2-Dipath and Proper 2-Dipath Colourings

by

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B.Sc., University of Victoria, 2009

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ABSTRACT

A *2-dipath k-colouring* of an oriented graph G is an assignment of k colours, $1, 2, \ldots, k$, to the vertices of G such that vertices joined by a directed path of length two are assigned different colours. The *2-dipath chromatic number* is the minimum number of colours needed in such a colouring. There are two possible models, depending on whether adjacent vertices must also be assigned different colours.

For both models of 2-dipath colouring we develop the basic theory, including characterizing the oriented graphs that can be 2-dipath coloured using a small number of colours, finding bounds on the 2-dipath chromatic number, determining the complexity of deciding the existence of a 2-dipath k-colouring, describing a homomorphism model, and showing how to determine the 2-dipath chromatic number of tournaments and bipartite tournaments.

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My fellow grad students, for providing endless hours of distraction, when work just wasn't a priority.

The students in the Assistance Centre, for providing me with countless opportunities to try and come up with a different way to think about math.

I have heard there are troubles of more than one kind.
Some come from ahead and some come from behind.
But I've bought a big bat. I'm all ready you see.
Now my troubles are going to have troubles with me!
Dr. Seuss

DEDICATION

To my Mother,
Who may not understand what I do, but at least now she can pronounce it!

Chapter 1

Preliminaries

The first idea of a colouring of a digraph neglected the orientation of the arcs, colourings only the underlying undirected graph. Although this method avoids much of the structure of digraphs, the idea still has merit. For example, Gallai [10] and Roy [22] independently proved that *if the underlying graph of a digraph G has chromatic number $\chi(G)$, then G has a directed path on at least $\chi(G)$ vertices.* This is the best possible result in the sense that the edges of any undirected graph G can be oriented so that the longest directed path in the resulting digraph has exactly $\chi(G)$ vertices. The underlying undirected graph of a tournament is complete, therefore the theorem of Gallai and Roy implies an earlier result of Rédei [21] which says that *every tournament has a directed Hamilton path.*

In 1995, Courcelle published a paper in which the first different notion of a colouring of a digraph was introduced [9]. The definition applies only to oriented graphs: directed graphs obtained from simple graphs by assigning an orientation to each edge. Courcelle defined oriented k-colouring of an oriented graph G as an assignment of the colours $1, 2, \ldots, k$ to the vertices of G such that adjacent vertices are assigned different colours and if there is arc from a vertex coloured i to a vertex coloured j, then there is no arc from a vertex coloured j to a vertex coloured i. This type of colouring assignment respects the orientation of the arcs.

Courcelle's original motivation was to demonstrate the expressive power of monadic second order logic in graph problems. Since it's origination, the concept of oriented colouring has grown into a well-developed area of study (see the paper by Sopena [23] for a survey). Also introduced in the same paper by Courcelle was the idea of oriented colourings that are also injective on the in-neighbourhood of each vertex. These have also developed into an area of study [6, 18, 20, 24], but less so than oriented colourings. The injective colouring of undirected graphs is also an active area of study, first considered by Hahn, Kratochvil, Sířan, and Sotteau [15].

The definition of oriented colouring implies that vertices of an oriented graph which are joined by a directed path of length two must be assigned different colours. Let x, y, z be a directed path of length two such that the vertices x and y are assigned colours i and $j \neq i$, respectively. Since the definition precludes an arc from a vertex coloured j to a vertex coloured i, the colour assigned to z can not be i. The vertices x and z are then assigned different colours. A lower bound on the number of colours required for an oriented colouring of an oriented graph G can be obtained from colourings that satisfy only the condition that vertices joined by a directed path of length two must be assigned different colours. This idea of *2-dipath colourings* was first considered by Min and Wang [19] in 2007. Their main result is that *at most seven colours are needed in a 2-dipath colouring of any orientation of a Halin graph.*

More general than 2-dipath colouring is the concept of $L(p, q)$-labelling of an oriented graph. In these labellings, the vertices are assigned integers $1, 2, \ldots, k$ in such a way that vertices joined by an arc are assigned colours i and j such that $|i - j| \geq p$, and vertices joined by a directed path of length two must be assigned colours i and j such that $|i - j| \geq q$. The originators of this idea, Chang and Liaw [8], were motivated by the corresponding concept for undirected graphs [13]. A survey of the many results in this area was done by Calamoneri [5]. Goncalves, Raspaud and Shalu [12] have also introduced a number of variations of $L(p, q)$-labellings of oriented graphs.

As defined above, a proper 2-dipath colouring of an oriented graph is an $L(1,1)$-labelling. Hence if G has no directed 4-cycle, the minimum number of colours needed in an oriented colouring in the square of G (as a digraph) is an upper bound for the minimum number of colours needed in a proper 2-dipath colouring of an oriented graph G. It is also possible to regard $L(0,1)$ labellings as 2-dipath colourings. In these colourings, adjacent vertices need not be assigned different colours, but vertices joined by a directed path of length two must be assigned different colours. The main goal of this thesis is to develop a theory of 2-dipath colourings of oriented graphs for both of these models.

The remainder of the thesis is organized as follows. After establishing the basic definitions we will need in Section 2, we spend the first half of the thesis looking at 2-dipath colourings without the added constraint of being a proper colouring. We begin by looking at clique covers as an alternative way to determine the 2-dipath chromatic number $\chi_2(G)$ of an oriented graph G. In Section 3.3 we provide a characterization of 2-dipath 1- and 2-colourable oriented graphs, including a colouring algorithm for oriented trees as well as a complete determination of χ_2 for oriented cycles. After, we look at the complexity of 2-dipath colourings in Section 3.4, determining 2-dipath k-colouring for $k \geq 3$ is $\mathcal{NP}$-complete. Determining whether or not an oriented graph has a 2-dipath 2-colouring is reduced to a 2-satisfiability problem in Section 3.5. Here it is shown that the decision of whether G has a 2-dipath 2-colouring is polynomial-time solvable.

In Chapter 4 we show the question of whether an oriented graph G has a 2-dipath k-colouring is equivalent to whether there is a homomorphism to some special digraph G_k. Section 4.1 establishes the structure of the set of digraphs G_k, $k \geq 1$ and establishes the link between 2-dipathcolourings and homomorphisms. The next chapter, Chapter 5, moves on to look at 2-dipath colourings of tournaments using a variety of methods, with the results in Section 5.2 employing clique coverings and matchings to determine the χ_2 for tournaments. Section 5.3 focuses solely on bipartite tournaments and uses an equivalence relation to obtain χ_2.

The consideration of proper 2-dipath colourings begins in Chapter 6, with Section 6.1 using many of the same methods used for 2-dipath colourings for proper 2-dipath colourings to prove the corresponding results. Section 6.2 gives a characterization of 2-dipath 2- and 3-colourable oriented graphs as well as a proper 2-dipath colouring algorithm for oriented trees. In this section we also make use of the relationship between proper 2-dipath colourings and $L(j,k)$-labelings to determine the proper 2-dipath chromatic number χ_2'. The rest of Chapter 6 provides a sequel for proper 2-dipath colourings to the first half of the thesis on 2-dipath colourings, with Section 6.3 giving a reduction of proper 2-dipath colouring to 2-satisfiability.

Chapter 7 gives a homomorphism model for proper 2-dipath colourings similar to that found in Chapter 4, with Section 7.3 determining the complexity of proper 2-dipath colouring by using homomorphisms. Finally Chapter 8 looks at proper 2-dipath colourings of tournaments. Since the proper 2-dipath chromatic number for tournaments T_n is just $|V| = n$, we restrict our attention to bipartite tournaments. Section 8.1 establishes an equivalence relation on bipartite tournaments that is used to determine the proper 2-dipath chromatic number of bipartite tournaments, completing the sequel to the first half of the thesis.

Chapter 2

Definitions

We will refer to Bondy and Murty's text [4] for the basic definitions and concepts of graph theory. This section covers general results and notations that will be used, that may not be found or may not be standard in introductory textbooks. Particular definitions that are used only in one part of the thesis are not given here. They will be introduced when needed. For this thesis we will use oriented graphs, a set of digraphs with no multiple arcs and no loops.

Definition 2.0.1. An *oriented graph* is an ordered pair $G = (V, E)$, where V is a finite set of objects called *vertices* and E is a set of ordered pairs of distinct vertices such that if $(x, y) \in E$, then $(y, x) \notin E$. The elements of E are called *arcs*.

An oriented graph can be viewed as being obtained from an undirected simple graph by assigning a direction to each edge. In that spirit, a *tournament* is an oriented complete graph, and a *bipartite tournament* is an oriented complete bipartite graph.

Definition 2.0.2. For an oriented graph G, define $U(G)$ as the underlying undirected graph, with vertex set $V(U(G)) = V(G)$ and edge set $E(U(G)) = \{xy : (x, y) \in E(G)\}$.

When referring to adjacencies, we will use $N^+(x)$ and $N^-(x)$ to denote the out- and in-neighbourhoods of vertex x, with $N(x) = N^+(x) \cup N^-(x)$.

Definition 2.0.3. An *oriented path* in an oriented graph G is a finite sequence of vertices $x_0, x_1, ..., x_k$ such that for $i = 1, 2, ..., k$ either $(x_{i-1}, x_i) \in E$ or $(x_i, x_{i-1}) \in E$. A *directed path* or *dipath* is an oriented path $x_0, x_1, ..., x_k$ such that either $(x_{i-1}, x_i) \in E, 1 \leq i \leq k$ or $(x_i, x_{i-1}) \in E, 1 \leq i \leq k$. The integer k is the *length* of the path. We use $\vec{P}_{k+1}$ to denote a directed path on $k + 1$ vertices. Such a path has length k.

A directed path of length two will sometimes be referred to as a 2-*dipath*. The *directed distance*, $\vec{d}(u,v)$, between two vertices $u, v \in V(G)$, is the shortest length of a directed path between them. For our purposes we will still be concerned when two vertices are joined by a directed path of length two, even if $(u,v) \in E(G)$ or $(v,u) \in E(G)$.

Definition 2.0.4. An *oriented cycle* is an oriented graph G is a finite sequence of vertices $x_0, x_1, ..., x_k$ such that for $i = 1, 2, ..., k$ either $(x_{i-1}, x_i) \in E$ or $(x_i, x_{i-1}) \in E$ and either $(x_0, x_k) \in E$ or $(x_k, x_0) \in E$. A *directed cycle* is an oriented cycle $x_0, x_1, ..., x_k$ such that either $(x_{i-1}, x_i) \in E, 1 \leq i \leq k$ and $(x_k, x_0) \in E$ or $(x_i, x_{i-1}) \in E, 1 \leq i \leq k$ and $(x_0, x_k) \in E$. The integer k is the *length* of the cycle. We use C_k to denote an oriented cycle of length k and $\vec{C}_k$ to denote a directed cycle of length k.

Definition 2.0.5. Let G be an oriented graph. The *auxiliary graph* of G is the graph G_A with vertex set $V(G_A) = V(G)$ and $xy \in E(G_A)$ if and only if the vertices x and y are joined by a directed path of length two in G.

Definition 2.0.6. For oriented graphs G and H, a homomorphism $G \rightarrow H$ is a function $f : V(G) \rightarrow V(H)$ such that $(f(x), f(y)) \in E(H)$ whenever $(x,y) \in E(G)$. Homomorphisms between oriented graphs preserve adjacencies, inducing a map between the arcs of G and those of H.

Example 2.0.7. Referring to Figure 2.1, let G be the oriented cycle of length six with $V(G) = \{u, v, w, x, y, z\}$, and let H be an oriented cycle of length four with $V(H) = \{a, b, c, d\}$. Let f_1 be the homomorphism $G \rightarrow H$ such that $f_1 : V(G) \rightarrow V(H)$, with $f_1(u) = d = f_1(y), f_1(v) = a = f_1(z), f_1(w) = b$ and $f_1(x) = c$. Let f_2 be the homomorphism $G \rightarrow H$ such that $f_2(u) = b = f_2(w), f_2(v) = a = f_2(z), f_2(x) = c$ and $f_2(y) = d$. The oriented graphs G and H as well as the homomorphisms f_1 and f_2 are shown in the figure.

Definition 2.0.8. A *2-dipath k-colouring* of a oriented graph G is a function $c : V(G) \rightarrow \{1, 2, ..., k\}$, such that if $\vec{d}(x,y) = 2$, then $c(x) \neq c(y)$ for all $x, y \in V(G)$. An oriented graph G is *2-dipath k-colourable* if there exists a 2-dipath k-colouring, c. The *2-dipath chromatic number* of G is the smallest k such that G is 2-dipath k-colourable, and is denoted $\chi_2(G)$.

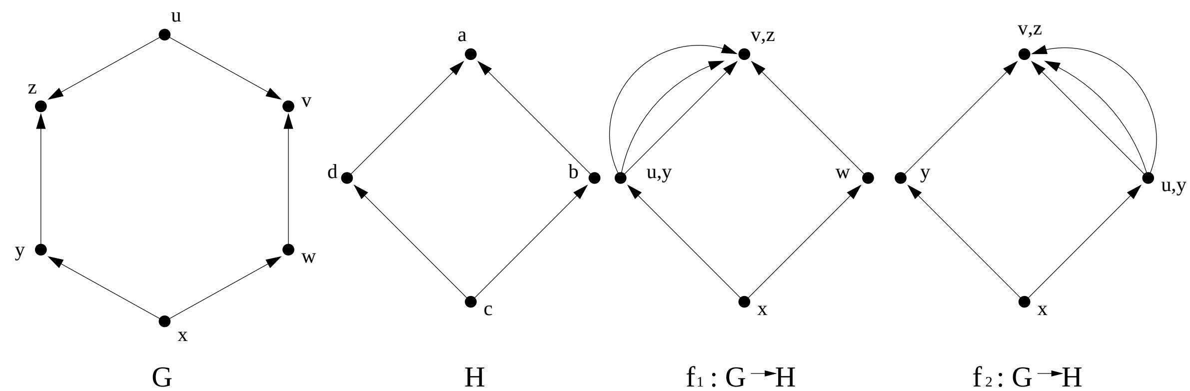

Figure 2.1: Homomorphisms from $G \to H$

A 2-dipath colouring of an oriented graph G need not also be a proper colouring of G. Adjacent vertices may be assigned the same colour, provided they are not also joined by a directed path of length two. Each 2-dipath k-colouring of an oriented graph G also corresponds to an ordered partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$, where V_i, the i^{th} *colour class,* is the *set of vertices which receive colour* i. Note that the V_i's are allowed to be empty. Additional properties of such a partition will be further discussed in terms of the homomorphism model presented in 4.1.

