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Author(s): Edwin Burmeister, Christopher Caton, A. Rodney Dobell, Stephen Ross
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THE "SADDLEPOINT PROPERTY" AND THE STRUCTURE OF DYNAMIC HETEROGENEOUS CAPITAL GOOD MODELS

BY EDWIN BURMEISTER, CHRISTOPHER CATON, A. RODNEY DOBELL, AND STEPHEN ROSS[1]

The topological properties of dynamic heterogeneous capital good models are examined, and it is found that the savings hypothesis crucially influences the dimension of the manifold consisting of the locus of backward solutions from stationary equilibrium. If not all capital gains are saved, the convergent manifold is generally of higher dimension than it is if no income from capital gains is spent on consumption. Accordingly, the characteristic equation for the associated linear system near stationary equilibrium may have more than half its roots with negative real parts, and thus in general the model does not possess a "regular saddlepoint property."

1. INTRODUCTION AND SUMMARY

MUCH RECENT WORK on the dynamics of heterogeneous capital models has emphasized that the unique stationary equilibrium for the models under study is a "saddlepoint" in price-quantity space. Generally one speaks of a saddlepoint in E^{2n} as a critical point with the property that the locus of backward solutions from equilibrium is a manifold of dimension n; thus, given any initial vector of capital stocks, there exists a unique initial price vector that will bring the system asymptotically to the stationary equilibrium.[2] *However, this saddlepoint property does not hold for many heterogeneous capital models*; we have found a class of counterexamples in which the equilibrium point is approached along a manifold of dimension greater than n. Thus, given an initial vector of per capita capital stocks, there exist many assignments of initial prices for which the system converges asymptotically to its unique stationary equilibrium.[3] This result, which is valid for many technologies not studied here, depends crucially upon the assumption that not all capital gains are saved.

This result suggests that general stability theorems may be difficult to obtain for dynamic heterogeneous capital good models because no model of our class

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[2] More precisely, existence of a unique price vector with this property depends also upon the manifold "covering" the capital space. We shall also follow a convenient precedent and call a stationary point for which the characteristic equation of the associated linear system has k roots with negative real part and $n - k$ roots with positive real part a stationary point of type k or a k-saddle. (See [**4**].)

[3] Hahn has conjectured that "it may be that there are a number of accumulation paths which converge" [**7**, p. 180] in a heterogeneous capital good model with "no saving out of wages and no consumption out of profit" [**7**, p. 176]. However, we do not use this savings assumption.

Likewise, Samuelson's work [**10** and **11**] indicates that the dynamic equilibrium of heterogeneous capital good models may be a "regular" saddlepoint of type n under very general assumptions, but *only* when all capital gains are saved. Also, in unpublished work Stiglitz has indicated that with finitely-lived capital goods the equilibrium point may not be a saddlepoint.

has a saddlepoint of type n, and hence none can ever be viewed as a "regular Hamiltonian system" resulting from an optimization problem. Given an initial vector of (per capita) capital stocks $k(0)$, however, it is still not true that the assignment of any arbitrary initial price vector $p(0)$ will yield asymptotic convergence to equilibrium. Furthermore, in a specific example given below, all paths which do not converge to equilibrium have the property that at least one price becomes zero in finite time, and hence such paths are revealed to be inconsistent with perfect competition.[4] The presence of this result is important because the models studied by Shell-Stiglitz [**12**] and Caton-Shell [**3**] (in which this property was previously demonstated) have two important special features: (i) total consumption equals total wage income, and (ii) the production possibility frontier in output space is not strictly concave. One might suspect that the finite time property depends crucially on (i), (ii), or both, but our example demonstrates that this is not the case.

2. THE TECHNOLOGICAL STRUCTURE OF A GENERAL CLASS OF HETEROGENEOUS CAPITAL GOOD MODELS

Consider a model with one (possibly composite) consumption good and n different capital goods. Denoting the outputs of these by $Y_0, Y_1, \ldots, Y_n$, respectively, the capital stocks by $K_1, \ldots, K_n$, and the labor supply by $L \equiv K_0$, there exists a production possibility frontier

$$Y_0 = T(Y_1, \ldots, Y_n; K_0, K_1, \ldots, K_n) \tag{1}$$

or, using lower case letters for per capita quantities,

$$y_0 = P(y_1, \ldots, y_n; 1, k_1, \ldots, k_n). \tag{2}$$

For mathematical simplicity we introduce the following assumption: *For any given $\bar{k}_i$'s > 0, the function T defines a strictly concave hypersurface in the space of admissible* $(y_0, y_1, \ldots, y_n)$. We emphasize, however, that our results do not depend upon this assumption as was indicated in an earlier and longer version of this paper. Note that the function T embodies all the usual static efficiency conditions of pure competition or Lerner-Lange socialism. In particular

$$\partial T/\partial Y_i \equiv T_i = -p_i \qquad (i = 1, \ldots, n) \tag{3}$$

are the competitive prices of the respective outputs, while

$$\partial T/\partial K_i \equiv T_{n+i+1} = w_i \qquad (i = 0, 1, \ldots, n) \tag{4}$$

are the competitive gross rental rates of the respective factor inputs; the lower case letters now indicate that prices and rental rates are in terms of the consumption good numeraire.

[4] This example satisfies all the usual neoclassical assumptions and also possesses the following properties: (i) All prices (in terms of the consumption good as numeraire) and capital stocks (per capita) can take on any value in the open interval $(0, \infty)$; (ii) momentary equilibrium is always unique for any assigned vectors of prices and capital stocks p and k respectively; (iii) the differential equations for p and k are always causal; and (iv) there exists a unique stationary equilibrium (p^*, k^*).

