w_1 (see Figure 3.6). Then the subdigraph induced by $\{v, w, v_1, w_1\}$ is a copy of Figure 3.3 (a), a contradiction. Therefore, v or w is in X.

One may wonder if there exists a vertex in X which only has witness triples of the second type. We show that no such vertex exists. Suppose $u \in X$ only has witness

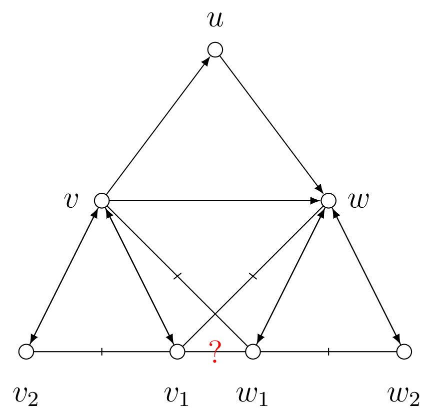


Figure 3.6: $v \in X$ or $w \in X$

triples of the second type. Let (u, v, w) be such a witness triple. We claim that v or w only has witness triples of the second type. We assume that $v \mapsto u \mapsto w$. (The proof for the case when $w \mapsto u \mapsto v$ is the same.) Then in order to avoid a copy of Figure 3.3 (d), we must have $v \mapsto w$. The following cases illustrate v or w is a vertex that only has witness triples of the second type:

- **Case 1:** $w \notin X$. There exist vertices w_1 and w_2 such that $w \leftrightarrow w_1$ and $w \leftrightarrow w_2$. Since $w_1 \in N^-(w)$ and $w_2 \in N^+(w)$, we know that w_1 and w_2 are joined by a single arc. We also know that v must in X. If v only has witness triples of the second type, then we are done. Otherwise, assume that (v, v_1, v_2) is a witness triple of the first type where $v \leftrightarrow v_2$ and the arcs between v and v_1 and between v_1 and v_2 are single. Since $\{u, w\} \subseteq N^+(v)$ and $v_2 \in N^-(v)$, there are arcs between u and v_2 and between w and v_2. Moreover, v_2 is adjacent to both w_1 and w_2. Since $\{v, u\} \subseteq N^-(w)$ and $\{w_1, w_2\} \subseteq N^+(w)$, there are arcs between uw_1, uw_2, vw_1 and vw_2. If v_2 and w_1 are connected by a single arc, then the subdigraph induced by $\{v, w, v_2, w_1\}$ is a copy of Figure 3.3 (a), a contradiction. Hence $v_2 \leftrightarrow w_1$. If v_2 is adjacent to w_2 by a single arc, then the subdigraph induced by $\{v, w, v_2, w_2\}$ is a copy of Figure 3.3 (a), a contradiction. Therefore we have $v_2 \leftrightarrow w_2$. Furthermore, we have $w \leftrightarrow v_2$, as otherwise $\{w, w_1, w_2, v_2\}$ induces a chordless cycle of length 4 in $S(D)$, which contradicts the fact that $S(D)$ is chordal. If $u \mapsto v_2$ or $v_2 \mapsto u$, then the subdigraph induced by $\{u, v, w, v_2\}$ is a copy of Figure 3.3 (b), a contradiction. Hence $u \leftrightarrow v_2$. Since v_1 is adjacent to v_2 by a single arc, and u and v_2 are connected by symmetric arcs, hence u is adjacent to v_1. Note that $u \in X$, so if $u \leftrightarrow v_1$, then $v_1 \leftrightarrow v_2$, a contradiction. If $u \mapsto v_1$ or $v_1 \mapsto u$, then (u, v_1, v_2) is a witness triple of the first type, also a contradiction (see Figure 3.4 (a)).

- **Case 2:** $w \in X$. Suppose that w has a witness triple of the first type. Let (w, w_1, w_2) be such a witness triple. Form the discussion in Case 1, we know that v must be in X. If v only has witness triples of the second type, then we are done. So, assume that (v, v_1, v_2) is a witness triple of the first type. Because $\{u, w\} \subseteq N^+(v)$ and $v_2 \in N^-(v)$, we have both u and w are adjacent to v_2. Suppose that $w \leftrightarrow v_2$. Then $u \leftrightarrow v_2$, as otherwise $\{u, v, w, v_2\}$ induces a copy of Figure 3.3 (b), a contradiction. Using the same discussion as in Case 1, we find that u and v_1 must be adjacent. If they are connected by symmetric arcs, then u is not a simplicial vertex of $S(D)$, a contradiction. If u and v_1 are joined by a single arc, then (u, v_1, v_2) is a witness triple of the second type, a contradiction (see Figure 3.4 (b)). Hence w and v_2 are joined by a single arc. Since w_2 is an in-neighbour as well as an out-neighbour of w, the quasi-transitivity implies v and v_2 are adjacent to w_2. If v is adjacent to w_2 by a single arc or v_2 is adjacent to w_2 by a single arc, then the subdigraph induced by $\{v, w, v_2, w_2\}$ is a copy of Figure 3.3 (a), a contradiction. Therefore, $v \leftrightarrow w_2$ and $v_2 \leftrightarrow w_2$. Since $u \in N^-(w)$ and $w_2 \in N^+(w)$, there is an arc between u and w_2. If this is a single arc, then $\{u, v, w, w_2\}$ induces a copy of Figure 3.3 (b), a contradiction. Hence, we have $u \leftrightarrow w_2$. Since w_1 and w_2 are adjacent, w_1 must be adjacent to u. If $u \mapsto w_1$ or $w_1 \mapsto u$, then (u, w_1, w_2) is a witness triple of the first type, a contradiction. On the other hand, the facts that $u \in X$ and $u \leftrightarrow w_2$ imply if $u \leftrightarrow w_1$, then $w_1 \leftrightarrow w_2$, a contradiction (see Figure 3.4 (c)).

