Central Configurations of the Curved N-Body Problem

by

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B.Sc., Sichuan University, 2010
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ABSTRACT

We extend the concept of central configurations to the N-body problem in spaces of nonzero constant curvature. Based on the work of Florin Diacu on relative equilibria of the curved N-body problem and the work of Smale on general relative equilibria, we find a natural way to define the concept of central configurations with the effective potentials. We characterize the ordinary central configurations as constrained critical points of the cotangent potential, which helps us to establish the existence of ordinary central configurations for any given masses. After these fundamental results, we study central configurations on $\mathbb{H}^2$, ordinary central configurations in $\mathbb{S}^3$, and special central configurations in $\mathbb{S}^3$ in three separate chapters. For central configurations on $\mathbb{H}^2$, we generalize the theorem of Moulton on geodesic central configurations, the theorem of Shub on the compactness of central configurations, the theorem of Conley on the index of geodesic central configurations, and the theorem of Palmore on the lower bound for the number of central configurations. We show that all three-body central configurations that form equilateral triangles must have three equal masses. For ordinary central configurations in $\mathbb{S}^3$, we construct a class of $\mathbb{S}^3$ ordinary central configurations. We study the geodesic central configurations of two and three bodies. Three-body non-geodesic ordinary central configurations that form equilateral triangles must have three equal masses. We also put into the evidence some other classes of central configurations. For special central configurations, we show that for any $N \geq 3$, there are masses that admit at least one special central configuration. We then consider the Dziobek special central configurations and obtain the central configuration equation in terms of mutual distances and volumes formed by the position vectors. We end the thesis with results concerning the stability of relative equilibria associated with 3-body special central configurations. We find that these relative equilibria are Lyapunov stable when confined to $\mathbb{S}^1$, and that they are linearly stable on $\mathbb{S}^2$ if and only if the angular momentum is bigger than a certain value determined by the configuration.

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Chapter 1

Introduction

1.1 A Brief History of the Curved N-body problem

The curved N-body problem aims to determine the motion of N point masses in spaces of constant Gaussian curvature $\kappa \neq 0$, namely spheres for $\kappa > 0$ and hyperbolic spheres for $\kappa < 0$, under the attractive force law given by the cotangent potential. This problem has its roots in the work of János Bolyai and Nikolai Lobachevsky, done in the 1830s. They independently had the idea of generalizing celestial mechanics to hyperbolic space [11, 53]. The analytic form of the potential was introduced by Ernest Schering for hyperbolic spheres in 1870 [77], and by Wilhelm Killing for spheres in 1873 [45]. In the case of the Kepler problem the potential is a harmonic function in the 3-dimensional (but not in the 2-dimensional) case. In the early 1900s, Heinrich Liebmann showed that every bounded orbit of the Kepler problem is closed [50]. These two properties are also true in the Newtonian N-body problem in the Euclidean space [9]. Thus the cotangent potential is considered to be the natural extension of the Newtonian potential to spaces of constant curvature. There were also other attempts to extend the Newtonian potential, however, they were short-lived [51].

After the rise of general relativity, the above results were almost forgotten. This problem attracted attention later from the point of view of quantum mechanics [78] and the theory of integrable dynamical systems [15]. This led to the rediscovery of the results mentioned above, sometimes with partial improvements. From the 1990s, this direction was also pursued by the Russian school of celestial mechanics, especially for the equations describing the motion of two bodies, which unlike in the Euclidean case are not integrable [12, 44, 46, 79]. Researchers mainly used the intrinsic coordinates of $\mathbb{S}^3$ ($\mathbb{H}^3$), which are hard to manipulate. Thus most of these researchers concentrated on the 2-dimensional case. For more details of the history, we refer the readers to [19].

Instead of working with the intrinsic coordinates, Florin Diacu wrote the equation of motions using extrinsic coordinates. For the 3-dimensional sphere, he used the coordinates of $\mathbb{R}^4$ and for the 3-dimensional hyperbolic sphere, he used the coordinates of $\mathbb{R}^{3,1}$, the Minkowski space. In this setup, the matrix Lie group $SO(4)$ ($SO(3,1)$) serves as the symmetry group. Therefore, this setup facilitated many studies on relative equilibria, and the rotopulsator solutions defined by Florin Diacu in the attempt to find a correspondent of the homographic motions of the Newtonian N-body problem [17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 56, 63, 71, 87, 88, 89, 91].

Another application of pursuing this problem, according to Florin Diacu [19], is deciding whether the physical space is elliptic, flat, or hyperbolic. This question was already asked by Lobachevsky and Gauss. They tried to determine the shape of space based on ideas from non-Euclidean geometry. However, they failed since the observation and measurement errors were larger than the potential deviation of the physical space from zero curvature [47]. There were other attempts. For instance, the so-called boomerang experiment analysed the cosmological background radiation [10]. All of them, however, failed to provide a definite answer on whether the physical space is curved or not.

Florin Diacu proposed a potential way to offer a solution to this problem: find stable orbits that exist only in, say, flat space, and then seek them in the universe through astronomical observations. In fact, a small step in this direction was already made by showing that the Lagrangian relative equilibria of the 3-body problem appear only in the Euclidean space for nonequal masses. It is well known that such orbits exist in the solar system, such as the equilateral triangles formed by the Sun, Jupiter, and any of the Trojan and the Greek asteroids [19, 29], and the equilateral triangle formed by Saturn, its large moon Tethys, and one of the two smaller moons, Telesto and Calypso. However, this discovery, according to Florin Diacu, only hints that the space is Euclidean for the solar-system scales. The motions of the asteroids are not exactly at the vertices of an equilateral triangle. As well, there might be other quasi-periodic motions in the curved space that are similar to the equilateral triangle relative equilibria.

1.2 Background and Motivation

Let $\mathbb{S}^3$ be the unit sphere in $\mathbb{R}^4$, and $\mathbb{H}^3$ be the unit hyperbolic sphere in $\mathbb{R}^{3,1}$. For both 4-dimensional linear spaces, we use coordinates (x, y, z, w). Given the positive masses $m_1, \ldots, m_N$ in $\mathbb{S}^3$ ($\mathbb{H}^3$), whose positions are described by the configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N) \in (\mathbb{S}^3)^N \big((\mathbb{H}^3)^N\big)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, we define the singularity set

$$\Delta = \begin{cases} \cup_{1\le i<j\le N}\{\mathbf{q} \in (\mathbb{S}^3)^N\,;\ d_{ij} = 0, \text{or } \pi\} & \text{in } \ \mathbb{S}^3; \\ \cup_{1\le i<j\le N}\{\mathbf{q} \in (\mathbb{H}^3)^N\,;\ d_{ij} = 0\} & \text{in } \ \mathbb{H}^3, \end{cases}$$

where d_{ij} be the distance between m_i and m_j in $\mathbb{S}^3$ ($\mathbb{H}^3$). For $\mathbf{q} \notin \Delta$, define the force function U ($-U$ being the potential function) as

$$U(\mathbf{q}) = \begin{cases} \sum_{1\le i<j\le N} m_i m_j \cot d_{ij}(\mathbf{q}) & \text{in } \mathbb{S}^3, \\ \sum_{1\le i<j\le N} m_i m_j \coth d_{ij}(\mathbf{q}) & \text{in } \mathbb{H}^3. \end{cases}$$

Define the kinetic energy as $T(\dot{\mathbf{q}}) = \sum_{1\le i\le N} \frac{1}{2} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$, $\dot{\mathbf{q}} = (\dot{\mathbf{q}}_1, ..., \dot{\mathbf{q}}_N)$. Then the curved N-body problem is given by the Lagrangian system on $T((\mathbb{S}^3)^N \setminus \Delta)$ $\big(T((\mathbb{H}^3)^N \setminus \Delta)\big)$, with

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T(\dot{\mathbf{q}}) + U(\mathbf{q}).$$

This setup was proposed by Florin Diacu. The advantage is that we can use the matrix Lie groups $SO(4), SO(3,1)$ as the symmetry groups. Since both symmetry groups are 6-dimensional, there are several kinds of relative equilibria, namely fixed points, positive elliptic relative equilibria, positive elliptic-elliptic relative equilibria, negative elliptic relative equilibria, negative hyperbolic relative equilibria, and negative elliptic-hyperbolic relative equilibria. Florin Diacu summarized his research on this topic in [25].

This thesis stems from the work of Florin Diacu on relative equilibria. It is well known that to find relative equilibria in the form of $\exp(\xi t)\mathbf{q}$ of a mechanical system with symmetry group G, where ξ is in the Lie Algebra of G, it suffices to find critical points of the effective potential U_ξ that depends on ξ. Thus the study of the various kinds of relative equilibria can be reduced to the study of the corresponding effective potentials. Surprisingly, I found that one effective potential is enough for all, namely, $U - \lambda I$, where $I = \sum_{i=1}^N m_i(x_i^2 + y_i^2)$. Therefore, the study of various kinds of relative equilibria can be unified by the study of the following equation,

$$\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, \quad i = 1, ..., N.$$

Recall that the solutions of a similar equation in the Newtonian N-body problem are called *central configurations.* We thus call the solutions of our equation *central configurations of the curved N-body problem.*

1.2.1 Central Configurations of the Newtonian N-body problem

The main purpose of this thesis is to extend the study of the central configurations of the Newtonian N-body problem to curved space. Recall that central configurations of the Newtonian N-body problem are solutions of the equation

$$\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, \; i = 1, ..., N,$$

where U is the Newtonian potential: $U(\mathbf{q}) = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$, $\mathbf{q} = (\mathbf{q}_1, ...\mathbf{q}_N) \in (\mathbb{R}^3)^N$ is the configuration, and $I = \sum_{i=1}^{N} m_i \mathbf{q}_i \cdot \mathbf{q}_i$ is the moment of inertia. The notation of central configurations was introduced by Pierre-Simon Laplace in 1789 [49]. While the first examples of central configurations were found by Euler and Lagrange earlier [36, 48]. By their time, all central configurations of three bodies were known, namely, three collinear central configurations found by Euler and two equilateral triangle central configurations found by Lagrange. A first systematic study of this concept appeared only in 1900, when Otto Dziobek published a fundamental paper on central configurations [35]. Since then, research in this direction has continued, showing that the central configurations are essential for understanding the motions of the Newtonian N-body problem.

From the mathematical point of view, central configuration equation is an algebraic system in $3N$ variables. It does not involve the time variable. However, each central configuration gives rise to simple, explicit solutions of the Newtonian N-body problem such that the configuration is similar to the initial configuration during the motion. These motions are given by *homothetic orbits* and *relative equilibria* [52]. If we compose these solutions we obtain *homographic orbits.* Central configurations also appear in other circumstances. For instance, when three or more bodies tend to a simultaneous collision, or they scatter to infinity, they tend asymptotically to a central configuration [75, 81]. Central configurations are also involved in the topological classification of the planar N-body problem [62, 84]

For $N \leq 3$, all solutions were known by the time of Lagrange. But for higher N, we do not know much about them. A basic question is called the Wintner-Smale problem: given N positive masses, is the number of central configurations finite, up to symmetries? Smale listed it as the 6-th problem for the 21-st century [86]. The case when $N = 4$ was solved only in 2006 by Hampton and Moeckel [58, 42], who showed that the number of central configurations is between 32 and 8472 for generic masses. The case when $N = 5$ was solved in 2012 by Albouy and Kaloshin [6, 61], who showed that the number of central configurations is finite for generic masses. For $N \geq 6$, even generic finiteness is open. However, if non-positive masses are allowed, there is a continuum of central configurations [40, 72]. There are also other interesting problems on this subject [4].

1.3 Summary and Organization

In Chapter 2, using Florin Diacu's setup, we derive the equations of motion of the curved N-body problem. Then by the obvious symmetry of the Hamiltonian, we find the corresponding first integrals with Noether's theorem. We also point out that in qualitative studies the value of the curvature is irrelevant and that only the sign matters. Thus we only need to study solutions on the unit sphere and the unit hyperbolic sphere.

In Chapter 3, we first introduce the concept of relative equilibrium from the viewpoint of geometric mechanics. The well-known theorem of Smale shows that to find a relative equilibrium is equivalent to finding a critical point of the corresponding effective potential. We then find the effective potentials corresponding to the six types of relative equilibria. Remarkably, they are of the same form. We also offer another proof using an elementary approach. The two equivalent methods show that the key to finding relative equilibria is to solve the equation

$$\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, \quad i = 1, ..., N.$$

We define the solutions of this equation as central configurations of the curved N-body problem. If the equations are satisfied only for some $\lambda \neq 0$, we call it an ordinary central configuration; if the equations are satisfied for $\lambda = 0$, we call it a special central configuration. No special central configuration exists on the hyperbolic sphere. In the end, we discuss the connection between central configurations and relative equilibria in more detail.

In Chapter 4, we first characterize the moment of inertia I in a geometric way. This justifies the notation of central configurations to a certain level and provides some geometric insight into the problem. We then define the equivalent classes of central configurations, an approach that is essential for counting central configurations. We collect some useful facts about central configurations in the last two sections. For instance, we show that any central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on a particular hyperbolic 2-sphere; we also show that any $\mathbb{S}^2$ central configuration can be found on a particular 2-sphere. These derivations reduce the study of central configurations to convenient settings.

In Chapter 5, we first characterize central configurations as critical points of some functions. More precisely, central configurations are the rest points of the gradient flow of some functions on some manifolds. Using this property, we are able to show the existence of ordinary central configurations for any given positive masses. We also find a convenient way to compute the Hessian of these critical points, and estimate their minimal nullity. However, there are masses that do not possess special central configurations. We define M_N as the subset of $\mathbb{R}_+^N$ for which there exist special central configurations. In the end, we extend the Wintner-Smale problem to the curved N-body problem.

The central configurations of the curved N-body problem are roughly divided into three main categories: central configurations in $\mathbb{H}^3$, ordinary central configurations in $\mathbb{S}^3$, and special central configurations in $\mathbb{S}^3$. The following three chapters are devoted to study these three categories.

In Chapter 6, we consider central configurations in $\mathbb{H}^3$. We start with several examples of central configurations and then write the equation $\nabla_{\mathbf{q}_i}U = \lambda\nabla_{\mathbf{q}_i}I$ in another form, which becomes useful later. We then generalize the result of Shub [80], which shows that the set of central configurations is compact for given masses. We generalize the celebrated Moulton's theorem in $\mathbb{H}^3$, that is, there are $N!/2$ geodesic central configurations for any N masses. Then we study the Hessian of these geodesic central configurations with an idea introduced by Conley [67] in the Newtonian N-body problem. With these preparations, we apply Morse theory to get a lower bound for the number of central configurations for generic masses. In the end, we study the non-geodesic central configurations in the 3-body case. We obtain a necessary condition, which implies that all equilateral triangle central configurations must have equal masses.

In Chapter 7, we study ordinary central configurations in $\mathbb{S}^3$. We start with several examples of ordinary central configurations and then write the equation $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ in another form, which becomes useful later. One class of examples is formed by $\mathbb{S}^3$ central configurations, which contrasts with what happens in $\mathbb{H}^3$ and shows the complexity of this problem in $\mathbb{S}^3$. We then turn to geodesic central configurations for two and three masses. We find that Moulton's theorem cannot be generalized directly. Surprisingly, there is a continuum of central configurations for two equal masses. For three masses, we study the inverse problem: for a given configuration on $\mathbb{S}^1$, find positive masses such that a geodesic central configuration exists. In the end, we study the non-geodesic central configurations in the 3-body case. We obtain a necessary condition, which implies that all equilateral triangle central configurations must have equal masses. This condition also helps us to find another class of central configurations.