Consider now given k_i's and given p_i's, say $p_i = \bar{p}_i > 0$ and $k_i = \bar{k}_i > 0$ $(i = 1, \ldots, n)$. Provided equations (3) can be satisfied with equality, clearly momentary (static) equilibrium is completely determined and is unique. Since by assumption T defines a strictly concave hypersurface in the space of admissible $(y_0, y_1, \ldots, y_n)$, the tangency conditions (3) uniquely determine the corresponding outputs $(\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n)$. Finally, since the w_i given by (4) depend only on $(\bar{y}_1, \ldots, \bar{y}_n)$ and $(\bar{k}_1, \ldots, \bar{k}_n)$, these too are uniquely determined.[5]

The dynamics of the system is determined by $2n$ differential equations in the k_i's and p_i's. We assume that the labor supply grows at the exponential rate g and that the ith capital good depreciates at the exponential rate δ_i; hence the capital accumulation equations are simply

$$\dot{k}_i = y_i - (g + \delta_i)k_i \qquad (i = 1, \ldots, n). \tag{5}$$

Likewise it is well-known that the capital accumulation process is efficient only if

$$\dot{p}_i = -w_i + (r_0 + \delta_i)p_i \qquad (i = 1, \ldots, n) \tag{6}$$

where r_0 is the common interest or profit rate; these familiar conditions are also necessary for profit maximization in a world of pure competition, perfect capital markets, and perfect myopic foresight. (We will omit the index t for time where no confusion is possible; however, when clarity is needed we will write $y_i(t)$, $k_i(t)$, etc.)

Our primary concern is the evolution of the system governed by equations (5) and (6). As shown above, both the y_i's and the w_i's are functions of $(p_1, \ldots, p_n)$ and $(k_1, \ldots, k_n)$. Thus *if* we can determine the interest or profit rate endogenously—i.e., if there exists a function

$$r_0 = r_0(p_1, \ldots, p_n; k_1, \ldots, k_n), \tag{7}$$

then (5) and (6) may be written in the causal, autonomous form

$$\begin{aligned} \dot{k}_i &= f^i(k_1, \ldots, k_n; p_1, \ldots, p_n), \\ \dot{p}_i &= h^i(k_1, \ldots, k_n; p_1, \ldots, p_n), \end{aligned} \qquad (i = 1, \ldots, n), \tag{8}$$

with initial conditions $k_i(0) = \bar{k}_i$ and $p_i(0) = \bar{p}_i$. We now turn to the question of when r_0 can be determined by an equation such as (7).[6]

[5] Of course, for some sets of exogenous $\bar{k}_i$'s and $\bar{p}_i$'s, conditions (3) may not have a solution with equalities, and in this case it becomes necessary to study "corner solutions." When the problem of "corner solutions" exists, the dynamics of the model cannot be expressed as causal differential equations of the usual kind, but rather the model must be studied as a more complex "dynamical system." Such problems are of interest, but for our purposes we wish to avoid them and to concentrate on the properties of a simpler system. Accordingly, we assume that for *any* positive values of the p_i's and k_i's, equations (3) can all be satisfied with equalities and hence momentary (static) equilibrium is always unique. In the examples given below this assumption is satisfied.

[6] Hahn's model [**6**] implicitly assumes that all capital gains are saved, so that the relationship equating realized investment to desired saving does not explicitly involve the yield r_0. It is this that accounts for the fact that only $n - 1$ prices may be chosen initially to be consistent with momentary equilibrium. So long as some income from capital gains is consumed, the saving equation can be solved directly for r_0, the common yield or profit rate on assets. We shall elaborate upon this observation below.

The basic flow equilibrium condition asserts that $\dot{V} = S$ where $\dot{V}$ is the realized change in the real value of capital (wealth) and S is desired saving.[7] On the assumption that the propensities to save from the three components of an income stream—capital gains, net rentals, and wages—are given constants s_c, s_r, and s_w, respectively, we obtain

(9) $$S = s_c \sum_{i=1}^{n} \dot{p}_i K_i + s_r \sum_{i=1}^{n} (w_i K_i - p_i \delta_i K_i) + s_w w_0 K_0$$

and

(10) $$\dot{V} = \frac{d}{dt}\left(\sum_{i=1}^{n} p_i K_i\right) = \sum_{i=1}^{n} \dot{p}_i K_i + \sum_{i=1}^{n} p_i \dot{K}_i.$$

Substituting $\dot{p}_i$ from (6), using $\dot{K}_i = Y_i - \delta_i K_i$, and manipulating the income identity $\Sigma_{i=0}^{n} p_i Y_i = \Sigma_{i=0}^{n} w_i K_i$, we find that (9) and (10) can be solved for r_0 provided $s_c < 1$. For example, if $0 \leqslant s_r = s_c < 1$ and $s_w = 0$, we may solve explicitly for

(11) $$r_0 = \frac{\sum_{i=1}^{n} (w_i K_i - p_i Y_i)}{(1 - s_c) \sum_{i=1}^{n} p_i K_i} = \frac{Y_0 - w_0 K_0}{(1 - s_c)V}$$

or, in per capita terms,

(12) $$r_0 = (y_0 - w_0)/(1 - s_c)v$$

where $v \equiv V/K_0$ is the per capita value of wealth in terms of the consumption good as numeraire. Since all the variables on the right-hand side of (12) are functions only of $(p_1, \ldots, p_n; k_1, \ldots, k_n)$, this relationship implicitly defines (7).

