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Abstract

A dominating function for a graph is a function from its vertex set into the unit interval so that the sum of function values taken over the closed neighbourhood of each vertex is at least one. We prove that any graph has a positive minimal dominating function and begin an investigation of the question: When are convex combinations of minimal dominating functions themselves minimal dominating?

1. Introduction.

A *dominating function* (DF) of a graph $G = (V,E)$ is a function $f : V \longrightarrow [0,1]$ such that for each vertex $v \in V$, $\sum_{u \in N[v]} f(u) \geq 1$, (where $N[v]$ denotes the closed neighbourhood of the vertex v). This concept extends the idea of a dominating set of a graph. If each value $f(u)$ of such a function is integral (*i.e.* $f(u) \in \{0,1\}$), then $S = \{v \mid f(v) = 1\}$ is a dominating set of G, *i.e.* every vertex w in $V - S$ is adjacent to at least one vertex of S.

Dominating functions also arise as solutions to the linear programming relaxation of the following 0–1 integer program (P):

$$\text{Min} \sum_{i=1}^{n} x_i$$

$$\text{s.t.} \qquad Nx \geq 1$$

$$x_i \in \{0,1\}, \qquad x = (x_1,\cdots,x_n).$$

Here, N denotes a neighbourhood matrix of the n–vertex graph G, *i.e.* $N = A + I$ where A is an adjacency matrix of G and I is the $n \times n$ identity matrix. The question of when a solution to the linear programming relaxation of (P), where $x_i \geq 0$, yields an optimal solution to the 0–1 integer program (P) defined above, was first studied by Farber in 1984 [5]. However, the first definition and study of the fractional dominating number $\gamma_f(G)$, (*i.e.* the minimum value of $\sum_{i=1}^{n} x_i$ in a solution x to the linear programming relaxation) was done by Hedetniemi, Hedetniemi and Wimer in 1987 [14]. Since then a considerable amount of work has been done on dominating functions and fractional domination numbers of graphs (*cf.* [1–4, 6–13, 15]).

In this paper there are two principal topics. In Section 2 we prove, with the help of a theorem on independent sets, that every graph has a minimal dominating function (MDF) all of whose values are nonzero and in the following section we begin an investigation of the following question (first raised by A. Majumdar [16]): When are convex combinations of MDFs themselves MDFs?

2. Existence of Positive Minimal Dominating Functions.

For DFs f, g of G, we write $f \leq g$ if for all $v \in V$, $f(v) \leq g(v)$. Further we write $f < g$ if $f \leq g$ and for some $v \in V$, $f(v) < g(v)$. A DF g of G is *minimal* if for all functions $f : V \longrightarrow [0,1]$ such that $f < g$, f is not dominating. For a DF f, we define the *boundary of* f, denoted by B_f, to equal $\left\{v \mid \sum_{u\in N[v]} f(u) = 1\right\}$ and the *positive set of* f, denoted by P_f, to equal $\{v \mid f(v) > 0\}$. A subset V_1 of V is said to *dominate* $V_2 \subseteq V$ if each $v \in V_2$ is in V_1 or adjacent to some vertex of V_1.

Proposition 1. (Fricke) *A* DF f *of* G *is an* MDF *if and only if* B_f *dominates* P_f.

The main result of this section asserts the existence of an MDF whose function values are positive for any graph G. We will need a preliminary result concerning independent sets. Let S be any vertex subset of G. For $s \in S$ define

$$Z_s = \{z \in V{-}S \mid N(z) \cap S = \{s\}\},$$

$$Y_s = \Big\{y \in Z_s \mid y \text{ is adjacent to no vertex of } \bigcup_{\substack{s' \in S \\ s' \neq s}} Z_{s'}\Big\},$$

$$X_s = \{x \in Y_s \mid \deg(x) < \deg(s)\}$$

and

$$T = \{t \in V{-}S \mid |N(t) \cap S| \geq 2\}.$$

Note that $\bigcup_{s \in S} Z_s = V - (S \cup T)$.