Definition 2.0.9. A *proper 2-dipath k-colouring* of an oriented graph G is a function $c : V(G) \to \{1, 2, ..., k\}$, such that if $\vec{d}(x, y) \leq 2$ then $c(x) \neq c(y)$ for all $x, y \in V(G)$. An oriented graph G is *proper 2-dipath k-colourable* if there exists a proper 2-dipath k-colouring c. The *2-dipath proper chromatic number* is the smallest k such that G is proper 2-dipath k-colourable, and is denoted $\chi_2'(G)$.

For any oriented graph G, $\chi_2(G)$ (or $\chi_2'(G)$) is also the minimum number of sets required to partition $V(G)$, so that there is no 2-dipath among any pair vertices in the same class, (or no dipath of length at most 2 for proper 2-dipath colourings).

Definition 2.0.10. An *oriented k-colouring* of an oriented graph G is a function $c : V(G) \to \{1, 2, ..., k\}$ such that:

- For every arc $(x, y) \in E(G), c(x) \neq c(y)$,
- If there is an arc from a vertex coloured i to a vertex coloured j, then there is no arc from any vertex coloured j to a vertex coloured i.

The *oriented chromatic number* of G denoted $\chi_o(G)$ is the minimum k for which an oriented k-colouring of G exists.

Oriented colourings are of interest because of their direct connection to proper 2-dipath colourings. In an oriented colouring, vertices at directed distance two must receive different colours, making every oriented colouring a proper 2-dipath colouring.

Definition 2.0.11. The square G^2 of an oriented graph G has vertex set $V(G^2) = V(G)$ and arc set $E(G^2) = E(G) \cup \{xy$:there exists z such that $xz, zy \in E(G)\}$. Note that if G has a $\vec{C}_4$, then G^2 is not an oriented graph.

This type of structure is useful when considering proper 2-dipath colourings, as it creates adjacencies in G^2 between vertices that were at distance 2 in G.

Chapter 3

2-dipath Colourings

3.1 Introduction

We begin by examining the relationships between 2-dipath colourings, proper 2-dipath colouring and oriented colourings. We use inequalities to exhibit the connections between the various chromatic numbers.

Proposition 3.1.1. *For an oriented graph G, $\chi_2(G) \leq \chi'_2(G) \leq 2\chi_2(G)$*

Proof. $\chi_2(G) \leq \chi'_2(G)$: Any 2-dipath proper colouring of a graph G is also a 2-dipath colouring.

$\chi'_2(G) \leq 2\chi_2(G)$: Consider a 2-dipath k-colouring of G with $k = \chi_2(G)$. Since no two vertices of the same colour are joined by a directed path of length two, for each i the set V_i of vertices of colour i can be partitioned into the two independent sets $V_i^+ = \{x : N^+(x) \cap V_i \neq \emptyset\}$ and $V_i^- = V_i - V_i^+$. Note that $V_i^- \supseteq \{x : N^-(x) \cap V_i \neq \emptyset\}$. Then $(V_1^+, V_1^-, V_2^+, V_2^-, \ldots, V_k^+, V_k^-)$ corresponds to a proper 2-dipath $2k$-colouring of G. □

Proposition 3.1.2. *Every oriented k-colouring of an oriented graph G is also a proper 2-dipath k-colouring of G.*

Proof. Let c be an oriented k-colouring of an oriented graph G. We need to show that all pairs of vertices x, y with $\vec{d}(x, y) \leq 2$ have $c(x) \neq c(y)$. Clearly by the first condition $c(x) \neq c(y)$ for $\vec{d}(x, y) = 1$. By the second condition, if $(x, y), (y, z) \in E(G)$, then x and z must be assigned different colours in any oriented colouring of G. In this way, vertices at directed distance at most two in G are assigned different colours by c, making c a proper 2-dipath colouring. □

Corollary 3.1.3. $\chi_2(G) \leq \chi_o(G)$.

We now relate 2-dipath colourings of oriented graphs to colourings of the auxiliary graph G_A.

Theorem 3.1.4. *Let G be an oriented graph. There is a 1-1 correspondence between the set of 2-dipath colourings of G and the set of proper colourings of G_A.*

Proof. ($\Rightarrow$) Let c be a 2-dipath colouring of G with k colours. For any two vertices $x, y \in V(G_A)$ with $xy \in E(G_A)$, we have that $\vec{d}(x, y) = 2$ in G, and so $c(x) \neq c(y)$, and c is a proper colouring of G_A with k colours.

($\Leftarrow$) Let c be a proper colouring of G_A with k colours. For any two vertices $x, y \in V(G)$ joined by a directed path of length two, we have $xy \in E(G_A)$, and so $c(x) \neq c(y)$, and c is a 2-dipath colouring of G with k colours. □

Corollary 3.1.5. *For an oriented graph G, $\chi_2(G) = \chi(G_A)$.*

We can use Theorem 3.1.4 to obtain a bound on $\chi_2(G)$ for an oriented graph G, by applying Brooks' Theorem for graph colourings. This gives us $\chi_2(G) = \chi(G_A)$ $\leq \Delta(G_A) + 1 \leq \Delta^-(G)^2 + \Delta^+(G)^2 + 1$, as G_A may be an odd cycle.

3.2 Clique Covering

We now explore further the relationship between an oriented graph G and its auxiliary graph G_A. We will examine the correspondence between 2-dipath colourings of an oriented graph and disjoint clique coverings of $\overline{G_A}$ the complement of its auxiliary graph. Later, we will look at a specific instance of clique coverings for the auxiliary graphs of tournaments.

Definition 3.2.1. Let G be a graph. A *clique covering* of G is a partition of $V(G)$ into disjoint sets, $\{V_1, V_2, ..., V_k\}$ such that the subgraph of G induced by each V_i is a complete subgraph of G. The minimum number of such sets in such a partition is the *clique covering number* of G, denoted $\omega'(G)$.

Note that not every edge need be included in one of the complete subgraphs in a clique covering of G. For a graph G we know $\chi(G) = \omega'(\overline{G})$.

Proposition 3.2.2. *For an oriented graph G with auxiliary graph G_A, $\chi_2(G) = \omega'(\overline{G_A})$*

Recall that for a graph G, $\omega(G)$ is the *clique number* of G, that is the largest number of vertices in a complete subgraph of G.

Corollary 3.2.3. *For an oriented graph G with auxiliary graph G_A, $\chi_2(G) \geq \omega(G_A)$.*

3.3 Characterizations

3.3.1 2-Dipath 1-Colourable Oriented Graphs

The following gives a characterization of oriented graphs which have 2-dipath colourings with one colour.

Fact 3.3.1. *An oriented graph G has a 2-dipath colouring with one colour if and only if $\vec{P}_3$ is not a subgraph of G.*

The absence of a $\vec{P}_3$ in G ensures that G does not contain a directed path of length two. Since no two vertices are joined by a directed path of length two, one colour suffices to colour the vertices of G. Alternatively, if such a subgraph does exist, there are two vertices in G joined by a 2-dipath which must receive different colours, making one colour no longer sufficient. Note the statement 3.3.1 can also be formulated as: an oriented graph G has a 2-dipath colouring if and only if there is no homomorphism $\vec{P}_3 \to G$. We will adopt this idea of homomorphism duality (see [17]) in the sequel.

Proposition 3.3.2. *An oriented graph G has a 2-dipath colouring with one colour if and only if $G_A = \overline{K_n}$, where $n = |V(G)|$.*

Example 3.3.3. The following are examples of oriented graphs which can be 2-dipath coloured with 1 colour:

- even length oriented cycles C_{2k} with alternating forwards/backwards arcs
- stars with all inwards or all outwards directed arcs
- oriented paths with alternating forwards/backwards arcs

Theorem 3.3.4. *Let G be an oriented graph. Then $\chi_2(G) = 1$ if and only if G is bipartite and there exists a bipartition (V_1, V_2) such that every arc has its origin in V_1 and its terminus in V_2.*

Proof. (⇒) Let G be an oriented graph with $\chi_2(G) = 1$. By definition, G has no directed path of length two. Since G has no 2-dipaths, G cannot contain an oriented odd cycle, and thus G is bipartite. For such an oriented graph G either the in-degree or the out-degree of any vertex must equal zero, and so such a partition (V_1, V_2) exists.

(⇐) Let G be a bipartite oriented graph with bipartition (V_1, V_2) as described. Clearly G has no dipath of length two, as every vertex in V_1 has in-degree zero and every vertex in V_2 has out-degree zero. In this way, a 2-dipath colouring of G only requires one colour, and $\chi_2(G) = 1$. □

3.3.2 2-Dipath 2-Colourable Oriented Graphs

We can also use the auxiliary graph for an oriented graph to determine if it can be 2-dipath coloured with two colours, based on whether or not the auxiliary graph is bipartite. In what follows, we give a characterization of 2-dipath 2-colourable graphs. We show further that $\chi_2(T) \leq 2$ for any oriented tree T, and give an efficient algorithm to find an optimal 2-dipath colouring of a given oriented tree T.

Corollary 3.3.5. *For an oriented graph G with auxiliary graph G_A, $\chi_2(G) = 2$ if and only if $\chi(G_A) = 2$.*

We now characterize the oriented graphs G with $\chi_2(G) = 2$. Observe that directed cycles of length not congruent to 0 (mod 4) can not be 2-dipath coloured with two colours, as their auxiliary graphs are not bipartite.

Definition 3.3.6. Let $\mathcal{F}_1$ be the set of directed odd cycles, $C_{2k+1},\ k \geq 1$, and let $\mathcal{F}_2$ be the set of oriented graphs constructed from an undirected odd cycle $C_{2k+1}, k \geq 1$ by replacing $xy \in E(C_{2k+1})$ by one of the 2-dipaths $x \to m_{xy} \to y$ or $y \to m_{yx} \to x\}$.

Example 3.3.7. Figure 3.1 illustrates the construction of an oriented graph $F \in \mathcal{F}_2$ from a directed C_5 using the Definition 3.3.6.

Lemma 3.3.8. *If $F \in \mathcal{F}_1 \cup \mathcal{F}_2$, then its auxiliary graph, F_A (as defined in 2.0.5), contains at least one odd cycle.*

Proof. Suppose $F \in \mathcal{F}_1$, and let the vertices of F be $v_1, v_2, \ldots, v_{2k+1}$ in cyclic order. Then, $v_1 v_3 \ldots, v_{2k+1} v_2, v_4, v_{2k} v_1$ is an odd cycle in F_A. Now suppose $F \in \mathcal{F}_2$, and let the vertices of F be $v_1, v_{1,2}, v_2, v_{2,3}, v_3, \ldots, v_{2k,2k+1}, v_{2k+1}, v_{2k+1,1}$ where for every i, either $v_i, v_{i,i+1}, v_{i+1}$ or its reverse is a directed path of length two, and subscripts are taken modulo $2k+1$. Then $v_1, v_2, \ldots v_{2k+1}, v_1$ is an odd cycle in F_A. □

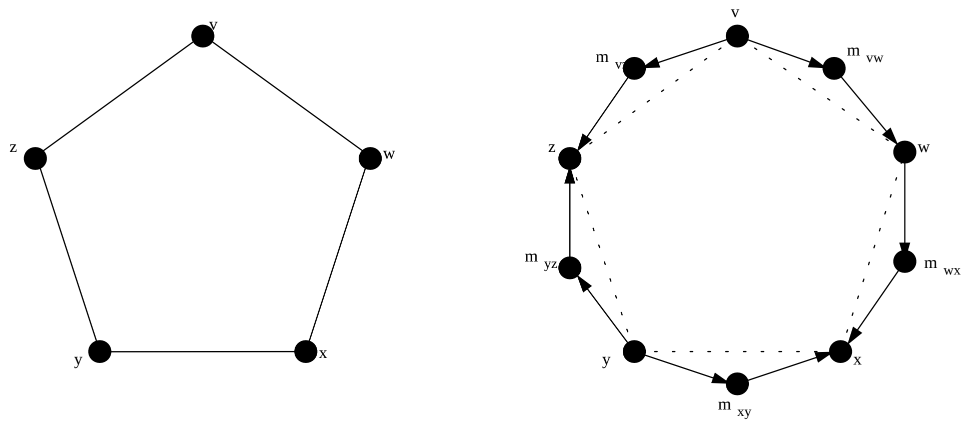

Figure 3.1: Construction of an element of $\mathcal{F}_2$ from C_5

Corollary 3.3.9. *Let $F \in \mathcal{F}$. Then $\chi_2(F) \geq 3$.*

Theorem 3.3.10. *For an oriented graph G, $\chi_2(G) = 2$ if and only if there is no $F \in \mathcal{F}_1 \cup \mathcal{F}_2$ such that there exists a homomorphism $F \to G$.*

Proof. ($\Rightarrow$) Let $\chi_2(G) = 2$ and take a 2-dipath 2-colouring of G. Suppose there exists $F \in \mathcal{F}$ such that there exists a homomorphism $h : F \to G$. Define a colouring of F by assigning each vertex $v \in V(F)$ the same colour as vertex $f(x)$. Since homomorphisms preserve arcs, this is a 2-dipath 2-colouring of F, a contradiction, (every $F \in \mathcal{F}$ has $\chi_2(F) \geq 3$ by 3.3.9).

($\Leftarrow$) Suppose there does not exist $F \in \mathcal{F}_1 \cup \mathcal{F}_2$, with $F \to G$, but $\chi_2(G) > 2$. By 3.3.5, G_A is not bipartite, implying that G_A contains an odd cycle C_{2k+1}, $k \geq 0$. Consider the vertices of $C_{2k+1} \in G_A$ in G. By the construction of G_A, every pair of adjacent vertices in G_A are joined by a directed path of length two in G. In this way, C_{2k+1} is formed from a closed walk of length $4k+2$ in G containing no maximal directed path of odd length. Thus, there exists a homomorphism from some $F = C_{4k+2} \in \mathcal{F}_2$ into G, a contradiction. ☐

Clearly since trees are acyclic and bipartite, their auxiliary graphs are bipartite. Therefore they do not admit homomorphisms from any $F \in \mathcal{F}$. We can also examine only the auxiliary graph of a tree to determine the 2-dipath chromatic number.

Definition 3.3.11. The eccentricity of a vertex v in a connected graph, (or oriented graph), G, is the maximum distance from v to another vertex $u \in V(G)$ in the underlying undirected graph.