Therefore, v or w only has witness triples of the second type. Without loss of generality, assume that v is such a vertex. Thus, if u only has witness triples of the second type, and (u, v, w) is such a witness triple, then we have v also only has witness triples of the second type. Moreover, there exists a vertex v' such that (v, v', w') is a witness triple of the second type and v' only has witness triples of the second type (it is possible that $u = v'$). Continuing this process, we get a sequence of vertices $u, v, v', ...,$ each of which only has witness triples of the second type, along with the witness triples $(u, v, w), (v, v', w'), ...$ of the second type. Clearly, this sequence will have a repeated vertex. It follows that there exists a cycle $\mathcal{C}_l = u_1, u_2, ..., u_l$ such that u_i only has witness triples of the second type, and (u_i, u_{i+1}, w_i) for each i is such a witness triple. Assume that such cycle $\mathcal{C}_l$ is chosen to have the shortest length. It is not hard to see that the arcs in $\mathcal{C}$ are all single. Suppose that $l \geq 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_l, u_1, w_l)$ are witness triples of the second type. Proposition 3.2 ensures that $\mathcal{C}_l$ is not a directed cycle. Without loss of generality, assume that $u_1 \mapsto u_2$ and $u_1 \mapsto u_l$. Then by the definition of witness triples, we have $w_1 \mapsto u_1, w_1 \mapsto u_2, u_1 \mapsto w_l$ and $u_l \mapsto w_l$. Since $w_1 \in N^-(u_1)$ and $u_l \in N^+(u_1)$, there is an arc between w_1 and u_l. If $w_1 \leftrightarrow u_l$, then (u_l, u_1, w_1) is a witness triple of the first type, a contradiction. If $u_l \mapsto w_1$, then the subdigraph induced by $\{u_1, u_l, w_1\}$ is a copy of Figure 3.3 (d), a contradiction. Hence, we have $w_1 \mapsto u_l$. But then (u_1, u_l, w_1) is a witness triple of the second type and we have a shorter sequence u_1, u_l, a contradiction. Suppose that $l = 2$. Then (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the second type, where both u_1 and u_2 only have witness triples of the second type. Without loss of generality, assume that $u_1 \mapsto u_2$. We have $w_1 \mapsto u_1, w_1 \mapsto u_2, u_1 \mapsto w_2$ and $u_2 \mapsto w_2$. Since $w_1 \in N^-(u_1)$ and $w_2 \in N^+(u_1)$, they are adjacent. It is not hard to see that $w_1 \mapsto w_2$, as otherwise $\{u_1, w_1, w_2\}$ is a copy of Figure 3.3 (d) if $w_2 \mapsto w_1$, or $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 3.3 (c) if $w_2 \leftrightarrow w_1$. At this point, we can see that (u, w_1, w_2) is a witness triple of the second type. Suppose that w_2 is not in X or has a witness triple of the first type. Then we know that $w_1 \in X$ and only has witness triples of the second type. Therefore, there is a vertex x_1, such that $x_1 \mapsto w_1$. Since $x_1 \in N^-(w_1)$ and $\{u_1, u_2, w_2\} \subseteq N^+(w_1)$, we know that x_1 is adjacent to u_1, u_2 and w_2. That means the subdigraph induced by $\{u_1, u_2, w_1, w_2, x_1\}$ is a semi-complete digraph. Using the same argument as in the proof of Theorem 3.1, we can show that $\{u_1, u_2, w_1, w_2\} \subseteq N^+(x_1)$, and (u_1, x_1, w_2) is a witness triple of the second type. Furthermore, since either w_2 is not in X or has a witness triple of the first type, x_1 only has witness triples of the second type. There exists a vertex x_2 such that $x_2 \mapsto x_1$. Since $x_2 \in N^-(x_1)$, we know that x_2 is adjacent to u_1, u_2, w_1 and w_2. Then $\{u_1, u_2, w_1, w_2, x_1, x_2\}$ induces a semi-complete subdigraph. In the similar way as in the proof for Theorem 3.1, we know that $\{u_1, u_2, w_1, w_2, x_1\} \subseteq N^+(x_2)$, and (u_1, x_2, w_2) is a witness triple of the second type (see Figure 3.5 (a)). Continuing with this way, we either find a forbidden subdigraph in D, or obtain distinct vertices $x_1, x_2, ...$ Since D is finite, the process will end up with a forbidden subdigraph, which is a contradiction. Therefore, assume that w_2 does not have witness triples of the first type. That means w_2 only has witness triples of the second type. There exists a vertex y_1 such that $w_2 \mapsto y_1$. Since $\{u_1, u_2, w_1\} \subseteq N^-(w_2)$ and $y_1 \in N^+(w_2)$, the subdigraph induced by $\{u_1, u_2, w_1, w_2, y_1\}$ is semi-complete and we can conclude that $\{u_1, u_2, w_1, w_2\} \subseteq N^-(y_1)$. Moreover, (u_1, w_1, y_1) is a witness triple of the second type. If $y_1 \notin X$ or y_1 has a witness triple of the first type, then we go back to those cases we just work on with. Hence, assume that y_1 only has witness triples of the second type. Thus, there exists a vertex y_2 such that $y_1 \mapsto y_2$. Then we know that y_2 is adjacent to each of u_1, u_2, w_1, w_2, y_1 and hence $\{u_1, u_2, w_1, w_2, y_1, y_2\}$

induces a semi-complete digraph. Moreover, from the proof of Theorem 3.1, we know that $\{u_1, u_2, w_1, w_2, y_1\} \subseteq N^-(y_2)$. Hence (u_1, w_1, y_2) is a witness triple of the second type and y_2 only has witness triples of the second type (see Figure 3.5 (b)). Similarly as above, this process will end up with a forbidden subdigraph, a contradiction. Therefore, no such cycle $\mathcal{C}_l$ exists.

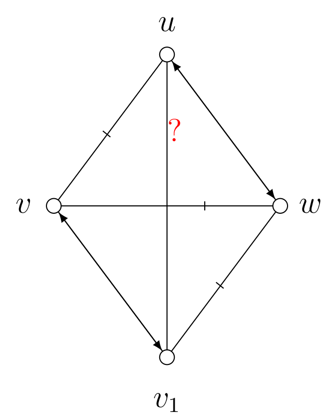


Figure 3.7: If $v \notin X$

From the above, we know that every vertex $u \in X$ has a witness triple of the first type. Suppose that (u, v, w) is a witness triple of the first type and $v \notin X$. Then by Lemma 1.1, there exists a simplicial vertex $v'(v \neq v')$ such that v' is in the component of v in $S(D) \setminus N[w]$. Since v and v' are in the same component, there is a vertex v_1 (possibly, $v_1 = v'$) such that $v \leftrightarrow v_1$. Because v is adjacent to both u and w, we have v_1 is also adjacent to both$u$ and w. Moreover, since $v_1 \in S(D) \setminus N[w]$, w and v_1 are connected by a single arc. Above all, we know that no matter $u \leftrightarrow v_1, u \mapsto v_1$ or $v_1 \mapsto u$, the subdigraph induced by $\{u, v, w, v_1\}$ is a copy of Figure 3.3 (a), a contradiction (See Fig 3.7). Therefore, $v \in X$. Moreover, v must also have a witness triple of the first type. Let (v, v', w') be such a witness triple and $v' \in X$. Keeping this process, we obtain a sequence of vertices $u, v, v', ... \in X$ along with witness triples $(u, v, w), (v, v', w'), ...$ of the first type. This sequence will have a repeated vertex because D is finite. We end up with either a semi-strict di-simplicial vertex, or a sequence of vertices $u_1, u_2, ..., u_r$ such that $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_r, u_1, w_r)$ are witness triples of the first type. Assume that such sequence is chosen where r is the least. Immediately, we have $u_i \leftrightarrow w_i$, and the arcs between u_i and u_{i+1} and between u_{i+1} and w_i are single arcs. Suppose that $r = 2$. Since $u_2 \leftrightarrow w_2$ and u_2 is adjacent to w_1, we know that w_1 and w_2 are adjacent. Then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 3.3 (a), a contradiction. Suppose that $r \geq 3$. Since $u_2 \leftrightarrow w_2$ and u_2 is adjacent to u_1, we know that u_1 and w_2 are adjacent. If they are joined by a single are, then (u_2, u_1, w_2) is a witness triple of the first type and there is a shorter sequence u_1, u_2, a contradiction. Therefore, $u_1 \leftrightarrow w_2$. Then u_1 and u_3 are adjacent because w_2 and u_3 are adjacent. Note that $u_1 \in X$, so if u_1 and u_3 are joined by symmetric arcs, then $u_3 \leftrightarrow w_2$, which contradicts the fact. If $u_1 \mapsto u_3$ or $u_3 \mapsto u_1$, then (u_1, u_3, w_2) is a witness triple of the first type and we yield a shorter sequence $u_1, u_3, ..., u_r$, a contradiction.

Therefore, D has a semi-strict di-simplicial vertex.