THEOREM 2. *Any graph* G *has a maximum independent set* S *with* $\bigcup_{s \in S} X_s = \emptyset$.

Proof. Suppose the theorem is false. Let S be a maximum independent set so that $|\bigcup_{s \in S} X_s|$ is minimum. We note that the subgraph induced by each Z_s is complete. (If z_1 were not adjacent to z_2 in Z_s, then $(S{-}\{s\}) \cup \{z_1, z_2\}$ is a larger independent set contrary to the definition of S.) For $u \in S$ with $X_u \neq \emptyset$, let $x \in X_u$ and form the new maximum independent set $S^* = (S \cup \{x\}) - \{u\}$. Form T^*, Z^*_s, Y^*_s and X^*_s from S^* as above. It will be shown that $C^* = \bigcup_{s \in S^*} X^*_s$ has fewer elements than $C = \bigcup_{s \in S} X_s$, a contradiction.

We first observe that $S^* \cap C^* = \emptyset$. Next we show that $T \cap C^* = \emptyset$. Consider a vertex $t \in T$. Then $t \in V - S^*$ and either $t \in T^*$ (and hence $t \notin C^*$) on t is adjacent to exactly one vertex, say v, in $S - \{u\}$ and t is adjacent to u. In the latter case, if t is adjacent to x, then $t \in T^*$ (and hence not in C^*). Thus, it remains to consider the case in which $N(t) \cap S^* = \{v\}$ *i.e.* $t \in Z_v^*$. But t is adjacent to $u \in Z_x^*$. Therefore by definition of Y_v^*, $t \notin Y_v^*$ and hence $t \notin X_v^*$. It follows that $T \cap C^* = \emptyset$.

We now show $(Z_s - Y_s) \cap C^* = \emptyset$ for all $s \in S - \{u\}$. Let $z \in Z_s - Y_s$ where $s \neq u$. This implies $z \in Z_s^*$ and z is adjacent to a vertex, say w, of $Z_{s'}$ where $s \neq s' \in S$.

Case (i). If $s' \neq u$, then z is adjacent to a vertex, in fact w, of $Z_{s'}^*$, and hence $z \notin X_s^*$ *i.e.* $z \notin C^*$.

Case (ii). If $s' = u$, then z is adjacent to say $z_u \in Z_u$. But (as noted above) Z_u induces a complete graph, hence z_u is adjacent to x. Further z_u is not adjacent to any other vertex of S^*. We conclude $z_u \in Z_x^*$ and z is adjacent to vertex $z_u \in Z_x^*$. Therefore $z \notin X_s^*$ which implies $z \notin C^*$ and we have proved $(Z_s - Y_s) \cap C^* = \emptyset$ for all $s \in S - \{u\}$.

Consider $z \in Z_u - Y_u$. Then $z \in Z_x^*$ (since Z_u is complete) and is adjacent to a vertex, say w, of $Z_{s'}$ where $u \neq s' \in S$. Since $w \in Z_{s'}^*$, $z \notin X_x^*$ and therefore $(Z_u - Y_u) \cap C^* = \emptyset$.

Consider $y \in Y_s - X_s$ where $s \neq u$. The degree condition $\deg(s) \leq \deg(y)$ which forces $y \notin X_s$, also implies $y \notin X_s^*$. Thus $(Y_s - X_s) \cap C^* = \emptyset$ for all $s \neq u$.

Finally, suppose $y \in Y_u - X_u$, then

$$\deg(y) \geq \deg(u). \tag{1}$$

If $y \in C^*$, then $y \in X_x^*$ and hence

$$\deg(y) < \deg(x). \tag{2}$$

But $x \in X_u$ and so

$$\deg(x) < \deg(u). \tag{3}$$

From (2) and (3) we deduce $\deg(y) < \deg(u)$ contrary to (1). Therefore $(Y_u - X_u) \cap C^* = \emptyset$.