Proposition 3.3.12. *For an oriented tree T, $\chi_2(T) \leq 2$.*

Proof 1. Let T be an oriented tree, and consider the auxiliary graph T_A. Since T is acyclic and bipartite, T_A is also bipartite. In this way, T_A is a forest and so by 3.3.10 the oriented tree T has $\chi_2(T) \leq 2$. □

Proof 2. We will prove the statement using induction on $n = |V|$. The statement is true for $n = 1$, a single vertex. Assume it is true when $n \leq k$, for some $k \geq 2$, and let T be an oriented tree on $k+1$ vertices. Since $k+1 \geq 2, T$ has at least one leaf. Take x to be a leaf of maximum eccentricity. Let s be the only neighbour of x in T, and let A be the set of leaves which are in-neighbours of s, and B be the set of leaves which are out-neighbours of s in T, ($x \in A \cup B$). Then $T' = T - (A \cup B)$ is an oriented tree and s is a leaf of T'. Take y to be the unique neighbour of s in T'. Since $|V(T')| < k+1, T'$ has a 2-dipath colouring c with two colours. If $(s, y) \in E(T')$, then a 2-dipath colouring of T is obtained by assigning every vertex in B the same colour as y by c, and every vertex in A the opposite colour. Alternatively, if $(y, s) \in E(T')$, then colour every vertex of A the same colour as y by c, and every vertex in B the opposite colour. □

Proof 2 gives rise to an inductive colouring algorithm, done by selecting and removing leaves and colouring the remaining oriented graph. The following is an alternate algorithm, which colours an oriented tree by selecting a vertex as a root, and systematically colouring it's neighbours.

Algorithm 3.3.13. *2-dipath colouring of an oriented tree.*

1. *Pick a vertex* $v \in V(T)$ *and set it as the root of* T *with* $c(v) = 1$*. Arrange all other vertices by increasing distance from the root, where distance is taken from the underlying graph.*

2. *Repeat over all coloured vertices with uncoloured neighbours at this step. Let* $u \in V(T)$ *be coloured, but have uncoloured neighbours.*
   - *If* $c(u) = 1$*, then colour the uncoloured vertices of* $N^+(u)$ *with colour* 1*, and colour all the uncoloured vertices of* $N^-(u)$ *with colour* 2*.*
   - *Otherwise, if* $c(u) = 2$*, then colour all its uncoloured in-neighbours with colour* 2*, and the uncoloured out-neighbours with colour* 1*.*

3. *Repeat over all coloured vertices* $x \in V(T)$ *which have uncoloured neighbours.*
   - *If c(x)=1, then colour all uncoloured vertices in* $N^+(x)$ *colour* 2 *and colour all uncoloured vertices in* $N^-(x)$ *colour* 1*.*
   - *Otherwise,* $c(x) = 2$ *and colour all uncoloured vertices in* $N^+(x)$ *colour* 1 *and colour all uncoloured vertices in* $N^-(x)$ *colour* 2*.*

4. *Repeat Steps 2 and then 3 until all vertices have been assigned a colour.*

We will now look at 2-dipath colourings of oriented cycles. We completely determine the 2-dipath chromatic number of all oriented cycles, for any given any number of vertices, n.

Fact 3.3.14. *For any oriented cycle* C*, the auxiliary graph* C_A *is the union of disjoint paths and cycles.*

Proof. Let C be an oriented cycle. Since any vertex in a cycle can be at directed distance two from at most two vertices, then $\Delta(C_A) \leq 2$. In this way, C_A is a union of disjoint paths and cycles. □

Lemma 3.3.15. *For a directed cycle* $G = \vec{C_n}$ *with* $n \equiv 1, 3 \pmod 4$ *we have that* $G_A \simeq U(\vec{C_n})$*, the underlying graph of the directed cycle* $\vec{C_n}$*.*

Proof. Let $\vec{C_n} = v_1, v_2, ..., v_n$, with $n \equiv 1, 3 \pmod 4$. Since $\vec{C_n}$ is directed, every pair of vertices at distance two in the underlying undirected graph C_n are also at directed distance two in $\vec{C_n}$. In this way, $G_A = v_1, v_3, ..., v_{n-2}, v_n, v_2, v_4, ..., v_{n-1}, v_1 = C_n$, a cycle on n vertices, and $U(\vec{C_n}) \equiv G_A$. □

The following gives examples of how to optimally colour various oriented cycles. Note that for a given n, the largest number of colours that will be required is to colour the directed cycle $\vec{C_n}$.

Example 3.3.16. For an oriented cycle C_n, if $n \equiv 0 \pmod 4$ then the cycle can be 2-dipath coloured using two colours, starting at any vertex working in one direction around the cycle, alternately colouring adjacent pairs of vertices colour 1 or colour 2. This is illustrated in Figure 3.2.

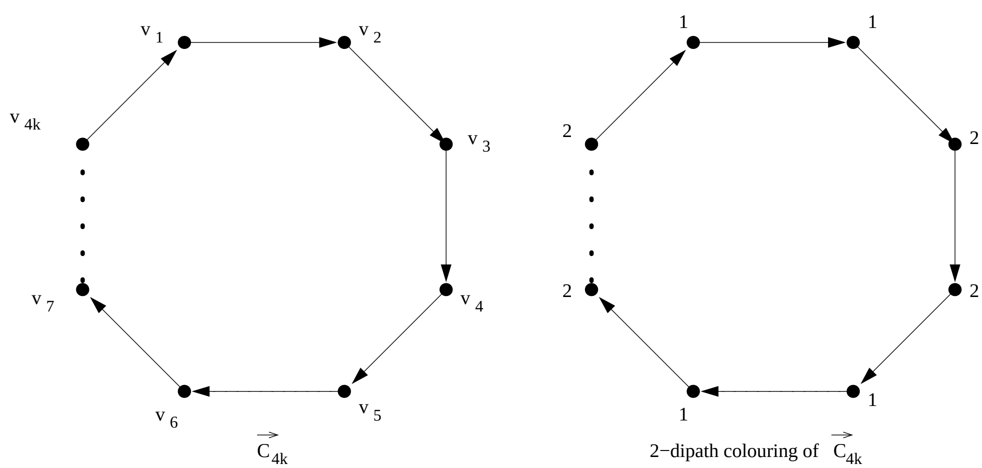

Figure 3.2: 2-dipath colouring of $\vec{C}_{4k}$

Example 3.3.17. For an oriented cycle C_n with $n \equiv 2 \pmod 4$, colour alternating pairs of vertices as before with colours 1 and 2, (starting at a given vertex and working in one direction around the cycle), and colour the final pair of uncolored vertices with colour 3, as shown in Figure 3.3.

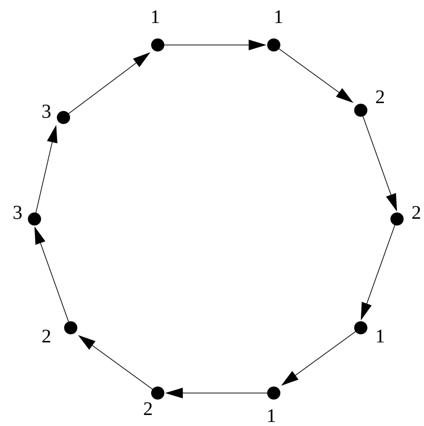

Figure 3.3: 2-dipath colouring of $\vec{C}_{10}$

Example 3.3.18. For an oriented cycle C_n with $n \equiv 1 \pmod 4$, again colour as before, moving one direction around the cycle colouring pairs of vertices colour 1, then colour 2, and then when only one vertex remains uncoloured, colour that final vertex colour 3. Figure 3.4 gives an example of such a colouring.

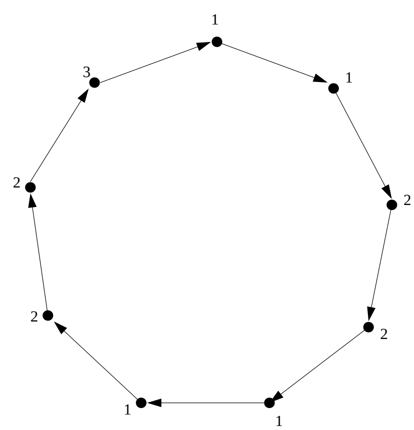

Figure 3.4: 2-dipath colouring of $\vec{C}_9$

Example 3.3.19. For an oriented cycle C_n with $n \equiv 3 \pmod 4$, (and $C_n \neq \vec{C}_3$), colour as before starting with colour 1 and moving around the cycle, until only three vertices remain uncoloured, (the last pair will have received colour 2). Colour the final three vertices as follows: the next vertex receives colour 1, and the final pair of vertices receive colour 3. A colouring of this type is given in Figure 3.5.

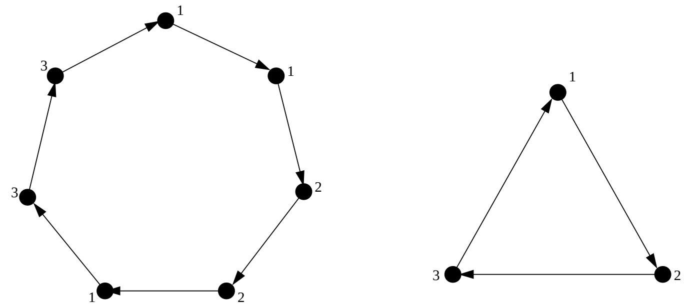

Figure 3.5: 2-dipath colouring of $\vec{C_7}$ and $\vec{C_3}$

Proposition 3.3.20. *For an oriented cycle $C = v_1, v_2, ..., v_n$ on n vertices:*

- $\chi_2(C) = 1 \Leftrightarrow n \equiv 0, 2 \pmod 4$ *and C does not contain* $\vec{P_3}$

- $\chi_2(C) = 2 \Leftrightarrow$
  1. $n \equiv 0 \pmod 4$ *and C contains a* $\vec{P_3}$*, or*
  2. $n \equiv 1, 3 \pmod 4$ *and* $C \ncong \vec{C_n}$*, or*
  3. $n \equiv 2 \pmod 4), C$ *contains a* $\vec{P_3}$*,* $d^-(v_i) = 0$ *or* $d^+(v_i) = 0$ *for some odd index i, and* $d^-(v_j) = 0$ *or* $d^+(v_j) = 0$ *for some even index j.*

- $\chi_2(C) = 3 \Leftrightarrow n \not\equiv 0 \pmod 4$ *and* $C \ncong \vec{C_n}$*.*

Proof. Together, 3.3.1 and the definition of χ_2 prove the first claim. Now consider the second proposed statement;

Suppose $\chi_2(C) = 2$.

- If $n \equiv 0 \pmod 4$, C can be coloured $v_1 = 1, v_2 = 1, v_3 = 2, v_4 = 2, v_5 = 1, v_6 = 1..., v_{n-1} = 2, v_n = 2$, ensuring that every pair of vertices at directed distance two in C receive different colours.

- If $n \equiv 1, 3 \pmod 4$, then C can not be a directed cycle, as by Lemma 3.3.15 the auxiliary graph of $\vec{C_n}$ is not bipartite.

- An oriented cycle satisfying the given conditions has an auxiliary graph which does not contain a cycle, and so its auxiliary graph by Fact 3.3.14 is the union of disjoint paths and can be coloured with two colours.

Let C be an oriented cycle with $n \not\equiv 0 \pmod 4$, which does not satisfy the above conditions. Then the auxiliary graph for C would contain either the odd cycle $v_1, v_3, ..., v_{n-1}$ or $v_2, v_4, ...v_n$. Therefore $\chi_2 \geq 3$. □

3.4 Complexity

Definition 3.4.1. For a graph G, with vertex set $V(G)$ and edge set $E(G)$, let V_1 and V_2 be two disjoint copies of $V(G)$, such that if $x \in V(G)$ then $x_1 \in V_1$ and $x_2 \in V_2$. Define $Bip(G)$ to be the oriented graph with vertex set $V(Bip(G)) = V_1 \cup V_2$, and edge set $E(Bip(G))$ constructed such that for any $xy \in E(G)$, we have $(x_1, x_2), (y_1, y_2), (y_2, x_1)$ and $(x_2, y_1) \in E(Bip(G))$.

Example 3.4.2. Figure 3.6 illustrates the construction of $Bip(G)$ from a given G.

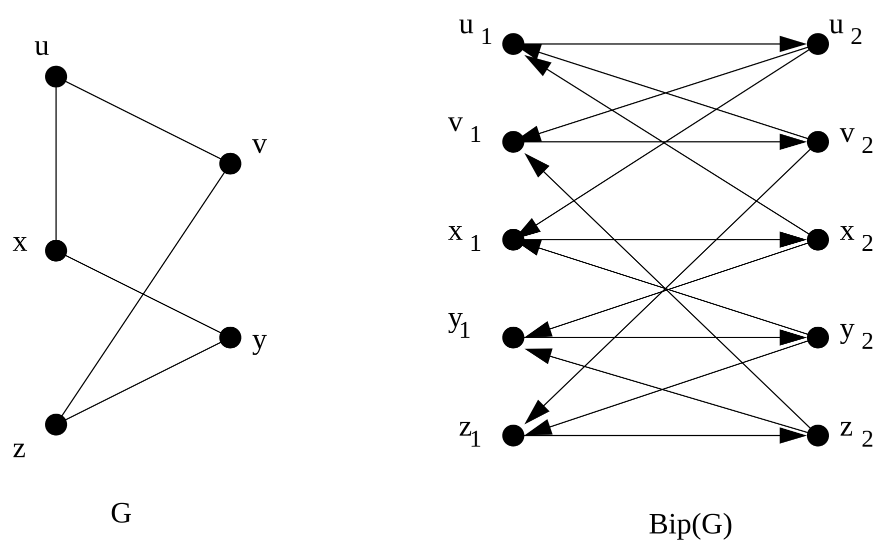

Figure 3.6: Construction of $Bip(G)$ from G

The following gives a method of determining whether a graph is k-colourable by determining whether or not $Bip(G)$ is 2-dipath k-colourable. This will help to prove that deciding if a given oriented graph G is 2-dipath k-colourable is $\mathcal{NP}$-complete, for any fixed $k \geq 3$.