In Chapter 8, we study special central configurations in $\mathbb{S}^3$, which are solutions of the system

$$\nabla_{\mathbf{q}_i} U = 0, \quad i = 1, ..., N.$$

We start with writing the equation in another useful form. Then we provide examples of special central configurations for any $N \geq 3$ masses, which shows that the mass set M_N is not empty when $N \geq 3$. One class of examples uses the special geometry of $\mathbb{S}^3$, a way first proposed by Florin Diacu [25]. We then consider special central configurations in higher dimensional spheres. Those special central configurations when the N particles span an $(N-1)$-dimensional linear space (or, an $(N-2)$-sphere) are of special interest. They are analogous to Dziobek central configurations of the Newtonian N-body problem [52]. We call them Dziobek special central configurations as well. For them, the central configuration equation can be nicely written in terms of the mutual distances and volumes formed by the position vectors. We then apply these equations to the physically interesting examples: three particles on $\mathbb{S}^1$, four particles on $\mathbb{S}^2$ and five particles in $\mathbb{S}^3$. In the end, we find the mass set M_3.

In Chapter 9, we study the stability of the relative equilibria associated with all special central configurations in the 3-body case, on $\mathbb{S}^1$ and $\mathbb{S}^2$. We first rewrite the equations of motions in spherical coordinates. Then we show that the relative equilibria are Lyapunov stable when confined to $\mathbb{S}^1$. When considered on $\mathbb{S}^2$, the linear stability of these motions depends on their angular momenta. For each central configuration, there is one critical value such that the motion is linearly stable if and only if the angular momentum is bigger than this value.

Finally, in Chapter 10, we draw some conclusions and maps some further directions of research.

Chapter 2

The Curved N-Body Problem

In this chapter, we derive the equations of motion of the curved N-body problem, then find the obvious first integrals.

2.1 Equations of Motion

In this section we introduce the N-body problem in spaces of constant nonzero curvature, which we will refer to as the *curved N-body problem*, in contrast to its analogue in Euclidean space, which we will call the *Newtonian N-body problem*. As in [19], we set the curved N-body problems in the unit 3-sphere and the unit hyperbolic 3-sphere as Hamiltonian systems in the Euclidean space $\mathbb{R}^4$ and in the Minkowski space $\mathbb{R}^{3,1}$, respectively, with holonomic constraints that restrict the motion of the bodies to these manifolds.

Vectors are all column vectors, but written as row vectors in the text. Recall that $\mathbb{R}^4$ and $\mathbb{R}^{3,1}$ are endowed with different inner products: for two vectors, $\mathbf{q}_1 = (x_1, y_1, z_1, w_1)$ and $\mathbf{q}_2 = (x_2, y_2, z_2, w_2)$, they are given by

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = x_1x_2 + y_1y_2 + z_1z_2 + \sigma w_1w_2,$$

where $\sigma = 1$ for the Euclidean space and $\sigma = -1$ for the Minkowski space. Then the unit sphere $\mathbb{S}^3$ and the unit hyperbolic sphere $\mathbb{H}^3$ are

$$\mathbb{S}^3 := \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + w^2 = 1\} \quad \text{and}$$
$$\mathbb{H}^3 := \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 - w^2 = -1,\ w > 0\},$$

respectively. We can merge these two manifolds into

$$\mathbb{M}^3 := \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + \sigma w^2 = \sigma, \text{ with } w > 0 \text{ for } \sigma = -1\}.$$

Denote by $\mathbf{m} \in \mathbb{R}_+^N$ the mass vector $(m_1, \ldots, m_N)$ where m_i is the mass of the ith particle. Given the positive masses $m_1, \ldots, m_N$, whose positions are described by the configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N) \in (\mathbb{M}^3)^N$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, we define the singularity set

$$\Delta = \cup_{1 \leq i < j \leq N} \{\mathbf{q} \in (\mathbb{M}^3)^N \,;\, \mathbf{q}_i = \pm \mathbf{q}_j\}.$$

Let d_{ij} be the geodesic distance between the point masses m_i and m_j, we define the force function U ($-U$ being the potential function) on $(\mathbb{M}^3)^N \setminus \Delta$ as

$$U(\mathbf{q}) := \sum_{1 \leq i < j \leq N} m_i m_j \mathrm{ctn} d_{ij},$$

where $\mathrm{ctn}(x)$ stands for $\cot(x)$ in $\mathbb{S}^3$ and $\coth(x)$ in $\mathbb{H}^3$. We would like to mention that there are many other choices of the potential, but this potential is coherent with the Newtonian N-body problem, see [7, 19]. We introduce two more notations, which unify the trigonometric and hyperbolic functions,

$$\mathrm{sn}(x) = \sin(x) \text{ or } \sinh(x), \quad \mathrm{csn}(x) = \cos(x) \text{ or } \cosh(x).$$

Then the distance d_{ij} is given by the expression $d_{ij} := \mathrm{arccsn}(\sigma \mathbf{q}_i \cdot \mathbf{q}_j)$, where $\mathrm{arccsn}(x)$ is the inverse function of $\mathrm{csn}(x)$. We define the kinetic energy as

$$T(\mathbf{p}) = \sum_{1 \leq i \leq N} \frac{1}{2} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i = \sum_{1 \leq i \leq N} \frac{1}{2} m_i^{-1} \mathbf{p}_i \cdot \mathbf{p}_i,$$

where $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$ is the momentum of m_i. We also denote the momentum of the particle system by

$$\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_N).$$

Then the curved N-body problem is given by the Hamiltonian system on $T^*((\mathbb{M}^3)^N \setminus \Delta)$, with

$$H(\mathbf{q}, \mathbf{p}) := T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q}).$$

Let us derive the equations of motion for the Hamiltonian system in $\mathbb{S}^3$. The

Hamiltonian is

$$H = \sum_{1\le i\le N} \frac{1}{2} m_i^{-1}\mathbf{p}_i\cdot\mathbf{p}_i - \sum_{1\le i<j\le N} m_i m_j \cot d_{ij}.$$

Here U is defined on $(\mathbb{S}^3)^N \setminus \Delta$, with the set of singularities $\Delta = \Delta^- \cup \Delta^+$, where

$$\begin{aligned}\Delta^- &:= \cup_{1\le i<j\le N}\{\mathbf{q}\in(\mathbb{S}^3)^N : \mathbf{q}_i = -\mathbf{q}_j\} \setminus \cup_{1\le i<j\le N}\{\mathbf{q}\in(\mathbb{S}^3)^N : \mathbf{q}_i = \mathbf{q}_j\},\\ \Delta^+ &:= \cup_{1\le i<j\le N}\{\mathbf{q}\in(\mathbb{S}^3)^N : \mathbf{q}_i = \mathbf{q}_j\} \setminus \cup_{1\le i<j\le N}\{\mathbf{q}\in(\mathbb{S}^3)^N : \mathbf{q}_i = -\mathbf{q}_j\}.\end{aligned}$$

We will call Δ^- the antipodal singularity set and Δ^+ the collision singularity set. Using constrained Hamiltonian dynamics, we get the equations describing the motion of the bodies,

$$\begin{cases}\dot{\mathbf{q}}_i = m_i^{-1}\mathbf{p}_i\\ \dot{\mathbf{p}}_i = \nabla_{\mathbf{q}_i}U - m_i^{-1}(\mathbf{p}_i\cdot\mathbf{p}_i)\mathbf{q}_i = \nabla_{\mathbf{q}_i}U - m_i(\dot{\mathbf{q}}_i\cdot\dot{\mathbf{q}}_i)\mathbf{q}_i\\ \mathbf{q}_i\cdot\mathbf{q}_i = 1, \quad \mathbf{p}_i\cdot\mathbf{q}_i = 0, \quad i = 1, ..., N,\end{cases}$$

where $\nabla_{\mathbf{q}_i}U$ stands for the gradient of U on the manifold $(\mathbb{S}^3)^N$. The gradient can be interpreted as the attractive force on $\mathbf{q}_i$ produced by all the other particles. The term $-m_i^{-1}(\mathbf{p}_i\cdot\mathbf{p}_i)\mathbf{q}_i$ can be viewed as the constraint force keeping the particles on the sphere. Thus we denote $\nabla_{\mathbf{q}_i}U$ and $\nabla_{\mathbf{q}_i}m_i m_j \cot d_{ij}$ by $\mathbf{F}_i$ and $\mathbf{F}_{ij}$, respectively. We have

$$\mathbf{F}_{ij} = \frac{-m_i m_j}{\sin^2 d_{ij}}\nabla_{\mathbf{q}_i} d_{ij} = \frac{-m_i m_j}{\sin^2 d_{ij}}\nabla_{\mathbf{q}_i}\cos^{-1}\mathbf{q}_i\cdot\mathbf{q}_j = \frac{m_i m_j}{\sin^3 d_{ij}}\nabla_{\mathbf{q}_i}\mathbf{q}_i\cdot\mathbf{q}_j.$$

The gradient of $\mathbf{q}_i\cdot\mathbf{q}_j$ on the manifold $(\mathbb{S}^3)^N$ can be computed as follows. We extend any function $f\colon (\mathbb{S}^3)^N \to \mathbb{R}$ to the ambient space $\bar{f}\colon (\mathbb{R}^4)^N \to \mathbb{R}$,

$$\bar{f}(\mathbf{q}) = f\left(\frac{\mathbf{q}_1}{\sqrt{\mathbf{q}_1\cdot\mathbf{q}_1}}, \cdots, \frac{\mathbf{q}_N}{\sqrt{\mathbf{q}_N\cdot\mathbf{q}_N}}\right).$$

Then $\bar{f}(\lambda\mathbf{q}) = \bar{f}(\mathbf{q})$ for $\lambda > 0$, i.e., $\bar{f}$ is a homogeneous function of degree zero. Let $\widetilde{\nabla}$ be the gradient in the ambient space and $\frac{\partial}{\partial n_i}$ the unit normal vector of the i-th unit sphere. Since $\frac{\partial \bar{f}}{\partial r_i}=0$, we obtain $(\widetilde{\nabla}_{\mathbf{q}_i}\bar{f})|_{(\mathbb{S}^3)^N}=\nabla_{\mathbf{q}_i}f+\frac{\partial \bar{f}}{\partial r_i}\frac{\partial}{\partial n_i}=\nabla_{\mathbf{q}_i}f$. Thus

$$\begin{aligned}\mathbf{F}_{ij}&=\frac{m_im_j}{\sin^3 d_{ij}}\widetilde{\nabla}_{\mathbf{q}_i}\frac{\mathbf{q}_i\cdot\mathbf{q}_j}{\sqrt{\mathbf{q}_i\cdot\mathbf{q}_i}\sqrt{\mathbf{q}_j\cdot\mathbf{q}_j}}=\frac{m_im_j}{\sin^3 d_{ij}}\frac{\sqrt{\mathbf{q}_i\cdot\mathbf{q}_i}\sqrt{\mathbf{q}_j\cdot\mathbf{q}_j}\mathbf{q}_j-\mathbf{q}_i\cdot\mathbf{q}_j\frac{\sqrt{\mathbf{q}_j\cdot\mathbf{q}_j}}{\sqrt{\mathbf{q}_i\cdot\mathbf{q}_i}}\mathbf{q}_i}{(\sqrt{\mathbf{q}_i\cdot\mathbf{q}_i}\sqrt{\mathbf{q}_j\cdot\mathbf{q}_j})^2}\\&=\frac{m_im_j[\mathbf{q}_j-\cos d_{ij}\mathbf{q}_i]}{\sin^3 d_{ij}}.\end{aligned}$$

Thus the equations of motion for the curved N-body problem in $\mathbb{S}^3$ are

$$\begin{cases}\dot{\mathbf{q}}_i=m_i^{-1}\mathbf{p}_i\\ \dot{\mathbf{p}}_i=\sum_{j=1,j\neq i}^{N}\frac{m_im_j[\mathbf{q}_j-\cos d_{ij}\mathbf{q}_i]}{\sin^3 d_{ij}}-m_i(\dot{\mathbf{q}}_i\cdot\dot{\mathbf{q}}_i)\mathbf{q}_i\\ \mathbf{q}_i\cdot\mathbf{q}_i=1,\quad \mathbf{p}_i\cdot\mathbf{q}_i=0,\quad i=1,...,N.\end{cases}$$

Gravitation law in $\mathbb{S}^3$. A mass m_2 at $\mathbf{q}_2\in\mathbb{S}^3$ attracts another mass m_1 at $\mathbf{q}_1\in\mathbb{S}^3$ ($\mathbf{q}_1\neq\pm\mathbf{q}_2$) along the minimal geodesic connecting the two points with a force whose magnitude is $\frac{m_1m_2}{\sin^2 d_{12}}$. More precisely,

$$\mathbf{F}_{12}=\frac{m_1m_2[\mathbf{q}_2-\cos d_{12}\mathbf{q}_1]}{\sin^3 d_{12}}.$$

Similarly, we can derive the equations of motion for the Hamiltonian system in $\mathbb{H}^3$. The Hamiltonian is

$$H=T(\mathbf{q},\mathbf{p})-U(\mathbf{q})=\sum_{1\le i\le N}\frac{1}{2}m_i^{-1}\mathbf{p}_i\cdot\mathbf{p}_i-\sum_{1\le i<j\le N}m_im_j\coth d_{ij}.$$

Here U is defined on $(\mathbb{H}^3)^N\setminus\Delta$, and the set of singularities is

$$\Delta:=\cup_{1\le i<j\le N}\{\mathbf{q}\in(\mathbb{H}^3)^N:\,\mathbf{q}_i=\mathbf{q}_j\}.$$

We interpret $\nabla_{\mathbf{q}_i}U$ and $\nabla_{\mathbf{q}_i}m_im_j\coth d_{ij}$ as $\mathbf{F}_i$ and $\mathbf{F}_{ij}$ respectively. Similar computations lead to

$$\mathbf{F}_{ij}=\frac{m_im_j[\mathbf{q}_j-\cosh d_{ij}\mathbf{q}_i]}{\sinh^3 d_{ij}},$$

and the equations of motion for the curved N-body problem in $\mathbb{H}^3$ are

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1}\mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j [\mathbf{q}_j - \cosh d_{ij} \mathbf{q}_i]}{\sinh^3 d_{ij}} + m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = -1, \quad \mathbf{p}_i \cdot \mathbf{q}_i = 0, \quad i = 1, ..., N. \end{cases}$$

Gravitation law in $\mathbb{H}^3$. A mass m_2 at $\mathbf{q}_2 \in \mathbb{H}^3$ attracts another mass m_1 at $\mathbf{q}_1 \in \mathbb{H}^3$ ($\mathbf{q}_1 \neq \mathbf{q}_2$) along the minimal geodesic connecting the two points with a force whose magnitude is $\frac{m_1 m_2}{\sinh^2 d_{12}}$. More precisely,

$$\mathbf{F}_{12} = \frac{m_1 m_2 [\mathbf{q}_2 - \cosh d_{12} \mathbf{q}_1]}{\sinh^3 d_{12}}.$$

Using the functions $\mathrm{sn}(x)$ and $\mathrm{csn}(x)$ introduced earlier, we can blend the two systems of equations into one system in $(\mathbb{M}^3)^N \setminus \Delta$ [19, 25],

$$\begin{cases} \dot{\mathbf{q}}_i = m_i^{-1}\mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j [\mathbf{q}_j - \mathrm{csn} d_{ij} \mathbf{q}_i]}{\mathrm{sn}^3 d_{ij}} - \sigma m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad \mathbf{p}_i \cdot \mathbf{q}_i = 0, \quad i = 1, ..., N. \end{cases} \tag{2.1}$$

Remark 1. *If we derive the equation of motion in $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$, where $\mathbb{S}^3_\kappa = \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 + w^2 = \kappa^{-1}\} \quad \kappa > 0$, and $\mathbb{H}^3_\kappa = \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 - w^2 = \kappa^{-1}, w > 0\} \quad \kappa < 0$, we would see that the gravitational law is*

$$\mathbf{F}_{12} = \frac{m_1 m_2 |\kappa|^{\frac{3}{2}} [\mathbf{q}_2 - csn|\kappa|^{\frac{1}{2}} d_\kappa(\mathbf{q}_1, \mathbf{q}_2)\mathbf{q}_1]}{sn^3 \left(|\kappa|^{\frac{1}{2}} d_\kappa(\mathbf{q}_1, \mathbf{q}_2) \right)},$$

[19, page 29], where $d_\kappa(\mathbf{q}_1, \mathbf{q}_2)$ is the distance between the two particles in $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$. Formally, it tends to the gravitational law in $\mathbb{R}^3$ when $\kappa \to 0$, which again shows that the potential is coherent with the Newtonian potential.