Similarly suppose $s_c = s_r = s_w \equiv s$ where $0 < s < 1$; then we have

(13) $$r_0 = [Y_0 - (1 - s)w_0 K_0]/(1 - s)V$$

or

(14) $$r_0 = [y_0 - (1 - s)w_0]/(1 - s)v,$$

and again r_0 is determined endogenously as a function of the p_i's and k_i's.

3. A SPECIFIC MODEL

We can now apply the above to a specific model. Let $n = 2$ and assume that the production possibility frontier is given by

(15) $$(Y_0^2 + Y_1^2 + Y_2^2)^{\frac{1}{2}} = K_0^{\alpha_0} K_1^{\alpha_1} K_2^{\alpha_2}$$

[7] This equilibrium condition is found in the writings of many neoclassical authors. An economic justification in the context of this model is given by Burmeister and Dobell [2, pp. 297–298 and 303–304]

where $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Thus equation (1) becomes

$$(16) \qquad Y_0 = T(Y_1, Y_2; K_0, K_1, K_2) = (K_0^{2\alpha_0} K_1^{2\alpha_1} K_2^{2\alpha_2} - Y_1^2 - Y_2^2)^{\frac{1}{2}}.$$

We may take the viewpoint that

$$(17) \qquad Z = K_0^{\alpha_0} K_1^{\alpha_1} K_2^{\alpha_2}$$

is the production function for an intermediate product Z. Then Z is "split" or "cracked" in accordance with

$$(18) \qquad Z = (Y_0^2 + Y_1^2 + Y_2^2)^{\frac{1}{2}},$$

i.e., Z is the radius of a ball in E^3 with its center at the origin; the radius of the ball is determined by factor endowments. Thus relative prices, which determine a hyperplane tangent to $(Y_0^2 + Y_1^2 + Y_2^2)^{\frac{1}{2}}$ when conditions (3) are satisfied, can assume *any* positive values for any positive k_i's. However, the production process given by (18) and its associated cost function are self-dual, so that

$$(19) \qquad (P_0^2 + P_1^2 + P_2^2)^{\frac{1}{2}} = P_Z$$

or

$$(20) \qquad (1 + p_1^2 + p_2^2)^{\frac{1}{2}} = P_Z/P_0 \equiv p_Z.$$

Of course, p_Z is simply the price of the intermediate product in terms of the consumption good as numeraire. Since the cost function for the Cobb-Douglas production function is

$$(21) \qquad \left(\frac{W_0}{\alpha_0}\right)^{\alpha_0} \left(\frac{W_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{W_2}{\alpha_2}\right)^{\alpha_2},$$

in equilibrium we have

$$(22) \qquad (1 + p_1^2 + p_2^2)^{\frac{1}{2}} = p_Z = \left(\frac{w_0}{\alpha_0}\right)^{\alpha_0} \left(\frac{w_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{w_2}{\alpha_2}\right)^{\alpha_2}.$$

In the next section we will see that the relationship (22) appears in many calculations.

In order to simplify the calculations which follow, we shall suppose $\alpha_1 = \alpha_2 \equiv \alpha$ and $\alpha_0 = 1 - 2\alpha$. We also adopt the "neoclassical savings assumption" $s_c = s_r = s_w \equiv s, 0 < s < 1$, and we assume that both capital goods depreciate at the common rate, δ. It should be noted that, given our assumptions, this model may behave as a one capital good, two-sector model which is known to be stable. In particular, this result follows when $k_1(t) = k_2(t)$ and $p_1(t) = p_2(t)$ for all t and "aggregation" is possible. Nevertheless, our model is continuous in the parameters α_1 and α_2, so certainly our qualitative results remain valid when α_1 and α_2 differ slightly, although in such cases *no* aggregation is possible. Indeed, computer calculations of the growth paths when $\alpha_1 \neq \alpha_2$ verify the natural global extension of an important local stability property discussed below, namely in a neighborhood of equilibrium three characteristic roots have negative real parts and only one has a positive real part.

Finally, it should be stressed again that although the technology we have specified allows us to avoid all "corner solution" problems, the assumption that such problems do not exist in general is quite unfounded. When such problems occur, of course, they are not important for local analysis provided the equilibrium point is interior. Likewise, uniqueness of both momentary (static) and dynamic equilibria cannot, in general, be ensured without strong regularity conditions such as those we have imposed.

4. DETERMINATION OF UNIQUE MOMENTARY AND DYNAMIC EQUILIBRIA

The production possibility frontier for this model can now be written as

$$Y_0 = [K_0^{2(1-2\alpha)} K_1^{2\alpha} K_2^{2\alpha} - Y_1^2 - Y_2^2]^{\frac{1}{2}} \tag{23}$$

or in intensive form

$$y_0 = (k_1^{2\alpha} k_2^{2\alpha} - y_1^2 - y_2^2)^{\frac{1}{2}}. \tag{24}$$

From (3) we know that

$$\partial y_0 / \partial y_i = -p_i \qquad (i = 1, 2). \tag{25}$$

Thus

$$\begin{aligned} p_1 &= -(k_1^{2\alpha} k_2^{2\alpha} - y_1^2 - y_2^2)^{-\frac{1}{2}}(-y_1) \\ &= y_1/y_0, \end{aligned} \tag{26}$$

and, similarly,

$$p_2 = y_2/y_0. \tag{27}$$

Combining these with (24), one obtains

$$\begin{aligned} y_0 &= \frac{k_1^\alpha k_2^\alpha}{p_Z}, \\ y_1 &= \frac{p_1 k_1^\alpha k_2^\alpha}{p_Z}, \\ y_2 &= \frac{p_2 k_1^\alpha k_2^\alpha}{p_Z}, \end{aligned} \tag{28}$$

where p_Z is defined by (20). Thus, for initial factor endowments and prices, the outputs of the consumption good and of the two capital goods are each uniquely determined.