$\square$

3.5 Extended Semi-complete Digraphs

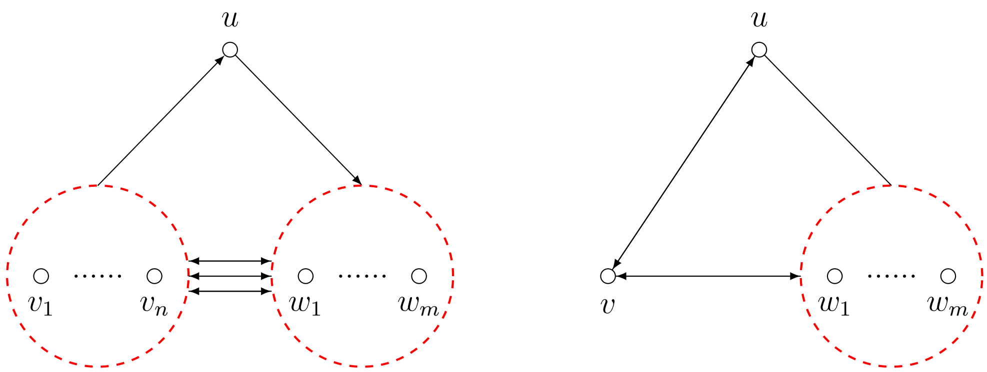
Semi-complete semi-strict chordal digraphs and quasi-transitive semi-strict chordal digraphs share the same list of forbidden subdigraphs. It turns out this holds for extended semi-complete semi-strict chordal digraphs.

Theorem 3.3. *Let D be an extended semi-complete digraph such that $S(D)$ is chordal. Then D is a semi-strict chordal digraph if and only if it does not contain any of the digraphs in Figure 3.3 as an induced subdigraph.*

Proof. Assume that D is obtained from the semi-complete digraph D' by replacing each vertex x with an independent set S_x. If D' contains a digraph F as a subdigraph, then D also contains F as subdigraph. Moreover, D' is a subdigraph of D. The necessity has been proved from Lemma 3.4. For the sufficiency, suppose that D does not contain any of the digraphs in Figure 3.3 as an induced subdigraph. We prove that D has a semi-strict di-simplicial vertex.

Since D does not contain any of the digraphs in Figure 3.3 as induced subdigraphs, and it is generated from the semi-complete digraph D', we know that D' does not contain any copies in Figure 3.3. Theorem 3.1 tells us that D' is a semi-strict chordal digraph and hence has a semi-strict di-simplicial vertex for every induced subdigraph. Assume that u' is a semi-strict di-simplicial vertex of D'. Then for every v' and w' in $V(D')$ such that $v' \to u'$ and $u' \to w'$, we have $v' \leftrightarrow w'$. Assume that $u \in S_{u'}$ and suppose that u' is only adjacent to other vertices by single arcs in D'. Then for any vertices $\{v', w'\} \subseteq V(D')$, $v' \mapsto u'$ and $u' \mapsto w'$ imply $v' \leftrightarrow w'$. We know that u is only adjacent to other vertices by single arcs in D. Moreover, for any vertices $\{v, w\} \subseteq V(D)$ where $v \in S_{v'}$ and $w \in S_{w'}$, if $v \mapsto u$ and $u \mapsto w$, then $v \leftrightarrow w$. Therefore, u is a semi-strict di-simplicial vertex of D (see Figure 3.8 (a)). Hence, assume that u' is adjacent to some vertices by symmetric arcs in D'. Use Q to denote those vertices that are adjacent to u' by symmetric arcs. Since u' is a semi-strict di-simplicial vertex in D', for every vertex $v' \in Q$, we have $v' \leftrightarrow u' \to w'$ or $v' \leftrightarrow u' \leftarrow w'$ implies $v' \leftrightarrow w'$. Consider the following two cases regarding to the extension of the vertices in Q.

Suppose that $|S_{v'}| = 1$ for every vertex v' in Q. Let $v \in S_{v'}$ be a vertex that is adjacent to u by symmetric arcs, then we know that v is adjacent to every vertex in D since it is in an independent set by itself. For any vertex $w \in S_{w'}$ that is adjacent to u regardless of the orientation, we have $v \leftrightarrow w$ (see Figure 3.8 (b)). Therefore, u is a semi-strict di-simplicial vertex of D.

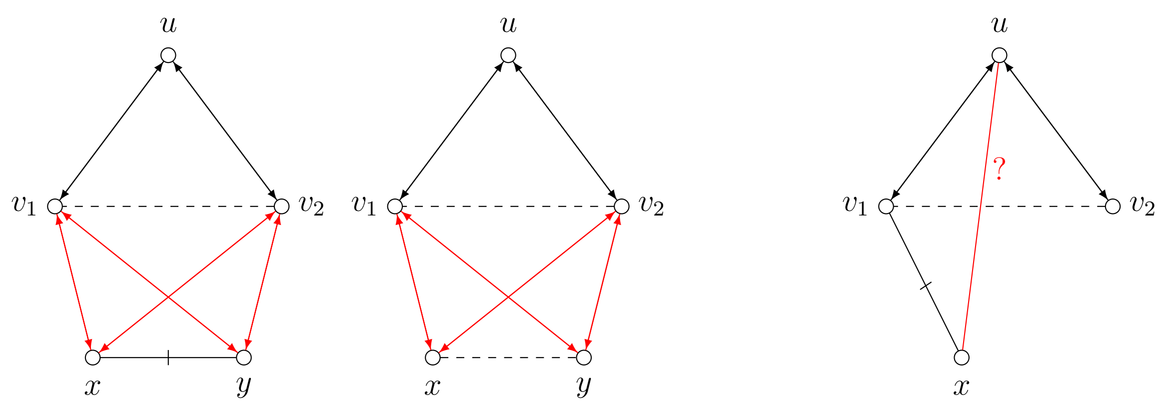


(a) u is only adjacent to single arcs (b) v is in an independent set by itself

Figure 3.8: An illustration of the proof of Theorem 3.3: to show u is a semi-strict di-simplicial vertex

Suppose that $|S_{v'}| > 1$ for some v' in Q. Let $v_1 \in S_{v'}$ and $v_2 \in S_{v'}$ be the two vertices in D. Clearly, v_1 and v_2 are not adjacent, and $u \leftrightarrow v_1$, $u \leftrightarrow v_2$. We claim that v_1 is a semi-strict di-simplicial vertex in D. We are proving our claim by contrapositive that assuming v_1 is not a semi-strict di-simplicial vertex in D. Suppose that v_1 is not a simplicial vertex in $S(D)$, then there exist vertices x and y such that $v_1 \leftrightarrow x$ and $v_1 \leftrightarrow y$ but either x and y are not adjacent or they are connected by a single arc. We know that $v_2 \leftrightarrow x$ and $v_2 \leftrightarrow y$. Then $\{v_1, v_2, x, y\}$ induced a chordless cycle of length 4 in $S(D)$, a contradiction (see Figure 3.9 (a)). Hence, assume that v_1 is a simplicial vertex in $S(D)$. By Lemma 3.2 and Proposition 3.1, v_1 has a witness triple of the first or the second type. In either type, there exists a vertex x such that x is adjacent to v_1 by single arc. Assume that $x \in S_{x'}$ where $x' \in V(D')$. Since v_1 and x are connected by a single arc but u and v_1 are connected by a symmetric arc, we know that x and u are not in the same independent set, hence they are adjacent. If $u \mapsto x$ or $x \mapsto u$, then we have $u' \leftrightarrow v'$, and the arcs between u' and x' and between v' and x' are single. Thus (u', x', v') is a wintess triple in the semi-complete digraph D', a contradiction to the fact that u' is semi-strict di-simplicial in D'. If $u \leftrightarrow x$, then we know that $u' \leftrightarrow x'$ in D', and so $v' \leftrightarrow x'$. Hence, v' and x' are adjacent in

D' by symmetric arcs and so v_1 and x in D must also be connected by symmetric arcs, a contradiction (see Figure 3.9 (b)). Therefore, v_1 has no witness triples, i.e. is a semi-strict di-simplicial vertex of D. $\square$



(a) $v_1 \notin X$ (b) $v_1 \in X$

Figure 3.9: An illustration of the proof of Theorem 3.3: to show v_1 is a semi-strict di-simplicial vertex

3.6 Locally Semi-complete Digraphs

In this section, we investigage the list of forbidden subdigraphs for locally semi-complete semi-strict chordal digraphs. As we shall see, the list is wider than the one for any of the previously studied classes.