Some researchers studied the curved N-body problem in $\mathbb{S}^3_\kappa$ and $\mathbb{H}^3_\kappa$ with curvature $\kappa \neq \pm 1$ [44]. For our purpose, this is not necessary since it has been shown in [19] that there are coordinate and time-rescaling transformations,

$$\mathbf{q}_i = |\kappa|^{-1/2}\mathbf{r}_i, \;\; i = \overline{1, N} \quad \text{and} \quad \tau = |\kappa|^{3/4} t,$$

which bring the systems from $\mathbb{S}^3_\kappa$ *and* $\mathbb{H}^3_\kappa$ *to systems to* $\mathbb{S}^3$ *and* $\mathbb{H}^3$*, respectively.*

2.2 First Integrals

The Hamiltonian of the curved N-body problem is invariant under the action of the rotation group $O(4)$ ($O(3,1)$). This fact leads to the six angular momentum integrals.

Recall that a 4×4 matrix A is in $O(4)$ if it keeps the inner product in the 4-dimensional Euclidean space, that is, if

$$A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot \mathbf{v}, \text{ for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^4.$$

It is a matrix Lie group and it has two components. The component containing I, the identity matrix, that is, those matrices with determinant one, is denoted by $SO(4)$. The tangent space at I, the Lie algebra of $O(4)$, is a 6-dimensional linear space and is denoted by $\mathfrak{s}o(4)$. A 4×4 matrix X is in $\mathfrak{s}o(4)$ if $X^T = -X$.

Recall that a 4×4 matrix A is in $O(3,1)$ if it keeps the inner product in the 4-dimensional Minkowski space, that is, if

$$A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot \mathbf{v}, \text{ for any } \mathbf{u}, \mathbf{v} \in \mathbb{R}^{3,1}.$$

It is a matrix Lie group with four components [66]. The two components with determinant one is denoted by $SO(3,1)$, and the one containing I is denoted by $SO^+(3,1)$. The tangent space at I, the Lie algebra of $O(3,1)$, is a 6-dimensional linear space and is denoted by $\mathfrak{s}o(3,1)$. A 4×4 matrix X is in $\mathfrak{s}o(3,1)$ if $\psi X^T \psi = -X$, where $\psi = \text{diag}(1,1,1,-1)$.

For the system in $\mathbb{S}^3$, $O(4)$ keeps the Hamiltonian. Let $\phi \in O(4)$. Extend this action to $T^*(\mathbb{S}^3)^N$ via

$$\phi(\mathbf{q}, \mathbf{p}) = (\phi\mathbf{q}_1, \ldots, \phi\mathbf{q}_N, \phi\mathbf{p}_1, \ldots, \phi\mathbf{p}_N).$$

Then

$$\begin{aligned} H(\phi(\mathbf{q},\mathbf{p})) &= \sum_{1\le i\le N} \frac{1}{2} m_i^{-1} \phi\mathbf{p}_i \cdot \phi\mathbf{p}_i - \sum_{1\le i<j\le N} m_i m_j \cot d(\phi\mathbf{q}_i, \phi\mathbf{q}_j) \\ &= \sum_{1\le i\le N} \frac{1}{2} m_i^{-1} \mathbf{p}_i \cdot \mathbf{p}_i - \sum_{1\le i<j\le N} m_i m_j \cot d(\mathbf{q}_i, \mathbf{q}_j) \\ &= H(\mathbf{q},\mathbf{p}). \end{aligned}$$

This action also preserves the symplectic form $\omega = d(\sum_{1\le i\le N} \mathbf{p}_i d\mathbf{q}_i)$. Similarly, we can verify that $O(3,1)$ is the symmetric group for the system in $\mathbb{H}^3$. These facts show:

Proposition 1. *Let ϕ be an element of the isometry group of $\mathbb{M}^3$, then $(\mathbf{q}(t), \mathbf{p}(t))$ solves the curved N-body problem if and only if $\phi\left(\mathbf{q}(t), \mathbf{p}(t)\right)$ does.*

Theorem 1 (Noether's Theorem)**.** *Let G be the symmetric group of $H(\mathbf{q},\mathbf{p})$ and ϕ_s be a one-parameter subgroup of G, $\chi(q) = \frac{d}{ds}|_{s=0}\phi_s(\mathbf{q})$. Then*

$$F(\mathbf{q},\mathbf{p}) = \mathbf{p} \cdot \chi(q)$$

is a first integral.

The Lie algebra $\mathfrak{so}(4)$ of the rotation group $O(4)$ is composed of the 4×4 skew-symmetric matrices. Thus we get

$$F(\mathbf{q},\mathbf{p}) = \mathbf{p} \cdot \chi(q) = \sum_i m_i \begin{bmatrix} \dot{w}_i & \dot{x}_i & \dot{y}_i & \dot{z}_i \end{bmatrix} \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} \begin{bmatrix} w_i \\ x_i \\ y_i \\ z_i \end{bmatrix},$$

which leads to the following six independent integrals of the system in $\mathbb{S}^3$,

$$\begin{aligned} \omega_{xy} &= \sum_{i=1}^N m_i(\dot{x}_i y_i - x_i \dot{y}_i), & \omega_{xz} &= \sum_{i=1}^N m_i(\dot{x}_i z_i - x_i \dot{z}_i), \\ \omega_{xw} &= \sum_{i=1}^N m_i(\dot{x}_i w_i - x_i \dot{w}_i), & \omega_{yz} &= \sum_{i=1}^N m_i(\dot{y}_i z_i - y_i \dot{z}_i), \\ \omega_{yw} &= \sum_{i=1}^N m_i(\dot{y}_i w_i - y_i \dot{w}_i), & \omega_{zw} &= \sum_{i=1}^N m_i(\dot{z}_i w_i - z_i \dot{w}_i). \end{aligned} \tag{2.2}$$

Similarly, we use the Lie algebra $\mathfrak{so}(3,1)$ to derive the first integrals of the system in $\mathbb{H}^3$. We find that it leads to (2.2) as well. These integrals were first found in [19, 25] by using wedge product. We will refer to them as *angular momentum integrals.*

Chapter 3

Relative Equilibria and Central Configurations

In this chapter, we first introduce the definition of relative equilibria in the context of mechanical systems with symmetry. We use Smale's theorem to show that finding relative equilibria of the curved N-body problem is equivalent to finding the critical points of the corresponding effective potentials. Surprisingly, the effective potentials corresponding to different relative equilibria have the same form, as we show with the help of some non-elementary arguments. We also prove this fact by an elementary approach. With this effective potential, we define central configurations and discuss the relationships between central configurations and motions of the curved N-body problem.

3.1 Relative Equilibria

In this section we introduce the relative equilibria of the curved N-body problem and classify these solutions into several classes. We then give examples of relative equilibria in each class.

We begin with some definitions for general mechanical systems.

Definition 1 ([83]). *A mechanical system with symmetry consists of a* 4*-tuple* (M, K, V, G) *where* M *is a manifold,* K *is the kinetic energy,* V *the potential energy and* G *a Lie group acting on* M *preserving* K *and* V *with all data smooth.*

For each $\boldsymbol{\xi}$ belonging to the Lie algebra $\mathfrak{g}$ of G, there is a vector field $\boldsymbol{\xi}_M$ on M, given by

$$\boldsymbol{\xi}_M(\mathbf{q}) := \left.\frac{d}{dt}\right|_{t=0} (\exp(\boldsymbol{\xi} t)\mathbf{q}) \,.$$

Here we denote by $\boldsymbol{\xi}_M(\mathbf{q})$ the vector at $\mathbf{q} \in M$, and by $\exp(\boldsymbol{\xi} t)\mathbf{q}$ the action of $\exp(\boldsymbol{\xi} t)$ on $\mathbf{q}$.

Definition 2 ([84])**.** *A solution of the mechanical system with symmetry* (M, K, V, G) *is called a relative equilibrium if it is also an integral curve of the vector field* $\boldsymbol{\xi}_M$*. In other words, a relative equilibrium is a solution in the form of* $\exp(\boldsymbol{\xi} t)\mathbf{q}$*. The curve* $\exp(\boldsymbol{\xi} t) \in G$ *is called a one-parameter subgroup of* G*.*

Let us return to the curved N-body problem.

Proposition 2. *A one-parameter subgroup of* $SO(4)$ *is of the form* $PA_{\alpha,\beta}(t)P^{-1}$*, with* $P \in SO(4)$ *and*

$$A_{\alpha,\beta}(t) = \begin{bmatrix} \cos\alpha t & -\sin\alpha t & 0 & 0 \\ \sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & \cos\beta t & -\sin\beta t \\ 0 & 0 & \sin\beta t & \cos\beta t \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}.$$

We call these rotations positive elliptic-elliptic if $\alpha \neq 0$ and $\beta \neq 0$, and positive elliptic if only one of them is zero. We call the corresponding relative equilibria *positive elliptic-elliptic relative equilibria* and *positive elliptic relative equilibria*, respectively.

Proposition 3. *A one-parameter subgroup of* $SO^+(3,1)$ *is of the form* $PB_{\alpha,\beta}(t)P^{-1}$ *or* $PC_\eta(t)P^{-1}$*, with* $P \in SO(3,1)$*, and*

$$B_{\alpha,\beta}(t) = \begin{bmatrix} \cos\alpha t & -\sin\alpha t & 0 & 0 \\ \sin\alpha t & \cos\alpha t & 0 & 0 \\ 0 & 0 & \cosh\beta t & \sinh\beta t \\ 0 & 0 & \sinh\beta t & \cosh\beta t \end{bmatrix}, C_\eta(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\eta t & \eta t \\ 0 & \eta t & 1-\eta t^2/2 & \eta t^2 \\ 0 & \eta t & -\eta t^2 & 1+\eta t^2/2 \end{bmatrix},$$

where $\alpha,\ \beta,\ \eta \in \mathbb{R}$*.*

Similarly, the negative elliptic, negative hyperbolic, negative elliptic-hyperbolic and parabolic transformations correspond to $\alpha \neq 0$ and $\beta = 0$, $\alpha = 0$ and $\beta \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$, and $\eta \neq 0$, respectively. We call the corresponding relative equilibria *negative elliptic relative equilibria, negative hyperbolic relative equilibria, negative elliptic-hyperbolic relative equilibria* and *parabolic relative equilibria*, respectively.

We can easily check that

$$A_{\alpha,\beta}(t) = \exp(\boldsymbol{\xi}_1 t), \;\; B_{\alpha,\beta}(t) = \exp(\boldsymbol{\xi}_2 t), \;\; C_\eta(t) = \exp(\boldsymbol{\xi}_3 t),$$

where $\boldsymbol{\xi}_1 \in \mathfrak{so}(4)$, $\boldsymbol{\xi}_2$, $\boldsymbol{\xi}_3 \in \mathfrak{so}(3,1)$, and

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{bmatrix}, \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \end{bmatrix}, \quad \boldsymbol{\xi}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\eta & \eta \\ 0 & \eta & 0 & 0 \\ 0 & \eta & 0 & 0 \end{bmatrix}.$$

Recall that Proposition 1 shows that for any ϕ in the isometry group, $(\mathbf{q}(t), \mathbf{p}(t))$ solves the curved N-body problem if and only if $\phi\,(\mathbf{q}(t), \mathbf{p}(t))$ does. Thus we cover all possible relative equilibria for the curved N-body problem if we define them in terms of the three normal forms of the one-parameter subgroup. To simplify the notation, we will denote initial positions without any argument and attach the argument t to functions depending on time.

Definition 3. *Let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ be a nonsingular initial configuration of the masses $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{R}_+^N$, $N \geq 2$, in $\mathbb{M}^3$, where the initial position vectors are $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$. Then a solution of the form*

$$\mathbf{q}(t) = Q(t)\mathbf{q} := (Q(t)\mathbf{q}_1, \ldots, Q(t)\mathbf{q}_N)$$

of system (2.1), with $Q(t)$ being $A_{\alpha,\beta}(t)$, $B_{\alpha,\beta}(t)$, or $C_\eta(t)$, is called a relative equilibrium.

The following fact was first proved in [19, 25]. For completeness, we also give the proof. Note that the result holds for any mechanical system in $\mathbb{H}^3$ with $O(3,1)$ symmetry.

Proposition 4. *There are no parabolic relative equilibria for the curved N-body problem in $\mathbb{H}^3$.*

Proof. Let $\mathbf{q}(t) = C_\eta(t)\mathbf{q}$ be a solution. Then

$$\mathbf{q}_i(t) = (x_i, y_i - \eta t z_i + \eta t w_i, \eta t y_i + (1 - \frac{\eta t^2}{2}) z_i + \frac{\eta t^2}{2} w_i, \eta t y_i - \frac{\eta t^2}{2} z_i + (1 + \frac{\eta t^2}{2}) w_i).$$

Thus direct computation leads to

$$\omega_{zw} = \sum_{i=1}^{N} m_i(z_i\dot{w}_i - w_i\dot{z}_i) = \sum_{i=1}^{N} m_i\eta(z_i - w_i)y_i - \eta t(\sum_{i=1}^{N} m_i(z_i - w_i)^2).$$

In $\mathbb{H}^3$, $x^2 + y^2 + z^2 = w^2 - 1$ and $w \geq 1$. Thus $w_i > z_i$, and $\sum_{i=1}^{N} m_i(z_i - w_i)^2 > 0$. Thus ω_{zw} could not be a constant, a contradiction which proves $C_\eta(t)\mathbf{q}$ could not be a solution. Thus there are no parabolic relative equilibria. □

Therefore there are only five types of relative equilibria. Florin Diacu gave a nice summary of his study on this topic in [19, 25], where he studied the criteria and the qualitative behaviour of these solutions, and gave many examples. The following examples of relative equilibria are based on his results.

Example 1. *In $\mathbb{S}^3$, let us place three equal masses $m_1 = m_2 = m_3 = \frac{13\sqrt{39}}{512}$ at $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\mathbf{q}_j = (x_j, y_j, z_j, w_j)$, $j = 1, 2, 3$, where*

$$x_j = \frac{1}{2}\cos\beta_j, \quad y_j = \frac{1}{2}\sin\beta_j, \quad z_j = \frac{\sqrt{3}}{2}, \quad w_j = 0, \quad \beta_j = \frac{2\pi j}{3}.$$

Then the computations show that $\mathbf{q}(t) = A_{1,0}(t)\mathbf{q}$ is a positive elliptic relative equilibrium and $\mathbf{q}(t) = A_{\sqrt{2},1}(t)\mathbf{q}$ is a positive elliptic-elliptic relative equilibrium.

Example 2. *In $\mathbb{H}^3$, let us place three equal masses $m_1 = m_2 = m_3 = \frac{8\sqrt{2}}{9}$ at $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, 2, 3$, where*

$$\begin{array}{llll} x_1 = 0, & y_1 = 0, & z_1 = 0, & w_1 = 1 \\ x_2 = 1, & y_2 = 0, & z_2 = 0, & w_2 = \sqrt{2} \\ x_3 = -1, & y_3 = 0, & z_3 = 0, & w_3 = \sqrt{2}. \end{array}$$

Then the computations show that $\mathbf{q}(t) = B_{1,0}(t)\mathbf{q}$ is a negative elliptic relative equilibrium, $\mathbf{q}(t) = B(t)_{0,1}\mathbf{q}$ is a negative hyperbolic relative equilibrium, and $\mathbf{q}(t) = B_{1/2,\sqrt{3}/2}(t)\mathbf{q}$ is a negative elliptic-hyperbolic relative equilibrium.

Though one can check the above statements by direct computations, we will give a better explanation of them later in this chapter. Notice that in each example the relative equilibria can be generated from the same initial configuration. Far from being a coincidence, this fact will be clarified soon.