It has been shown that C^* does not intersect

$$\bigcup_{s \in S} [(Y_s - X_s) \cup (Z_s - Y_s)] \cup T \cup S^*.$$

Therefore $C^* \subseteq (C - \{x\}) \cup \{u\}$. But by (3), $u \notin C^*$ and we conclude $|C^*| < |C|$, the required contradiction. ∎

Theorem 3. *Any graph* G *has an* MDF *satisfying* $P_f = V$.

Proof. Let S be a maximum independent set of G which satisfies Theorem 2. We use notation as above. If $|V| = n$ define

$$f(z) = \frac{1}{n^2} \quad \text{for} \quad z \in \bigcup_{s \in S} Z_s ,$$

$$f(t) = \frac{1}{n^4} \quad \text{for} \quad t \in T$$

and for each $s \in S$ define $f(s)$ such that $\sum_{u \in N[s]} f(u) = 1$

We show that f is an MDF with $P_f = V$. If $s \in S$, then $f(s) \le 1$ and $\sum_{u \in N(s)} \le \frac{n-1}{n^2}$. Therefore

$$1 = f(s) + \sum_{u \in N(s)} f(u) \le f(s) + \frac{n-1}{n^2}\,, \tag{4}$$

which implies $f(s) > 0$. Thus f is a function from $V \longrightarrow [0,1]$ such that $P_f = V$. It remains to show f is dominating.

Suppose that $z \in Z_s - Y_s$ for some $s \in S$ and s joins m vertices of T. Recall that $Z_s \cup \{s\}$ induces a complete subgraph and z is adjacent to at least one vertex of $\bigcup_{\substack{s' \in S \\ s' \neq s}} Z_{s'}$. Therefore

$$\sum_{u \in N[z]} f(u) \ge \sum_{u \in N[s]} f(u) - \frac{m}{n^4} + \frac{1}{n^2}$$

$$= 1 + \frac{1}{n^2}\left[1 - \frac{m}{n^2}\right] > 1.$$

Next let $y \in Y_s$ for some $s \in S$ where s is adjacent to m vertices of T and y is adjacent to m' vertices of T. Since $X_s = \emptyset$, $\deg(y) \ge \deg(s)$ and this implies $m' \ge m$. Therefore

$$\sum_{u \in N[y]} f(u) = \sum_{u \in N[s]} f(u) - \frac{m}{n^4} + \frac{m'}{n^4}$$

$$= 1 + \frac{m' - m}{n^4} > 1.$$

Finally, if $t \in T$, t is adjacent to at least two vertices say s_1, s_2 of S and

$$\sum_{u \in N[t]} f(u) \geq f(s_1) + f(s_2) + f(t)$$

$$\geq 2\left[1 - \left[\frac{n-1}{n^2}\right]\right] + \frac{1}{n^4} \quad \text{(by (4))}$$

$$= 2 - \frac{2}{n} + \left[\frac{2}{n^2} + \frac{1}{n^4}\right] > 1.$$

Therefore f is a dominating function and since $B_f = S$ dominates $P_f = V$, f is a MDF. ∎

We observe that the maximum property of S is not essential for the assignment of function values performed in Theorem 3, which constructed a positive MDF. The same construction may be performed provided that S satisfies the following conditions:

(i) S is maximal independent,

(ii) Z_s induces a complete subgraph for each $s \in S$

and (iii) $\bigcup_{s \in S} X_s = \emptyset$.

A set S with these three properties may be constructed with the following polynomial time algorithm. Hence for any graph, a positive MDF may be found in polynomial time.

