SOME FAMILIES OF GENERATING FUNCTIONS FOR THE JACOBI POLYNOMIALS

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Abstract

The authors derive a unification and generalization of several families of linear, bilinear, and bilateral generating functions for the classical Jacobi polynomials, which were given in a number of earlier works on the subject. They also show how the results considered here can be extended further to hold true for the product of two Jacobi polynomials of different arguments. Relevant historical remarks and observations, and connections with other works on applications of some of these generating functions, are presented rather briefly.

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1. Introduction

For the classical Jacobi polynomials (*cf.*, *e.g.*, Szegö [27, p. 68])

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \tag{1}$$

Jacobi's generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n = 2^{\alpha+\beta} R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta} \tag{2}$$
$$\left(R := (1-2xt+t^2)^{\frac{1}{2}}\right)$$

is well-known. (For several interesting proofs of the classical result (2), the interested reader is referred (among other places) to Szegö [27, Section 4.4], Rainville [15, Section 140], Carlitz [5], Askey [2], Foata and Leroux [9], Srivastava [18], and Parnes and Ekhad [13].) Another remarkable generating function for the Jacobi polynomials, which is due essentially to Feldheim [7], has the *elegant* form:

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\,\beta-n)}(x)\, t^n = \left\{1+\tfrac{1}{2}(x+1)t\right\}^{\alpha}\left\{1+\tfrac{1}{2}(x-1)t\right\}^{\beta}, \tag{3}$$

which does indeed follow easily from the definition (1).

Over a decade ago, Gupta [10] gave three *bilateral* generating functions for the Jacobi polynomials, each of which involved the Gaussian hypergeometric ${}_2F_1$ function. In terms of the Lauricella triple hypergeometric series F_4, F_8, and F_7 (which, in the notations used by Saran [16], are F_E, F_G, and F_S, respectively) (see, *e.g.*, Srivastava and Karlsson [21, pp. 42-43]), these *main* results of Gupta's paper [10] may be recalled here as follows:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(x)\ {}_2F_1(\lambda+n, a; b; y)\, t^n$$
$$= F_E\left[\lambda,\lambda,\lambda,a,\mu,\mu;\ b,\ \alpha+1,\ \beta+1;\ y,\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t\right]; \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha-n,\,\beta-n)}(x)\ {}_2F_1(\lambda+n, a; b; y)\, t^n$$
$$= F_G\left[\lambda,\lambda,\lambda,a,-\alpha,-\beta;\ b,\mu,\mu;\ y,\ -\tfrac{1}{2}(x+1)t,\ -\tfrac{1}{2}(x-1)t\right]; \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha-n,\beta-n)}(x)\ {}_2F_1(a,b;\ \mu+n;\ y)\, t^n$$
$$= F_S\left[a,\lambda,\lambda,b,-\alpha,-\beta;\ \mu,\mu,\mu;\ y,\ -\tfrac{1}{2}(x+1)t,\ -\tfrac{1}{2}(x-1)t\right], \tag{6}$$

where, and throughout the present paper, $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\cdots\}). \end{cases}$$

Much more general results than the bilateral generating functions (4), (5), and (6) were given earlier by (i) Srivastava and Daoust [19] who derived a multivariable bilateral generating function involving a general class of polynomials and the (Srivastava-Daoust) generalized Lauricella functions; (ii) Panda [12] who provided a further extension of the aforementioned work of Srivastava and Daoust [19], and (iii) Rahman [14] who considered generating functions for the Jacobi polynomials involving an essentially arbitrary sequence of complex numbers. The results of each of these papers were reproduced subsequently in a treatise on the subject of generating functions by Srivastava and Manocha [22, Chapter 2, Problems 57, 59, and 14 (ii)]. The main object of this paper is to present some unifications and generalizations of the generating functions considered in each of these (and many other) earlier works.