3.2 Relative Equilibria and the Effective Potentials

We use Smale's theorem to show that finding various relative equilibria of the curved N-body problem is equivalent to finding the critical points of effective potentials. These criteria, though equivalent to the ones given by Florin Diacu [19, 25], are quite different from them in form and will be essential in defining the concept of central configurations.

Theorem 2 (Smale, [84])**.** *Suppose* (M, K, V, G) *is a mechanical system with symmetry and* $\boldsymbol{\xi} \in \mathfrak{g}$. *Then* $\exp(\boldsymbol{\xi} t)\mathbf{q}$ *is a relative equilibrium if and only if* $\mathbf{q}$ *is a critical point of the real valued function on* M *which sends* $\mathbf{q}$ *into* $V(\mathbf{q}) - K(\boldsymbol{\xi}_M(\mathbf{q}), \boldsymbol{\xi}_M(\mathbf{q}))$, *the effective potential corresponding to* $\boldsymbol{\xi}$.

Recall that the relative equilibria in $\mathbb{S}^3$ are in the form of $\exp(\boldsymbol{\xi}_1 t)\mathbf{q}$, and the relative equilibria in $\mathbb{H}^3$ are in the form of $\exp(\boldsymbol{\xi}_2 t)\mathbf{q}$. Let us find the corresponding effective potentials.

Theorem 3. *Let* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, *be a nonsingular configuration in* $\mathbb{S}^3$. *Then* $\exp(\boldsymbol{\xi}_1 t)\mathbf{q} = A_{\alpha,\beta}(t)\mathbf{q}$ *is a relative equilibrium if and only if this configuration satisfies the equation*

$$\frac{\beta^2 - \alpha^2}{2} \nabla_{\mathbf{q}_i} \left(\sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \right) = \nabla_{\mathbf{q}_i} U(\mathbf{q}), \ i = 1, ..., N.$$

Let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, *be a nonsingular configuration in* $\mathbb{H}^3$. *Then* $\exp(\boldsymbol{\xi}_2 t)\mathbf{q} = B_{\alpha,\beta}(t)\mathbf{q}$ *is a relative equilibrium if and only if this configuration satisfies the equations*

$$-\frac{\alpha^2 + \beta^2}{2} \nabla_{\mathbf{q}_i} \left(\sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \right) = \nabla_{\mathbf{q}_i} U(\mathbf{q}), \ i = 1, ..., N.$$

Proof. In Chapter 2 we have seen that the curved N-body problem problem is a mechanical system with $O(4)$ ($O(3,1)$) symmetry. Thus Smale's theorem applies. The action of $\exp(\boldsymbol{\xi}_i t)$ is

$$\exp(\boldsymbol{\xi}_i t)\mathbf{q} = (\exp(\boldsymbol{\xi}_i t)\mathbf{q}_1, ..., \exp(\boldsymbol{\xi}_i t)\mathbf{q}_N).$$

Thus the vector fields generated by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ on $(\mathbb{S}^3)^N$ and $(\mathbb{H}^3)^N$ are simply $\boldsymbol{\xi}_1\mathbf{q} = (\boldsymbol{\xi}_1\mathbf{q}_1, ..., \boldsymbol{\xi}_1\mathbf{q}_N)$ and $\boldsymbol{\xi}_2\mathbf{q} = (\boldsymbol{\xi}_2\mathbf{q}_1, ..., \boldsymbol{\xi}_2\mathbf{q}_N)$, respectively.

Recall that the kinetic energy is $K(\dot{q}, \dot{q}) = \sum_{i=1}^N \frac{1}{2} m_i \dot{q}_i \cdot \dot{q}_i$. In $\mathbb{S}^3$, using the fact $\mathbf{q}_i \cdot \mathbf{q}_i = 1$, we obtain

$$\begin{aligned}
K(\boldsymbol{\xi}_1\mathbf{q}, \boldsymbol{\xi}_1\mathbf{q}) &= \sum_{i=1}^N \frac{1}{2} m_i \boldsymbol{\xi}_1\mathbf{q}_i \cdot \boldsymbol{\xi}_1\mathbf{q}_i \\
&= \sum_{i=1}^N \frac{1}{2} m_i (-\alpha y_i, \alpha x_i, -\beta w_i, \beta z_i) \cdot (-\alpha y_i, \alpha x_i, -\beta w_i, \beta z_i) \\
&= \sum_{i=1}^N \frac{1}{2} m_i \left(\alpha^2 (x_i^2 + y_i^2) + \beta^2 (z_i^2 + w_i^2)\right) \\
&= \sum_{i=1}^N \frac{1}{2} m_i \left(\alpha^2 (x_i^2 + y_i^2) + \beta^2 (1 - x_i^2 - y_i^2)\right) \\
&= \frac{\alpha^2 - \beta^2}{2} \sum_{i=1}^N m_i (x_i^2 + y_i^2) + \frac{\beta^2}{2} \sum_{i=1}^N m_i.
\end{aligned}$$

In $\mathbb{H}^3$, we obtain

$$\begin{aligned}
K(\boldsymbol{\xi}_2\mathbf{q}, \boldsymbol{\xi}_2\mathbf{q}) &= \sum_{i=1}^N \frac{1}{2} m_i \boldsymbol{\xi}_2\mathbf{q}_i \cdot \boldsymbol{\xi}_2\mathbf{q}_i \\
&= \sum_{i=1}^N \frac{1}{2} m_i (-\alpha y_i, \alpha x_i, \beta w_i, \beta z_i) \cdot (-\alpha y_i, \alpha x_i, \beta w_i, \beta z_i) \\
&= \sum_{i=1}^N \frac{1}{2} m_i \left(\alpha^2 (x_i^2 + y_i^2) + \beta^2 (w_i^2 - z_i^2)\right) \\
&= \sum_{i=1}^N \frac{1}{2} m_i \left(\alpha^2 (x_i^2 + y_i^2) + \beta^2 (1 + x_i^2 + y_i^2)\right) \\
&= \frac{\alpha^2 + \beta^2}{2} \sum_{i=1}^N m_i (x_i^2 + y_i^2) + \frac{\beta^2}{2} \sum_{i=1}^N m_i.
\end{aligned}$$

Recall that $-U$ is the potential. Thus ignoring the constant, the effective potentials with respect to $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are

$$V_{\boldsymbol{\xi}_1}(\mathbf{q}) = -U(\mathbf{q}) - \sum_{i=1}^{N} \frac{m_i}{2}(\alpha^2 - \beta^2)(x_i^2 + y_i^2),$$
$$V_{\boldsymbol{\xi}_2}(\mathbf{q}) = -U(\mathbf{q}) - \sum_{i=1}^{N} \frac{m_i}{2}(\alpha^2 + \beta^2)(x_i^2 + y_i^2).$$

Thus $\exp(\boldsymbol{\xi}_i t)\mathbf{q}$ is a relative equilibrium if and only if $\mathbf{q}$ is a critical point of these effective potentials, which is equivalent to the stated two equations. This remark completes the proof. □

It is remarkable that the effective potentials depend on the parameters α, β in such a manner, which is due to the fact that the spheres are 3-dimensional. The consequence is that a critical point of the potential $-U(\mathbf{q}) - \sum_{i=1}^{N} \frac{m_i}{2}(\alpha_1^2 - \beta_1^2)(x_i^2 + y_i^2)$ is also a critical point of $-U(\mathbf{q}) - \sum_{i=1}^{N} \frac{m_i}{2}(\alpha_2^2 - \beta_2^2)(x_i^2 + y_i^2)$, as long as $\alpha_1^2 - \beta_1^2 = \alpha_2^2 - \beta_2^2$. In other words, once we obtain one relative equilibrium $A_{\alpha_1,\beta_1}\mathbf{q}$, then $A_{\alpha_2,\beta_2}\mathbf{q}$ is automatically a relative equilibrium as long as $\alpha_1^2 - \beta_1^2 = \alpha_2^2 - \beta_2^2$. Thus there is no need to separate the study of relative equilibria into five categories.

3.3 An Elementary Approach

In this section, we show a result equivalent to Theorem 3 by an elementary approach. Let $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, be a nonsingular configuration and $Q(t)\mathbf{q}$ a relative equilibrium, where $Q(t)$ is $A_{\alpha,\beta}(t)$ or $B_{\alpha,\beta}(t)$. Again, to simplify the notation, we will denote initial positions and velocities without any argument and attach the argument t to functions depending on time.

We first substitute $\mathbf{q}_i(t) = Q(t)\mathbf{q}_i,\ i = 1, ..., N$, into equations (2.1) and obtain

$$m_i\ddot{Q}(t)\mathbf{q}_i = \nabla_{\mathbf{q}_i}U(t) - \sigma m_i[\dot{Q}(t)\mathbf{q}_i \cdot \dot{Q}(t)\mathbf{q}_i]Q(t)\mathbf{q}_i,\ i = 1, ..., N.$$

Since U is invariant under the isometry group, it is easy to see that $Q^{-1}(t)\nabla_{\mathbf{q}_i}U(t) = \nabla_{\mathbf{q}_i}U$. Multiplying to the left by $Q^{-1}(t)$ yields

$$m_iQ^{-1}(t)\ddot{Q}(t)\mathbf{q}_i = \nabla_{\mathbf{q}_i}U - \sigma m_i[\dot{Q}(t)\mathbf{q}_i \cdot \dot{Q}(t)\mathbf{q}_i]\mathbf{q}_i. \tag{3.1}$$

Theorem 4. *Let* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N), \mathbf{q}_i = (x_i, y_i, z_i, w_i),\ i = 1, ..., N,$ *be a nonsingular configuration in* $\mathbb{S}^3$. *Then* $A_{\alpha,\beta}(t)\mathbf{q}$ *is a relative equilibrium if and only if this configuration satisfies the equations*

$$m_i(\beta^2 - \alpha^2)\begin{bmatrix} x_i(w_i^2 + z_i^2) \\ y_i(w_i^2 + z_i^2) \\ -z_i(x_i^2 + y_i^2) \\ -w_i(x_i^2 + y_i^2) \end{bmatrix} = \nabla_{\mathbf{q}_i} U,\ i = 1, ..., N. \tag{3.2}$$

Proof. Using the fact that $A_{\alpha,\beta}(t) = \exp(\boldsymbol{\xi}_1 t)$ and that $\exp(\boldsymbol{\xi}_1 t)$ and $\boldsymbol{\xi}_1$ commute, straightforward computations show that

$$A_{\alpha,\beta}^{-1}(t)\ddot{A}_{\alpha,\beta}(t) = \operatorname{diag}(-\alpha^2, -\alpha^2, -\beta^2, -\beta^2),$$
$$\dot{A}_{\alpha,\beta}(t)\mathbf{q}_i \cdot \dot{A}_{\alpha,\beta}(t)\mathbf{q}_i = \alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2).$$

Substituting these expressions into equations (3.1), we obtain that

$$m_i \begin{bmatrix} -\alpha^2 x_i \\ -\alpha^2 y_i \\ -\beta^2 z_i \\ -\beta^2 w_i \end{bmatrix} = \nabla_{\mathbf{q}_i} U - m_i[\alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2)] \begin{bmatrix} x_i \\ y_i \\ z_i \\ w_i \end{bmatrix}, \quad i = 1, ..., N.$$

Using in the above equations the identity $\mathbf{q}_i \cdot \mathbf{q}_i = 1$, we can conclude that

$$x_i\left[-\alpha^2 + \alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2)\right] = x_i(\beta^2 - \alpha^2)(z_i^2 + w_i^2),$$
$$y_i\left[-\alpha^2 + \alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2)\right] = y_i(\beta^2 - \alpha^2)(z_i^2 + w_i^2),$$
$$z_i\left[-\beta^2 + \alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2)\right] = -z_i(\beta^2 - \alpha^2)(x_i^2 + y_i^2),$$
$$w_i\left[-\beta^2 + \alpha^2(x_i^2 + y_i^2) + \beta^2(z_i^2 + w_i^2)\right] = -w_i(\beta^2 - \alpha^2)(x_i^2 + y_i^2).$$

Then we are led to equations (3.2), a remark that completes the proof. □

Theorem 5. *Let* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N),\ \mathbf{q}_i = (x_i, y_i, z_i, w_i),\ i = 1, ..., N,$ *be a nonsingular configuration in* $\mathbb{H}^3$. *Then* $B_{\alpha,\beta}(t)\mathbf{q}$ *is a relative equilibrium if and only if this* *configuration satisfies the equations*

$$-m_i(\alpha^2+\beta^2)\begin{bmatrix} x_i(w_i^2-z_i^2)\\ y_i(w_i^2-z_i^2)\\ z_i(x_i^2+y_i^2)\\ w_i(x_i^2+y_i^2)\end{bmatrix} = \nabla_{\mathbf{q}_i}U, \quad i=1,...,N. \tag{3.3}$$

Proof. Using the fact that $B_{\alpha,\beta}(t)=\exp(\boldsymbol{\xi}_2 t)$ and that $\exp(\boldsymbol{\xi}_2 t)$ and $\boldsymbol{\xi}_2$ commute, straightforward computations show that

$$B^{-1}_{\alpha,\beta}(t)\ddot{B}_{\alpha,\beta}(t) = \mathrm{diag}(-\alpha^2,-\alpha^2,\beta^2,\beta^2),$$
$$\dot{B}_{\alpha,\beta}(t)\mathbf{q}_i\cdot\dot{B}_{\alpha,\beta}(t)\mathbf{q}_i = \alpha^2(x_i^2+y_i^2)-\beta^2(z_i^2-w_i^2).$$

Substituting these results into equations (3.1), we obtain

$$m_i\begin{bmatrix}-\alpha^2x_i\\-\alpha^2y_i\\ \beta^2z_i\\ \beta^2w_i\end{bmatrix} = \nabla_{\mathbf{q}_i}U + m_i[\alpha^2(x_i^2+y_i^2)-\beta^2(z_i^2-w_i^2)]\begin{bmatrix}x_i\\y_i\\z_i\\w_i\end{bmatrix}, \quad i=1,...,N.$$

Using in the above equations the identity $\mathbf{q}_i\cdot\mathbf{q}_i=-1$, we can conclude that

$$\begin{aligned}
x_i\left[-\alpha^2-\alpha^2(x_i^2+y_i^2)+\beta^2(z_i^2-w_i^2)\right] &= x_i(\alpha^2+\beta^2)(z_i^2-w_i^2),\\
y_i\left[-\alpha^2-\alpha^2(x_i^2+y_i^2)+\beta^2(z_i^2-w_i^2)\right] &= y_i(\alpha^2+\beta^2)(z_i^2-w_i^2),\\
z_i\left[\beta^2-\alpha^2(x_i^2+y_i^2)+\beta^2(z_i^2-w_i^2)\right] &= -z_i(\alpha^2+\beta^2)(x_i^2+y_i^2),\\
w_i\left[\beta^2-\alpha^2(x_i^2+y_i^2)+\beta^2(z_i^2-w_i^2)\right] &= -w_i(\alpha^2+\beta^2)(x_i^2+y_i^2).
\end{aligned}$$

Then we are led to equations (3.3), a remark that completes the proof. □

Theorem 3 and the above two theorems are equivalent. For example, in $\mathbb{S}^3$, define $f(x,y,z,w)=x^2+y^2$ as a function from $\mathbb{S}^3$ to $\mathbb{R}$. To find the gradient of f, we employ the trick used to derive $\nabla_{\mathbf{q}_i}\mathbf{q}_i\cdot\mathbf{q}_j$ in Chapter 2. Extend f to a homogeneous function $\bar{f}$ of degree zero in the ambient space $\mathbb{R}^4$,

$$\bar{f}(x,y,z,w) := \frac{x^2+y^2}{x^2+y^2+z^2+w^2}.$$

Let $\widetilde{\nabla}$ be the gradient in the ambient space, and $\frac{\partial}{\partial n}$ be the unit normal vector of the unit sphere. Since $\frac{\partial \bar{f}}{\partial r} = 0$, we obtain $(\widetilde{\nabla} \bar{f})|_{\mathbb{S}^3} = \nabla f + \frac{\partial \bar{f}}{\partial r}\frac{\partial}{\partial n} = \nabla f$. Thus straightforward computations show that

$$\nabla f(x,y,z,w) = 2\left(x(w^2+z^2), y(w^2+z^2), -z(x^2+y^2), -w(x^2+y^2)\right).$$

Hence we can conclude that

$$\nabla_{\mathbf{q}_i} I(\mathbf{q}) = 2m_i\left(x_i(w_i^2+z_i^2), y_i(w_i^2+z_i^2), -z_i(x_i^2+y_i^2), -w_i(x_i^2+y_i^2)\right).$$

Thus the right hand side of (3.2) is $\frac{\beta^2-\alpha^2}{2}\nabla_{\mathbf{q}_i}\left(\sum_{i=1}^N m_i(x_i^2+y_i^2)\right)$. Theorem 3 matches Theorem 4. Similarly, in $\mathbb{H}^3$,

$$\nabla_{\mathbf{q}_i} I(\mathbf{q}) = 2m_i\left(x_i(w_i^2-z_i^2), y_i(w_i^2-z_i^2), z_i(x_i^2+y_i^2), w_i(x_i^2+y_i^2)\right).$$

Thus Theorem 3 also matches Theorem 5.