Equations (4) enable us to evaluate the gross rental rates of the factor inputs at any point of time:

$$\begin{aligned} w_i &= \partial y_0 / \partial k_i \qquad (i = 1, 2), \\ w_0 &= \partial Y_0 / \partial K_0. \end{aligned} \tag{29}$$

It can be shown that (29) yields

$$\text{(30)}\qquad \begin{aligned} w_1 &= \alpha k_1^{\alpha-1}k_2^{\alpha}p_Z, \\ w_2 &= \alpha k_1^{\alpha}k_2^{\alpha-1}p_Z, \qquad \text{and} \\ w_0 &= (1-2\alpha)k_1^{\alpha}k_2^{\alpha}p_Z, \end{aligned}$$

which, in turn, satisfy (22).

We now seek the solutions to the equations

$$\dot{k}_i = 0,$$

and

$$\dot{p}_i = 0 \qquad (i = 1, 2).$$

Combining (5), (6), (14), (28), and (30), we require

$$\text{(31)}\qquad \dot{k}_i = \frac{p_i k_1^{\alpha} k_2^{\alpha}}{p_Z} - (g+\delta)k_i = 0$$

and

$$\text{(32)}\qquad \dot{p}_i = p_i(r_0 + \delta) - \alpha\frac{k_1^{\alpha}k_2^{\alpha}}{k_i}p_Z = 0,$$

where the common net own-rate of return is

$$r_0 = \frac{k_1^{\alpha}k_2^{\alpha}[1 - (1-s)(1-2\alpha)p_Z^2]}{p_Z(1-s)(p_1k_1 + p_2k_2)}.$$

Now, from the symmetry which has been introduced for computational convenience, it is obvious that $k_1^* = k_2^* \equiv k^*$ and $p_1^* = p_2^* \equiv p^*$ where the starred variables represent equilibrium values. Thus in equilibrium (33) and (34) will hold:

$$\text{(33)}\qquad \dot{k} = pk^{2\alpha}(1 + 2p^2)^{-\frac{1}{2}} - (g+\delta)k = 0$$

and

$$\text{(34)}\qquad \begin{aligned} \dot{p} = {} & \frac{k^{2\alpha}(1+2p^2)^{-\frac{1}{2}}[1 - (1-s)(1-2\alpha)(1+2p^2)]}{2(1-s)k} \\ & - \alpha k^{2\alpha-1}(1+2p^2)^{\frac{1}{2}} + p\delta = 0. \end{aligned}$$

From (33)

$$k^{2\alpha-1}(1+2p^2)^{-\frac{1}{2}} = (g+\delta)/p;$$

substituting this in (34), one obtains

$$\frac{g+\delta}{2p(1-s)}[1 - (1-s)(1-2\alpha)(1+2p^2)] - \frac{\alpha(g+\delta)(1+2p^2)}{p} + p\delta = 0,$$

from which one finally derives

$$(35)\qquad p^* = \sqrt{\frac{s(g+\delta)}{2(1-s)g}},\qquad k^* = \left(\sqrt{\frac{2(g+\delta)(g+s\delta)}{s}}\right)^{1/(2\alpha-1)}.$$

The $\alpha_1 \neq \alpha_2$ case is a straightforward generalization of the above but is not necessary for our purposes. Thus equilibrium values for prices and capital stocks are defined and uniquely determined in terms of the parameters of the system.

5. ANALYSIS OF LOCAL STABILITY

In this section we examine the stability properties of the equilibrium point of the system. The analysis will reveal that the particular saddlepoint property found in previous heterogeneous capital good models is not generally valid, i.e., there are not always n roots with positive real part and n with negative real part in the neighborhood of the equilibrium point. On the contrary, in our example there are, in fact, three stable directions of approach to equilibrium and only one unstable direction.

We begin by expanding the system linearly about the equilibrium point (p^*, k^*). Considerations of symmetry yield a perturbed system of the form

$$(36)\qquad \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ e & f & j & h \\ f & e & h & j \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

where

$$\xi \equiv \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \begin{bmatrix} p^* \\ p^* \end{bmatrix},$$

$$\eta \equiv \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} - \begin{bmatrix} k^* \\ k^* \end{bmatrix},$$

$$a \equiv \left.\frac{\partial \dot{p}_1}{\partial p_1}\right|_{(p^*,k^*)} = \left.\frac{\partial \dot{p}_2}{\partial p_2}\right|_{(p^*,k^*)},$$

and

$$b \equiv \left.\frac{\partial \dot{p}_1}{\partial p_2}\right|_{(p^*,k^*)} = \left.\frac{\partial \dot{p}_2}{\partial p_1}\right|_{(p^*,k^*)},$$

with obvious notation for the remaining terms.

The partitioned property of the system matrix can be exploited to simplify the computation of both the eigenvectors and the characteristic roots of the linear system. Notice first that the eigenvectors are of the two types:

$$(37)\qquad \begin{bmatrix} 1 \\ 1 \\ \beta_1 \\ \beta_1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \beta_2 \\ \beta_2 \end{bmatrix}; \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ \theta_3 \\ -\theta_3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ \theta_4 \\ -\theta_4 \end{bmatrix}.$$