Algorithm PositiveMDFSet

Input: graph G=(V,E)

Output: a set S subset of V satisfying conditions (i),(ii),(iii).

```
1.  Let  S  be any maximal independent set in G.
2.  changed ← true
3.  while changed do
4.    if NOT(Zs induces a complete subgraph for each s in S) then
5.      let <Zs> be a graph which is not complete
6.      let u,v be two nonadjacent vertices in Zs
7.      let S′ ← S − {s} ∪ {u,v}
8.      let S be any maximal independent set containing S′
9.    else if ∪Xs ≠ ∅ then
10.     let s be any vertex for which Xs ≠ ∅
11.     let u ∈ Xs
12.     let S′ ← S − {s} ∪ {u}
13.     let S be any maximal independent set containing S′
14.        else changed ← false
15.        od
16. end PositiveMDFSet
```

It is clear from the proof of Theorem 2 that Algorithm PositiveMDFSet constructs a set S satisfying conditions (i), (ii) and (iii). Each iteration of the while-loop (lines 3–14) either produces a maximal independent set that is strictly larger than the set which was constructed during the previous iteration or produces a maximal independent set having the same size as the set constructed during the previous iteration but for which $|\cup X_s|$ is strictly smaller.

Thus, the while-loop must terminate after at most $O(N^2)$ iterations for a graph $G = (V,E)$ with $|V| = N$ vertices. Since both lines 4–8 and lines 9–13 can be executed in at most $O(N^2)$ steps, it follows that the complexity of Algorithm PositiveMDFSet is no worse than $O(N^4)$. We suspect, however, that an improved complexity bound is possible.

In particular, an $O(N)$ algorithm can be constructed for producing a positive MDF for an arbitrary tree T. However, this algorithm uses an altogether different approach. Details of this algorithm will appear in a subsequent paper.

3. Convexity of Minimal Dominating Functions

The *value* of a DF f of G is defined to be $\sum_{u \in V} f(v)$. The work of this section was motivated by the following interpolation question: Given MDFs f, g of G with values t_1, t_2 respectively and any t satisfying $t_1 < t < t_2$; Does there exist an MDF of G with value t? A. Majumdar [16] noticed that in some cases convex combinations are suitable.

Suppose that f and g are DFs of G for $\lambda \in (0,1)$, define $h_\lambda : V \longrightarrow [0,1]$ by

$$h_\lambda(v) = \lambda f(v) + (1-\lambda)g(v) \quad \text{for each} \quad v \in V. \tag{5}$$

Using (5), it is elementary to show that h_λ is a DF of G and that if f, g have values t_1, t_2 respectively and $t_1 < t < t_2$, by a suitable choice of $\lambda \in (0,1)$, h_λ has value t. However if f, g are MDFs, h_λ is not always minimal. We are thus led to study the relation $\mathcal{R}$ on the set $\mathcal{F}$ of MDFs of G, defined by: $f\mathcal{R}g$ if and only if h_λ is an MDF for all $\lambda \in (0,1)$. The remainder of the section begins this investigation.

Theorem 4. *For* MDFs f, g *of* G, $f\mathcal{R}g$ *if and only if* $B_f \cap B_g$ *dominates* $P_f \cup P_g$.

Proof. It will be proved that $B_{h_\lambda} = B_f \cap B_g$ and $P_{h_\lambda} = P_f \cup P_g$. The result is then immediate from Proposition 1. If $v \notin P_f \cup P_g$, then

$f(v) = g(v) = h_\lambda(v) = 0$. If, say, $v \in P_f$, then $h_\lambda(v) \geq \lambda f(v) > 0$. Thus $P_{h_\lambda} = P_f \cup P_g$.

Suppose $v \in B_f \cap B_g$. Then

$$\sum_{u \in N[v]} h_\lambda(u) = \sum_{u \in N[v]} \lambda f(u) + (1-\lambda)g(u)$$

$$= \lambda \sum_{u \in N[v]} f(u) + (1-\lambda) \sum_{u \in N[v]} g(u)$$

$$= \lambda + (1-\lambda) = 1.$$

A similar calculation shows $\sum_{u \in N[v]} h_\lambda(u) > 1$ for $u \notin B_f \cap B_g$ and hence $B_{h_\lambda} = B_f \cap B_g$. ■

We now consider the question of existence of MDFs which relate in $\mathcal{R}$ to every other MDF. The MDF g of G is called a *universal MDF* if for all $f \in \mathcal{F}$ and all $\lambda \in (0,1)$, $h_\lambda \in \mathcal{F}$. The following proposition enables us to obtain classes of graphs which have universal MDFs.