3.4 Central Configurations and the Associated Relative Equilibria

We are now motivated to study the equation

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i}[\sum_{i=1}^N m_i(x_i^2+y_i^2)], i = 1, ..., N.$$

Recall that in the Newtonian N-body problem, solutions of such equation are called central configurations [62, 90]. We introduce central configurations in $\mathbb{S}^3$ and $\mathbb{H}^3$. We will also isolate a particular class of central configurations that corresponds to fixed-point solutions in $\mathbb{S}^3$, but which don't exist in $\mathbb{H}^3$. Finally, we discuss the motions of the curved N-body problem related to central configurations, namely relative equilibria and homothetic motions.

Definition 4. *Consider N point masses $m_1, \dots, m_N$ in $\mathbb{M}^3$ at $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$. The moment of inertia of the particle system is the function*

$$I(\mathbf{q}) := \sum_{i=1}^N m_i(x_i^2+y_i^2).$$

Definition 5. *Assume that the point masses* $m_1, \ldots, m_N$ *in* $\mathbb{M}^3$ *have the nonsingular positions given by the vector* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$. *Then* $\mathbf{q}$ *is a central configuration of the curved* N*-body problem in* $\mathbb{M}^3$ *if it solves the equations*

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}), \ i = 1, ..., N, \tag{3.4}$$

where I *is the moment of inertia and the constant* $\lambda \in \mathbb{R}$ *can be viewed as a Lagrangian multiplier. We will further refer to these conditions as the central configuration equations.*

Explicitly, the central configuration equations (3.4) are

$$\sum_{j\neq i, j=1}^{N} \frac{m_j m_i \mathbf{q}_j}{\mathrm{sn}^3 d_{ij}} - \sum_{j\neq i, j=1}^{N} \frac{m_j m_i \mathrm{csn} d_{ij}}{\mathrm{sn}^3 d_{ij}} \mathbf{q}_i = \lambda \nabla_{\mathbf{q}_i} I, \ i = 1, ..., N. \tag{3.5}$$

Proposition 5. *The* i*-th equation of the central configuration equations* (3.5) *holds if and only if there is a constant* θ_i *such that*

$$\sum_{j\neq i, j=1}^{N} \frac{m_j m_i \mathbf{q}_j}{sn^3 d_{ij}} - \theta_i \mathbf{q}_i = \lambda \nabla_{\mathbf{q}_i} I. \tag{3.6}$$

Proof. Assume that (3.6) holds. Multiply $\mathbf{q}_i$ to the both sides of (3.6). Since $\mathbf{q}_i \cdot \mathbf{q}_j = \sigma \mathrm{csn} d_{ij}$, $\mathbf{q}_i \cdot \mathbf{q}_i = \sigma$, and $\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} I = 0$, we obtain $\theta_i = \sum_{j\neq i, j=1}^{N} \frac{m_j m_i \mathrm{csn} d_{ij}}{\mathrm{sn}^3 d_{ij}}$. Thus (3.6) is equivalent to the i-th equation of (3.5). □

The following class of central configurations exists in $\mathbb{S}^3$ only [19, 25].

Definition 6. *Consider the masses* $m_1, \ldots, m_N$ *in* $\mathbb{S}^3$. *Then a configuration* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, *is called a special central configuration if it is a critical point of the force function* U, *i.e.*

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = 0, \ i = 1, ..., N.$$

In other words, $\mathbf{F}_i = 0, i = 1, ..., N$. *To avoid any confusion, we will call ordinary central configurations those central configurations that are not special.*

Here is one remark on terminology. These special central configurations were introduced in [19, 25] under the name of *fixed points*. Given such a configuration $\mathbf{q}$, we see with the help of Theorem 4 that $A_{0,0}(t)\mathbf{q}$ is an associated relative equilibrium, which is a *fixed-point solution*: $\mathbf{q}(t) = \mathbf{q}$, $\mathbf{p}(t) = 0$. This explains the old terminology. Let us introduce some new terminology as well.

Definition 7. *A central configuration* $\mathbf{q}$ *of the curved* N*-body problem is called*

- *a geodesic central configuration if it is lying on a geodesic;*
- *an* $\mathbb{S}^2$ *central configuration if it is lying on a great 2-sphere;*
- *an* $\mathbb{H}^2$ *central configuration if it is lying on a great hyperbolic 2-sphere;*
- *an* $\mathbb{S}^3$ *central configuration if it is not lying on any great 2-sphere;*
- *an* $\mathbb{H}^3$ *central configuration if it is not lying on any great hyperbolic 2-sphere.*

Central configurations will play an important role in the study of the curved N-body problem. They influence the topology of the integral manifolds [55, 84]. Now we discuss the connection between them and the motions of the curved N-body problem. Let

$$\begin{aligned}\mathbb{S}^1_{xy} :=& \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 = 1, z = w = 0\},\\ \mathbb{S}^1_{zw} :=& \{(x, y, z, w) \in \mathbb{R}^4 | z^2 + w^2 = 1, x = y = 0\},\\ \mathbb{H}^1_{zw} :=& \{(x, y, z, w) \in \mathbb{R}^4 | z^2 - w^2 = -1, x = y = 0\}.\end{aligned}$$

Lemma 1. *On* $(\mathbb{S}^3)^N$,

$$\nabla_{\mathbf{q}_i} I = 0 \text{ if and only if } \mathbf{q}_i \in \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw},$$

On $(\mathbb{H}^3)^N$,

$$\nabla_{\mathbf{q}_i} I = 0 \text{ if and only if } \mathbf{q}_i \in \mathbb{H}^1_{zw}.$$

Proof. On $(\mathbb{S}^3)^N$, recall that

$$\nabla_{\mathbf{q}_i} I = 2m_i(x_i(w_i^2 + z_i^2), y_i(w_i^2 + z_i^2), -z_i(x_i^2 + y_i^2), -w_i(x_i^2 + y_i^2)).$$

On one hand, if $\nabla_{\mathbf{q}_i} I$ is a zero vector, then

$$(x_i(w_i^2 + z_i^2))^2 + (y_i(w_i^2 + z_i^2))^2 = (x_i^2 + y_i^2)(w_i^2 + z_i^2)^2 = 0,$$

which means that $\mathbf{q}_i \in \mathbb{S}^1_{xy}$ or $\mathbb{S}^1_{zw}$. On the other hand, it is easy to see that if $\mathbf{q}_i \in \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, then $\nabla_{\mathbf{q}_i} I = 0$.

On $(\mathbb{H}^3)^N$, recall that

$$\nabla_{\mathbf{q}_i} I = 2m_i(x_i(w_i^2 - z_i^2), y_i(w_i^2 - z_i^2), z_i(x_i^2 + y_i^2), w_i(x_i^2 + y_i^2)).$$

Again, on one hand, if $\nabla_{\mathbf{q}_i} I$ is a zero vector, then

$$(x_i(w_i^2 - z_i^2))^2 + (y_i(w_i^2 - z_i^2))^2 = (x_i^2 + y_i^2)(w_i^2 - z_i^2)^2 = 0,$$

which means that $x_i = y_i = 0$, since $w_i^2 - z_i^2 = 1 + x_i^2 + y_i^2 \neq 0$. Thus we obtain that $\mathbf{q}_i \in \mathbb{H}^1_{zw}$. On the other hand, it is easy to see that if $\mathbf{q}_i \in \mathbb{H}^1_{zw}$, then $\nabla_{\mathbf{q}_i} I = 0$. This remark completes the proof. □

Corollary 1. *Consider a central configuration* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, *in* $\mathbb{M}^3$. *Let* λ *be the constant in the central configuration equation* $\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q})$.

- *If* $\mathbf{q}$ *is an ordinary central configuration in* $\mathbb{S}^3$, *then it gives rise to a one-parameter family of relative equilibria:* $A_{\alpha,\beta}(t)\mathbf{q}$ *with* $\lambda = \frac{\beta^2 - \alpha^2}{2}$.

- *If* $\mathbf{q}$ *is in* $\mathbb{H}^3$, *then it gives rise to a one-parameter family of relative equilibria:* $B_{\alpha,\beta}(t)\mathbf{q}$ *with* $\lambda = -\frac{\alpha^2 + \beta^2}{2}$.

- *If* $\mathbf{q}$ *is a special central configuration in* $\mathbb{S}^3$ *and not all the particles are on* $\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, *then it gives rise to a one-parameter family of relative equilibria:* $A_{\alpha,\beta}(t)\mathbf{q}$ *with* $0 = \beta^2 - \alpha^2$.

- *If* $\mathbf{q}$ *is a special central configuration in* $\mathbb{S}^3$ *and all the particles are on* $\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, *then it gives rise to a two-parameter family of relative equilibria:* $A_{\alpha,\beta}(t)\mathbf{q}$ *with* $\alpha, \beta \in \mathbb{R}$.

Before proving the corollary, let us make the following remark on terminology. In the literature, the concept of relative equilibria stands for both the central configurations and the rigid motions associated to them [55, 84]. In this thesis, however, we use the term relative equilibrium only for the associated motion.

Proof. The first two claims are obvious by Theorem 3. If $\mathbf{q}$ is a special central configuration in $\mathbb{S}^3$, then by Theorem 3, $A_{\alpha,\beta}(t)\mathbf{q}$ is an associated relative equilibrium if and only if $\frac{\beta^2 - \alpha^2}{2} \nabla_{\mathbf{q}_i} I = 0$ for $i = 1, ..., N$.

There are two possibilities: first, if there exists some $\mathbf{q}_i$ with $\nabla_{\mathbf{q}_i} I \neq 0$, that is, there is some $\mathbf{q}_i \notin \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, then $0 = \beta^2 - \alpha^2$, i.e., there is a one-parameter family of relative equilibria associated to the special central configuration $\mathbf{q}$: $A_{\alpha,\beta}(t)\mathbf{q}$ with $0 = \beta^2 - \alpha^2$; second, if $\nabla_{\mathbf{q}_i} I = 0$ for all i, that is, $\mathbf{q}_i \in \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$ for all i, then there is no limitation for α, β, i.e., there is a two-parameter family of relative equilibria associated to the special central configuration $\mathbf{q}$: $A_{\alpha,\beta}(t)\mathbf{q}$ with $\alpha, \beta \in \mathbb{R}$. This remark completes the proof. □

Remark 2. *The reader may notice a gap in the proof. For a central configuration in $\mathbb{H}^3$, we don't have a one-parameter family of relative equilibria, as claimed, unless we can show that the value of λ is always negative. This fact will be proved in Chapter 5.*

Let us notice that while 3-dimensional central configurations of the Newtonian N-body problem do not have associated relative equilibria [90], all central configurations of the curved N-body problem have associated relative equilibria.

Now it is easy to explain what happens in Examples 1 and 2 of the first section. In Example 1, we can check that the given configuration $\mathbf{q}$ is a central configuration in $\mathbb{S}^3$ with $\lambda = -\frac{1}{2}$. Then we obtain the positive elliptic and positive elliptic-elliptic relative equilibria from it by letting $\frac{\beta^2-\alpha^2}{2} = -\frac{1}{2}$. Similarly, in Example 2, the given configuration $\mathbf{q}$ is a central configuration in $\mathbb{H}^3$ with $\lambda = -\frac{1}{2}$, and we obtain the negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic relative equilibria from it by letting $-\frac{\alpha^2+\beta^2}{2} = -\frac{1}{2}$.

In the family of relative equilibria associated to one central configuration, there are motions of different characteristics. In $\mathbb{S}^3$, the relative equilibria can be positive elliptic and positive elliptic-elliptic. In $\mathbb{H}^3$, they can be negative elliptic, negative hyperbolic, and negative elliptic-hyperbolic. Furthermore, these rigid motions can be periodic or quasi-periodic. For an ordinary central configuration in $\mathbb{S}^3$, the intersections of the hyperbola $\lambda = \frac{\beta^2-\alpha^2}{2}$ and the line $\beta = k\alpha$, $k \in \mathbb{Q}$ in the $\alpha\beta$ plane give periodic motions; otherwise, the motions are quasi-periodic. For a special central configuration in $\mathbb{S}^3$ that not all particles are on $\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, the relative equilibria are always periodic. If $\mathbf{q}$ is on $\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, then any points on the line $\beta = k\alpha$, $k \in \mathbb{Q}$ in the $\alpha\beta$ plane give periodic motions; otherwise, the motions are quasi-periodic. For an ordinary central configuration in $\mathbb{H}^3$, the relative equilibria are periodic if and only if $\beta = 0$.

However, unlike in the Newtonian N-body problem, central configurations do not

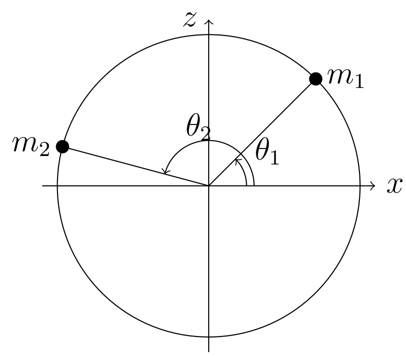

Figure 3.1: A central configuration of two bodies on $\mathbb{S}^1_{xz}$

provide us with *homothetic solutions*, which occur only in vector spaces, since they require similarity [90]. Actually, since there is no centre of masses, it makes no sense to talk about homothetic solutions. For a special central configuration, if we set the particles at rest at $t = 0$, then we obtain a fixed-point solution. For an ordinary central configuration, let us look at the following simple example.

Example 3. *Consider a two-body central configuration on* $\mathbb{S}^1_{xz} := \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + z^2 = 1, y = w = 0\}$. *Let us place two masses* $\mathbf{m} = (m_1, m_2)$ *at* $\mathbf{q}_i = (x_i, z_i) = (\cos\theta_i, \sin\theta_i), i = 1, 2$. *It will be proved in Chapter 7 that if equation*

$$m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0 \tag{3.7}$$

is satisfied, then the configuration is a central configuration, see Figure 3.1. At $t = 0$, *set the bodies at rest. To find the solution* $\mathbf{q}(t)$, *note that the angular momentum* ω_{xz} *is*

$$\omega_{xz} = \sum_{i=1}^{2} m_i(\dot{x}_i z_i - x_i \dot{z}_i) = \sum_{i=1}^{2} m_i(\cos^2\theta_i\dot{\theta}_i + \sin^2\theta_i\dot{\theta}_i) = \sum_{i=1}^{2} m_i\dot{\theta}_i = 0.$$

Thus we get $m_1\theta_1(t) + m_2\theta_2(t) = m_1\theta_1 + m_2\theta_2$. *Obviously, they will collide at some point* $\bar{\theta}$. *Then* $m_1\bar{\theta} + m_2\bar{\theta} = m_1\theta_1 + m_2\theta_2$, *so we obtain* $\bar{\theta} = \frac{m_1\theta_1 + m_2\theta_2}{m_1 + m_2}$.