Theorem 3.4.3. *A graph G is k-colourable if and only if $Bip(G)$ is 2-dipath colourable with k colours.*

Proof. ($\Rightarrow$) Suppose G is k-colourable, and consider such a colouring c of $V(G)$. For $x \in V(G)$ and $x_1, x_2 \in V(Bip(G))$, let $c(x_1) = c(x_2) = c(x)$. We need to show that for every pair of vertices $u, v \in Bip(G)$, with $\vec{d}(u,v) = 2$, we have $c(u) \neq c(v)$. For any 2-dipath in $Bip(G)$, by construction, both ends are in either $V(G_1)$ or $V(G_2)$. Without loss of generality, $u, v \in V(G_1)$. Then $u = x_1$ and $v = y_1$ with $\vec{d}(x_1, y_1) = 2$. There then exists $z_2 \in V(G_2)$ with either (x_1, z_2) and $(z_2, y_1) \in E(Bip(G))$ or (y_1, z_2) and $(z_2, x_1) \in E(Bip(G))$. We consider these two cases separately.

<u>**Case 1**</u>: If $(x_1, z_2) \in E(Bip(G))$, then by construction $z_2 = x_2$, as x_2 is the only out-neighbour of x_1 in $Bip(G)$. In this way, $xy \in E(G)$ as (x_1, x_2) and $(x_2, y_1) \in E(Bip(G))$, and so $c(x) \neq c(y)$ implies that $c(x_1) \neq c(y_1)$, and c extends to give a 2-dipath colouring of $Bip(G)$ with k colours.

<u>**Case 2**</u>: If $(y_1, z_2) \in E(Bip(G))$, then by construction $z_2 = y_2$, as y_2 is the only out-neighbour of y_2 in $Bip(G)$. As in Case 1, $xy \in E(G)$, so $c(x) \neq c(y)$ implies that $c(x_1) \neq c(y_1)$ in $Bip(G)$ and c is a 2-dipath colouring of $Bip(G)$ with k colours.

($\Leftarrow$) Suppose $Bip(G)$ is 2-dipath colourable with k colours. Take such a colouring c of $V(Bip(G))$ and extend it to $V(G)$; that is $c(x) = c(x_1)$, for $x \in V(G)$ and $x_1 \in V(Bip(G))$. For any $x, y \in G$ with $xy \in E(G)$ we have that x_1 and y_1 are joined by a directed path of length 2 in $Bip(G)$, and so $c(x_1) \neq c(y_1)$ implying that $c(x) \neq c(y)$ in G, giving a proper k-colouring of G. $\square$

Theorem 3.4.3 allows us to test whether or not a given graph G is k-colourable by testing if $Bip(G)$ is k-colourable. Since deciding k-colourability of a graph is $\mathcal{NP}$-complete for each fixed $k \geq 3$, [11] and since the oriented graph $Bip(G)$ can be constructed in time $\mathcal{O}(|E(G)|)$, we have the following corollary:

Corollary 3.4.4. *The problem of deciding if a given oriented graph has a 2-dipath k-colouring is $\mathcal{NP}$-complete for any fixed $k \geq 3$.*

Corollary 3.4.5. *The problem of deciding if a given oriented graph has a 2-dipath 2-colouring is Polynomial.*

Proof. By 3.3.5, a graph is 2-dipath colourable if and only if G_A is bipartite. The problem of determining if a graph is 2-dipath colourable with two colours then reduces to determining whether or not G_A is bipartite. $\square$

Alternatively we can establish the complexity of 2-dipath colourings with two colours by a transformation to 2-satisfiability. In the next section we use this approach to give an alternative algorithm to decide if a given oriented graph has a 2-dipath 2-colouring.

3.5 2-Satisfiability

In this section we reduce the problem of deciding if a given oriented graph has a 2-dipath 2-colouring to the problem of deciding if a collection of 2-literal Boolean expressions can all be made true at the same time. The direct relationship between 2-dipath 2-colouring and 2-satisfiability allows us to apply the algorithms that exist for 2-satisfiability to 2-dipath 2-colourings. There are two efficient algorithms that can be used to determine 2-satisfiablity in polynomial time, one using first-order resolution [14], and the other using strongly connected components of an implication graph [1].

Definition 3.5.1. [11] Let $B = \{b_1, b_2, ..., b_m\}$ be a set of Boolean variables. A *truth assignment* for B is a function $t : B \to \{0, 1\}$. If $t(b) = 1$, then we say the literal b is true. Otherwise if $t(b) = 0$, we say b is false. Note that a literal b is true if and only if $\bar{b}$ is false. A *clause* over B is a set of literals from B, such as $\{b_1, \bar{b}_2, b_3\}$, representing the disjunction of those literals, ie $b_1 \vee \bar{b}_2 \vee b_3$. A clause is *satisfied* if and only if at least one of the literals is true under that truth assignment. If C is a set of clauses, C is *satisfiable* if and only if there is some truth assignment that simualtaneously satisfies all clauses in C.

Definition 3.5.2. If B is a set of clauses, each of which has size at most two and B has a satisfying truth assignment, we say that B is 2-Satisfiable.

We will use the two colours, $0, 1$ (or False, True) as the possible values of a truth assignment (or colouring). The vertices of an oriented graph will be used as the Boolean variables and the set of two variable clauses will be made up of pairings of vertices at distance two in G. If all clauses can simultaneously be satisfied, then the graph in question can be 2-dipath coloured with two colours. With this construction, we get another proof that 2-dipath colouring with two colours can be determined in polynomial time.

We denote the set of all variables to then be $V(G)$, as the context will make it clear whether we mean the set of vertices, or the set of Boolean variables.

Definition 3.5.3. For an oriented graph G, with vertex set $V(G)$, define $B_G = \{x \veebar y : x, y \in V(G)$ and x and y are joined by a directed path of length $2\} \cup \{x \vee \bar{x} : x$ is not an end of any 2-dipath$\}$, where $x \veebar y$, is x *exclusive or* y, evaluating as true when exactly one of x or y is true. Note that $x \veebar y$ can be written using the conjunction of two disjunctions as follows; $(x \vee y) \wedge (\bar{x} \vee \bar{y})$.

In this way, B_G is the collection of two variable clauses we wish to satisfy. A satisfying truth assignment will give us a colouring of $V(G)$ with colours 0 and 1. The clauses $x \vee \bar{x}$ are trivially satisfiable and are included to assure every vertex is assigned a colour.

Theorem 3.5.4. *An oriented graph G is 2-dipath colourable with two colours if and only if B_G is 2-satisfiable.*

Proof. ($\Rightarrow$) Suppose a dirgraph G can be 2-dipath coloured with two colours. Take such a two colouring c of G, with colours 0 and 1, and consider B_G. For any clause $(x \veebar y) \in B_G$, the directed distance between x and y in G is $\vec{d}(x, y) = 2$, and so $c(x) \neq c(y)$ making every $x \veebar y$ evaluates as true.

($\Leftarrow$) Suppose that B_G is 2 satisfiable, and consider a satisfying assignment c of 0 and 1 to the vertices of G. By construction, any pair of vertices $x, y \in G$ with $\vec{d}(x, y) = 2$ has $x \veebar y \in B_G$. Therefore, under c, we have that $c(x) \neq c(y)$, making c is a 2-dipath colouring of G with two colours. □

Corollary 3.5.5. *For an oriented graph G, the decision of whether G has a 2-dipath 2-colouring is polynomial-time solvable.*

Chapter 4

Homomorphism Model

4.1 The Graph G_k

In this section, we will define a set of oriented graphs G_k, $k \geq 1$, such that an oriented graph G is 2-dipath k-colourable if and only if there is a homomorphism of $G \to G_k$.

Definition 4.1.1. Let G_k be the oriented graph defined in the following way: $V(G_k) = \{u = (u_0; u_1, u_2, ..., u_k) : 1 \leq u_0 \leq k$ and $u_i \in \{+, -\}, 1 \leq i \leq k\}$. $E(G_k) = \{(u, v) : u_0 = i$ and $v_0 = j$, with $u_j = +$ and $v_i = -\}$. For every vertex $u \in G_k$, define u_0 as the *index* of u.

Example 4.1.2. If $u = (3; +, +, -, +)$ and $v = (2; +, -, -, -,)$, then (u, v) is an arc of G_4, as $u_0 = 3$, $v_0 = 2$, $u_2 = +$ and $v_3 = -$.

Example 4.1.3. Figures 4.1 and 4.2 show the oriented graphs G_1, G_2 and G_3.

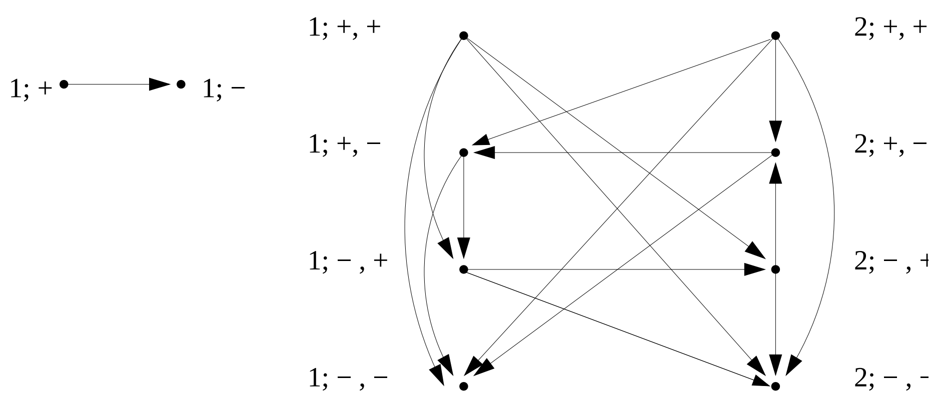

Figure 4.1: G_1 and G_2

4.2 Properties of G_k

Each oriented graph G_k has $k \cdot 2^k$ vertices, (k sets of all 2^k possible sequences of length k using $+, -$), $deg^+(v) + deg^-(v) = k \cdot 2^k/2 = k \cdot 2^{k-1}$, and $|E(G)| = (k \cdot 2^k) \cdot (k \cdot 2^{k-1})/2 = k^2 \cdot 2^{2k-2}$. In this section we examine how the properties of G_k are helpful towards determining the 2-dipath chromatic number of other oriented graphs using homomorphisms into G_k.

Definition 4.2.1. Let G_k be the the graph as defined in 4.1.1. Define $V_i \subset V(G_k)$, $i \in \{1, 2, ..., k\}$ as the collection of vertices $v \in V(G_k)$ with $v_0 = i$. Then $|V_i| = 2^k$.

Claim 4.2.2. *The subdigraph of G_k induced by each set V_i is a bipartite tournament with each independent set being of size 2^{k-1}.*

Proof. By definition, two vertices $u = (i, u_1, u_2, \ldots, u_k)$ and $v = (i, v_1, v_2, \ldots, v_k) \in V_i$ of G_k are adjacent if and only if $\{u_i, v_i\} = \{+, -\}$. Since there are 2^{k-1} vertices $(i, u_1, u_2, \ldots, u_k)$ with $u_i = +$ and an equal number with $u_i = -$, the claim follows. ☐

Using 4.2.2 we get that among any three vertices chosen from a single set V_i, at least one pair are not connected by an arc. In this way, any tournament $T \subset G_k$ must have $|V(T)| \leq 2k$, as no more than two vertices can belong to each set V_i.

Proposition 4.2.3. *There exists a homomorphism $G_k \to G_{k+1} \backslash V_i$ for any $i \in \{1, 2, ..., k+1\}$.*

Proof. If $i = k+1$, then the homomorphism is clear; mapping every vertex of G_k, to one of the vertices of G_{k+1} with the same first k positions, (there will be two choices). If $i \neq k+1$, proceed in the following way. For every $v \in G_k$, with $v_0 \neq i$, map v to $v' \in G_{k+1} \backslash V_i$, such that $v_0 = v'_0,\ v_1 = v'_1, ..., v_{i-1} = v'_{i-1},\ v_i = v'_{k+1},\ v_{i+1} = v'_{i+1}, ..., v_k = v'_k$. For $v \in G_k$ with $v_0 = i$, map v to $v' \in G_{k+1} \backslash V_i$ with $v'_0 = k+1$ and $v_1 = v'_1,\ v_2 = v'_2, ..., v_{i-1} = v'_{i-1},\ v_i = v'_{k+1},\ v_{i+1} = v'_{i+1}, ..., v_k = v'_k$. Suppose that $(u, v) \in E(G_k)$ with $u_0 = j,\ v_0 = l,\ u_l = +$ and $v_j = -$, we need to show that $(u', v') \in E(G_{k+1}$.

__**Case1**__: $i = j \neq k$: If $i = j$, then $u'_0 = k+1$, $u'_l = u_l = +$ and $v_j \to v'_{k+1} = v_l = -$, with $(u', v') \in E(G_{k+1} \backslash V_1)$.

__**Case2:**__ $j \neq i = l$: If $i = l$, then $v'_0 = k+1, v'_j = v_j -, u'_0 = k+1$, and $u_l \to u'_{k+1} = u_l = +$, with $(u', v') \in E(G_{k+1} \backslash V_i)$.

__**Case3**__: $i = j = l$: If $i = j = l$, then $u'_0 = k+1$ $v'_0 = k+1$, as well as $u_l \to u'_{k+1} = u_l = +$ and $v_j \to v'_{k+1} = v_j = -$, making $(u', v') \in E(G_{k+1} \backslash V_i)$.

<u>Case4:</u> $j \neq i \neq l$: If $j \neq i \neq l$, then $v'_0 = l, u'_0 = j, v_j = v'_j = -$ and $u_l = u'_l = +$, making $(u', v') \in E(G_{k+1} \backslash V_i)$.
In this way, for any arc $(u, v) \in E(G_k)$, the images u', v' also have $(u', v') \in E(G_{k_1} \backslash V_i)$.
□

Corollary 4.2.4. *There exists a homomorphism $G_k \to G_n$, for any $n \geq k$.*

Proposition 4.2.5. *There exists a homomorphism $G_{k+1} \backslash V_i \to G_k$ for any $i \in \{1, 2, ..., k\}$.*

Proof. Consider $G_{k+1} \backslash V_i$, for some $i \in \{1, 2, ..., k+1\}$, delete the i^{th} term v_i in every vertex sequence, and relabel as in 4.2.3. If $i \neq k+1$, then relabel all vertices v with $v_0 = k+1$, as $v'_0 = i, v'_1 = v_1, v'_2 = v_2, ..., v'_{i-1} = v_{i-1}, v'_i = v_{k+1}, v'_{i+1} = v_{i+1}, ..., v'_k = v_k$. Identify every pair of vertices with the same vertex sequence in each of the remaining k sets of vertices with the same index, (as they have the same in and out neighbours). This leaves k classes, with 2^k vertices in each class, clearly isomorphic to G_k. □

Lemma 4.2.6. *For $u, v \in V(G_k)$, with $u = (i; u_1, u_2, ..., u_k) \in V_i$ and $v = (j; v_1, v_2, ..., v_k) \in V_j$, $i \neq j$, there is no directed path of length two between u and v if and only if $u_l = v_l$, for all l, $1 \leq l \leq k$.*

Proof. ($\Rightarrow$) Suppose that u and v are not joined by a directed path of length two, but that $u_l \neq v_l$ for some $l, 1 \leq l \leq k$. Let $u_l = +$ and $v_l = -$. There is a vertex $w \in V(G_k)$ with $w_0 = l, w_i = -$ and $w_j = +$. In this way, $(u, w), (w, v) \in E(G_k)$ and u, v are joined by a directed path of length two, a contradiction.