By substitution into the original system, (36), the eigenvectors of the first type yield

$$\lambda = (a + b) + \beta(c + d)$$

and

$$\lambda\beta = (e + f) + \beta(j + h),$$

and upon eliminating β we obtain an equation for two of the characteristic roots,

$$(38)\qquad [\lambda - (a + b)][\lambda - (j + h)] = (c + d)(e + f);$$

similarly for the second type of eigenvector we obtain

$$(39)\qquad [\lambda - (a - b)][\lambda - (j - h)] = (c - d)(e - f).$$

The discriminant of equation (38) is given by

$$\begin{aligned} \Delta &= [(a + b) + (j + h)]^2 - 4[(a + b)(j + h) - (c + d)(e + f)] \\ &= [(a + b) - (j + h)]^2 + 4(c + d)(e + f) \\ &\geqslant 4(c + d)(e + f). \end{aligned}$$

Now,

$$\begin{aligned} c + d &\equiv \left.\frac{\partial \dot{p}_1}{\partial k_1}\right|_{(p^*,k^*)} + \left.\frac{\partial \dot{p}_1}{\partial k_2}\right|_{(p^*,k^*)} \\ &= -\frac{(2\alpha - 1)(g + \delta)}{2k^*p^*(1 - s)}\frac{s\delta}{g}, \end{aligned}$$

and

$$\begin{aligned} e + f &\equiv \left.\frac{\partial \dot{k}_1}{\partial p_1}\right|_{(p^*,k^*)} + \left.\frac{\partial \dot{k}_1}{\partial p_2}\right|_{(p^*,k^*)} \\ &= \frac{k^*}{p^*}\frac{(g + \delta)(1 - s)g}{g + s\delta}. \end{aligned}$$

Therefore,

$$(40)\qquad (c + d)(e + f) = \frac{(1 - 2\alpha)(1 - s)g(g + \delta)\delta}{g + s\delta} > 0.$$

It follows that the roots of equation (38) are both real and distinct. By a similar calculation

$$(c - d)(e - f) = \frac{2\alpha(g + s\delta)(g + \delta)}{s} > 0, \tag{41}$$

which implies that the discriminant of (39) is also positive and, hence, it too has real and distinct roots.

To determine the signs of the roots we need only note that the sign of the product of the roots of (38) is given by

$$\begin{aligned} \operatorname{sign} \lambda_1\lambda_2 &= \operatorname{sign}\{(a + b)(j + h) - (c + d)(e + f)\} \\ &= \operatorname{sign}\left\{-\left[\frac{g(g + \delta)}{(g + s\delta)} + g\right][(g + \delta)(2\alpha - 1)] \right. \\ &\quad \left. - \frac{(1 - 2\alpha)(1 - s)g(g + \delta)\delta}{(g + s\delta)}\right\} \\ &= \operatorname{sign}\{(1 - 2\alpha)(g + \delta)g(2g + (s + 1)\delta) \\ &\quad -(1 - 2\alpha)(1 - s)g(g + \delta)\delta\} \\ &= \operatorname{sign}\{2g + (s + 1)\delta - (1 - s)\delta\} \\ &= \operatorname{sign}\{2g + 2s\delta\} = (+). \end{aligned} \tag{42}$$

It follows that both roots have the same sign. Since the sum of the roots is given by

$$\begin{aligned} \lambda_1 + \lambda_2 &= (a + b) + (j + h) \\ &= -\frac{g(g + \delta)}{(g + s\delta)} - g + (g + \delta)(2\alpha - 1) < 0, \end{aligned} \tag{43}$$

both roots are negative. The product of the roots of equation (39), however, is given by

$$\begin{aligned} \operatorname{sign}(\lambda_3\lambda_4) &= \operatorname{sign}\{(a - b)(j - h) - (c - d)(e - f)\} \\ &= \operatorname{sign}\left\{\frac{1}{s}2\alpha(g + s\delta)[-(g + \delta)] - \frac{1}{s}2\alpha(g + s\delta)(g + \delta)\right\} \\ &= (-), \end{aligned} \tag{44}$$

and, hence, one root will be negative and one will be positive. Thus we have proved that three of the four characteristic roots are negative and only one is positive.

Since the roots of (38) and (39) are distinct, it follows from (37) that the eigenvectors are all linearly independent. Consequently, the four eigenvectors span E^4 and the solution to the differential system local to (p^*, k^*) may be written as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = c_1 X_1 e^{\lambda_1 t} + c_2 X_2 e^{\lambda_2 t} + c_3 X_3 e^{\lambda_3 t} + c_4 X_4 e^{\lambda_4 t}, \tag{45}$$

where X_i is the eigenvector associated with λ_i, the c_i are constants determined by the initial conditions, and λ_4 is the positive root. Hence for any initial (p_1, p_2, k_1, k_2) such that the deviation vector (ξ, η) lies in the subspace spanned by the vectors associated with the negative characteristic roots,

$$\begin{bmatrix} 1 \\ 1 \\ \beta_1 \\ \beta_1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ \beta_2 \\ \beta_2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ \theta_3 \\ -\theta_3 \end{bmatrix},$$

the system will converge to (p^*, k^*). What is required is simply that (ξ, η) be orthogonal to the independent direction provided by the eigenvector of the positive root,

$$X_4 = \begin{bmatrix} 1 \\ -1 \\ \theta_4 \\ -\theta_4 \end{bmatrix}.$$

Moreover, these local properties generalize: the subspace spanned by the three stable eigenvectors, X_1, X_2, and X_3 will be tangent in E^4 at the stationary point, (p^*, k^*), to the three dimensional manifold along which solutions converge to equilibrium. A proof of this global dynamic property is given in Section 7 below.