In some sense, if $\mathbf{q}(t)$ *is always a central configuration, we may call it a homothetic orbit. Notice that* (3.7) *implies that* $\mathbf{q}$ *is not a central configuration if* $\mathbf{q}_1$ *and* $\mathbf{q}_2$ *are in the same quadrant. It is easy to construct examples for which* $\bar{\theta} \in (0, \pi/2)$. *Thus it is impossible that* $\mathbf{q}(t)$ *is always a central configuration. For example, let* $\mathbf{m} = (2, 1)$ *and* $(\theta_1, \theta_2) = (15°, 135°)$. *Then* $m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 2\sin 30° + \sin 270° = 0$. *This is a central configuration at* $t = 0$, *but the bodies collide at* $\bar{\theta} = \frac{215° + 135°}{3} = 55°$.

Thus in general, we could not expect any homothetic motions from a central configuration. There are exceptions: the highly symmetric equilateral triangle central configurations for three equal masses on a 2-sphere lead to motions where $\mathbf{q}(t)$ is always similar (from the viewpoint of Euclidean geometry in $\mathbb{R}^3$) to $\mathbf{q}(0)$ and remains a central configuration for all t [18]. We will discuss the details of these central configurations later, see Chapter 6 and Chapter 7.

Chapter 4

Central Configurations

We are now motivated to study central configurations. In this chapter we prove some basic facts about them. We first give them physical description, which justifies the notion of central configurations, then we define equivalent classes of central configurations. In the last two sections, we collect some lemmas and theorems which would be quite useful in our later investigations.

4.1 A Physical Description of Central Configurations

It turns out that the moment of inertia I possesses a geometric meaning, which brings some insight into this problem and provides a physical description of central configurations.

Recall that

$$\mathbb{S}^1_{zw} = \{(x,y,z,w) \in \mathbb{R}^4 | z^2 + w^2 = 1, x = y = 0\},$$
$$\mathbb{H}^1_{zw} = \{(x,y,z,w) \in \mathbb{R}^4 | z^2 - w^2 = -1, x = y = 0\}.$$

Lemma 2. *If $A = (x,y,z,w)$ is a point in $\mathbb{S}^3$, then $z^2 + w^2 = \cos^2 d(A, \mathbb{S}^1_{zw})$. If $A = (x,y,z,w)$ is a point in $\mathbb{H}^3$, then $-z^2 + w^2 = \cosh^2 d(A, \mathbb{H}^1_{zw})$, where $d(A, \mathcal{M}) := \inf_{B \in \mathcal{M}} d(A,B)$, with A, B representing points and $\mathcal{M}$ being a smooth manifold.*

Proof. View A as a vector in $\mathbb{R}^4$. Denote by $\mathbb{R}^3_A$ the 3- (or 2-) dimensional subspace spanned by A, $e_z = (0,0,1,0)$, and $e_w = (0,0,0,1)$. Denote by $\mathbb{R}^2_{zw}$ the 2-dimensional subspace spanned by e_z and e_w.

In $\mathbb{S}^3$, the minimal geodesic connecting A and $\mathbb{S}^1_{zw}$ is on the great 2-sphere $\mathbb{S}^2_A = \mathbb{R}^3_A \cap \mathbb{S}^3$. Let $\theta = d(A, \mathbb{S}^1_{zw})$. Then $A = A_v + A_h \in (\mathbb{R}^2_{zw})^\perp \oplus \mathbb{R}^2_{zw}$ with $||A_v|| = \sin\theta$ and $||A_h|| = \cos\theta$. Hence, we obtain

$$\cos^2 d(A, \mathbb{S}^1_{zw}) = ||A_h||^2 = ||(A \cdot e_z)e_z + (A \cdot e_w)e_w||^2 = ||ze_z + we_w||^2 = z^2 + w^2.$$

In $\mathbb{H}^3$, the minimal geodesic connecting A and $\mathbb{H}^1_{zw}$ is on the great hyperbolic 2-sphere $\mathbb{H}^2_A = \mathbb{R}^3_A \cap \mathbb{H}^3$. Let $\theta = d(A, \mathbb{H}^1_{zw})$. Then similarly we have $A = A_v + A_h \in (\mathbb{R}^2_{zw})^\perp \oplus \mathbb{R}^2_{zw}$ with $||A_v|| = \sinh\theta$ and $||A_h|| = \cosh\theta$. Hence, we obtain

$$\begin{aligned}\cosh^2 d(A, \mathbb{H}^1_{zw}) &= ||A_h||^2 = ||\frac{A \cdot e_z}{e_z \cdot e_z} e_z + \frac{A \cdot e_w}{e_w \cdot e_w} e_w||^2 \\ &= ||ze_z - (-w)e_w||^2 = |(ze_z + we_w) \cdot (ze_z + we_w)| \\ &= |z^2 - w^2| = -z^2 + w^2.\end{aligned}$$

□

Theorem 6. *A nonsingular configuration* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N),\ \mathbf{q}_i = (x_i, y_i, z_i, w_i),\ i = 1, ..., N,$ *in* $\mathbb{M}^3$ *is a central configuration if and only if*

$$\begin{aligned}\nabla_{\mathbf{q}_i} U(\mathbf{q}) &= \lambda m_i \sin[2d(\mathbf{q}_i, \mathbb{S}^1_{zw})] \nabla_{\mathbf{q}_i} d(\mathbf{q}_i, \mathbb{S}^1_{zw}),\ i = 1, ..., N,\ \text{in } \mathbb{S}^3, \\ \nabla_{\mathbf{q}_i} U(\mathbf{q}) &= \lambda m_i \sinh[2d(\mathbf{q}_i, \mathbb{H}^1_{zw})] \nabla_{\mathbf{q}_i} d(\mathbf{q}_i, \mathbb{H}^1_{zw}),\ i = 1, ..., N,\ \text{in } \mathbb{H}^3,\end{aligned} \tag{4.1}$$

where $\lambda \in \mathbb{R}$ *is a constant.*

Proof. By Lemma 2, we obtain $x_i^2 + y_i^2 = \sin^2 d(\mathbf{q}_i, \mathbb{S}^1_{zw})$ in $\mathbb{S}^3$ and $x_i^2 + y_i^2 = \sinh^2 d(\mathbf{q}_i, \mathbb{H}^1_{zw})$ in $\mathbb{H}^3$. Thus

$$I = \sum_{1 \le i \le N} m_i \sin^2 d(\mathbf{q}_i, \mathbb{S}^1_{zw}) \text{ in } \mathbb{S}^3, \quad I = \sum_{1 \le i \le N} m_i \sinh^2 d(\mathbf{q}_i, \mathbb{H}^1_{zw}) \text{ in } \mathbb{H}^3.$$

Then the central configuration equation (3.4) can be written as (4.1). □

By definition, special central configurations are special arrangements of the particles such that the force on each particles cancels. By the above theorem, ordinary central configurations are special arrangements of the particles with the property that the gravitational force produced on each particle by all the others particles points towards the geodesic $\mathbb{S}^1_{zw}$ ($\mathbb{H}^1_{zw}$) and is proportional to $m_i \sin[2d(\mathbf{q}_i, \mathbb{S}^1_{zw})]$ ($m_i \sinh[2d(\mathbf{q}_i, \mathbb{H}^1_{zw})]$).

Recall that in the Newtonian N-body problem, central configurations are those arrangements of particles such that all $\mathbf{F}_i$ are pointing towards the centre of mass [90]. In the curved N-body problem, instead of a point, all $\mathbf{F}_i$ are pointing towards a geodesic. Furthermore, it will be shown in the last section of this chapter that all central configurations in $\mathbb{H}^3$ can be found on a submanifold $\mathbb{H}^2_{xyw} := \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 - w^2 = -1, z = 0\}$, and that all ordinary $\mathbb{S}^2$central configurations can be found on a submanifold $\mathbb{S}^2_{xyz} := \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 = 1, w = 0\}$. The intersection of $\mathbb{H}^2_{xyw}$ and $\mathbb{H}^1_{zw}$ is $(0, 0, 0, 1)$, and the intersections of $\mathbb{S}^2_{xyz}$ and $\mathbb{S}^1_{zw}$ are $(0, 0, \pm 1, 0)$. It is easy to see that the minimal path connecting $\mathbf{q}_i$ on $\mathbb{H}^2_{xyw}$ ($\mathbb{S}^2_{xyz}$) and the geodesic $\mathbb{H}^1_{zw}$ ($\mathbb{S}^1_{zw}$) lies on the two submanifolds. Thus we can say that for all central configurations in $\mathbb{H}^3$, all $\mathbf{F}_i$ are pointing towards one point; for all ordinary $\mathbb{S}^2$ central configurations, all $\mathbf{F}_i$ are pointing towards one of two points. The vector fields $\nabla(x^2 + y^2)$ on the two submanifolds are sketched in Figure 4.1.

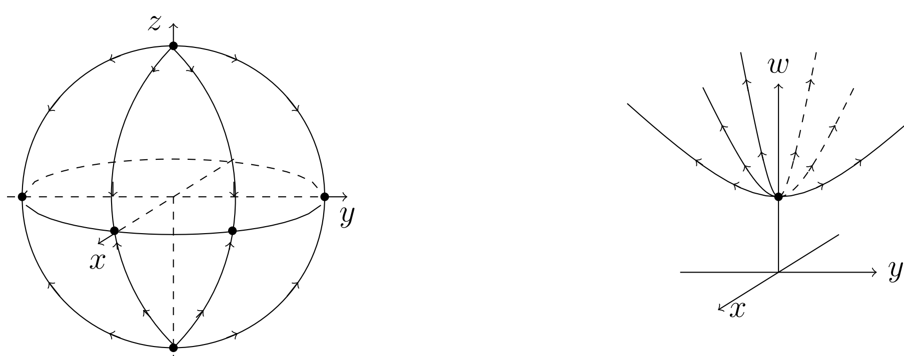

Figure 4.1: $\nabla(x^2 + y^2)$ on $\mathbb{S}^2_{xyz}$ and $\mathbb{H}^2_{xyw}$

4.2 Equivalent Central Configurations

In this section we find a way to count central configurations. Since the inertia I is not invariant under all rotations, it follows that central configurations are only invariant under a subgroup of the symmetry group. Since the space is not homogeneous, it follows that there is no way to scale the general central configurations. This fact makes it easy to show the existence of a continuum of central configurations where the configurations change the size. We illustrate this phenomenon with one example in the end.

Recall that central configurations in the Newtonian N-body problem are invariant under translations, rotations, reflections and scaling [90]. In the curved N-body problem, U is invariant under the symmetry group $O(4)$ or $O(3,1)$. We can check by the formula of $\nabla_{\mathbf{q}_i} U$ that $\nabla_{\mathbf{q}_i} U|_{\mathbf{q}'=\chi\mathbf{q}} = \chi \nabla_{\mathbf{q}_i} U|_{\mathbf{q}}$ for any χ in the symmetry group. Though the inertia I is not invariant under all elements of the symmetry group, it is invariant under a subgroup $O(2)\times O(2)$ $(O(2)\times O(1,1))$. Let $\chi = (\chi_1, \chi_2) \in O(2)\times O(2)$ $(O(2)\times O(1,1))$. The action is

$$\chi\mathbf{q} = (\chi\mathbf{q}_1, ..., \chi\mathbf{q}_N),\ \chi\mathbf{q}_i = (\chi_1(x_i, y_i)^T, \chi_2(z_i, w_i)^T).$$

We can easily check that $\nabla_{\mathbf{q}_i} I|_{\mathbf{q}'=\chi\mathbf{q}} = \chi \nabla_{\mathbf{q}_i} I|_{\mathbf{q}}$ by using the formula of $\nabla_{\mathbf{q}_i} I$ or Lemma 2.

There is no other obvious transform that keeps the central configuration equation. Also note that the inertia I is not involved in the equation for special central configurations. Thus we introduce the following definition.

Definition 8. *Let* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $\ i = 1, ..., N$, *and* $\mathbf{q}' = (\mathbf{q}'_1, \dots, \mathbf{q}'_N)$, $\mathbf{q}'_i = (x'_i, y'_i, z'_i, w'_i)$, $\ i = 1, ..., N$, *be two central configurations in* $\mathbb{M}^3$.

- *If they are special central configurations in* $\mathbb{S}^3$, *then they are equivalent if there is* $\chi \in SO(4)$, *such that* $\mathbf{q} = \chi\mathbf{q}'$.
- *If they are ordinary central configurations, then they are equivalent if there is* $\chi = (\chi_1, \chi_2) \in SO(2)\times SO(2)$ $(SO(2)\times SO(1,1))$, *such that* $\mathbf{q} = \chi\mathbf{q}'$.

Notice that we use $SO(2)\times SO(2)$ $(SO(2)\times SO(1,1))$ instead of $O(2)\times O(2)$ $(O(2)\times O(1,1))$. We adopt this definition to keep consistency with the critical point characterization of central configurations, which will be introduced in Chapter 5.

Based on in Example 1, we give the following continuum of central configurations.

Example 4 (Lagrangian central configuration on $\mathbb{S}^2_{xyz}$)**.** *Recall that* $\mathbb{S}^2_{xyz} = \{(x, y, z, w) \in \mathbb{S}^3 : w = 0\}$. *Let three equal masses* $\mathbf{m} = (1, 1, 1)$ *be at*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \mathbf{q}_j = (x_j, y_j, z_j, w_j),\ j = 1, 2, 3,$$

$$x_j = \sqrt{1-c^2}\cos\beta_j, \quad y_j = \sqrt{1-c^2}\sin\beta_j, \quad z_j = c, \quad w_j = 0, \quad \beta_j = \frac{2\pi(j-1)}{3},$$

where c could have any value between -1 and 1, see Figure 4.2. By symmetry, we see that $\nabla_{\mathbf{q}_i} U$ is pointing towards the north pole if $c > 0$, or towards the south pole if $c < 0$. Comparing with Figure 4.1, we get that there must be some constant λ such *that* $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ *for* $1 \leq i \leq 3$. *Note that* $d_{12} = d_{13} = d_{23}$, *which is reminiscent of the 3-body central configuration in the Newtonian* N*-body problem found by Lagrange [90]. We will call them Lagrangian central configurations.*

By the convention we introduced, rotating the central configurations in the xy*-plane does not lead to new central configurations, and the rotated ones still remain on the original 2-sphere; rotating them in the* zw*-plane does not lead to new central configurations either, although they will not remain on the original 2-sphere. Though these central configurations, for different value of* c*, are similar in some sense, there does not exist an element in* $SO(2) \times SO(2)$ *to relate any two of them. Thus we see that there is a continuum of central configurations.*

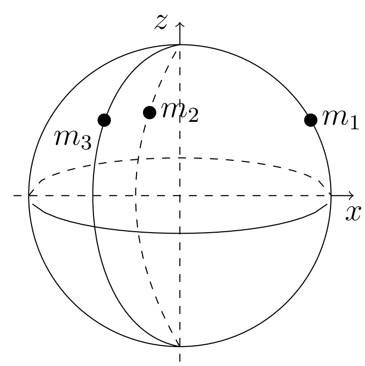

Figure 4.2: Lagrangian central configurations on $\mathbb{S}^2_{xyz}$

In Chapter 5, we will see that, for any given masses, there is a continuum of central configurations.

4.3 Some Useful Properties of Central Configurations

In the following two sections, we collect some lemmas and theorems that would be useful in the study of central configurations. We first prove a property that is analogous to the relationship $\sum_{i=1}^{N} m_i \mathbf{q}_i = 0$ for central configurations of the Newtonian N-body problem [62]. Then we prove results on the non-existence of special central configurations in $\mathbb{H}^3$ and in hemispheres of $\mathbb{S}^3$, provided that at least one body is not on the boundary of the hemisphere.