($\Leftarrow$) Suppose that $u_l = v_l$ for all $l, 1 \leq l \leq k$, and that u and v are joined by a directed path of length two. Let w be the vertex connecting u and v, such that $(u, w), (w, v) \in E(G_k)$, with $w = (w_0 = m; w_1, w_2, ..., w_k)$. Since $(u, w) \in E(G_k)$, $u_m = +$ and $w_i = -$, as well as $w_j = +$ and $v_m = -$, and so $u_m \neq v_m$, a contradiction. □

Proposition 4.2.7. $\chi_2(G_k) = k$.

Proof. $\chi_2(G_k) \leq k$. Consider colouring G_k with k colours by assigning colour i to all vertices $v \in V_i$. Suppose there are two vertices, u, v with $u_0 = i = v_0, 1 \leq i \leq k$, that are connected by a 2-dipath in G_k. Let w be the internal vertex, then $w_i = -$, to receive an arc from a vertex in V_i, as well as a $w_i = +$ to be able to send an arc to class i, a contradiction.

$\chi_2(G_k) \geq k$. Consider a colouring of G_k with fewer than k colours. Since $|V(G_k)| = k \cdot 2^k$, partitioning the vertices of G_k into less than k classes places more than 2^k vertices in a single part, with no two at directed distance two. By the Pigeonhole Principle, there exists a colour i such that more than 2^k vertices are assigned colour i. Hence there exist three vertices $u, v \in V_i$ and $w \in V_j$ such that $i \neq j$ and there is no directed path of length two joining any two of u, v and w. However, by Lemma 4.2.6 there is a directed path of length two joining either u and w, or v and w, a contradiction. Therefore, the largest size of a colour class with no two vertices at directed distance two is then 2^k, and G_k cannot be coloured with fewer than k colours. □

Theorem 4.2.8. *An oriented graph G has a 2-dipath colouring with k colours if and only if there exists a homomorphism $G \to G_k$.*

Proof. (⇒) Suppose G has a *2-dipath colouring* c with the k colours, $1, 2, \ldots k$. For each $v \in V(G)$ with colour i, let $v_0 = i$. Next, if a vertex v has an in-neighbour from colour class j, set $v_j = -$, or if a vertex has an out-neighbour in colour class j, set $v_j = +$. Note that no vertex v can have both an in-neighbour u and an out-neighbour w from the same colour class, as u and w would then be at directed distance two and would not be able to receive the same colour by c. In the remaining positions of each vertex sequence, arbitrarily assign a " $+$ " or " $-$ ". By construction mapping vertices of G to the vertex in G_k with the same associated sequence is the desired homomorphism.

(⇐) Suppose there exists a homomorphism $f : G \to G_k$. Since G_k is an oriented graph, any vertices at directed distance two in G map to distinct vertices in G_k also connected by a dipath of length two. A 2-dipath colouring of G_k then gives a similar colouring of G by colouring vertex $v \in V(G)$ with colour i if and only if $f(v) \in V_i$. □

Corollary 4.2.9. *If there exists a homomorphism $G \to H$ and H has a 2-dipath colouring with k colours, then so does G.*

Proof. Given that H has a 2-dipath colouring with k colours, by Theorem 4.2.8 there is a homomorphism $H \to G_k$. Since homomorphisms compose, we have a homomorphism $G \to G_k$, which again by 4.2.8 tells us that G has a 2-dipath colouring with k colours. □

Corollary 4.2.10. *If an oriented graph G can be 2-dipath coloured with k colours, then it can be 2-dipath coloured with n colours for any $n \geq k$.*

Proof. If G can be 2-dipath coloured with k colours, then by 4.2.8 there is a homomorphism $G \to G_k$. By 4.2.3 there is a homomorphism $G_k \to G_{k+1}$. By induction there is a homomorphism $G_k \to G_n$ for any $n \geq k$. Finally, by composition of homomorphisms, there is a homomorphism from $G \to G_n$. $\square$

Here is a proof of the alternate form of 3.3.1 from Section 3.3, which makes use of composition of homomorphisms and the oriented graph G_1.

Fact 3.3.1 Let $\vec{P_3}$ be a directed path on three vertices. An oriented graph G has a 2-dipath colouring with one colour if and only if there is no homomorphism $\vec{P_3} \to G$.

Proof. ($\Rightarrow$) Suppose that G can be 2-dipath coloured with one colour. By 4.2.8 there is a homomorphism $G \to G_1$. If there is a homomorphism $\vec{P_3} \to G$ then by composition of homomorphisms, there is a homomorphism $\vec{P_3} \to G_1$, a contradiction.

($\Leftarrow$) Suppose that there is no homomorphism $\vec{P_3} \to G$, then G does not contain any 2-dipaths between two vertices. Every vertex can then be coloured with the same colour, giving a 2-dipath colouring of G with one colour. $\square$

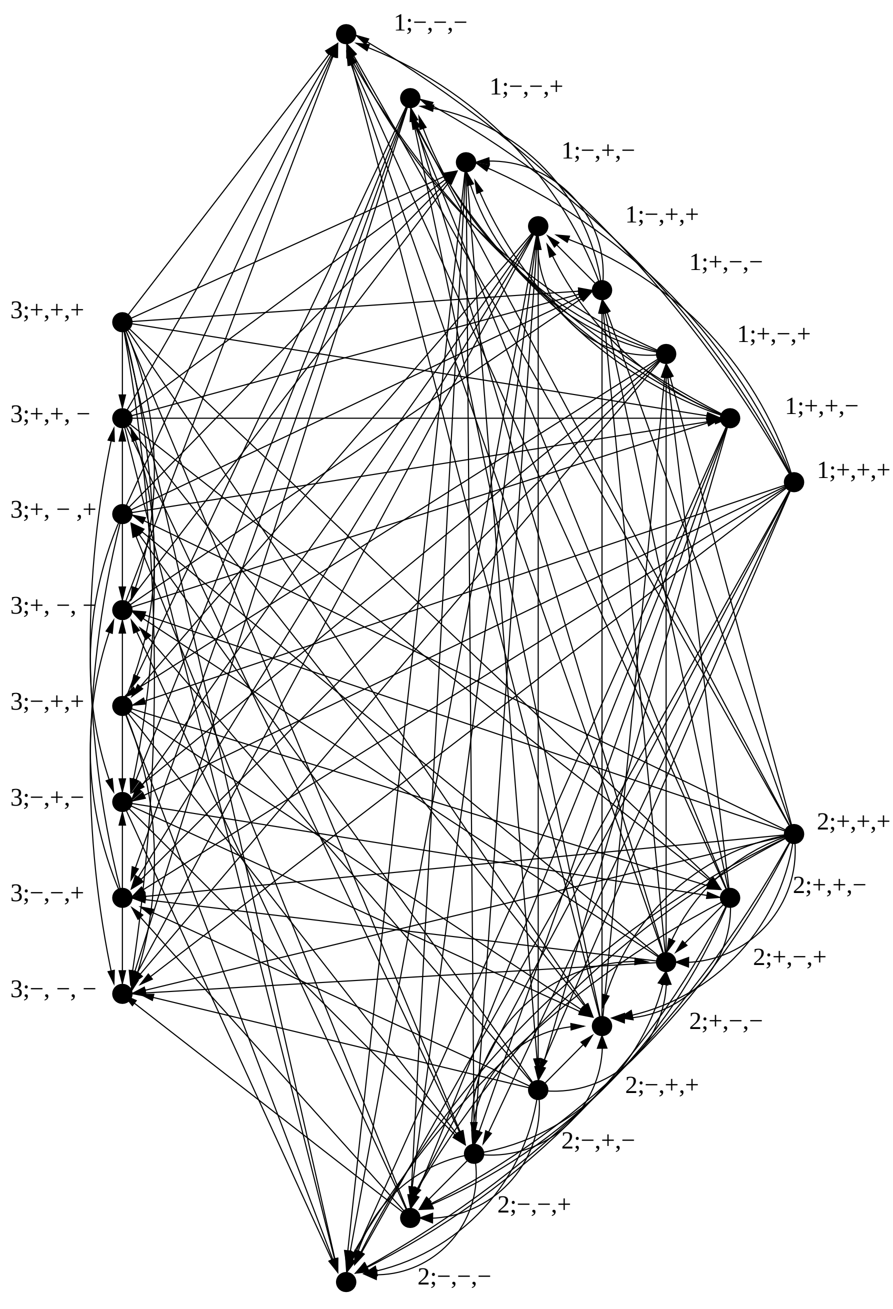

Figure 4.2: The oriented graph G_3

Chapter 5

2-Dipath Colourings of Tournaments

In this chapter we explore 2-dipath colourings of tournaments. We first examine bounds for the 2-dipath chromatic number of tournaments. We then examine the relationship between the 2-dipath chromatic number of tournaments and various other properties, including independence, matchings and coverings. Section 5.3 looks at 2-dipath colourings of bipartite tournaments.

5.1 2-Dipath Colouring of Tournaments

This section looks further at 2-dipath colourings of tournaments via their auxiliary graphs. Claim 4.2.2 says that among any three vertices of G_k, there is a pair that are not adjacent. For this reason, the 2-dipath colouring of any oriented three cycle requires at least two colours.

Lemma 5.1.1. *For a 2-dipath colouring of a tournament T_n, at most two vertices $x, y \in V(T_n)$ can received the same colour.*

Proof. Let u, v, and w be three vertices of a tournament T, the subtournament of T induced by $\{u, v, w\}$ is either transitive or a directed three cycle. In either case, some pair of these vertices are joined by a directed path of length two, and hence must be assigned different colours. □

Corollary 5.1.2. *Let T be a tournament. The independence number of its auxiliary graph T_A, satisfies $\alpha(T_A) \leq 2$.*

Proof. Suppose that $\alpha(T_A) \geq 3$, then there are vertices $x, y, z \in V(T_A)$ such that $xy, yz, xz \notin E(T_A)$. However, as in the proof of 5.1.1, given any three vertices of a tournament T, at least two of them are joined by a dipath of length two and so at least one of xy, yz or xz is an edge of T_A. □

Corollary 5.1.3. *For a tournament T_n on n vertices, $\lceil n/2 \rceil \leq \chi_2(T_n) \leq n$.*

Definition 5.1.4. Let D_1 and D_2 be disjoint oriented graphs. The *join of D_1 and D_2* is the oriented graph with vertex set $V(D_1) \cup V(D_2)$ and arc set $E(D_1) \cup E(D_2) \cup \{(x, y) : x \in V(D_1) \text{ and } y \in V(D_2)\}$.

Lemma 5.1.5. *Let $k \geq 1$. There exists a tournament T_k on the k vertices with $\chi_2(T_k) = k$.*

Proof. For odd values of k, take the tournament T on k vertices, $0, 1, 2, \ldots, k-1$ with $(u, v) \in E(G)$ if and only if $v - u \equiv 1, 2, \ldots, {}^{(k-1)}/_2 (\text{mod } k)$. Every pair of vertices u, v are joined by a directed path of length two in T, as every value $1, 2, ..., k-1$ is attainable as the sum of two of the values $1, 2, ..., {}^{(k-1)}/_2 (\text{mod } k)$. In this way, the auxiliary graph of T_k is complete and $\chi_2(T_k) = k$. For even values of k, take a tournament T on $k-3$ vertices as described above and a directed cycle on three vertices $\vec{C_3}$. Let T_k be the join of C_3 and T. Clearly every pair of vertices in T are joined by a path of length two, as are every pair of vertices in C_3. Furthermore, every pair of vertices (u, v) with $u \in V(\vec{C_3})$ and $v \in V(T)$, are joined by a directed path of length two, with $(u, w) \in E(\vec{C_3}), (w, v) \in E(T_k)$.

□

Theorem 5.1.6. *For any $n \geq 1$ and any k, such that $\lceil n/2 \rceil \leq k \leq n$, there is a tournament T_n on n vertices with $\chi_2(T_n) = k$.*

Proof. Take a tournament T on k vertices as in 5.1.5 with $\chi_2(T) = k$. Construct a tournament T_n on n vertices from T by replacing $n - k$ of the vertices $v \in V(T)$ with pairs of vertices $\{v_1, v_2\}$ joined by the arc (v_1, v_2) such that v_1 and v_2 have the same neighbourhoods in the rest of T_n. In this way, $N^+(v_1) \cap N^-(v_2) = \emptyset$ and $N^-(v_1) \cap N^+(v_2) = \emptyset$ and there is no directed path of length two between any such pair, but there is a 2-dipath between each such vertex and any other vertex in T_n.

A 2-dipath colouring of T with k colours then extends to a 2-dipath colouring of T_n with k colours, with every pair v_1, v_2 receiving the same colour. The tournament T_n cannot be coloured with fewer than k colours, as $\chi_2(T) = k$, and all 2-dipaths in T are preserved in T_n. Therefore $\chi_2(T_n) = k$. □

Example 5.1.7. The following is a demonstration of how you can construct a tournament T_n with $\chi_2(T_n) = k$, for $(n > k \geq n/2)$. The example is done for $n = 7$ and $k = 5$, where we begin by taking a tournament T_5 on 5 vertices, with $\chi_2(T_5) = 5$.