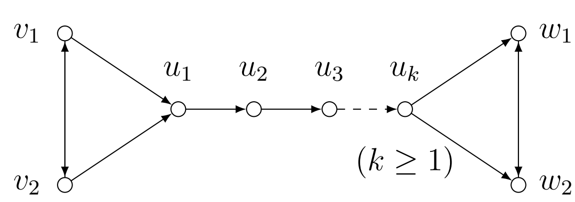


Figure 3.10: Forbidden subdigraph for locally semi-complete semi-strict digraphs

Theorem 3.4. *Let D be a locally semi-complete digraph. Suppose that $S(D)$ is chordal. Then D is a semi-strict chordal digraph if and only if it does not contain any of the digraphs in Figure 2.4, Figure 3.3 or Figure 3.10 as an induced subdigraph.*

Proof. For the necessity, Lemma 3.4 shown that if a digraph contains any of the subdigraphs in Figure 3.3, then it cannot be a semi-strict chordal digraph. It is easy to check that there is a witness triple for every vertex in the digraphs in Figure 2.4 and Figure 3.10 and hence none of them is a semi-strict chordal vertex. Therefore, no semi-strict chordal digraph contains any of the subdigraphs in Figure 2.4 or Figure 3.10 as an induced subdigraph.

Suppose that D does not contain any copy in Figure 2.4, Figure 3.3 or Figure 3.10 as an induced subdigraph. We prove by contradiction that D has a semi-strict di-simplicial vertex. So assume that none of the vertices in D is semi-strict di-simplicial. Let u be the vertex in X. Then by Proposition 3.1, u has witness triples of the first, the second or the third type.

We claim that if u only has witness triples of the second or the third type and (u, v, w) is such a witness triple, then v or w only has witness triples of the second or the third type. By the definition of witness triples of the second or the third type, we have either $v \mapsto u \mapsto w$ or $w \mapsto u \mapsto v$, and either v is adjacent to w by a single arc, or they are not adjacent. Without loss of generality, assume that $v \mapsto u \mapsto w$. The proof for the other case is similar. If one of v or w only has witness triples of the second or the third type, then we are done. Otherwise, they are either not in X or they have a witness triple of the first type. In the other word, there exist vertices v_1, v_2, w_1 and w_2 (they may be the same vertex), such that $v \leftrightarrow v_2$, $v — v_1$, $w \leftrightarrow w_2$, $w — w_1$, and the arcs between v_1 and v_2 and between w_1 and w_2 are single. Since $\{u, v_2\} \subseteq N^+(v)$ and $\{u, w_2\} \subseteq N^-(w)$, u is adjacent to v_2 and w_2. We prove the claim by the cases according to the type of (u, v, w). Suppose that (u, v, w) is a witness triple of the third type. Then v is not adjacent to w. If $v_2 = w_2$, then since $\{v, w\} \subseteq N^-(v_2)$, there is an arc between v and w, a contradiction. If v_2 is adjacent to w or v is adjacent to w_2, then v and w must be adjacent, a contradiction. If v_2 is adjacent to w_2, then v must be adjacent to w_2, which we just showed is not possible. Hence v is not adjacent to w_2, and v_2 is not adjacent to both w and w_2. If $u \to v_2$, then $\{v_2, w\} \subseteq N^+(u)$ and v_2 and w are adjacent, a contradiction. Hence $v_2 \mapsto u$. If $w_2 \to u$, then $\{v, w_2\} \subseteq N^-(u)$ and v and w_2 must be adjacent, also a contradiction. Therefore, we have $u \mapsto w_2$ (see Figure 3.11 (a)). But then $\{u, v, w, v_2, w_2\}$ induces a copy of Figure 3.10, a contradiction. Suppose that (u, v, w) is a witness triple of the second type. If $w \mapsto v$, then $\{u, v, w\}$ induces a copy of Figure 3.3 (d), a contradiction. Hence, we have $v \mapsto w$. Since $\{u, v_2\} \subseteq N^+(v)$, they are connected. If $v_2 = w_2$, then we have $u \leftrightarrow v_2$, or as otherwise $\{u, v, w, v_2\}$ induces a copy of Figure 3.3 (b) and get a contradiction. Therefore, $u \neq v_1$. Since v_1 is adjacent to v_2, we have u and v_1 are connected. If they are connected by a single arc, then (u, v_1, v_2) is a witness triple of the first type, a contradiction. Since $u \in X$, if $u \leftrightarrow v_1$, then we have $v_1 \leftrightarrow v_2$, also a contradiction. Therefore, $v_2 \neq w_2$ and so w and v_2 are not connected by symmetric arcs. Since $\{w, v_2\} \subseteq N^+(v)$, they are adjacent by a single arc. Suppose that $u \not\to v_2$. Then $v_2 \mapsto w$, as otherwise $\{u, v, w, v_2\}$ induces a copy of Figure 3.3 (c). Since $\{v, v_2, w_2\} \subseteq N^-(w)$, w_2 is adjacent to both v and v_2. If any of these arcs is single, then $\{v, w, v_2, w_2\}$ induces a copy of Figure 3.3 (a). Therefore, we have $v \leftrightarrow w_2$ and $v_2 \leftrightarrow w_2$. If u is adjacent to w_2 by a single arc, then the subdigraph induced by $\{u, v, w, w_2\}$ is a copy of Figure 3.3 (b). Hence $u \leftrightarrow w_2$. Moreover, we have $u \neq w_1$. Since u and w_1 are either both in-neighbours or both out-neighbours of w_2, they are connected. If $u \not\to w_1$, then (u, w_1, w_2) is a witness triple of the first type, a contradiction. Note that $u \in X$, so if $u \leftrightarrow w_1$, then $w_1 \leftrightarrow w_2$, this contradicts the assumption (see Figure 3.11 (b)). Suppose that $u \leftrightarrow v_2$. Then because u and v_1

are both in-neighbours or both out-neighbours of v_2, they are connected. If u and v_1 are joined by a single arc, then (u, v_1, v_2) is a witness triple of the first type, a contradiction. If $u \leftrightarrow v_1$, then $u \in X$ implies that $v_1 \leftrightarrow v_2$, a contradiction (see Figure 3.11 (c)). Above all, we have showed the correctness of our claim.