6. GENERALIZATION TO n CAPITAL GOODS

The results of Section 5 also serve as a counterexample to any conjecture that in a heterogeneous capital good model, given the n initial capital stocks, there always exists a *unique* choice of the remaining free prices that will allow the system to converge to the equilibrium Golden Age growth path. In our above example there is, in fact, an entire three dimensional manifold containing paths that converge to the equilibrium point. Given an initial $(k_1(0), k_2(0))$ we can choose either of the other prices, say $p_1(0)$, and a value for the remaining price, $p_2(0)$, can be found that will enable the system to converge to (p^*, k^*).[8]

This property is shared by all models that allow consumption out of capital gains but not by models where all gains are saved. One way to see the distinction between implications of the two savings hypotheses is to rewrite (5) and (6) as

$$\dot{k} = F(p, k)$$

and

$$\dot{p} = H(p, k, r_0)$$

[8] Strictly speaking we have not verified that the three dimensional manifold may be extended globally in a non-pathological manner, but it seems highly unlikely that the dimensionality of the manifold will be altered as we move from (p^*, k^*). Calculations have verified this global property for particular examples, and it is proved for a special case in the next section.

where $k = (k_1, \ldots, k_n)$ and $p = (p_1, \ldots, p_n)$. To close the model we need an equation which allows us to solve for the common interest rate r_0. When consumption out of capital gains is allowed, the equilibrium condition $S = \dot{V}$ can be solved directly for

$$r_0 = r_0(p, k)$$

which allows us to write the dynamic equations in the causal form of (8). Thus, for example, (14) implies (7).

On the other hand, when capital gains are all saved, the savings hypothesis takes the form $\Psi(p, k) = 0$ and *does not* contain r_0 explicitly. Differentiating the function $\Psi(p, k)$ allows us to again find a function $r_0 = r_0(p, k)$, as in (7). Notice, however, that $\Psi(p, k) = 0$ imposes *one additional* restriction on the economy. In particular, given the initial capital stocks, $k(0)$, not all choices of $p(0)$ are admissible, only those for which $\Psi(p(0), k(0)) = 0$.

Consequently, when all capital gains are saved, the system is restricted to motions along a manifold of dimension $2n - 1$, and if n initial capital stocks are given, only $n - 1$ prices are open to choice. It is not surprising, then, that in the specialized two capital good models that have employed such savings hypotheses, there is a unique choice of the initial price vector which allows convergence to the equilibrium growth path. Given the $n + 1(=3)$ restrictions in such models, the specification of $(k_1(0), k_2(0))$ leaves only one price free. Once $p_1(0)$ is chosen, the savings relation determines $p_2(0)$ and future development of the economy. In effect, then, it would be impossible to have convergence along a manifold unless the economy were actually stable.

When capital gains enter into the savings relation in a meaningful way, we actually gain a degree of freedom since we now no longer are imposing any restriction on the initial price choices. As a result, we are free to choose both $p_1(0)$ and $p_2(0)$, although only a restricted set of such choices will allow convergence.

In a general n capital good model where not all capital gains are saved, we can freely choose all of the n initial prices. Furthermore, we are led to conjecture that for a wide class of models there will be $n + 1$ stable directions of approach to equilibrium and $n - 1$ unstable ones, and the stationary point will be approached along paths lying in a manifold of dimensionality $n + 1$; in other words, having chosen any one initial price, there exists a unique choice of the remaining $n - 1$ prices for which the model converges.

This conjecture is consistent with the work of Samuelson [**11**] who has proved that if consumption is given by a general consumption function of the form

$$y_0 = \varepsilon h(k_1, \ldots, k_n; \dot{k}_1, \ldots, \dot{k}_n),$$

then for ε sufficiently small, the model behaves like a closed von Neumann model and has the usual (generalized) saddlepoint of type n. However, this analogy to the von Neumann model does *not* follow when capital gains terms involving $\dot{p}_i$'s enter the $h(\cdot)$ consumption function since $\dot{p}_i = d(-T_i)/dt$, $y_i = \dot{k}_i + (g + \delta_i)k_i$ (and thus $\dot{p}_i$ depends upon terms which include $\ddot{k}_i$'s).

7. GLOBAL RESULTS AND NONCOMPETITIVE PATHS

In the example studied in Sections 3, 4, and 5 we have shown that, given an initial vector of per capita capital stocks $k(0)$, there are many possible initial price vectors $p(0)$ which will yield asymptotic convergence to equilibrium, but it is still not true that any arbitrary $p(0)$ will lead to such convergence. We turn now to the question of how the system develops when it is off the convergent manifold. It is proved below that all paths not converging to equilibrium will have the property that at least one price becomes zero in finite time, and hence such paths will be revealed (eventually) to be inconsistent with perfect competition, free disposability of capital goods, and equilibrium in the markets for capital assets. (See, e.g., [**3**, pp. 19–20] for the details of this argument.) Given the fiction of Walrasian markets for all periods into the future, the economy will never follow such errant paths, and convergence to equilibrium will be assured.

For the purposes of this section alone, we define two new variables, $k \equiv k_1/k_2$ and $p \equiv p_1/p_2$. We can then study the development of the model in the (k, p) plane of Figure 1.[9] For convenience we also set α_1 and α_2 equal although the argument of this section depends upon this equality in no essential way and the generalization is trivial. As before $p_Z = (1 + p_1^2 + p_2^2)^{\frac{1}{2}}$ and is, in effect, the price of undifferentiated

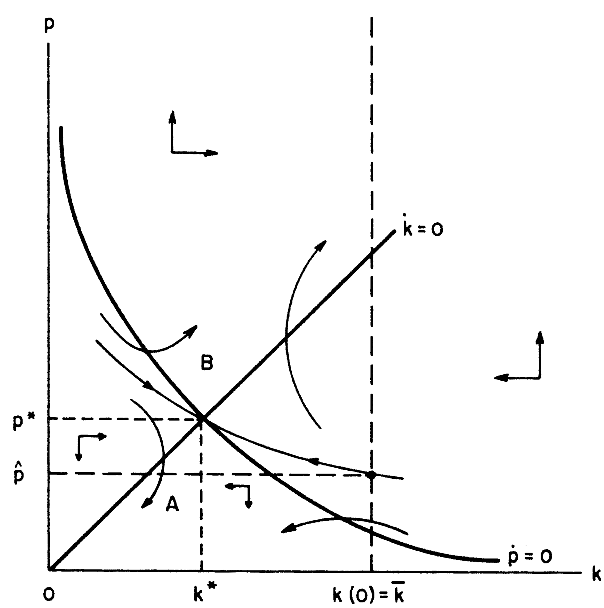

FIGURE 1.