2. Bilateral Generating Functions

The generating functions (1) and (2), and their numerous generalizations and variations considered in the mathematical literature, were unified and extended by Srivastava and Singhal [24] in the form:

$$\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\,\beta+\mu n)}(x)\, t^n$$
$$= (1+\xi)^{\alpha+1}(1+\eta)^{\beta+1}\left[1-\lambda\xi-\mu\eta-(\lambda+\mu+1)\xi\eta\right]^{-1}, \tag{7}$$

where ξ and η satisfy the equations:

$$(x+1)^{-1}\xi = (x-1)^{-1}\eta = \frac{1}{2}t(1+\xi)^{\lambda+1}(1+\theta)^{\mu+1}. \tag{8}$$

It may be of interest to remark in passing that the Srivastava-Singhal generating function (7) was applied recently by Chen and Ismail [6] in order to determine the asymptotic behaviour of the Jacobi polynomials:

$$P_n^{(\alpha+\lambda n,\, \beta+\mu n)}(x)$$

when $n \to \infty$, and α, β, λ, μ, and x remain fixed. Strehl [26], on the other hand, has presented an interesting combinatorial proof of the Srivastava-Singhal result (7).

For a suitably bounded sequence $\{\Omega_n\}_{n=0}^{\infty}$ of essentially arbitrary complex numbers, we shall prove the general result:

$$\sum_{n=0}^{\infty} \Omega_n\, P_n^{(\alpha+\lambda n,\, \beta+\mu n)}(x)\, t^n$$
$$= \sum_{\ell,m=0}^{\infty} \Omega_{\ell+m} \frac{(\alpha+1)_{(\lambda+1)(\ell+m)}\, (\beta+1)_{(\mu+1)(\ell+m)}}{(\alpha+1)_{\lambda\ell+(\lambda+1)m}\, (\beta+1)_{(\mu+1)\ell+\mu m}} \tag{9}$$
$$\cdot \frac{\left\{\frac{1}{2}(x+1)t\right\}^{\ell}}{\ell!} \frac{\left\{\frac{1}{2}(x-1)t\right\}^{m}}{m!},$$

provided that each member of (9) exists.

Proof. Denoting, for convenience, the first member of the generating function (9) by Δ, and applying the definition (1), we have

$$\Delta := \sum_{n=0}^{\infty} \Omega_n\, P_n^{(\alpha+\lambda n,\, \beta+\mu n)}(x)\, t^n$$
$$= \sum_{n=0}^{\infty} \Omega_n\, t^n \sum_{k=0}^{n} \binom{\alpha+(\lambda+1)n}{n-k} \binom{\beta+(\mu+1)n}{k}$$
$$\cdot \left(\frac{x-1}{2}\right)^{k} \left(\frac{x+1}{2}\right)^{n-k},$$

which, upon inversion of the order of summation, yields

$$\Delta = \sum_{k=0}^{\infty} \left(\frac{x-1}{x+1}\right)^k \sum_{n=k}^{\infty} \Omega_n \binom{\alpha+(\lambda+1)n}{n-k} \binom{\beta+(\mu+1)n}{k} \left\{\frac{1}{2}(x+1)t\right\}^n$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \Omega_{n+k} \binom{\alpha+(\lambda+1)(n+k)}{n} \binom{\beta+(\mu+1)(n+k)}{k}$$

$$\cdot \left\{\frac{1}{2}(x+1)t\right\}^n \left\{\frac{1}{2}(x-1)t\right\}^k,$$

which would lead us at once to the second member of (9) if we express each of the combinatorial coefficients in terms of the Pochhammer symbols.

This evidently completes the proof of (9) under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, the result (9) holds true (at least as a relation between *formal* power series) for those values of the various parameters and variables involved for which each member of the assertion (9) exists.

For $\lambda = \mu = 0$, (9) immediately yields (*cf.* Rahman [14]; see also Srivastava and Manocha [22, p. 148, Problem 14 (ii)])

$$\sum_{n=0}^{\infty} \Omega_n \, P_n^{(\alpha,\beta)}(x)\, t^n$$
$$= \sum_{\ell,m=0}^{\infty} \Omega_{\ell+m} \frac{(\alpha+1)_{\ell+m}(\beta+1)_{\ell+m}}{(\alpha+1)_\ell(\beta+1)_m} \frac{\left\{\frac{1}{2}(x-1)\,t\right\}^\ell}{\ell!} \frac{\left\{\frac{1}{2}(x+1)\,t\right\}^m}{m!}. \tag{10}$$

Gupta's bilateral generating function (4) is an obvious *further* special case of the known result (10) when

$$\Omega_n = \frac{(\lambda)_n(\mu)_n}{(\alpha+1)_n(\beta+1)_n} \; {}_2F_1(\lambda+n,\, a;\, b;\, y)$$
$$\left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right).$$