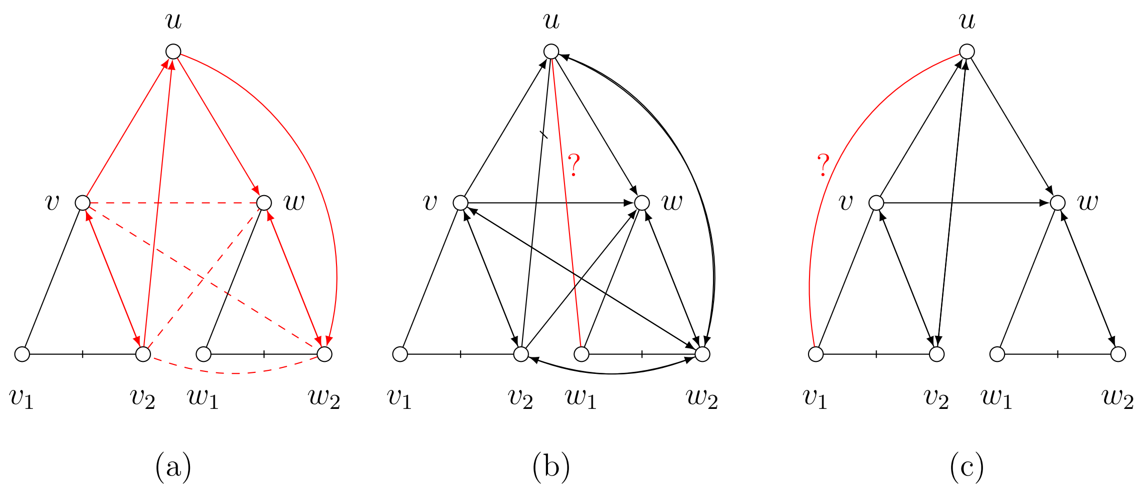


Figure 3.11: u only has the second or the third type

We proved that if there is a vertex u which only has witness triples of the second or the third type and (u, v, w) is such a witness triple, then at least one of v or w only has witness triples of the second or the third type. Due to the symmetry property of such a witness triple, we assume without loss of generality that v is a vertex that only has witness triples of the second or the third type. Then there exists a witness triple (v, v', w') such that v' only has witness triples of the second or the third type. Continuing this way, we obtain a sequence of vertices $u, v, v', ...,$ such that all of them only have witness triples of the second or the third type. Since D is finite, this sequence will have a repeated vertex. There exists a sequence $u_1, u_2, ..., u_l$ such that $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_l, u_1, w_l)$ are witness triples of the second or the third type, and u_i only has witness triples of the second or the third type. Assume that such a sequence is chosen so that l is the least. Clearly, u_i is adjacent to u_{i+1} by a single arc. Assume that $u_1 \mapsto u_2$. Then we have $w_1 \mapsto u_1$. We prove by contrapositive that such sequence does not exist by considering the value of the length l.

- **Case 1:** $l \geq 3$. Then $(u_1, u_2, w_1), (u_2, u_3, w_2)$ are witness triples of the second or the third type. If $u_1 \leftrightarrow u_3$, then (u_1, u_2, u_3) is a witness triple of the first type, which contradics with the assumption. Therefore, either u_1 and u_3 are not adjacent, or they are connected by a single arc. If $u_2 \mapsto u_3$, then (u_2, u_1, u_3) is a witness triple of the second or the third type. Hence there is a shorter sequence u_1, u_2, which is a contradiction. If $u_3 \mapsto u_2$, then we have $u_2 \mapsto w_2$. If $u_1 \leftrightarrow w_2$, then (u_1, u_2, w_2) is a witness triple of the first type, a contradiction. If $u_1 \not\leftrightarrow w_2$, then (u_2, u_1, w_2) is a witness triple of the second or the third type, and we again have a shorter sequence u_1, u_2 and obtain a contradiction.

- **Case 2:** $l = 2$. Then $(u_1, u_2, w_1), (u_2, u_1, w_2)$ are witness triples of the second or the third type. And we have $w_1 \mapsto u_1$, $u_2 \mapsto w_2$.

  - Suppose that both (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the third type. Then u_2 is not adjacent to w_1, and u_1 is not adjacent to w_2. If $w_1 \to w_2$, then there is an arc between u_1 and w_2, which is a contradiction. If $w_2 \mapsto w_1$, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Fig 2.4, a contradiction. Therefore, w_1 and w_2 are not connected. We claim that w_1 or w_2 only has witness triples of the second or the third type. Suppose that this is not the case. Then w_1 and w_2 are either not a simplicial vertices of $S(D)$, or have a witness triple of the first type. In either case, there exist vertices w_3 and w_4 such that $w_1 \leftrightarrow w_3$ and $w_2 \leftrightarrow w_4$. It is not hard to see that $w_3 \neq w_4$. If u_2 and w_3 are adjacent, then u_2 should be adjacent to w_1, which contradicts our assumption. If u_1 and w_4 are adjacent, then u_1 should be adjacent to w_2, which is also a contradiction. Hence u_2 is not adjacent to w_3, and u_1 is not adjacent to w_4. Since $\{u_1, w_3\} \subseteq N^+(w_1)$, and $\{u_2, w_4\} \subseteq N^-(w_2)$, we have u_1 is adjacent to w_3 and u_2 is adjacent to w_4. If $u_1 \to w_3$, then since $\{u_2, w_3\} \subseteq N^+(u_1)$, u_2 and w_3 are adjacent, a contradiction. If $w_4 \to u_2$, then u_1 is adjacent to w_4, a contradiction. Therefore, we have $w_3 \mapsto u_1$ and $u_2 \mapsto w_4$. If w_1 is adjacent to w_4, or w_2 is adjacent to w_3, then w_1 must be adjacent to w_2, a contradiction. If w_3 is adjacent to w_4, then w_1 must be adjacent to w_4, which is not possible. Hence, w_1 is not adjacent to w_4, w_2 is not adjacent to w_3, and w_3 is not adjacent to w_4. But then the subdigraph induced by $\{u_1, u_2, w_1, w_2, w_3, w_4\}$ is a copy of Figure 3.10 (see Figure 3.12 (a)), a contradiction. Hence, w_1 or w_2 only has witness triples of the second or the third type.

We now work on the cases when at least one of w_1 and w_2 only has witness triples of the second or the third type. First, assume that exactly one of w_1 and w_2 only has witness triples of the second or the third type. By symmetry we may assume that w_1 is the vertex. Immediately, we know that there exist two vertices x_1 and y_1 such that $x_1 \mapsto w_1$ and $w_2 \leftrightarrow y_1$. Since u_1 is not adjacent to w_2, we have u_1 is not adjacent to y_1. Since $\{u_2, y_1\} \subseteq N^-(w_2)$, they are adjacent, and the only possible connection between them is $u_2 \mapsto y_1$, as otherwise we will have u_1 and y_1 are adjacent, which is impossible. Since u_2 and w_1 are not adjacent and $x_1 \mapsto w_1$, the only possible cases between u_2 and x_1 are either $u_2 \mapsto x_1$ or they are not adjacent. If $u_2 \mapsto x_1$, then there must be symmetric arcs between u_1 and x_1, as otherwise (u_1, w_1, x_1) or (u_1, u_2, x_1) induces a copy of Figure 3.3 (d), or (u_1, u_2, w_1, x_1) induces a copy of Figure 2.4. But then (u_1, u_2, x_1) is a witness triple of the first type, which contradicts the fact. Therefore, u_2 is not adjacent to x_1, and the only possible cases between u_1 and x_1 are either $x_1 \mapsto u_1$ or they are not adjacent. If $x_1 \mapsto u_1$, then (u_1, u_2, x_1) is a witness triple of the third type and x_1 is not adjacent to w_2. Since w_2 is not only has witness triples of the second or the third type, we know that x_1 must only have witness triples of the second or the third type. Suppose that x_1 and u_1 are not adjacent. If $x_1 \rightarrow w_2$, then x_1 is adjacent to u_2, a contradiction. If $w_2 \mapsto x_1$, then $\{u_1, u_2, w_1, w_2, x_1\}$ induces a copy of Figure 2.4, a contradiction. Therefore, x_1 is not adjacent to w_2. Then we know that x_1 is not adjacent to y_1, as otherwise x_1 must be adjacent to w_2. Moreover, we have w_1 and y_1 are not adjacent. Assume now that x_1 has not only witness triples of the second or the third type. Then there exists a vertex x_2' such that $x_1 \leftrightarrow x_2'$. Since x_1 is not adjacent to u_1, u_2, w_2 and y_1, we know that x_2' is also not adjacent to the verictes u_1, u_2, w_2 and y_1. Since $\{w_1, x_2'\} \subseteq N^+(x_1)$, they are adjacent. Moreover, they can only be adjacent as $x_2' \mapsto w_1$, as otherwise u_1 must be adjacent to x_2'. At this point, $\{u_1, u_2, w_1, w_2, x_1, y_1, x_2'\}$ induces a copy of Figure 3.10 (see Figure 3.12 (b)), a contradiction. Therefore, x_1 only has witness triples of the second or the third type. In a short conclusion, we just show that no matter what kind of arc between u_1 and x_1, x_1 only has witness triples of the second or the thrid type. Therefore, there exists a vertex x_2 such that $x_2 \mapsto x_1$. We can use the above technique to show that x_2 only has witness triples of the second or the third type. Continueing this process, we either find a forbidden subdigraph in D, or obtain distinct vertices $x_1, x_2, \ldots.$ Since D is finite, this process will end up with a copy of one of the forbidden subdigraphs and get a contradiction.

