CIRCLE PROBABILITIES

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(American Mathematical Monthly Problem E3401)

E3401. *Proposed by James A. Davis, Michael Kerckhove, and J. Van Bowen, University of Richmond, VA.*

Suppose n points are independently chosen at random on the perimeter of a circle. What is the probability that all the points lie in some semicircle?

Solution by Bruce R. Johnson, University of Victoria, Victoria, B.C., Canada. More generally, we will show that the n points chosen independently at random on the perimeter of a circle of radius r will all lie in some arc of length $p\pi r$ with probability $n(p/2)^{n-1}$, where p is a fixed parameter such that $0 < p \leq 1$. The answer to the proposed problem is given by the special case $p = 1$.

With probability one no two of the n randomly chosen points will coincide; so these points will partition the perimeter of the circle into n arcs with the chosen points as endpoints of the arcs. We distinguish the n points by labeling them from 1 to n arbitrarily, and for each $j \in \{1,2,\cdots,n\}$ we let the arc extending from point j counterclockwise to the next chosen point on the perimeter be called arc j. Since positive parameter p is no larger than 1, at most one of the n arcs will be longer than $(2-p)\pi r$; so the events $A_1,A_2,\cdots,A_n$ are mutually exclusive, where A_j denotes the event that arc j is longer than $(2-p)\pi r$. Hence, by the additive property of probability

$$\begin{aligned} P(\text{the } n \text{ chosen points lie in some arc of length } p\pi r) &= P(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= \sum_{j=1}^{n} P(A_j). \end{aligned}$$

Also, for each j the event A_j will occur if and only if each of the $n-1$

distances measured from point j counterclockwise around the perimeter to each of the other $n-1$ chosen points is between $(2-p)\pi r$ and $2\pi r$. Since these $n-1$ distances are distributed independently and uniformly over the interval $(0,2\pi r)$, it follows that

$$P(A_j) = (p/2)^{n-1}.$$

Therefore,

$$P(\text{the } n \text{ chosen points lie in some arc of length } p\pi r) = n(p/2)^{n-1}.$$