For $\lambda = \mu = -1$, the generating function (9) reduces readily to the following (presumably new) form:

$$\sum_{n=0}^{\infty} \Omega_n \, P_n^{(\alpha-n,\,\beta-n)}(x)\, t^n$$

$$= \sum_{\ell,m=0}^{\infty} \Omega_{\ell+m}(-\alpha)_\ell\,(-\beta)_m \,\frac{\left\{-\frac{1}{2}(x+1)\,t\right\}^\ell}{\ell!}\;\frac{\left\{-\frac{1}{2}(x-1)\,t\right\}^m}{m!}. \tag{11}$$

Gupta's bilateral generating functions (5) and (6) would follow easily as *further* special cases of (11) when we set

$$\Omega_n = \frac{(\lambda)_n}{(\mu)_n}\;{}_2F_1(\lambda+n,\,a;\,b;\,y) \qquad (n \in \mathbb{N}_0)$$

and

$$\Omega_n = \frac{(\lambda)_n}{(\mu)_n}\;{}_2F_1(a,\,b;\,\mu+n;\,y) \qquad (n \in \mathbb{N}_0),$$

respectively, and identify the resulting triple series on the right-hand sides as known hypergeometric functions of the three variables

$$y, \qquad -\frac{1}{2}(x+1)\,t, \qquad -\frac{1}{2}(x-1)\,t.$$

Both (10) and (11), with Ω_n expressed in terms of a (Srivastava-Daoust) generalized hypergeometric function of s complex variables

$$z_1, \cdots, z_s,$$

were actually deduced by Srivastava and Daoust [19] in 1971 from a *single* bilateral generating function for a general class of polynomials (see also Srivastava and Manocha [22, p. 191, Problems 57 (ii) and 57 (iii)]). Thus, as we remarked in the preceding section, the bilateral generating functions (4), (5), and (6) are very specialized versions of the case $s = 1$ of the aforementioned results of Srivastava and Daoust [19].

Next, setting

$$\Omega_n = \frac{\prod_{j=1}^{r} (c_j)_{n\rho_j}}{\prod_{j=1}^{s} (d_j)_{n\sigma_j}}$$

$$(\rho_j > 0 \;\; (j = 1, \cdots, r); \;\; \sigma_j > 0 \;\; (j = 1, \cdots, s); \;\; n \in \mathbb{N}_0)$$

and interpreting the resulting right-hand side as a (Srivastava-Daoust) generalized Kampé de Fériet function (*cf.*, *e.g.*, Srivastava and Karlsson [21, p. 37]), we find from (9) that

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r} (c_j)_{n\rho_j}}{\prod_{j=1}^{s} (d_j)_{n\sigma_j}} P_n^{(\alpha+\lambda n,\, \beta+\mu n)}(x)\, t^n$$

$$= F^{r+2:0;0}_{s+2:0;0}\left(\begin{matrix} [c_1:\rho_1],\cdots,[c_r:\rho_r], & [\alpha+1:\lambda+1,\ \lambda+1], \\ [d_1:\sigma_1],\cdots,[d_s:\sigma_s], & [\alpha+1:\lambda,\ \lambda+1], \end{matrix}\right. \tag{12}$$

$$\left.\begin{matrix} [\beta+1:\mu+1,\ \mu+1]:—;—; \\ [\beta+1:\mu+1,\ \mu]:—;—; \end{matrix}\ \frac{1}{2}(x+1)\,t,\ \frac{1}{2}(x-1)\,t\right).$$

In particular, the generating functions (10) and (11) when

$$\Omega_n = \frac{\prod_{j=1}^{r} (c_j)_n}{\prod_{j=1}^{s} (d_j)_n} \qquad (n \in \mathbb{N}_0) \tag{13}$$

would reduce to the following elegant forms:

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r} (c_j)_n}{\prod_{j=1}^{s} (d_j)_n} P_n^{(\alpha,\beta)}(x)\, t^n$$

$$= F^{r+2:0;0}_{s:1;1}\left[\begin{matrix} \alpha+1,\ \beta+1, c_1,\cdots,c_r: & —;\ —; \\ d_1,\cdots,d_s: & \alpha+1;\ \beta+1; \end{matrix}\ \frac{1}{2}(x-1)t,\ \frac{1}{2}(x+t)t\right]; \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r} (c_j)_n}{\prod_{j=1}^{s} (d_j)_n} P_n^{(\alpha-n,\, \beta-n)}(x)\, t^n$$

$$= F^{r:1;1}_{s:0;0}\left[\begin{matrix} c_1,\cdots,c_r: -\alpha;\ -\beta; \\ d_1,\cdots,d_s: —;\ —; \end{matrix}\ -\frac{1}{2}(x+1)t,\ -\frac{1}{2}(x-1)t\right]. \tag{15}$$

Both (14) and (15), and their numerous special or confluent cases, are fairly well known (see, for example, Srivastava and Manocha [22, Chapter 2, Equation 2.6 (31) and

Problem 12] where references to relevant earlier works are also available). In fact, (14) was deduced by Srivastava and Daoust [20] from a generalized hypergeometric identity which they proved. [See also a recent work of Srivastava *et al.* [23] for numerous applications and extensions of (14).]

3. Bilinear Extensions

Making use of Bateman's *product formula* (*cf.* [4, p. 123]):

$$P_n^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) = \left(\frac{2}{x+y}\right)^n \cdot \sum_{k=0}^{n} \frac{(\alpha+\beta+2k+1)k!\,\Gamma(\alpha+\beta+k+1)}{(n-k)!\,\Gamma(\alpha+\beta+n+k+2)} \cdot \frac{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{\Gamma(\alpha+k+1)\,\Gamma(\beta+k+1)}\; P_k^{(\alpha,\beta)}(x)\; P_k^{(\alpha,\beta)}(y), \tag{16}$$

it is not difficult to prove the following bilinear extension of the generation function (9):

$$\sum_{\ell,m=0}^{\infty} \Omega_{\ell+m} \frac{(\alpha+1)_{(\lambda+1)(\ell+m)}\,(\beta+1)_{(\mu+1)(\ell+m)}}{(\alpha+1)_{\lambda\ell+(\lambda+1)m}\,(\beta+1)_{(\mu+1)\ell+\mu m}} \cdot \frac{\left\{\frac{1}{4}(1+x)(1+y)\,t\right\}^{\ell}}{\ell!}\,\frac{\left\{\frac{1}{4}(1-x)(1-y)\,t\right\}^{m}}{m!}$$
$$= \sum_{n=0}^{\infty} \frac{n!}{(\alpha+1)_n\,(\beta+1)_n} \sum_{k=0}^{\infty} \Omega_{n+k}\frac{t^{n+k}}{k!} \cdot \frac{(\alpha+1)_{(\lambda+1)(n+k)}\,(\beta+1)_{(\mu+1)(n+k)}\,\big(\alpha+\beta+(\lambda+\mu)(n+k)+1\big)_n}{\big(\alpha+\beta+(\lambda+\mu)(n+k)+1\big)_{2n}\,\big(\alpha+\beta+(\lambda+\mu)(n+k)+2n+2\big)_k} \cdot P_n^{(\alpha+\lambda(n+k),\,\beta+\mu(n+k))}(x)\; P_n^{(\alpha+\lambda(n+k),\,\beta+\mu(n+k))}(y). \tag{17}$$

Corresponding to the generating function (10), the special case $\lambda = \mu = 0$ of (17) immediately yields the bilinear formula:

$$\sum_{\ell,m=0}^{\infty} \Omega_{\ell+m} \frac{(\alpha+1)_{\ell+m}\,(\beta+1)_{\ell+m}}{(\alpha+1)_\ell\,(\beta+1)_m} \cdot \frac{\left\{\frac{1}{4}(1-x)(1-y)\,t\right\}^\ell}{\ell!} \frac{\left\{\frac{1}{4}(1+x)(1+y)\,t\right\}^m}{m!}$$

$$= \sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n\,(\beta+1)_n\,(\alpha+\beta+1)_{2n}} P_n^{(\alpha,\beta)}(x)\; P_n^{(\alpha,\beta)}(y) \cdot \sum_{k=0}^{\infty} \Omega_{n+k} \frac{(\alpha+1)_{n+k}\,(\beta+1)_{n+k}}{(\alpha+\beta+2n+2)_k} \frac{t^{n+k}}{k!}, \tag{18}$$