Secondly, suppose that both w_1 and w_2 only have witness triples of the second or the third type. Then there exists a vertex y_1 such that $w_2 \mapsto y_1$.

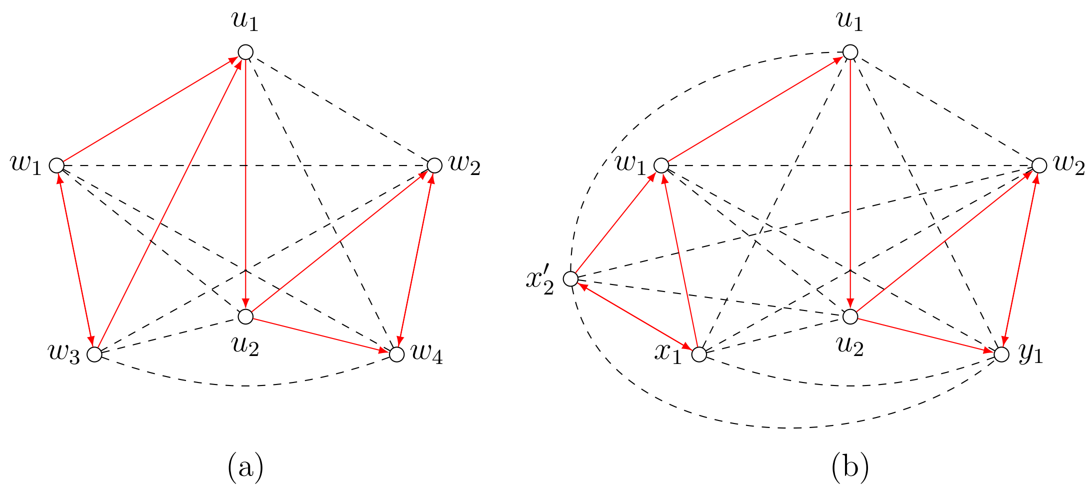


Figure 3.12: $l = 2$ with both of the third type

If $u_1 \to y_1$, then u_1 is adjacent to w_2, a contradiction. If $y_1 \mapsto u_1$, then we need to have $u_2 \leftrightarrow y_1$, as otherwise $\{u_1, u_2, y_1\}$ or $\{u_2, w_2, y_1\}$ induces a copy of Figure 3.3 (d), or $\{u_1, u_2, w_2, y_1\}$ induces a copy of Figure 2.4. But then (u_2, w_2, y_1) is a witness triple of the first type, a contradiction. Hence u_1 and y_1 are not adjacent. Furthermore, if $y_1 \to u_2$, then u_1 is adjacent to y_1. Therefore, we have either $u_2 \mapsto y_1$, or there is no arc between u_2 and y_1. If $u_2 \mapsto y_1$, then (u_2, u_1, y_1) is a witness triple of the third type. Since w_1 only has witness triples of the second or the third type, and if y_1 has not only witness triples of the second or the third type, we are back to the previous case. Therefore, at this case, assume that y_1 only has witness triples of the second or the third type. If u_2 and y_1 are not adjacent, then u_2, w_2 is a sequence of length two along with the witness triples (u_2, w_2, u_1) and (w_2, u_2, y_1) that we can also work on with. Then if y_1 not only has witness triples of the second or the third type, we are again back to the previous case. Therefore, for the above two cases, y_1 only has witness triples of the second or the third type. Hence, there exists a vertex y_2 such that $y_1 \mapsto y_2$ and we can follow the above discussion to show that

y_2 only has witness triples of the second or the third type. Keeping this discussion, we will eventually end up with a forbidden subdigraph and get a contradiction. Therefore, (u_1, u_2, w_1) and (u_2, u_1, w_2) cannot both be the witness triples of the third type.

– Suppose that either (u_1, u_2, w_1) or (u_2, u_1, w_2) is of the second type. Without loss of generality, assume that (u_1, u_2, w_1) is a witness of the second type such that $u_1 \mapsto u_2$. By the definition of witness triples, we have $w_1 \mapsto u_1, w_1 \mapsto u_2, u_2 \mapsto w_2$, and u_1 is not adjacent to w_2. If $w_1 \to w_2$, then u_1 is adjacent to w_2, which is not possible. If $w_2 \mapsto w_1$, then $\{u_2, w_1, w_2\}$ induces a copy of Figure 3.3 (d), a contradiction. Hence w_1 and w_2 are not adjacent. Then we have (u_2, w_1, w_2) as a witness triple of the third type. Therefore, w_1 or w_2 only has witness triples of the second or the third type. Suppose that w_1 be such a vertex. Then there exists a vertex x_1 such that $x_1 \mapsto w_1$.

  We assume that w_2 is not in X or it has a witness triple of the first type. Then there is a vertex y_1 such that $w_2 \leftrightarrow y_1$. Since u_1 is not adjacent to w_2, we have u_1 is not adjacent to y_1. Since $\{u_2, y_1\} \subseteq N^-(w_2)$, they are adjacent. Moreover, we have $u_2 \mapsto y_1$. If w_1 is adjacent to y_1, then w_1 must be adjacent to w_2, a contradiction. Therefore, y_1 and w_1 are not adjacent. If $x_1 \to w_2$, then w_1 and w_2 are adjacent, a contradiction. If $w_2 \mapsto x_1$, then we have $u_2 \leftrightarrow x_1$, as otherwise $\{u_2, w_1, x_1\}$ or $\{u_2, w_2, x_1\}$ induces a copy of Figure 3.3 (d), or $\{u_2, w_1, w_2, x_1\}$ induces a copy of Figure 2.4. But then (u_2, w_1, x_1) is a witness triple of the first type, a contradiction. Hence w_2 is not adjacent to x_1. If $u_2 \to x_1$, then w_2 and x_1 are adjacent, a contradiction. Therefore, there are only two possible cases between u_2 and x_1, either $x_1 \mapsto u_2$, or they are not adjacent. Suppose that x_1 and

u_2 are not adjacent. Then (w_1, u_2, x_1) and (u_2, w_1, w_2) are witness triples of the third type and both w_1 and u_2 only have witness triples of the second or third type. Then we are back to the previous case which has been taken care of. Suppose that $x_1 \mapsto u_2$. Then since $\{u_1, x_1\} \subseteq N^-(u_2)$, they are adjacent. Because u_1 only has witness triples of the second or the third type, u_1 and x_1 cannot be adjacent by symmetric arcs. If $u_1 \mapsto x_1$, then $\{u_1, w_1, x_1\}$ induces a copy of Figure 3.3 (d), a contradiction. Hence $x_1 \mapsto u_1$. At this point, we have (u_1, u_2, x_1) as a witness triple of the second type and by our assumption about w_2, we know that x_1 only has witness triples of the second or the third type. Then there exists a vertex x_2 such that $x_2 \mapsto x_1$ (see Figure 3.13 (a)). Keeping the same discussion, we have the result that x_2 only has witness triples of the second or the third type. At the end, the process will end up with a copy of those forbidden subdigraphs and get a contradiction.