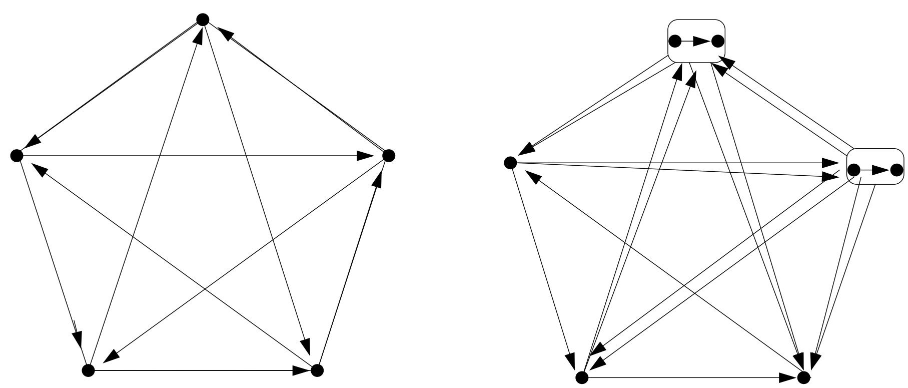

Figure 5.1: Construction of $\chi_2(T_7) = 5$

5.2 Clique Coverings and Matchings of Tournaments

In a 2-dipath colouring of a tournament, at most 2 vertices can be assigned the same colour, therefore $\omega'(\overline{T}_A) \leq 2$. We apply the ideas in 3.2.1 in light of this additional information.

Theorem 5.2.1. *The minimum number of colours required to 2-dipath colour a tournament T equals the minimum number of disjoint vertices and edges, or K_1's and K_2's required to partition the vertex set of $\overline{T_A}$.*

Proof. A partition of $\overline{T_A}$ into K_1's and K_2's gives a 2-dipath colouring of T by assigning each such clique a different colour. Any two vertices joined by a directed path of length two would be adjacent in T_A and therefore not adjacent in $\overline{T}_A$, and hence would receive different colours. Since $\overline{T_A}$ has no three cycles, the rest of the result follows from 3.2.1. □

We now show that the smallest number of K_1's and K_2's needed to cover the vertices of a graph G equals the number of edges in a maximum matching, plus the number of remaining unmatched vertices. For a matching M of a graph G, the minimum number of K_1's and K_2's is then $|M| + (n - 2 \cdot |M|)$, where $|M|$ is the number of K_2's and $n - 2 \cdot |M|$ is the number of vertices not covered by those $|M|$ disjoint edges.

Lemma 5.2.2. *For a graph G, the minimum number of disjoint K_1's and K_2's required to cover $V(G)$ equals the size of a maximum matching, plus the number of vertices not saturated by the matching.*

Proof. Clearly every covering of G by disjoint K_1's and K_2's gives a matching of G, by only considering the K_2's. In order to minimize the number of such cliques required to cover G, we maximize the number of K_2's. The maximum number of disjoint K_2's in such a covering, is then equal to the size of a maximum matching of G. Adding the remaining vertices of G not saturated by the matching as K_1's gives the minimum number of cliques required. □

Proposition 5.2.3. *Let T be a tournament with n vertices. Then $\chi_2(T) = n - |M|$, where M is a maximum matching in $\overline{T_A}$.*

Proof. By 5.2.1 we know that $\chi_2(T)$ equals the minimum number of disjoint K_1's and K_2's required to cover $V(T_A)$. Using Lemma 5.2.2, we can relate these cliques to a maximum matching in T_A, plus the remaining uncovered vertices. Combining these gives us the desired result; $\chi_2(T) = |M| + (n - 2 \cdot |M|) = n - |M|$. □

Corollary 5.2.4. *For a tournament T on n vertices, the auxiliary graph $\overline{T_A}$ has a perfect matching if and only if $\chi_2(T) = n/2$.*

Proof. ($\Rightarrow$) Let $\overline{T_A}$ have a perfect matching M, with $|M| = n/2$. Colour T in the following way; $c(x) = c(y)$ if and only if $xy \in E(M)$. Then $xy \in E(M)$ implies that $xy \notin E(T_A)$, therefore x, y are not joined by a 2-dipath in T. Hence we have a 2-dipath colouring. Furthermore, by 5.1.3 a tournament cannot be 2-dipath coloured with less than $n/2$ colours. Therefore $\chi_2(T) = n/2$.

($\Leftarrow$) Consider a 2-dipath colouring c of T with $n/2$ colours. By 5.1.1 there can be at most two vertices in each colour class, and since there are $n/2$ colours, each colour class contains exactly two vertices. Clearly if $c(x) = c(y)$ for $x, y \in E(T)$, then $xy \notin E(T_A)$. Taking every edge in $\overline{T_A}$ corresponding to a colour class of c, gives a matching M of $\overline{T_A}$ with $|M| = n/2$. Hence, M is a perfect matching. □

Proposition 5.2.3 implies that for a tournament T, $\chi_2(T)$ can be computed in polynomial time, by using Edmonds Algorithm [14] to find a maximum matching in T_A.

5.3 Equivalence Relations on Bipartite Tournaments

The same idea of matching and clique covers can be applied to bipartite tournaments, with a slight modification in the initial setup. A bipartite tournament, $T(m,n)$ has two independent sets A and B, with $|A| = n$ and $|B| = m$. The notation $T(A,B)$ is also sometimes used to denote such a tournament.

Definition 5.3.1. Define the equivalence relation Θ on the vertex set $V(G)$ of an oriented graph G in the following way;

$$\text{For } u, v \in V(G), u\Theta v \Leftrightarrow (N^+(u) = N^+(v)) \wedge (N^-(u) = N^-(v)).$$

For bipartite tournaments $T(m,n)$, if $u\Theta v$ for $u, v \in V(T(m,n))$, then u and v are in the same independent set of $T(m,n)$ and u, v are guaranteed to not be joined by a dipath of length two in $T(m,n)$, as a 2-dipath would require a third vertex which was an in-neighbour of one and an out-neighbour of the the other.

Definition 5.3.2. Let $T(m,n)$ be a bipartite tournament. The quotient of T with respect to Θ is the bipartite tournament $T' = T/\Theta$ with vertex set $V(T') = \{[x] : x \in V(T)\}$, the set of equivalence classes of Θ, and $([x],[y]) \in E(T')$ if and only if $(x,y) \in E(T)$.

Proposition 5.3.3. *Let $T = T(m,n)$ be a bipartite tournament with independent sets A and B, and let $T' = T/\Theta$. Then T' is a bipartite tournament with bipartition (A', B'), where $A' = \{[x] : x \in A\}$ and $B' = \{[x] : x \in B\}$.*

Corollary 5.3.4. *If $T = T(m,n)$ is a bipartite tournament with independent sets A and B and $T'(A', B') = T/\Theta$, then every two vertices in A' are joined by a directed path of length two, and every two vertices in B' are joined by a directed path of length two.*

Proof. Without loss of generality, let $u, v \in A'$. By definition of Θ, either $N^+(u) \cap N^-(v) \neq \emptyset$ or $N^+(v) \cap N^-(u) \neq \emptyset$. In either case, u and v are joined by a directed path of length two. □

Proposition 5.3.5. *Let $T'(A', B')$ be the quotient of the bipartite tournament $T = T_{m,n}$ with bipartition (A, B) taken with respect to Θ. Then, $T'_A \cong K_{|A'|} \cup K_{|B'|}$.*

Proof. By the previous result, the subgraph of T'_A induced by A' is a clique, and similarly for B'. No vertex in A' can be joined to a vertex in B' by a directed path path of length two (in either direction). The result follows as these two structures have disjoint auxiliary graphs. □

Given that in a 2-dipath colouring of the oriented graph $T'(A', B')$ there are at most two vertices in any colour class, we can apply the same results in clique coverings and matching as were used for tournaments. By the construction of T'_A, among any three vertices, there wiill be at least one edge, and so $\overline{T'_A}$ will be triangle-free.
Any colouring of T' can be used to give a colouring of $T(m, n)$ by colouring all vertices of $T(m, n)$ in the same equivalence class θ_i the colour the vertex θ_i receives in the colouring of $T' = ({}^{T(m,n)}/_{\Theta})$. Alternatively, the equivalence relation Θ can also be used to determine the 2-dipath chromatic number of a bipartite tournament without using clique coverings or matchings.

Corollary 5.3.6. *Let T be a bipartite tournament with bipartition (A, B) and let $T'(A', B') = T/\Theta$. Then, $\chi_2(T) = max\{|A'|, |B'|\}$.*

Chapter 6

Proper 2-dipath Colourings

Recall that for a 2-dipath colourings, two vertices $x, y \in V(G)$ are required to receive different colours if $\vec{d}(x, y) = 2$. Changing this definition slightly allows the us to consider 2-dipath colourings that are also proper colourings. Using Definition 2.0.8 we were able to establish a characterization of 2-dipath 1- and 2-colourable oriented graphs, a homomorphism model for these colourings and results concerning the 2-dipath chromatic number of tournaments. In this chapter, we examine which of these results also hold for proper 2-dipath colourings.

6.1 Proper 2-dipath Colourings

Definition 6.1.1. Let G'_A be the undirected auxiliary graph of an oriented graph G defined by $V(G'_A) = V(G)$ and $E(G'_A) = \{xy : \vec{d}_G(x, y) \leq 2\}$.

Recall that $U(G)$ denotes the undirected underlying graph of the directed graph G.

Fact 6.1.2. *For an oriented graph G, $G'_A = U(G^2)$.*

For the remainder of this document, we will refer to $U(G^2)$ instead of G'_A.

Theorem 6.1.3. *There is a 1-1 correspondence between the set of proper 2-dipath k-colourings of an oriented graph G and the set of k-colourings of $U(G^2)$.*

Proof. ($\Rightarrow$) Let c be a proper 2-dipath colouring of G with k colours. For any two vertices $x, y \in V(U(G^2))$ with $\vec{d}(x, y) \leq 2$ we have $xy \in E(U(G^2))$, with $c(x) \neq c(y)$, making c a proper colouring of $U(G^2)$ with k colours.

(⇐) Let c be a proper colouring of $U(G^2)$ with k colours. For any two vertices $x, y \in V(G)$ with $d(x, y) \leq 2$ we have $xy \in E(U(G^2))$, with $c(x) \neq c(y)$ making c a proper 2-dipath colouring of G with k colours. □

Corollary 6.1.4. *For an oriented graph G, $\chi'_2(G) = \chi(U(G^2))$.*

Proof. (⇒) Let c be a proper 2-dipath colouring of G, and consider a pair of vertices x, y adjacent in $U(G^2)$. For $xy \in E(U(G^2)), \vec{d}(x, y) \leq 2$ in G so x and y receive different colours by c, making c a proper colouring of $U(G^2)$.

(⇐) For any two vertices $x, y \in V(G)$, with $\vec{d}(x, y) \leq 2$, we have $xy \in V(U(G^2))$. By the definition of G^2, in a proper colouring c of $U(G^2)$ we have $c(x) \neq c(y)$ for any such pair x, y. In this way, c is also a proper 2-dipath colouring of G. □

The same idea of finding $\chi_2(G)$ by finding a clique covering of G_A can be used to find $\chi'_2(G)$ by determining $\omega'(\overline{U(G^2)})$. By Corollary 6.1.4, bounds for $\chi(U(G^2))$ are bounds for $\chi'_2 G)$. In particular, $\chi'_2(G) \leq 1 + \Delta(U(G^2)) \leq 1 + \Delta^+(G) + \Delta^-(G) + \Delta^+(G)^2 + \Delta^-(G)^2$.

Corollary 6.1.5. *For an oriented graph G with $U(G^2)$, $\chi'_2(G) = \omega'(\overline{U(G^2)})$*

Proof. Since $\chi(H) \geq \omega'(\overline{H})$ for any undirected graph H, the result follows from Corollary 6.1.4. □

Corollary 6.1.6. *For an oriented graph G with auxiliary graph $U(G^2)$, $\chi'_2(G) \geq \omega(U(G^2))$.*

6.2 Characterizations

6.2.1 Proper 2-Dipath 2-Colourable Oriented Graphs

This section gives a characterization of the oriented graphs which have proper 2-dipath colourings with two colours. By definition, these are all oriented graphs which do not contain a directed path of length two. Note that the only graphs which can be proper 2-dipath coloured with one colour are oriented graphs without any edges.

Proposition 6.2.1. *An oriented graph G has a proper 2-dipath colouring with two colours if and only if G does not contain $\vec{P}_3$ as a subgraph.*

Proof. ($\Rightarrow$) If G has a proper 2-dipath colouring with two colours, then clearly G cannot contain a $\vec{P}_3$, as it requires three colours in a proper 2-dipath colouring.

($\Leftarrow$) If G contains no directed path of length two, then $G^2 = G$. Since every orientation of an odd cycle contains a directed path of length two, G does not contain an oriented odd cycle. Therefore, G is an orientation of a bipartite graph. Since $G^2 = G, \chi_2'(G) = \chi(U(G^2)) \leq 2$. □

Corollary 6.2.2. *For an oriented graph G, $\chi_2'(G) = 2$ if and only if $\chi_2(G) = 1$.*

Proof. By Corollary 6.2.1, the set of oriented graphs with $\chi_2' = 2$ is identical to the set of oriented graphs with $\chi_2 = 1$ and at least one edge. Therefore, the results in Section 3.3 for 2-dipath 1-colourable oriented graphs also completely describe the oriented graphs with $\chi_2' = 2$. □

6.2.2 Proper 2-Dipath 3-Colourable Oriented graphs

We notice that any oriented graph G containing a dipath of length at least two, $\vec{P}_3$, must have $\chi_2'(G) \geq 3$. In what follows, we restrict the length of a longest dipath to be exactly two. The following is a result for all oriented graphs having longest dipath with length two, using some results of G. J. Chang et al. [7]. We begin with a definition to give us the necessary background information to be able to apply these results.

Definition 6.2.3. [7] For $j, k \in \mathbb{Z}^+, j \geq k$, we define an $L(j,k)$-labeling of an oriented graph G to be a function $f : V(G) \to \{k : k \in \mathbb{Z}, k \geq 0\}$, such that $|f(x) - f(y)| \geq j$ for all $(x,y) \in E(G)$, and $|f(x) - f(y)| \geq k$ for all $\vec{d}(x,y) = 2$ in G. The minimum of the maximum label value used, taken over all $L(j,k)$-labelings of an oriented graph G is the $\vec{\lambda}_{j,k}$-number, and is denoted $\vec{\lambda}_{j,k}(G)$.

Note that a proper 2-dipath colouring of an oriented graph G is a $L(1,1)$-labelling of G, as only the labels assigned to adjacent vertices and vertices at distance two must be unique.