Proposition 5. *If the* MDF g *satisfies* $B_g = V$ *and for all* $f \in \mathcal{F}$, B_f *dominates* V, *then* g *is a universal* MDF.

Proof. For $f \in \mathcal{F}$, $B_g \cap B_f = B_f$ which dominates $V \supseteq P_f \cup P_g$. The result follows from Theorem 4. ■

Theorem 6. *The path* P_n $(n \geq 1)$, *the cycle* C_n $(n \geq 3)$, *the complete bipartite graph* $K_{m,n}$ $(m,n \geq 1)$, *the* n*-vertex wheel* W_n $(n \geq 4)$ *and the* *complete graph* K_n $(n \geq 5)$ *all have universal* MDFs.

Proof. It is easy to verify that for any n, the path P_n has an MDF g with $B_g = V$ by assigning to consecutive vertices in the path, suitable consecutive elements from the sequence $100100100\cdots$ and it remains to prove that for all $f \in \mathcal{F}$, B_f dominates V. Suppose this is false and for $f \in \mathcal{F}$, B_f does not dominate $\{v\}$ where $v \in V$. It is obvious that an end-vertex of a tree T is in the boundary of any MDF of T. Therefore end-vertices and their neighbours are dominated by the boundary of any MDF. It follows that P_n contains a subpath with vertex sequence v_2, v_1, v, v_3, v_4. Since B_f dominates P_f, $f(v) = 0$ and by the dominating property, say, $f(v_1) > 0$. Vertex $v_1 \notin B_f$ (since v is undominated), hence $f(v_1) + f(v_2) > 1$. But B_f dominates v_1 $(\in P_f)$ and so $v_2 \in B_f$, which implies the contradiction $f(v_1) + f(v_2) \leq 1$. Therefore g is a universal MDF by Proposition 5.

The function f which assigns $1/3$ to each vertex is an MDF g of C_n satisfying $B_g = V$. A similar argument to that used for paths shows for all $f \in \mathcal{F}$, B_f dominates V and hence g is a universal MDF.

Suppose that $K_{m,n}$ has defining independent sets $\{a_1, \cdots, a_m\}$ and $\{b_1, \cdots, b_n\}$. Define the MDF g by $g(a_i) = (n-1)/(mn-1)$ for $i = 1, \cdots, m$ and $g(b_j) = (m-1)/(mn-1)$ for $j = 1, \cdots, n$. This function satisfies $B_g = V$. Without losing generality suppose $f \in \mathcal{F}$ is such that b_1 is not dominated by B_f. Then $f(b_1) = 0$ and $\sum_{i=1}^{m} f(a_i) > 1$. Moreover no $a_i \in B_f$ and hence some $b_j \in B_f$. This implies $\sum_{i=1}^{m} f(a_i) \geq 1$, a contradiction. Therefore B_f dominates V and g is a universal MDF by Proposition 5.

The proofs for W_n and K_n are omitted. ∎

For $S \subsetneq V$, let f_S be defined by $f_S(u) = 1$ if $u \in S$ and $f_S(u) = 0$ otherwise.

PROPOSITION 7. *Let* g *be an* MDF *such that* B_g *does not dominate vertex* v *and let* S *be any minimal dominating set containing* v. *Then* $g \not\mathcal{R} f_S$.

Proof. Since $v \in S$, $f_S(v) = 1$, *i.e.* $v \in P_{f_S} \subsetneq P_{f_S} \cup P_g$, $B_g \cap B_{f_S} \subsetneq B_g$ which does not dominate v. Therefore, by Theorem 4, $g \not\mathcal{R} f_S$. ■

COROLLARY. *If* g *is a universal* MDF, *then* B_g *dominates* V.