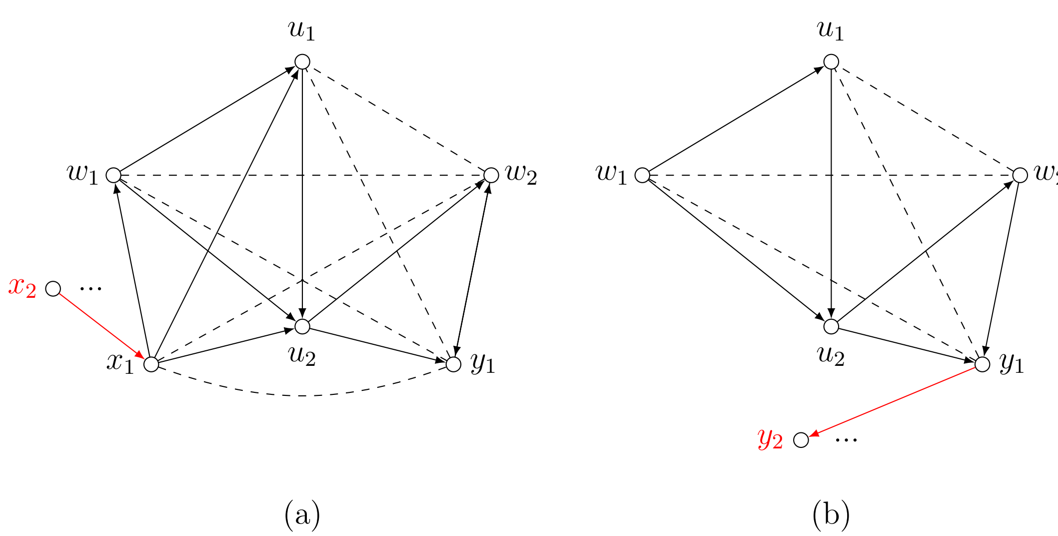


Figure 3.13: $l = 2$ with one of the second and one of the third type

Therefore, assume that w_2 only has witness triples of the second or the third type. There exists a vertex y_1 such that $w_2 \mapsto y_1$. If u_2 and y_1 are not adjacent, then (u_2, w_2, u_1) and (w_2, u_2, y_1) are witness triples of the third type where u_2 and w_2 only have witness triples of the second or the third type, which is the previous case that we just showed is impossible. If $y_1 \mapsto u_2$, then $\{u_2, w_2, y_1\}$ induces a copy of Figure 3.3 (d), a contradiction. If $u_2 \leftrightarrow y_1$, then (u_2, w_2, y_1) is a witness triple of the first type, a contradiction. We know from the above that the only possible left for u_2 and y_1 is $u_2 \mapsto y_1$. At this point, we have (u_2, w_2, u_1) as a witness triple of the third type and (w_2, u_2, y_1) as a witness triple of the second type, where each of u_2 and w_2 only has witness triples of the second or the third type. If y_1 is not only have witness triples of the second or the third type, then we are back to our previous discussion (y_1 here plays the same role as w_2 in the previous discussion). Therefore, y_1 only has witness triples of the second or the third type, and there exists a vertex y_2 such that $y_1 \mapsto y_2$ (see Figure 3.13 (b)). Continuing this process, we either find a forbidden subdigraph in D, or obtain distinct vertices $y_1, y_2, ...$ Since D is finite, the process will end up with a forbidden subdigraph, a contradiction.

– Suppose that both (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the second type. Assume by symmetry that $u_1 \mapsto u_2$. Then we have $w_1 \mapsto u_1, w_1 \mapsto u_2, u_1 \mapsto w_2$ and $u_2 \mapsto w_2$. If $w_1 \leftrightarrow w_2$, then $\{u_1, u_2, w_1, w_2\}$ induces a copy of Figure 3.3 (c), a contradiction. If $w_2 \mapsto w_1$, then the subdigraph induced by $\{u_1, w_1, w_2\}$ is a copy of Figure 3.3 (d), a contradiction. Hence, either $w_1 \mapsto w_2$ or w_1, w_2 are not adjacent. In either case, we have (u_1, w_1, w_2) as a witness triple of the second or the third type. We know that w_1 or w_2 only has witness triples of the second or the third type. Suppose that w_1 be such a vertex, and w_2 is not only has witness triples of the second or the third type. Then there exists a vertex x_1 such that $x_1 \mapsto w_1$. If $u_2 \leftrightarrow x_1$, then (u_2, w_1, x_1) is a witness triple of the first type, a contradiction. If $u_2 \mapsto x_1$, then $\{u_2, w_1, x_1\}$ induces a copy of Figure 3.3 (d). If u_2 and x_1 are not adjacent, then (w_1, u_2, x_1) is a witness triple of the third type and (u_2, w_1, w_2) is a witness triple of the second or the third type, and we are back to the previous cases. Hence, we have $x_1 \mapsto u_2$. Since $\{u_1, x_1\} \subseteq N^-(u_2)$, they must be adjacent. If $u_1 \leftrightarrow x_1$, then (u_1, w_1, x_1) is a witness triple of the first type, a contradiction. If $u_1 \mapsto x_1$, then $\{u_1, x_1, w_1\}$ induces a copy of Figure 3.3 (d), a contradiction. If $x_1 \mapsto u_1$, then x_1 and w_2 cannot be adjacent by symmetric arcs, as otherwise $\{u_1, u_2, w_2, x_1\}$ induces a copy of Figure 3.3 (c), a contradiction. Thus, either x_1 and w_1 are not adjacent or they are connected by a single arc, and we have (u_1, w_2, x_1) as a witness triple of the second or the third type. Since w_2 is not only has witness triples of the second or the third type, we know that x_1 only has witness triples of the second or the third type. Then there exists a vertex x_2 such that $x_2 \mapsto x_1$. We can similary show that x_2 only has witness triples of the second or the third type. Continuing this process, we will end up with a forbidden subdigraph and obtain a contradiction. Hence, assume that w_2 only has witness triples of the second or the third type. Then there exists a vertex y_1 such that $w_2 \mapsto y_1$. If $u_2 \leftrightarrow y_1$, then (u_2, w_2, y_1) is a witness triple of the first type, a contradiction. If $y_1 \mapsto u_2$, then $\{u_2, w_2, y_1\}$ induces a copy of Figure 3.3 (d), a contradiction. If u_2 and y_1 are not adjacent, then since (w_2, u_2, y_1) is a witness triple of the third type and (u_2, w_2, u_1) is a witness triple of the second type, we are back to the previous case which is not possible. Hence, we have $u_2 \mapsto y_1$. If u_1 and y_1 are not adjacent, we are again back to the previous case because (u_2, u_1, y_1) is a witness triple of the third type. If $u_1 \leftrightarrow y_1$, then (u_1, u_2, y_1) is a witness triple of the first type, a contradiction. If $y_1 \mapsto u_1$, then $\{u_1, u_2, y_1\}$ induces a copy of Figure 3.3 (d), a contradiction. If $u_1 \mapsto y_1$, then (u_2, u_1, y_1) is a witness triple of the second type. If y_1 is not in X or has a witness triple of the first type, then we are back to the previous case. Hence y_1 only has witness triples of the second or the third type. Thus, no matter if $w_1 \mapsto w_2$ or they are not adjacent, we have y_1 as a vertex that only has witness triples of the second or the third type. Moreover, w_1 and y_1 are not adjacent by symmetric arcs. Therefore, there exists a vertex y_2 such that $y_1 \mapsto y_2$. We are keeping this process, and since $V(D)$ is finite, the process will end up with a forbidden subdigraph and yield a contradiction.