Proposition 6.2.4. *For an oriented graph G, $\vec{\lambda}_{1,1}(G) = \chi'_2(G) - 1$.*

Theorem 6.2.5. *[7] Let G be an oriented bipartite graph with a directed path of length 3 but no directed path on four vertices. Then $\vec{\lambda}_{j,k}(G) = j + k$.*

Theorem 6.2.6. *[7] Let G be an oriented non-bipartite graph with a directed path of length 3 but no directed path on four vertices. Then $\vec{\lambda}_{j,k}(G) = 2j$.*

Theorem 6.2.7. *Let G be an oriented graph with a directed path of length three, but no directed path of length four. Then $\chi'_2(G) = 3$.*

Proof. We consider two cases; one for bipartite oriented graphs, and the other for oriented non-bipartite graphs, and apply the corresponding theorems from [7], using $j = k = 1$.

<u>**Case 1**</u>: If G is an oriented bipartite graph, then by 6.2.1 we get that $\vec{\lambda}_{1,1}(G) = j+k = 1 + 1 = 2$. Applying Lemma 6.2.4 to this equality gives us $\chi'_2(G) = \vec{\lambda}_{1,1}(G) + 1 = 3$, as required.

<u>**Case 2**</u>: If G is non-bipartite, then by 6.2.6 we have $\vec{\lambda}_{1,1}(G) = 2j = 2$. Again applying Lemma 6.2.4 we get $\chi'_2(G) = \vec{\lambda}_{1,1} + 1 = 3$, as required. □

We conclude this section by proving that every oriented tree is proper 2-dipath 3-colourable. The argument implies an efficient algorithm to optimally proper 2-dipath colour a given oriented tree.

Proposition 6.2.8. *For an oriented tree $T, \chi_2'(T) \leq 3$.*

Proof. If T does not contain a dipath of length greater than 2, then by **??** $\chi(T) \leq 2$. If T has a dipath of length greater than 2, then we will prove the result using induction on $n = |V(T)|$. The statement is true for every oriented path of length two. Assume it is true for $n \leq k$ for some $k \geq 2$, and let T be an oriented tree on $k+1$ vertices. Since $k+1 > 2$, T has at least one leaf; take x to be a leaf of maximum eccentricity. Let s be the only neighbour of x in T, and let $A = N^-(s)$ and $B = N^+(s), (x \in A \cup B)$. Define a new tree, $T' = T - (A \cup B)$, with s as a leaf of T', with unique neighbour y in T'. Since $A \cup B \neq |V(T')| \leq k$, and so T' has a proper 2-dipath colouring c with three colours. Without loss of generality, let $c(s) = 1$ and $c(y) = 2$. If $(s, y) \in E(T')$, then a proper 2-dipath colouring of T is obtained by assigning all vertices of B colour 1 ($c(B) = 1$) and every vertex in A colour 3 ($c(A) = 3$). Alternatively, if $(y, s) \in E(T')$, then colour every vertex of A colour 1 ($c(A) = 1$) and every vertex in B colour 3, ($c(B) = 3$). This extension of c gives a proper 2-dipath colouring of T as required. □

The proof of Proposition 6.2.8 implies the following algorithm to optimally proper 2-dipath colour a given oriented tree T.

Algorithm 6.2.9. *Proper 2-dipath colouring of an oriented tree.*

1. *Pick any vertex $v \in V(T)$, and let it be the root of T, setting $c(v) = 1$.*
2. *Repeat over all coloured vertices $u \in V(T)$ with uncoloured neighbours;*
    - *If c(u)=1, colour the uncoloured vertices in $N^+(u)$ with colour 2, and the uncoloured vertices in $N^-(u)$ with colour 3.*
    - *If c(u) = 2, then colour the uncoloured vertices in $N^+(u)$ with colour 3, and the uncoloured vertices in $N^-(u)$ with colour 1.*
    - *If c(u) = 3, then colour the uncoloured vertices in $N^+(u)$ with colour 1, and the uncoloured vertices in $N^-(u)$ with colour 2.*
3. *Repeat step 2 until all vertices have been assigned a colour.*

6.3 2-Satisfiability

The auxiliary graph $U(G^2)$ can be used to efficiently test whether $\chi'_2(G) = 2$, as can Proposition 6.2.1. In this section we describe how reduction to 2-satisfiability can also be used for this purpose.

Definition 6.3.1. For an oriented graph G, let $B'_G = \{x \veebar y : \vec{d}(x,y) \leq 2$ for $x, y \in V(G)\} \cup \{x \vee \neg x : x$ is not incident with an arc$\}$, where $x \veebar y$ is x exclusive or y.

Again, as was the case for 2-dipath colourings, B'_G gives a collection of two variable clauses representing adjacencies and 2-dipaths in G. These variables, (or vertices) can be given a truth assignment (or colour), of $0, 1$, in turn providing a truth value for every clause in B'_G. An assignment of $0, 1$ to the variables (vertices), making B_G 2-satisfiable is also proper 2-dipath colouring of the oriented graph G, with colours $0, 1$.

Theorem 6.3.2. *An oriented graph G has a proper 2-dipath colouring with two colours if and only if B_G is 2-satisfiable.*

Proof. ($\Rightarrow$) Suppose a digraph G has a proper 2-dipath colouring with two colours. Take such a two colouring c of G with colours 0 and 1, and consider B'_G. For any $x \veebar y \in B_G$ we have that $\vec{d}(x,y) \leq 2$ in G, so $c(x) \neq c(y)$ and every $x \veebar y$ evaluates as true. In this way, the conjunction of B'_G is 2-satisfiable.

($\Leftarrow$) Suppose that B'_G is 2-satisfiable. Consider a satisfying truth assignment c of 0 and 1 to the variables/vertices of G. By construction, any $x, y \in G$ with $\vec{d}(x,y) \leq 2$ has $x \veebar y \in B'_G$, under c, we have that $c(x) \neq c(y)$, making c a proper 2-dipath colouring of G with two colours. □

Corollary 6.3.3. *For an oriented graph G, deciding whether or not G has a proper 2-dipath colouring with two colours is polynomial time solvable.*

Chapter 7

Homomorphism Model

7.1 The Graph G'_k

Definition 7.1.1. The oriented graph G'_k is defined in the following way;

$V(G'_k) = \{u = (u_0; u_1, u_2, ..., u_k) : 1 \leq u_0 \leq k, u_i \in \{+, -\}$, for $i \neq u_0$, and $u_i = \cdot$ for $i = u_0.\}$

$E(G) = \{(u, v) : u_0 = i, v_0 = j$ with $v_i = -$ and $u_j = +\}$

For any vertex $v \in V(G'_k)$, $v_0 = i$ is defined as the index of v, (with v having a dot in the i^{th} position). In this way, there are no arcs among vertices with the same index.

Example 7.1.2. For $u = (1; \cdot, -, +, -)$ and $v = (2; +, \cdot, -, -)$, we have that $(v, y) \in E(G'_k)$, as $u_0 = 1, v_0 = 2$, with $u_2 = -$ and $v_1 = +$.

For $k \geq 2$, each oriented graph G'_k has $k \cdot 2^{k-1}$ vertices, (k sets of all 2^{k-1} possibilities of $k-1$ strings of two values; $+, -$). For any vertex $v \in V(G'_k)$, $deg(v) = (k-1) \cdot 2^{k-1}/2 = (k-1) \cdot 2^{k-2}$, as any vertex v will give or receive arcs from half of all other vertices u, with $u_0 \neq v_0$. In this way, we get $|E(G'_k)| = k \cdot 2^{k-1} \cdot (k-1) \cdot 2^{k-2}/2 = \binom{k}{2} \cdot 2^{2k-3}$.

Definition 7.1.3. Let G'_k be the graph as defined in 7.1.1. Define $V_i \subseteq V(G'_k), 1 \leq i \leq k$ as the collection of vertices $v \in V(G'_k)$, with $v_0 = i$; i.e. $V_i = \{v : v \in V(G'_k) \wedge v_0 = i\}$. (The size of each set V_i is $|V_i| = 2^{k-1}$, the number of $(k-1)$-sequences using $+$ or $-$.)

Note that, by construction, the subgraph of G'_k induced by each V_i is edgeless.

Figure 7.1 show the oriented graphs G'_1, G'_2 and G'_3.

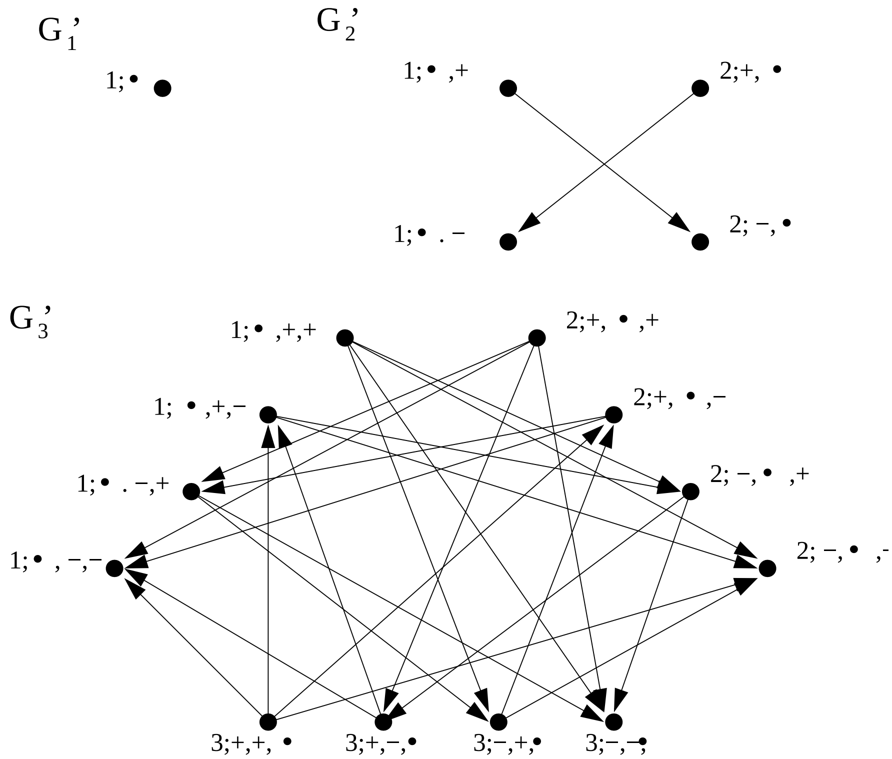

Figure 7.1: G'_1, G'_2 and G'_3

7.2 Properties of G'_k

Proposition 7.2.1. *There exists a homomorphism $G'_k \to G'_{k+1} \backslash V_i$ for any $i \in \{1, 2, ..., k\}$.*

Proof. For $i = k+1$ there is a homomorphism $G'_k \to G'_{k+1} \backslash V_i$, by mapping every vertex of G'_k to one with the same initial $k+1$ terms in the vertex sequence. If $i \neq k+1$, then relabel the vertices of $G'_k \backslash V_i$ in the following way. For every $v \in G'_{k+1} \backslash V_i$, relabel v_{i+1} as v_i, v_{i+2} as $v_i + 1$,..., v_{k+1} as v_k, and if $v_0 \geq i$, relabel v_0 as $v_0 - 1$. Again, as was the case with $i = k+1$, there is a mapping of vertices of G'_k into $G'_{k+1} \backslash V_i$, by mapping vertices to those with like sequences. ☐

Corollary 7.2.2. *There exists a homomorphism $G'_k \to G'_n$ for any $n \geq k$.*

Proposition 7.2.3. *There exists a homomorphism $G'_{k+1} \backslash V_i \to G_k$, for any $i \in \{1, 2, ..., k\}$.*

Proof. Take every $v \in V(G'_{k+1} \backslash V_i)$ and delete the i^{th} position, making the length of the vertex sequences k. Reorganize and relabel these vertices as in 7.2.1. Identifying vertices in each V_j with the same vertex sequence leaves an oriented graph on $k \cdot 2^{k-1}$ vertices which is isomorphic, (by construction), to G'_k. □

Lemma 7.2.4. *For an oriented graph G'_k, there are no dipaths of length two connecting any two vertices $x, y \in V_i$, for $1 \leq i \leq k$.*

Proof. Suppose for some $x, y \in V_i, 1 \leq i \leq k$ that $\vec{d}(x, y) = 2$. There is some vertex $w \in V(G'_k) - V_i$, such that (x, w) and $(w, y) \in E(G'_k)$. Since $(x, w) \in E(G'_k)$, we have that $w_i = -$ as w receives an arc from a vertex in V_i. Similarly, since $(w, y) \in E(G'_k)$, we get that $w_i = +$, as w sends an arc to a vertex in V_i. Since any position in a vertex sequence has exactly one sign, no such w exists. □

Lemma 7.2.5. *For $u, v \in V(G'_k)$, with $u = (i; u_1, u_2, ..., u_k) \in V_i$ and $v = (j; v_1, v_2, ..., v_k) \in V_j$, $i \neq j$, there is no directed path of length two between u and v if and only if $u_l = v_l$, for all l, $1 \leq l \leq k$.*

Proof. ($\Rightarrow$) Suppose that u and v do not have a directed path of length two, but that $u_l \neq v_l$ for some $l, 1 \leq l \leq k$. Let $u_l = +$ and $v_l = -$. There is a vertex $w \in V(G'_k)$ with $w_0 = l, w_i = -$ and $w_j = +$. In this way, $(u, w), (w, v) \in E(G'_k)$ and u, v are joined by a directed path of length two, a contradiction.

($\Leftarrow$) Suppose that $u_l = v_l$ for all $l, 1 \leq l \leq k$, and that u and v are at directed distance two. Let w be the vertex connecting u and v, such that $(u, w), (w, v) \in E(G'_k)$, with $w = (w_0 = m; w_1, w_2, ..., w_k)$. Since $(u, w) \in E(G'_k), u_m = +$ and $w_i = -$, as well as $w_j = +$ and $v_m = -$, and so $u_m \neq v_m$; a contradiction. □

Proposition 7.2.6. $\chi'_2(G'_k) = k$.

Proof. $\chi'_2(G'_k) \leq k$. Consider a colouring of G'_k by assigning the vertices of every V_i, colour i. By Lemma 7.2.4, there are no 2-dipaths among the vertices of these sets, and by construction no edges. In this way, every pair of vertices x, y with $\vec{d}(x, y) \leq 2$ lie in different colour classes, making this a proper 2-dipath k-colouring of G'_k.