[9] This figure, originally suggested by Atkinson [**1**], is identical to Figure 1 of [**3**], but as we discuss below, the interpretation and analysis of the figure is quite different.

output. Now

$$\frac{\dot{k}}{k} = \frac{\dot{k}_1}{k_1} - \frac{\dot{k}_2}{k_2} \tag{46}$$
$$= \frac{k_1^\alpha k_2^\alpha}{p_Z}\left[\frac{p_1}{k_1} - \frac{p_2}{k_2}\right] = \frac{p_2 k_1^{\alpha-1} k_2^\alpha}{p_Z}(p - k),$$

and hence $\dot{k} = 0$ requires $k = p$. Likewise, using (30),

$$\frac{\dot{p}}{p} = \frac{\dot{p}_1}{p_1} - \frac{\dot{p}_2}{p_2}$$
$$= (r_0 + \delta) - \frac{w_1}{p_1} - (r_0 + \delta) + \frac{w_2}{p_2} \tag{47}$$
$$= \frac{w_2}{p_2} - \frac{w_1}{p_1} = \frac{w_1}{p_1}(pk - 1),$$

so that $\dot{p} = 0$ requires that $pk = 1$. To determine $\dot{k}$ and $\dot{p}$ given k and p, we must also know one of k_1 or k_2, and one of p_1 or p_2. But we can say that above the $\dot{k} = 0$ ray k is rising, and below it k is falling; similarly, above the $\dot{p} = 0$ hyperbola p is rising, and below it p is falling. Consequently, we can examine some *qualitative properties* of the system in (p, k) space. In particular, the intersection of the $\dot{k} = 0$ and $\dot{p} = 0$ curves defines a saddlepoint equilibrium, and there exists a unique locus of p and k values along which the system converges in (p, k) space.[10]

In fact, if we are on this stable arm in Figure 1, then it will also be the case that the global system in (p_1, p_2, k_1, k_2) space will be stable, but this does not follow directly from the observation that the ratio system in (p, k) space is stable. It does follow from the analysis of local stability in Section 5, where we showed that the system in E^4 has only a single unstable direction. These global properties are verified further by the numerical results presented in Section 8. In the Caton-Shell model, since one functional restriction on the prices (at any moment of time) held as a direct consequence of the savings hypothesis, a particular value of p implied particular values for p_1 and p_2 and convergence in the ratio space insured that the global system also converged. In our model, however, it is at least a priori possible that even though the price ratio has converged the individual prices may not have done so, and, thus, we require the additional analysis of the other

[10] This statement and the preceding paragraph must be interpreted with great care. Clearly, the complete behavior of a four-dimensional system in general cannot be depicted by a two-dimensional figure. Suppose, however, we take as given one of the k_i's and one of the p_i's at time $t = 0$, say $k_1(0) = \bar{k}_1 > 0$ and $p_1(0) = \bar{p}_1 > 0$. Likewise, assume $k(0) \equiv k_1(0)/k_2(0) = \bar{k} > 0$ is given (which, of course, implies that $k_2(0) = \bar{k}_2 > 0$ is also known).

As stated, the *qualitative behavior* of the system is illustrated in Figure 1. The initial condition $\bar{k}$ determines a vertical line, and there exists one and only one value of $p(0)$ on this line, namely $p(0) = \hat{p}$, such that $(p(t), k(t))$ asymptotically approach (p^*, k^*) equilibrium.

Alternatively, let the initial conditions $\bar{k}_1, \bar{k}_2$, and $\bar{p}_1$ be given; then there exists a $p_2(0) = \bar{p}_2 > 0$ (such that $\bar{p}_1/\bar{p}_2 = \hat{p}$) for which the system converges. In other words, the manifold containing convergent paths is (globally) of dimension $n + 1 (= 3)$ in $2n (= 4)$-dimensional space (p_1, p_2, k_1, k_2).

sections to exclude this possibility. Now consider those paths that are off the stable arm of Figure 1.

We wish to prove that on any path not tending to equilibrium, some price will become zero in finite time. But any non-convergent path at some finite time must enter either region A (where $\dot{k} < 0$, $\dot{p} < 0$) or B (where $\dot{k} > 0$, $\dot{p} > 0$), as the directional arrows in Figure 1 indicate. (This result follows from the fact that the adjustment coefficients for equations (46) and (47), i.e.,

$$\frac{p_2 k_1^{\alpha} k_2^{\alpha-1}}{p_Z} \quad \text{and} \quad \frac{w_1}{p_2},$$

respectively, are appropriately bounded; see, e.g., footnote 11 below.) Therefore, we need only show that for any trajectory entering region A, the price ratio p becomes zero in finite time, and this conclusion is easily provided.[11] By symmetry, for trajectories entering region B, $1/p$ becomes zero in finite time.

In conclusion, we have shown that the violation of non-negativity in finite time holds for some neoclassical models with *strictly* concave transformation surfaces and that this result does not depend on either of two specific properties of earlier models, i.e., (i) the equality of total consumption and total wage income, or (ii) the existence of ruled segments on the production possibility frontier in output space.