or, equivalently,

$$\sum_{\ell,m=0}^{\infty} \frac{\Omega_{\ell+m}}{(\alpha+1)_\ell\,(\beta+1)_m} \frac{\left\{\frac{1}{4}(1-x)(1-y)\,t\right\}^\ell}{\ell!} \frac{\left\{\frac{1}{4}(1+x)(1+y)\,t\right\}^m}{m!}$$

$$= \sum_{n=0}^{\infty} \frac{n!\,(\alpha+\beta+1)_n}{(\alpha+1)_n\,(\beta+1)_n\,(\alpha+\beta+1)_{2n}} P_n^{(\alpha,\beta)}(x)\; P_n^{(\alpha,\beta)}(y) \cdot \sum_{k=0}^{\infty} \frac{\Omega_{n+k}}{(\alpha+\beta+2n+2)_k} \frac{t^{n+k}}{k!}, \tag{19}$$

which can indeed be proven *directly* from (10) by appealing to Bateman's product formula (16).

Similarly, the special case $\lambda = \mu = -1$ of (17) would correspond to the following bilinear extension of the generating function (11):

$$\sum_{\ell,m=0}^{\infty} \Omega_{\ell+m}\,(-\alpha)_\ell\,(-\beta)_m \frac{\left\{\frac{1}{4}(1+x)(1+y)\,t\right\}^\ell}{\ell!} \frac{\left\{\frac{1}{4}(1-x)(1-y)\,t\right\}^m}{m!}$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(\alpha+1)_n\,(\beta+1)_n} \sum_{k=0}^{\infty} \frac{\Omega_{n+k}}{(2k-\alpha-\beta)_n\,(k-\alpha-\beta-1)_k} \frac{t^{n+k}}{k!} \cdot P_n^{(\alpha-n-k,\,\beta-n-k)}(x)\; P_n^{(\alpha-n-k,\,\beta-n-k)}(y), \tag{20}$$

which can also be proven *directly* from (11) by applying Bateman's product formula (16).

The bilinear generating function (19) was proven, *without* using Bateman's product formula (16), by Rahman [14]. Its special case when the coefficients Ω_n are given by (13) may be recalled here in the form (*cf.* Rahman [14], Khan [11], and Srivastava and Manocha [22, p. 167, Problem 14]):

$$F^{r:0;0}_{s:1;1}\left[\begin{matrix} c_1,\cdots,c_r : \text{———};\ \text{———}; \\ d_1,\cdots,d_s : \alpha+1;\ \beta+1; \end{matrix}\ \frac{1}{4}(1-x)(1-y)\,t,\ \frac{1}{4}(1+x)(1+y)\,t\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!\,(\alpha+\beta+1)_n \prod_{j=1}^{r}(c_j)_n}{(\alpha+1)_n\,(\beta+1)_n\,(\alpha+\beta+1)_{2n} \prod_{j=1}^{s}(d_j)_n} \tag{21}$$

$$\cdot\, P_n^{(\alpha,\beta)}(x)\; P_n^{(\alpha,\beta)}(y)$$

$$\cdot\ {}_rF_{s+1}\left[\begin{matrix} c_1+n,\cdots,c_r+n; \\ \alpha+\beta+2n+2,\ d_1+n,\cdots,d_s+n; \end{matrix}\ t\right] t^n,$$

whose *further* special case when

$$r=2, \qquad s=0, \qquad c_1=\gamma, \quad \text{and} \quad c_2=\delta$$

was proven by Feldheim [8] (and, subsequently, by Al-Salam [1]). Indeed, upon *further* setting

$$\gamma = \frac{1}{2}(\alpha+\beta+1) \quad \text{and} \quad \delta = \frac{1}{2}(\alpha+\beta+2),$$

Feldheim's formula would reduce readily to Bailey's bilinear generating function for the Jacobi polynomials (*cf.*, *e.g.*, Srivastava and Manocha [22, p. 167, Problem 13]; see also Bailey [3], Stanton [25], and Srivastava [17]).

The bilinear generating function (20) does not seem to have appeared elsewhere. However, in terms of hypergeometric functions, its special case when the coefficients Ω_n are given by (13) would not yield as elegant a result as the familiar bilinear generating function (21).

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