In summary of the above discussion, we can conclude that no such sequence exist. Hence, for every vertex u in X, we know that u always has a witness triple of the first type. Assume that (u, v, w) is a witness triple of the first type. We claim that $v \in X$. Suppose that $v \notin X$. Then by Lemma 1.1, there exists a vertex v' in the component of v in $S(D) \setminus N[w]$ such that $v' \in X$, $v \neq v'$. Clearly, there is a path joining v and v'. Let vertex v_1 in that path such that $v \leftrightarrow v_1$ (it is a possible that $v_1 = v'$). Since v_1 is adjacent to v by symmetric arcs and u is adjacent to v, we have u and v_1 are adjacent. Since u is adjacent to w by symmetric arcs and u is adjacent to v_1, we have w and v_1 are adjacent. Since $v_1 \in S(D) \setminus N[w]$, v_1 and w are connected by a single arc. Then no matter what kind of arc between u and v_1 is, the subdigraph induced by $\{u, v, w, v_1\}$ is a copy of Figure 3.3 (a), a contradiction. Therefore, $v \in X$.

Given a vertex $u \in X$, then there exists a vertex $v \in X$ such that (u, v, w) is a witness triple of the first type. Moreover, there exists a vertex $v' \in X$ such that (v, v', w') is a witness triple of the first type. By the same discussion, there is a vertex $v'' \in X$ such that (v', v'', w'') is a witness triple of the first type. Continuing this process, we obtain a sequence of vertices $u, v, v', v''... \in X$. Clearly, this sequence will have a repeated vertex. Therefore, we either find a semi-strict di-simplicial vertex, or have a sequence of vertices $u_1, u_2, ..., u_r \in X$ such that $(u_1, u_2, w_1), (u_2, u_3, w_2), ..., (u_r, u_1, w_r)$ are witness triples of the first type. Assume that r is the shortest length of such a sequence. We show that such sequence does not exist by showing that r does not exist. By the definition of witness triples, u_i and w_i are connected by symmetric arcs, and the arcs between u_i and u_{i+1} and between u_{i+1} and w_i are single. Suppose that $r = 2$. Then (u_1, u_2, w_1) and (u_2, u_1, w_2) are witness triples of the first type. Since $u_1 \leftrightarrow w_1$ and u_1 is adjacent to w_2, we know that w_1 is adjacent to w_2. Then no matter $w_1 \mapsto w_2, w_2 \mapsto w_1$ or $w_1 \leftrightarrow w_2$, the subdigraph induced by $\{u_1, u_2, w_1, w_2\}$ is a copy of Figure 3.3 (a), a contradiction. Suppose that $r \geq 3$. There exist witness triples $(u_1, u_2, w_1), (u_2, u_3, w_2), ...$ of the first type. Since $u_2 \leftrightarrow w_2$ and u_2 is adjacent to u_1, then u_1 is adjacent to w_2. If $u_1 \leftrightarrow w_2$, then u_1 is adjacent to u_3 because w_2 is adjacent to u_3. If u_1 is adjacent to u_3 by a single arc, then (u_1, u_3, w_2) is a witness triple of the first type and there is a shorter sequence $u_1, u_3, ..., u_r$, a contradiction. Secause $u_1 \in X$, if u_1 is adjacent to u_3 by symmetric arcs, then $u_3 \leftrightarrow w_2$, a contradiction to the fact. Suppose that u_1 and w_2 are connected by a single arc, then (u_2, u_1, w_2) is a witness triple of the first type and we have a sequence u_1, u_2 of length two, a contradiction.

Therefore, D has a semi-strict di-simplicial vertex. □

Chapter 4

Conclusion and Future Study

We have studied in this thesis chordal digraphs and semi-strict chordal digraphs, within several well-structured digraph classes (e.g., quasi-transitive, extended semi-complete and locally semi-complete digraphs). We have given a forbidden subdigraph characterization of them for each of these digraph classes. It remains an open problem to characterize the general chordal digraphs and semi-strict chordal digraphs in terms of forbidden subdigraphs. The list of forbidden subdigraphs for either chordal digraphs or semi-strict chordal digraphs appears to be much larger than the union of the lists which we have given in the previous chapters. Finding the complete list of forbidden subdigraphs seems challenging.

The notion of knotting graphs is an attempt to study the general structure of chordal digraphs. A similar notion was used by Gallai [8] for the study of comparability graphs. Gallai [8] has found a characterization of comparability graphs in terms of absent of certain structures in the knotting graphs. It is possible that chordal digraphs can also be characterized in this way.

At least for now we know that if none of the group in a knotting graph has all vertices of degree at most one, then the corresponding digraph is not chordal. In the future, we will work on the other directions such as what is the obstruction for a knotting graph so that the corresponding digraph is not chordal. We conjecture that it may be a circuit or a closed walk.

Chordal digraphs can be recognized in $O(n^2m)$ time [18]. This can be done by recursively finding a di-simplicial vertex if one exists. A similar approach shows that semi-strict chordal digraphs can also be recognized in time $O(n^2m)$. It remains an open problem for devising linear time recognition algorithms for chordal digraphs and for semi-strict chordal digraphs.

Bibliography

[1] J. Bang-Jensen and G. Gutin, **Classes of Directed Graphs**, Springer Monographs in Mathematics. ISSN 1439-7382 (2018).

[2] J. Bang-Jensen and J. Huang, *Quasitransitive digraphs*, Graph Theory 20 (1995) 141-161.

[3] S. Benzer, *On the topology of the genetic fine structure*, Proc. Natl. Acad. Sci, USA 45 (1959) 1607-1620.

[4] J.E. Cohen, *Food Webs and Niche Space*, Princeton University Press, Princeton (1978)

[5] S. Das, M. Sen, A.B. Roy and D.B. West, *Interval digraphs: an analogue of interval graphs*, Graph Theory 13 (1989) 189-202.

[6] G.A. Dirac, *On rigid graphs*, Abh. Math. Sem. Univ. Hamburg 25 (1961) 71-76.

[7] T. Feder, P. Hell, J. Huang, and A. Rafiey, *Interval graphs, adjusted interval graphs and reflexive list homomorphisms*, Discrete Appl. Math 160 (2012) 697-707.

[8] T. Gallai, Transitiv orientierbare graphen, Acta Math. Acad. Sci. Hungar 18 (1967) 25-66.

[9] M.C. Golumbic, **Algorithmic Graph Theory and Perfect Graphs**, Academic Press, New York (1980).

[10] A. Hajnal and J. Surányi, *Über die Auflösung von Graphen in vollständige Teilgraphen*, Ann. Univ. Sci. Budapest Eötvös, Sect. Math. 1 (1958) 113-121.

[11] F. Harary, J.A. Kabell and F.R. McMorris, *Subtree acyclic digraphs*, Ars Combin. 34 (1992) 93-95.

[12] L. Haskins and D.J. Rose, *Toward characterization of perfect elimination digraphs*, SIAM J. Comput. 2 (1973) 217-224.

[13] P. Hell and C. Hernández-Cruz, *Strict chordal and strict split digraphs*, Discrete Appl. Math. 216 (2017) 609-617.

[14] D.J. Kleitman, *A note on perfect elimination digraphs*, SIAM J. Comput. 3 (1974) 280-282.

[15] F.R. McMorris and H.M. Mulder, *Subpath acyclic digraphs*, Discrete Math. 154 (1996) 189-201.

[16] D. Meister and J.A. Telle, *Chordal digraphs*, Theoret. Comput. Sci. 463 (2012) 73-83.

[17] D.J. Rose, *Triangulated graphs and the elimination process*, Journal of Mathematical Analysis and Applications 32 (1970) 597-609.

[18] D.J. Rose and R.E. Tarjan, *Algorithmic aspects of vertex elimination on directed graphs*, SIAM J. Comput. 34 (1978) 176-197.

[19] D.J. Rose, R.E. Tarjan and G.S. Lueker, *Algorithmic aspects of vertex elimination in graphs*, SIAM J. Comput. 5 (1976) 266-283.