Theorem 7. *Let* $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$, $\mathbf{q}_i = (x_i, y_i, z_i, w_i)$, $i = 1, ..., N$, *be an ordinary central configuration. Then we have the relationships*

$$\sum_{i=1}^{N} m_i x_i z_i = \sum_{i=1}^{N} m_i x_i w_i = \sum_{i=1}^{N} m_i y_i z_i = \sum_{i=1}^{N} m_i y_i w_i = 0. \tag{4.2}$$

Proof. We first prove (4.2) in $\mathbb{S}^3$. Let $\mathbf{v}_{i1} = (z_i, 0, -x_i, 0)$. Take the inner product of both sides of the i-th equation of (3.5) with $\mathbf{v}_{i1}$. Since

$$\mathbf{q}_j \cdot \mathbf{v}_{i1} = z_i x_j - x_i z_j, \quad \mathbf{q}_i \cdot \mathbf{v}_{i1} = 0, \quad \nabla_{\mathbf{q}_i} I \cdot \mathbf{v}_{i1} = 2m_i x_i z_i (\mathbf{q}_i \cdot \mathbf{q}_i) = 2m_i x_i z_i,$$

we obtain $\sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{\sin^3 d_{ij}}(z_i x_j - x_i z_j) = 2\lambda m_i x_i z_i$. Summing over all i leads to

$$2\lambda \sum_{i=1}^{N} m_i x_i z_i = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{\sin^3 d_{ij}}(z_i x_j - x_i z_j) = 0.$$

Since $\mathbf{q}$ is an ordinary central configuration, we have $\lambda \neq 0$. Thus we obtain $\sum_{i=1}^{N} m_i x_i z_i = 0$. The other relationships in $\mathbb{S}^3$ can be obtained by considering the inner product of (3.5) with

$$\mathbf{v}_{i2} = (w_i, 0, 0, -x_i), \quad \mathbf{v}_{i3} = (0, z_i, -y_i, 0), \quad \mathbf{v}_{i4} = (0, w_i, 0, -y_i).$$

The relationships in $\mathbb{H}^3$ can be obtained by considering the inner product of (3.5) with

$$\mathbf{v}_{i1} = (z_i, 0, -x_i, 0), \ \mathbf{v}_{i2} = (w_i, 0, 0, x_i), \quad \mathbf{v}_{i3} = (0, z_i, -y_i, 0), \quad \mathbf{v}_{i4} = (0, w_i, 0, y_i),$$

a remark that completes the proof. □

An obvious application of equations (4.2) is that of showing with little computational effort why certain configurations are not ordinary central configurations.

The following two theorems on the non-existence of special central configurations were first proved by Florin Diacu [19, 25]. For completeness, we reproduce the proofs here.

Theorem 8. *There are no special central configurations in* $\mathbb{H}^3$ *for any* $\mathbf{m} \in \mathbb{R}_+^N$.

Proof. For any given configurations $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$ of N masses in $\mathbb{H}^3$, assume that

$w_1 \leq w_2 ... \leq w_N$. Since $\cosh r > 1$ for $r \neq 0$, the w-component of $\mathbf{F}_N$ is

$$\sum_{i=1}^{N-1} \frac{w_i - \cosh d_{Ni} w_N}{\sinh^3 d_{Ni}} < \sum_{i=1}^{N-1} \frac{w_i - w_N}{\sinh^3 d_{Ni}} < 0.$$

Thus $\mathbf{q}$ can not be a special central configurations, a remark that completes the proof. □

Theorem 9. *There are no special central configurations for any* $\mathbf{m} \in \mathbb{R}_+^N$ *in any closed hemisphere of* $\mathbb{S}^3$ *(i.e. a hemisphere that contains its boundary), as long as at least one body does not lie on the boundary.*

Proof. Let $\mathbf{q}$ be a configuration that lies in a closed hemisphere of $\mathbb{S}^3$ and such that there is at least one body not on the boundary. Since special central configurations are still special central configurations after any rotation in $SO(4)$, we may assume that $\mathbf{q}$ is within the hemisphere $w \geq 0$ and there is some $w_i > 0$. Suppose that $0 \leq w_1 \leq w_2 \leq ... \leq w_N$ and $w_N > 0$. Consider the w-component of $\mathbf{F}_1$,

$$\mathbf{F}_{1w} = \sum_{j=2}^{N} \frac{m_j m_1 (w_j - \cos d_{1j} w_1)}{\sin^3 d_{1j}}.$$

Then $\mathbf{F}_{1w}$ must be positive since $w_j - \cos d_{1j} w_1 > w_j - w_1 \geq 0$ for each j, and $w_N - \cos d_{1N} w_1 > 0$. Thus $\mathbf{q}$ can not be a special central configurations, a remark that completes the proof. □

By Lemma 1, we have the following direct property.

Proposition 6. *A central configuration* $\mathbf{q}$ *on* $\mathbb{S}_{xy}^1 \cup \mathbb{S}_{zw}^1$ *is a special central configuration.*

4.4 Some Useful Properties of Ordinary Central Configurations

This section focuses on lower dimensional ordinary central configurations, namely geodesic central configurations, $\mathbb{S}^2$ central configurations and $\mathbb{H}^2$ central configurations. We show that any central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}_{xyw}^2$, and that any geodesic central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^1_{xw}$. We also show that any $\mathbb{S}^2$ central configuration in $\mathbb{S}^3$ can be found on $\mathbb{S}^2_{xyz}$, and that any geodesic central configuration in $\mathbb{S}^3$ is equivalent to some central configuration on $\mathbb{S}^1_{xz}$.

Recall that a (hyperbolic) 2-sphere means a sphere (hyperbolic sphere) isometric to the unit sphere (hyperbolic sphere) in $\mathbb{R}^3$ ($\mathbb{R}^{2,1}$). This object is the non-empty intersection of $\mathbb{M}^3$ with a three-dimensional linear subspace: $\{(x, y, z, w) \in \mathbb{R}^4 | ax + by + cz + dw = 0\}$ [14]. Similarly, a geodesic is the non-empty intersection of a (hyperbolic) 2-sphere with a two-dimensional linear subspace.

Lemma 3. *Assume that the intersection of $\mathbb{S}^3$ ($\mathbb{H}^3$) and the three dimensional linear space $V = \{(x, y, z, w) \in \mathbb{R}^4 | ax+by+cz+dw = 0\}$ is non-empty. Let $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N)$ ($N \geq 2$) be a nonsingular configuration on the (hyperbolic) 2-sphere: $V \cap \mathbb{M}^3$. If $\nabla_{\mathbf{q}_i} I$ are not all zero, then $\nabla_{\mathbf{q}_i} I \in V, i = 1, ..., N$, if and only if $a = b = 0$ or $c = d = 0$.*

Proof. Recall that

$$\nabla_{\mathbf{q}_i} I = 2m_i \left(x_i(w_i^2 + \sigma z_i^2), y_i(w_i^2 + \sigma z_i^2), -\sigma z_i(x_i^2 + y_i^2), -\sigma w_i(x_i^2 + y_i^2)\right).$$

Then $\nabla_{\mathbf{q}_i} I \in V, i = 1, ..., N$ if and only if

$$\begin{aligned} 0 &= ax_i(w_i^2 + \sigma z_i^2) + by_i(w_i^2 + \sigma z_i^2) - c\sigma z_i(x_i^2 + y_i^2) - d\sigma w_i(x_i^2 + y_i^2) \\ &= (ax_i + by_i)(w_i^2 + \sigma z_i^2) - \sigma(cz_i + dw_i)(x_i^2 + y_i^2) \\ &= (ax_i + by_i)(w_i^2 + \sigma z_i^2) + \sigma(ax_i + by_i)(x_i^2 + y_i^2) \\ &= (ax_i + by_i). \end{aligned}$$

Then we also have $cz_i + dw_i = 0$. Consider the matrix $A := \begin{bmatrix} a & b & c & d \\ a & b & 0 & 0 \\ 0 & 0 & c & d \end{bmatrix}$. Then $(\mathbf{q}_1, ..., \mathbf{q}_N) \in \ker A$. Since $\mathbf{q}_i$ and $\mathbf{q}_j$ are linearly independent, we obtain rank(ker A) ≥ 2, which implies that rank$A = 1$. Therefore, we have either $a = b = 0$ or $c = d = 0$. □

Now we turn to central configurations in $\mathbb{H}^3$. The following result has been published in [92]. Define $\mathbb{H}^2_{xyw} = \{(x, y, z, w) \in \mathbb{H}^3 : z = 0\}$.

Theorem 10. *Each central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^2_{xyw}$.*

Proof. We first show that all central configurations in $\mathbb{H}^3$ must lie on a hyperbolic 2-sphere. Then we show that there is some action $\chi \in SO(2) \times SO(1,1)$ which transforms that hyperbolic sphere to $\mathbb{H}^2_{xyw}$. Thus by the definition of equivalent central configurations, each central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^2_{xyw}$.

Consider the hyperbolic 2-sphere: $\mathbb{H}^2_\phi := \{(x,y,z,w) \in \mathbb{R}^4 | z\cosh\phi - w\sinh\phi = 0\} \cap \mathbb{H}^3$. The intersection is not empty, since the linear subspace and $\mathbb{H}^3$ share the point $(0,0,\sinh\phi,\cosh\phi)$. We show that each central configuration will be confined to only one such hyperbolic 2-sphere.

Assume that this is not the case. Suppose that there is a central configuration $\mathbf{q} = (\mathbf{q}_1, \cdots, \mathbf{q}_N)$ with $\mathbf{q}_i \in \mathbb{H}^2_{\phi_i}$, $\phi_1 \geq \phi_i$ for $i \neq 1$ and there is at least one i such that $\phi_1 > \phi_i$. Then $\mathbf{q}_i$ can be written as $(x_i, y_i, \rho_i \sinh\phi_i, \rho_i\cosh\phi_i)$ with $\rho_i > 0$ since $w_i = \rho_i\cosh\phi_i > 0$. By Lemma 3, $\nabla_{\mathbf{q}_1} I$ is in the linear subspace $\{(x,y,z,w) \in \mathbb{R}^4 | z\cosh\phi_1 - w\sinh\phi_1 = 0\}$. In order to have a central configuration, $\nabla_{\mathbf{q}_1} U$ must be in the linear subspace, i.e., $\mathbf{F}_{1z}\cosh\phi_1 - \mathbf{F}_{1w}\sinh\phi_1 = 0$, where $\mathbf{F}_{1z}$ and $\mathbf{F}_{1w}$ stand for the z-component and w-component of $\mathbf{F}_1$, respectively. However, using the explicit form of $\mathbf{F}_1$, we get

$$\begin{aligned}
&\mathbf{F}_{1z}\cosh\phi_1 - \mathbf{F}_{1w}\sinh\phi_1 \\
&= \sum_{i=2}^{N} m_i m_1 \left(\frac{z_i - z_1\cosh d_{i1}}{\sinh^3 d_{i1}} \cosh\phi_1 - \frac{w_i - w_1\cosh d_{i1}}{\sinh^3 d_{i1}} \sinh\phi_1 \right) \\
&= \sum_{i=2}^{N} m_i m_1 \frac{\rho_i \sinh\phi_i\cosh\phi_1 - \rho_i\cosh\phi_i\sinh\phi_1 - \cosh d_{i1}(z_1\cosh\phi_1 - w_1\sinh\phi_1)}{\sinh^3 d_{i1}} \\
&= \sum_{i=2}^{N} m_i m_1 \frac{\rho_i\sinh(\phi_i - \phi_1)}{\sinh^3 d_{i1}} < 0,
\end{aligned}$$

since $\phi_i \leq \phi_1$ for $i \neq 1$ and there is at least one i such that $\phi_i < \phi_1$.

Thus any central configuration must lie on only one such hyperbolic sphere, say $\mathbb{H}^2_\phi$. Let

$$\chi = (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix}) \in SO(2) \times SO(1,1).$$

Since $\begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} \rho_i\sinh\phi \\ \rho_i\cosh\phi \end{bmatrix} = \begin{bmatrix} 0 \\ \rho_i \end{bmatrix}$, $\chi(\mathbb{H}^2_\phi) = \mathbb{H}^2_{xyw}$. This calculation completes the proof. ☐

The above theorem can be generalized to a class of mechanical systems in $\mathbb{H}^3$. Consider a mechanical system in $\mathbb{H}^3$ with the kinetic energy $K = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$, and the force function $U = \sum_{1\le i<j\le N} m_i m_j f(d_{ij})$ ($-U$ being the potential), where $f(x)$ is a smooth function on $(0,\infty)$. Then this mechanical system has $O(3,1)$ symmetry and we can similarly define central configurations as the solutions of $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i}(x_i^2 + y_i^2)$ and define equivalent central configurations.

Corollary 2. *For a mechanical system in $\mathbb{H}^3$ as above, if $f'(x)$ is never zero, then each central configuration is equivalent to some central configuration on $\mathbb{H}^2_{xyw}$.*

Proof. By direct computation, we get $\mathbf{F}_{ij} = -m_i m_j f'(d_{ij}) \frac{\mathbf{q}_j - \cosh d_{ij} \mathbf{q}_i}{\sinh d_{ij}}$. Since $f'(d_{ij})$ is always positive or negative, it is easy to see that all the arguments in the proof of the above theorem hold for this mechanical system. This remark completes the proof. ☐

Thus there are no $\mathbb{H}^3$ central configurations. However, as we shall see later, there exist both special $\mathbb{S}^3$ central configurations and ordinary $\mathbb{S}^3$ central configurations. Thus the set of central configurations in $\mathbb{S}^3$ is richer and more interesting than in $\mathbb{H}^3$.

Define $\mathbb{H}^1_{xw} = \{(x,y,z,w) \in \mathbb{H}^3 : y = z = 0\}$.

Corollary 3. *Each geodesic central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^1_{xw}$.*

Proof. By Theorem 10, every geodesic central configuration is equivalent to some geodesic central configuration on $\mathbb{H}^2_{xyw}$. A geodesic on $\mathbb{H}^2_{xyw}$ is the non-empty intersection of a 2-dimensional linear space V and $\mathbb{H}^2_{xyw}$. Suppose that V is defined by $\{ax + by + dw = 0\}$. Suppose that a central configuration $\mathbf{q}$ is on $V \cap \mathbb{H}^2_{xyw}$. Then $\nabla_{\mathbf{q}_i} U$ lies in V for all i. It implies that each $\nabla_{\mathbf{q}_i} I$ belongs to V. Similar to Lemma 3, it is easy to show that it is sufficient and necessary to require $d = 0$.

Then any geodesic central configuration is equivalent to some central configuration on a geodesic $\{(x,y,w) \in \mathbb{H}^2_{xyw} | ax + by = 0\}$. It is easy to see that there is some element in $SO(2) \times SO(1,1)$ to transform the geodesic to $\mathbb{H}^1_{xw}$. This remark completes the proof. ☐

Now we discuss the $\mathbb{S}^2$ ordinary central configurations and geodesic ordinary central configurations in $\mathbb{S}^3$. Define $\mathbb{S}^2_{xzw} := \{(x,y,z,w) \in \mathbb{S}^3 : y = 0\}$, $\mathbb{S}^1_{xz} := \{(x,y,z,w) \in \mathbb{S}^3 : w = y = 0\}$ and recall that $\mathbb{S}^2_{xyz} := \{(x,y,z,w) \in \mathbb{S}^3 : w = 0\}$.

Theorem 11. *Any $\mathbb{S}^2$ ordinary central configuration is equivalent to some ordinary central configuration on $\mathbb{S}^2_{xyz}$ or on $\mathbb{S}^2_{xzw}$. Furthermore, there is a one-to-one correspondence between the central configurations on $\mathbb{S}^2_{xyz}$ and the central configurations on $\mathbb{S}^2_{xzw}$.*

Proof. Lemma 3 implies that any $\mathbb{S}^2$ ordinary central configuration is either on $\mathbb{S}^3 \cap \{ax+by=0\}$ or on $\mathbb{S}^3 \cap \{cz+dw=0\}$. It is easy to see that there is some element in $SO(2)\times SO(2)$ that would transform these 2-spheres to either $\mathbb{S}^2_{xyz}$ or $\mathbb{S}^2_{xzw}$. Thus any $\mathbb{S}^2$ ordinary central configuration is equivalent to some ordinary central configuration on $\mathbb{S}^2_{xyz}$ or on $\mathbb{S}^2_{xzw}$.