8. NUMERICAL RESULTS

To confirm the above analysis and to explore the sensitivity of these results to the choice of parameters, a computer was used to generate representative trajectories for the model under alternative types of specifications. The numerical results may be quickly summarized.

(i) For various selections of the parameter values, it has been possible to impose three initial conditions upon a fourth order problem and select the fourth

[11] From equation (47)

$$\dot{p} = p\left[\frac{w_2}{p_2} - \frac{w_1}{p_1}\right] = \frac{w_1}{p_2}(pk - 1),$$

and since $(pk - 1)$ is negative and falling in the interior of region A, it is sufficient to show that w_1/p_2 is bounded away from zero in region A. By direct calculation

$$(*) \qquad \frac{w_1}{p_2} = \alpha\frac{p_Z}{p_2}\frac{k_1^{\alpha}k_2^{\alpha}}{k_1} = \alpha\frac{p_Z}{p_2}k^{\alpha-1}k_2^{2\alpha-1}.$$

We know that p_Z is not less than p_2, and k is bounded above in region A. Also, k_2 is bounded above, as shown by the following argument. From (31)

$$\dot{k}_i = \frac{p_i}{p_Z}k_1^{\alpha}k_2^{\alpha} - (g + \delta)k_i,$$

and since $1 > 2\alpha$, $\dot{k}_i < 0$ for k_i sufficiently large. Hence for any initial $k_i(0)$ there is a $\bar{k}_i$ such that $k_i(t) < \bar{k}_i < \infty$, $i = 1, 2$. Thus all three non-constant terms on the right-hand side of (*) are positive and bounded away from zero, and accordingly so is w_1/p_2.

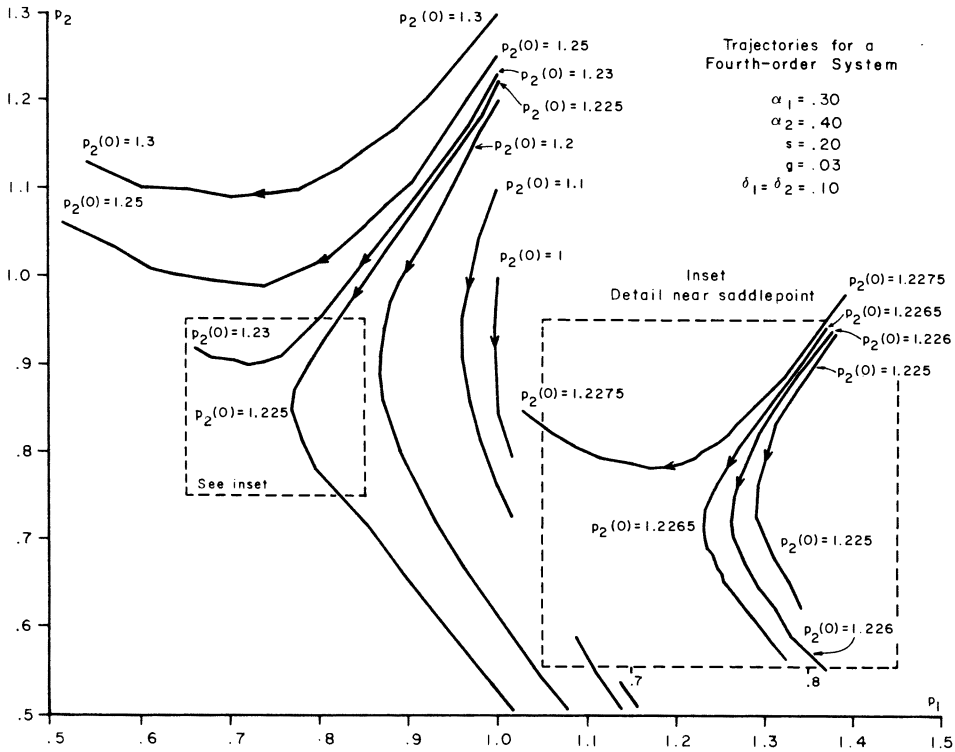

FIGURE 2.

initial condition so as to achieve convergence to the stationary equilibrium of the system.

(ii) The solution procedure (which determined the unique value for the fourth component of the vector of initial values associated with any selection of the other three components) was significantly simplified by virtue of the fact that values departing only slightly from the correct one lead very quickly to violation of non-negativity constraints. Apart from simplifying the computations, this result is itself of economic importance.

The situation is well depicted in Figure 2 which shows trajectories in the (p_1, p_2) space for given initial values of k_1 and k_2 and the arbitrary selection of one further initial condition $p_1(0) = 1$. Figure 2 indicates that the present fourth order example possesses a saddlepoint in the price space, so that for any choice of one initial price, there will be some unique initial value of the other price that will drive the system to equilibrium. Thus, although the local analysis undertaken in Section 5 employed the simplifying assumption that the capital elasticities were equal, the numerical results demonstrate that such symmetry is not essential to our conclusions.

9. CONCLUDING REMARK

This paper has shown that many dynamic heterogeneous capital good models share a common structure. However, that structure is so complex it seems doubtful that general theorems will be obtained easily—at least unless one is willing to deal with a restricted set of models by, e.g., specifying the technology and the savings hypothesis as in the first Hahn model [6]. For example, we have proved that reasonable economic models may have stationary equilibria which are not saddlepoints of type *n*, and, accordingly, such models can never be viewed as "regular Hamiltonian systems."

University of Pennsylvania,

University of Pennsylvania,

University of Toronto,

and

University of Pennsylvania

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