Proof. Immediate. ■

The next result enables us to demonstrate the existence of universal MDFs whose boundaries do not contain all vertices (as required by Proposition 5) and graphs, all of whose MDFs are universal, *i.e.* $\mathcal{R} = \mathcal{F} \times \mathcal{F}$.

Let u be a vertex of graph H. By a *complete addition to* H *at* u we mean the identification of u and a vertex of some complete graph with at least two vertices.

PROPOSITION 8. *Let* H *be any graph. Form* G *from* H *by making one or more complete additions to* H *at* u, *for each vertex* u *of* H. *Then each* MDF *of* G *is universal.*

Proof. If $a_1 \in V(G) - V(H)$, then $N[a_1] = \{a_1, a_2, \cdots, a_n, u\}$ where $u \in V(H)$ and $n \geq 1$. Suppose $f \in \mathcal{F}$ and $a_1 \notin B_f$. Then $\sum_{v \in N[a_1]} f(v) = \sum_{i=1}^{n} f(a_i) + f(u) > 1$ and hence for some i, $f(a_i) > 0$. But $N[a_j] = N[a_1]$ for each $j \in 1, \cdots, n$ and so $a_j \notin B_f$. Further $\sum_{v \in N[u]} f(v) \geq \sum_{v \in N[a_1]} f(v) > 1$ and so $u \notin B_f$. Therefore B_f does not dominate a_i which implies (using Proposition 1) f is not minimal. We conclude $V(G) - V(H) \subseteq B_f$. Thus, for any pair $f, g \in \mathcal{F}$, $B_f \cap B_g$ contains $V(G) - V(H)$ which is a dominating set of G and hence $B_f \cap B_g$ dominates $P_f \cup P_g$. By Theorem 4, $f\mathcal{R}G$, therefore each $f \in \mathcal{F}$ is universal.

Corollary. *Any* MDF *of a tree whose end vertices form a dominating set, is universal.*

The final result concerns non-existence of universal MDFs.

Proposition 9. *Let* G *be a graph which contains a vertex* v *such that for every* $u \in N[v]$ *there exists a* MDF f_u *such that* B_{f_u} *does not dominate* u. *Then* G *does not have a universal* MDF.

Proof. Suppose g is a universal MDF of G and let $v \in V(G)$ satisfy the hypothesis above. $B_g \cap B_{f_v}$ does not dominate v and hence $g(v) = 0$ (for otherwise $v \in P_g \cup P_{f_v}$). Similarly, for each $u \in N[v]$, $B_h \cap B_{f_u}$ does not dominate u and hence $g(u) = 0$. But then

$$\sum_{u \in N[v]} g(u) = 0,$$

so that g is not a DF, a contradiction. ■

Corollary. *If* G *is vertex transitive, then* G *has a universal iff for every* MDF f *of* G, B_f *dominates* V.

Proof. Let G be vertex transitive; suppose G is r–regular. If B_f dominates V(G) for every MDF f, then $g = \frac{1}{r+1}$ is universal, for $B_g = V$, hence $B_g \cap B_f$ dominates $V(G) = P_g \cup P_f$ (Theorem 4).

Conversely, let f be a MDF of G such that B_f does not dominate V(G); say, B_f does not dominate v. Since G is vertex transitive, there exists, for every $u \in V(G)$ and in particular for every $u \in N[v]$, a MDF f_u of G such that B_{f_u} does not dominate u. By the proposition, G does not have a universal MDF. ■

Let G be the circulant formed by adding edges $\{i,i+5\}$ for $i = 1,\cdots,5$ to the cycle with vertex sequence $1,\cdots,10$. Then for example the function f which is 1 on $\{1,3,6,8\}$ and 0 elsewhere is an MDF with $B_f = \{4,5,9,10\}$ which does not dominate V. By the corollary G has no universal MDF.

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