$\chi'_2(G'_k) \geq k$. Consider colouring G'_k with fewer than k colours. Since $|V(G'_k)| = k \cdot 2^{k-1}$, partitioning the vertices of G'_k into less than k classes, places more than 2^{k-1} vertices into some class. Therefore there exist vertices u, v, w such that $u, v \in V_i$ and $w \in V_j$, $j \neq i$. By Lemma 7.2.5 and the construction of G'_k, some two of u, v, w are joined by an arc or a 2-dipath. In this way, no set of vertices of size bigger than 2^{k+1} can be assigned the same colour, making $\chi_2'(G_k') \geq k$. □

Theorem 7.2.7. *An oriented graph G has a proper 2-dipath colouring with k colours if and only if there exists a homomorphism $G \to G_k'$.*

Proof. ($\Rightarrow$) Suppose G has a proper 2-dipath colouring c with k colours. Assign each of the k colour classes labels of $1, 2, ..., k$ and consider building vertex sequences for each of the vertices in $V(G)$. For each $v \in V(G)$ with colour i, let $v_0 = i$ and $v_i = \cdot$. Next, if v has an in-neighour in colour class j, set $v_j = +$, or if v has an out-neighbour in colour class j, set $v_j = -$. Note: as was the case for 2-dipath colourings, no vertex in $V(G)$ can have an in-neighbour and an out-neighbour which receive the same colour under c, as they would be at directed distance two. Finally, in the remaining unassigned positions, arbitrarily distribute $+/-$'s to complete every vertex's $k+1$ sequence. A homomorphism can then be constructed from $G \to G_k'$, by mapping a vertex in $V(G)$ to the vertex in $V(G_k')$ with the same vertex sequence.

($\Leftarrow$) Suppose there is a homomorphism $G \to G_k'$. Since G_k' is simple and loop-free, any vertices $x, y \in V(G)$ with $d(x, y) = 1$ or 2, are mapped to distinct vertices in G_k' with adjacencies preserved. A proper 2-dipath colouring c of G_k' with k colours is then also a proper 2-dipath colouring of G with k colours. □

Corollary 7.2.8. *If there exists a homomorphism $G \to H$, and H has a proper 2-dipath colouring with k colours, then so does G.*

Proof. Given that H has a proper 2-dipath colouring with k colours, there is a homomorphism $H \to G_k'$ by 7.2.7. Using composition of homomorphisms, we get a homomorphism $H \to G_k'$, which again by 7.2.7 tells us that G has a proper 2-dipath colouring with k colours. □

Corollary 7.2.9. *For an oriented graph G, if G has a proper 2-dipath colouring with k colours, then it has a proper 2-dipath colouring with $n \geq k$ colours.*

Proof. If G has a proper 2-dipath colouring with k colours, then by 7.2.7 there exists a homomorphism $G \to G_k'$. We know also using 7.2.2 and composition of homomorphisms that there is a homomorphism $G \to G_n'$ for any $n \geq k$. Therefore, there is a proper 2-dipath colouring of G with n colours, for any $n \geq k$. □

7.3 Complexity

We have already seen using a reduction to 2-satisfiability that the existence of a proper 2-dipath colouring with 2 colours can be decided in polynomial time. We now use homomorphisms and the oriented graph G'_k to determine the complexity of deciding the existence of a proper 2-dipath colourings with k colours, for any fixed $k \geq 3$.

Definition 7.3.1. An oriented graph G is *smooth* if for every $v \in V(G)$, $d^-(v) > 0$ and $d^+(v) > 0$.

The following is a consequence of [3] which proves a conjecture by Bang-Jensen, Hell and MacGillivray [2].

Theorem 7.3.2. *[3] If the oriented graph G contains a smooth oriented graph H with two directed cycles of relatively prime lengths, then homomorphism to G is $\mathcal{NP}$-complete.*

Proposition 7.3.3. *For any oriented graph G'_k, with $k \geq 4$, homomorphism to G'_k is $\mathcal{NP}$-complete.*

Proof. For $k \geq 4$, by construction G'_k contains directed cycles of lengths $3, 4, ..., k$ among the vertices without all $+$ or all $-$ sequences, (as those have either $d^+(v) = 0$ or $d^-(v) = 0$). In this way, such a graph H as required by Theorem 7.3.2 can be obtained from G'_k by deleting all vertices of in-degree 0 or out-degree 0, (vertices with all $+$ or all $-$ vertex sequences). In this way, homomorphism to G'_k is $\mathcal{NP}$-complete. □

Corollary 7.3.4. *For any fixed $k \geq 4$, proper 2-dipath colouring with k-colours is $\mathcal{NP}$-complete.*

In remains to determine the complexity for proper 2-dipath colouring with three colours. We will show the equivalent result that homomorphism to G_3 is $\mathcal{NP} - complete$. We need the following definition and a theorem by Hell and Nešetřil from [16], (also see [17]).

Definition 7.3.5. For a fixed oriented graph G, and a given oriented graph I with special vertices u, v. Define G^* to be the digraph with vertex-set $V(G^*) = V(G)$ and $(x, y) \in E(G^*)$ if and only if there exists a homomorphism $f : I \to G$ such that $f(u) = x$ and $f(v) = y$. This is called the *indicator construction*, with respect to the *indicator* (H, u, v).

Example 7.3.6. Consider $G = G_3'$ and the oriented graph H, in Fig. 7.2 with specified vertices u, v. In Figure 7.3 we apply the indicator construction with respect to (H, u, v) to G.

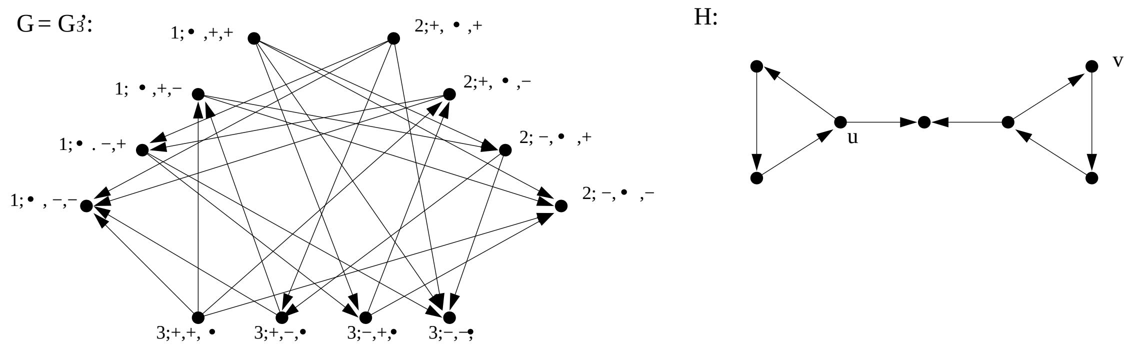

Figure 7.2: G_3' and given oriented graph H

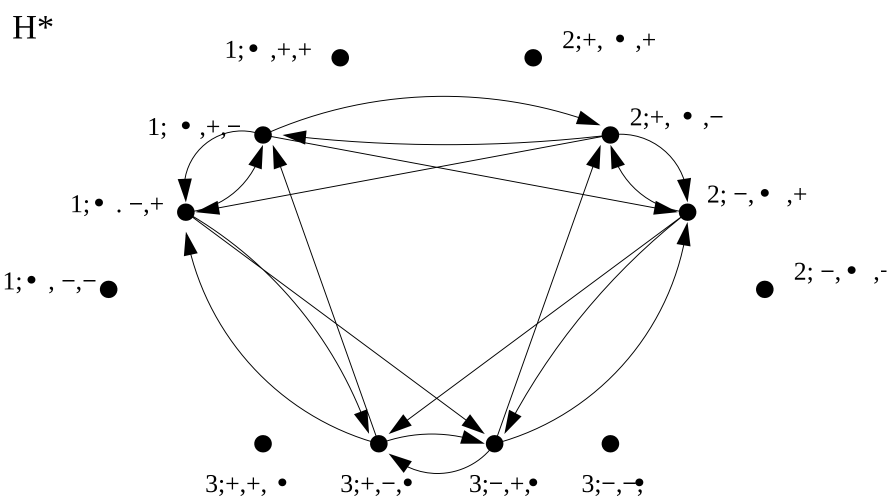

Figure 7.3: G^* with $G = G_3'$

Clearly removing the isolated vertices of G^* in Fig.7.3 gives a smooth oriented graph, which satisfies the criteria for Theorem 7.3.2.

Theorem 7.3.7. *[16] Let G be a fixed digraph, and G^* the result of applying the indicator construction with respect to some (I, u, v) to G. If homomorphism to G^* is $\mathcal{NP}$-complete, then so is homomorphism to G.*

Theorem 7.3.8. *Homomorphism to G_3' is $\mathcal{NP}$-complete.*

Proof. Let $G = G_3'$, and consider G^*, the result of applying the indicator construction with respect to (H,u,v) as shown in Figure 7.3. Removing the isolated vertices of G^* gives a smooth digraph that satisfies the requirements of Theorem 7.3.7, with directed cycles of both length two and three. Therefore homomorphism to G^* is $\mathcal{NP} - complete$, implying that homomorphism to $G = G_3'$ is $\mathcal{NP} - complete$. □

Theorem 7.3.9. *For every fixed $k \geq 3$, proper 2-dipath k-colouring is $\mathcal{NP}$-complete.*

Proof. The proof for this theorem follows directly from the statements in Proposition 7.3.3 and Theorem 7.3.8 □

Chapter 8

Bipartite Tournaments

Since for a tournament T_n on n vertices, $\chi'_2(T_n) = n$, we will only be considering proper 2-dipath colourings of bipartite tournaments.

8.1 Proper 2-Dipath Colouring of Bipartite Tournaments

Recall the equivalence relation Θ defined in 5.3.1. Any two vertices in the same equivalence class are neither adjacent nor joined by a 2-dipath, and hence can be assigned the same colour. Recall also the quotient oriented graph $T' = ({}^{T(m,n)}/_{\Theta})$. As before, T' is a bipartite tournament.

Proposition 8.1.1. *For a bipartite tournament $T(m,n)$, with $T' = ({}^{T(m,n)}/_{\Theta})$, $\chi'_2(T(m,n)) = \chi'_2(T') = |V(T')|=$ the number of equivalence classes of $\Theta(T(m,n))$.*

Proof. By the construction of T', for any two vertices $\Theta_i, \Theta_j \in V(T')$, $\vec{d}(\Theta_i, \Theta_j) \leq 2$. In this way the auxiliary graph for T' is complete, requiring $\chi'_2(T') \geq |V(T')|$. $\square$

Corollary 8.1.2. *For a bipartite tournament $T(m,n)$, $\chi'_2(T(m,n)) = \chi'_2(T')$.*

Proof. Since T' is a subdigraph of $T(m,n)$, $\chi'_2(T(m,n)) \geq \chi'_2(T')$. Consider a proper 2-dipath colouring c of T' that uses $\chi'_2(T')$ colours. A colouring of $T(m,n)$ with the same number of colours can be obtained by assigning all vertices of $T(m,n)$ in the equivalence class $[x]$, the colour $c([x])$. Hence, $\chi'_2(T(m,n)) = \chi'_2(T')$. $\square$

Bibliography

[1] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer Monographs in Mathematics. Springer-Verlag, London, 2nd edition, 2009.

[2] J. Bang-Jensen, P. Hell, and G. MacGillivray. Hereditarily hard H-colouring problems. *Discrete Math.*, 138:75–92, 1995.

[3] L. Barto, M. Kozik, and T. Niven. The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell). *SIAM J. Comput.*, 38(5):1782–1802, 2008/09.

[4] J. A. Bondy and U. S. R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springerlink, 3rd corrected printing edition, 2008.

[5] T. Calalmoneri. The $L(h,k)$-labelling problem: A survey and annotated bibliography. *The Computer Journal*, 49:585–608, 2006.

[6] R. J. Campbell. Reflexive injective oriented colourings. Master's thesis, University of Victoria, Mathematics and Statistics, University of Victoria, Victoria, Canada, 2009.

[7] G. Chang, J. Chen, D. Kuo, and S. Liaw. Distance-two labelings of digraphs. *Discrete Applied Mathematics*, 155:1007–1013, January 2007.

[8] G. Chang and S. C. Liaw. The $L(2,1)$-labeling problems on ditrees. *Ars Combinoria*, 66:23–31, 2003.

[9] B. Courcelle. The monadic second order logic of graphs vi: On several presentations of graphs by relational structures. *Discrete Applied Mathematics*, 54:117–149, 1994.

[10] T. Gallai. *On directed paths and circuits.* 115-118. Academic Press, New York, 1968.

[11] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theorey of $\mathcal{NP}$-Completeness.* A Series of Books in the Mathematical Sciences. Bell Laboratories, 1979.

[12] D. Goncalves and A. Raspaud. On oriented labelling parameters. In K. G. Subramanian, K. Rangarajan, and Madhavan Mukund, editors, *Formal Models, languages and application*, volume 66. World Scientific, London, 2006.

[13] J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM J. Discrete Math.*, 5:586–595, 1992.

[14] R. Grimaldi. *Discrete and Combinatorial Mathematics; An Applied Introduction.* Pearson Education, fifth edition, 2004.

[15] G. Hahn, J. Kratochvil, J. Sířan, and D. Sotteau. On the injective chromatic number of graphs. *Discrete Mathematics*, 256:179–192, 2002.

[16] P. Hell and J. Nesetril. On the complexity of H-colouring. *Journal of Combinatorial Theory Series B*, 48(1), February 1990.

[17] P. Hell and J. Nesetril. *Graphs and Homomorphisms.* Number 28 in Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Great Clarendon Street, Oxford OX2 6DP, 2004.

[18] G. MacGillivray, A. Raspaud, and J. Swarts. *Injective Oriented Colourings*, volume 5911, pages 262–272. Springer, Berlin, Lecture Notes in Comput. Sci.

[19] C. Min and W. Wang. The 2-dipath chromatic number of Halin graphs. *Information Processing Letters*, 99:47–53, March 2006.

[20] A. Raspaud and E. Sopena. Good and semi-strong colourings of oriented planar graphs. *Information Processing Letters*, 51:171–174, 1994.

[21] L. Rédei. Ein kombinatorischer satz. *Acta Lit Sci Szeged*, 7:39–43, 1934.

[22] B. Roy. Nombre chromatique et plus longs chemins d'un graphe. *Rev. Francaise Automat. Recherche Opérationelle Sér. Rouge*, 1:127–132, 1967.

[23] E. Sopena. Oriented graph colouring. *Discrete Mathematics*, 229:359–369, 2001.

[24] J. Swarts. *The complexity of digraph homomorphisms: Local tournaments, Injective Homomorphisms and Polymorphisms*. PhD thesis, University of Victoria, Mathematics and Statistics, University of Victoria, Victoria, Canada, 2008.