Let $\mathbf{q}$ be a central configuration on $\mathbb{S}^2_{xyz}$, i.e. $\nabla_{\mathbf{q}_i}U(\mathbf{q}) - \lambda\nabla_{\mathbf{q}_i}I(\mathbf{q}) = 0, i = 1,...,N$. Consider the orthogonal transformation $\varphi(x_i,y_i,z_i,w_i) = (z_i,w_i,x_i,y_i)$. Then $\mathbf{q}' = (\mathbf{q}'_1,...,\mathbf{q}'_N) = \varphi\mathbf{q} = (\varphi\mathbf{q}_1,...,\varphi\mathbf{q}_N)$ is a configuration on $\mathbb{S}^2_{xzw}$. Note that $\mathbf{q}'_i = (x'_i,y'_i,z'_i,w'_i) = (z_i,w_i,x_i,y_i)$

$$I(\mathbf{q}') = \sum_{i=1}^{N} m_i({x'}_i^2 + {y'}_i^2) = \sum_{i=1}^{N} m_i(1 - x_i^2 - y_i^2) = \sum_{i=1}^{N} m_i - I(\mathbf{q}),$$

and $U(\mathbf{q}') = U(\mathbf{q})$. Using the formulas of ∇U and ∇I, we obtain that $\nabla U(\mathbf{q}') = \varphi\nabla U(\mathbf{q})$ and $\nabla I(\mathbf{q}') = -\varphi\nabla I(\mathbf{q})$. Here $\nabla U(\mathbf{q}')$ and $\nabla I(\mathbf{q}')$ mean the gradient of U and I at $\mathbf{q}'$ respectively. Thus $\mathbf{q}'$ satisfies the central configuration equation $\nabla_{\mathbf{q}_i}U(\mathbf{q}') + \lambda\nabla_{\mathbf{q}_i}I(\mathbf{q}') = 0$, i=1,...,N. This remark completes the proof. □

The proof of the following statement is similar to that of Corollary 3. it is easy to prove the following corollary:

Corollary 4. *Each ordinary geodesic central configuration in $\mathbb{S}^3$ is equivalent to some central configuration on $\mathbb{S}^1_{xz}$.*

Chapter 5

Existence and Basic Problems

There are many interesting questions about central configurations. Fixing the masses $\mathbf{m} \in \mathbb{R}_+^N$, we may ask whether central configurations exist and, if so, how many are there up to equivalent classes? In this chapter, by interpreting ordinary central configurations as critical points of functions related to U, we show that ordinary central configurations always exist for given masses $\mathbf{m}$. But this is not the case for special central configurations. We also discuss the computation of the Hessian of central configurations since they are critical points. In the end, we formulate the Winter-Smale problem for the curved N-body problem.

5.1 The Gradient Flow on S_c

By definition, special central configurations are critical points of U. The key idea of this section is to interpret ordinary central configurations as critical points of U restricted to S_c, which is a subset of $(\mathbb{M}^3)^N$. We also discuss the structure of the set S_c.

Recall that any central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^2_{xyw} = \{(x, y, z, w) \in \mathbb{H}^3 : z = 0\}$ and that any geodesic central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^1_{xw} = \{(x, y, z, w) \in \mathbb{H}^3 : y = z = 0\}$. From now on, unless specified otherwise, we use $\mathbb{H}^2$ to indicate $\mathbb{H}^2_{xyw}$, and $\mathbb{H}^1$ for $\mathbb{H}^1_{xw}$. We only study central configurations on $\mathbb{H}^2$. Then two central configurations $\mathbf{q}, \mathbf{q}'$ on $\mathbb{H}^2$ are equivalent if there is some element χ in $SO(2)$ such that $\chi\mathbf{q} = \mathbf{q}'$. The action is defined by $\chi\mathbf{q} = (\chi\mathbf{q}_1, ..., \chi\mathbf{q}_N)$, $\chi\mathbf{q}_i = (\chi(x_i, y_i)^T, w_i)$. The expression of ∇I is

$$\nabla_{\mathbf{q}_i} I(\mathbf{q}) = 2m_i \left(x_i w_i^2, y_i w_i^2, w_i(x_i^2 + y_i^2) \right).$$

Now the following properties are obvious.

Proposition 7. – *A special central configuration in $\mathbb{S}^3$ is a critical point of the function $U : (\mathbb{S}^3)^N \setminus \Delta \to \mathbb{R}$.*

– *An ordinary central configuration $\bar{\mathbf{q}}$ in $\mathbb{S}^3$ is a critical point of the function $U(\mathbf{q}) - \lambda I(\mathbf{q}) : (\mathbb{S}^3)^N \setminus \Delta \to \mathbb{R}$, where λ is some constant depending on $\bar{\mathbf{q}}$.*

– *A central configuration $\bar{\mathbf{q}}$ on $\mathbb{H}^2$ is a critical point of the function $U(\mathbf{q}) - \lambda I(\mathbf{q}) : (\mathbb{H}^2)^N \setminus \Delta \to \mathbb{R}$, where λ is some constant depending on $\bar{\mathbf{q}}$.*

In this chapter, let M be the matrix $\mathrm{diag}(m_1, m_1, m_1, m_1, ..., m_N, m_N, m_N, m_N)$. Introduce a metric in $(\mathbb{R}^4)^N$ $\left((\mathbb{R}^{3,1})^N\right)$:

$$\langle \mathbf{q}, \mathbf{q} \rangle = \sum_{i=1}^{N} m_i \mathbf{q}_i \cdot \mathbf{q}_i = \mathbf{q} \cdot M\mathbf{q}.$$

For ordinary central configurations we have

$$\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle = \lambda \langle M^{-1}\nabla I, M^{-1}\nabla I \rangle.$$

Proposition 8. *Let $\mathbf{q}$ be an ordinary central configuration, then the value of λ in the central configuration equation is $\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle}$. For central configurations on $\mathbb{H}^2$, we have $\lambda < 0$.*

Proof. Since $\mathbf{q}$ is an ordinary central configuration, $\nabla I \neq 0$ and $\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle \neq 0$. Thus the value of λ for an ordinary central configuration $\mathbf{q}$ is $\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I \rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle}$.

On $\mathbb{H}^2$, direct computation leads to

$$\langle M^{-1}\nabla I, M^{-1}\nabla I \rangle = \sum_{i=1}^{N} 4m_i(x_i^2 w_i^4 + y_i^2 w_i^4 - (x_i^2 + y_i^2)^2 w_i^2) = \sum_{i=1}^{N} 4m_i(x_i^2 + y_i^2) w_i^2,$$

$$\begin{aligned}\langle M^{-1}\nabla U, M^{-1}\nabla I\rangle &= \sum_{i=1}^{N} m_i \left(\sum_{j\neq i}^{N} m_j \frac{\mathbf{q}_j - \cosh d_{ij}\mathbf{q}_i}{\sinh^3 d_{ij}} \cdot (x_i w_i^2, y_i w_i^2, (x_i^2+y_i^2)w_i)\right)\\ &= \sum_{1\le i<j\le N} m_i m_j \frac{(x_i x_j + y_i y_j)(w_i^2+w_j^2) - w_i w_j (x_i^2+y_i^2+x_j^2+y_j^2)}{\sinh^3 d_{ij}}\\ &= \sum_{1\le i<j\le N} m_i m_j \frac{(w_i w_j - \cosh d_{ij})(w_i^2+w_j^2) - w_i w_j (w_i^2+w_j^2-2)}{\sinh^3 d_{ij}}\\ &= \sum_{1\le i<j\le N} m_i m_j \frac{-\cosh d_{ij}(w_i^2+w_j^2) + (w_i^2+w_j^2) - (w_i^2+w_j^2) + 2w_i w_j}{\sinh^3 d_{ij}}\\ &= \sum_{1\le i<j\le N} m_i m_j \frac{(w_i^2+w_j^2)(1-\cosh d_{ij}) - (w_i - w_j)^2}{\sinh^3 d_{ij}} < 0.\end{aligned}$$

Here we used the identities $\cosh d_{ij} = w_i w_j - (x_i x_j + y_i y_j)$ and $x_i^2 + y_i^2 - w_i^2 = -1$. This remark completes the proof. □

This proposition fills the gap in the proof of Corollary 1. We denote by S_c the set $\{\mathbf{q} \in (\mathbb{S}^3)^N \setminus \Delta \,|\, I(\mathbf{q}) = c\}$ (respectively $\{\mathbf{q} \in (\mathbb{H}^2)^N \setminus \Delta \,|\, I(\mathbf{q}) = c\}$).

Proposition 9. *In the case of $\mathbb{H}^2$, $I^{-1}(c)$ is homeomorphic to a $(2N-1)$-dimensional sphere for each positive value of c.*

Proof. Consider the homemorphism $\pi : \mathbb{R}^2 \to \mathbb{H}^2$, $\pi(x,y) = (x,y,w)$. The map induces a homemorphism from $(\mathbb{R}^2)^N$ to $(\mathbb{H}^2)^N$, which we still denote by π. Thus the function $I : (\mathbb{H}^2)^N \to \mathbb{R}$ induces a function $\bar{I} : (\mathbb{R}^2)^N \to \mathbb{R}$ by $\bar{I}(\mathbf{q}') = I(\pi\mathbf{q}')$, where $\mathbf{q}'$ is a point in $(\mathbb{R}^2)^N$. It is easy to see that $\bar{I}^{-1}(c)$ is homeomorphic to a $(2N-1)$-dimensional sphere for each positive value of c and $\bar{I}^{-1}(c)$ is homeomorphic to $I^{-1}(c)$. This remark completes the proof. □

In the case of $\mathbb{S}^3$, consider the map $I : (\mathbb{S}^3)^N \to [0, \sum_{i=1}^N m_i]$ for given masses $m_1, ..., m_N$. Let $\mathcal{M}, \mathcal{N}$ be differentiable manifolds and $f\colon \mathcal{M} \to \mathcal{N}$ a differentiable map. Recall that f is called a *submersion* at $x \in \mathcal{M}$ if its differential, $Df_x\colon T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$, is surjective. In this case, x is called a *regular point*, otherwise, x is a *critical point*. A point $y \in \mathcal{N}$ is a *regular value* if the set $f^{-1}(y)$ consists of regular points, otherwise, y is a *critical value*.

Proposition 10. *The set of critical points of the map $I : (\mathbb{S}^3)^N \to [0, \sum_{i=1}^N m_i]$ is $\left(\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}\right)^N$ and the critical values are $\sum_{i=1}^N m_i\mu_i$, where μ_i is 0 or 1. If c is not a critical value, then $I^{-1}(c)$ is a smooth manifold.*

Proof. Suppose that $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ is a critical point, then

$$\nabla_{\mathbf{q}_1} I(\mathbf{q}) = \cdots = \nabla_{\mathbf{q}_N} I(\mathbf{q}) = \mathbf{0}.$$

By Lemma 1, this is true if and only if $\mathbf{q} \in \left(\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}\right)^N$. That is, $x_i^2 + y_i^2$ equals 0 or 1. Thus the critical values are $c = \sum_{i=1}^N m_i \mu_i$, where μ_i is 0 or 1. If c is not a critical value, then by the regular value theorem [43], $I^{-1}(c)$ is a smooth manifold, a remark that completes the proof. □

The existence of these critical values makes the topology of $I^{-1}(c)$ complicated in $\mathbb{S}^3$. We do not discuss this topic for now. Note that in the central configuration equation, the value λ can be also interpreted as a Lagrange multiplier. More precisely, Consider the restricted function:

$$U: \ S_c \to \mathbb{R}.$$

Then we have the following result.

Proposition 11. *Assume that* $\mathbf{q}$ *does not lie on* $\mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$. *Then the vectorfield*

$$X = M^{-1}\nabla U - M^{-1}\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I\rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I\rangle}\nabla I$$

is the gradient of $U|_{S_c}$, *the restriction of* $U(\mathbf{q})$ *on the set* S_c, *with respect to the metric* $\langle\cdot,\cdot\rangle$. *Moreover, the restpoints of this vectorfield are exactly the ordinary central configurations in* S_c. *Special central configurations in* $\mathbb{S}^3$ *are the restpoints of the gradient of* $U(\mathbf{q})$.

Proof. Since $\mathbf{q} \notin \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, it is not a critical point of the map I. Then near $\mathbf{q}$, S_c is locally a submanifold and the vector field is well defined. Since $\langle X, M^{-1}\nabla I\rangle = 0$, the vector field is tangential to S_c. For any $\mathbf{v} \in T_{\mathbf{q}} S_c$, we have $\langle \mathbf{v}, M^{-1}\nabla I \ \rangle = 0$, thus

$$\langle X, \mathbf{v}\rangle = \langle M^{-1}\nabla U, \mathbf{v}\rangle = dU\mathbf{v},$$

where dU is the differential of U. Thus X is the gradient flow of U on S_c. The other statements are self-clear, a remark that completes the proof. □

Besides helping to show the existence of ordinary central configurations, this proposition also helps in other ways. The gradient flow preserves certain submanifolds

ifolds of S_c. For example, the intersection of S_c and $\mathbb{S}^2_{xyz}$ is invariant under the flow. On $\mathbb{S}^2_{xyz}$, we have

$$\nabla_{\mathbf{q}_i} w_i = \nabla_{\mathbf{q}_i} \frac{w_i}{\sqrt{x_i^2 + y_i^2 + z_i^2 + w_i^2}} = (-w_i x_i, -w_i y_i, -w_i z_i, 1 - w_i^2) = (0, 0, 0, 1).$$

It is easy to get $\langle X, M^{-1}\nabla w\rangle = 0$. Then a critical point of U restricted to $S_c \cap (\mathbb{S}^2_{xyz})^N$ is also a critical point of $U|_{S_c}$, thus a central configuration. In other words, we can confine the bodies on $\mathbb{S}^2_{xyz}$ to search for ordinary central configurations, a fact that has been verified in Theorem 11.

In the Newtonian N-body problem, the set $\sum_{i=1}^N m_i \mathbf{q}_i = 0$ is invariant under the corresponding gradient flow. One may wonder if this is the case for the set defined by (4.2) for the curved N-body problem. Let

$$F : (\mathbb{M}^3)^N \to \mathbb{R}^4, F(\mathbf{q}) = (\sum_{i=1}^N m_i x_i z_i, \sum_{i=1}^N m_i y_i z_i, \sum_{i=1}^N m_i x_i w_i, \sum_{i=1}^N m_i y_i w_i).$$

Let $\mathcal{M} := F^{-1}(0)$. We are interested in whether $\mathcal{M} \cap S_c$ is invariant under the flow X. Note that if $\mathcal{M} \cap S_c$ is invariant, then $\mathcal{M} = \cup_c(\mathcal{M} \cap S_c)$ is also invariant. Generally, $\mathcal{M}$ is not invariant under the gradient flow X. Thus though the critical points of $U|_{S_c}$ always lie on $\mathcal{M}$, a critical point of U restricted to $S_c \cap \mathcal{M}$ may not be a critical point of $U|_{S_c}$. In other words, we can't confine the bodies on $\mathbb{S}_c \cap \mathcal{M}$ to search for ordinary central configurations. Let us look at a simple example.

Example 5. *Consider three masses on $\mathbb{H}^1$. It is easy to see that $\mathbb{H}^1$ is invariant. In this case, $F : (\mathbb{H}^1)^3 \to \mathbb{R}, F(\mathbf{q}) = \sum_{i=1}^3 m_i x_i w_i$. It is easy to see that $\mathcal{M}$ is homemorphic to $\mathbb{R}^2$. If $\mathcal{M}$ is invariant under the flow X, then we have $dFX = \langle M^{-1}\nabla F, X\rangle = 0$. Direct computation leads to*

$$dF = \sum_{i=1}^3 m_i(w_i dx_i + x_i dw_i),$$

$$dFM^{-1}\nabla U = \sum_{1\le i<j\le 3} m_i m_j \frac{2(x_i w_j + x_j w_i) - 2\cosh d_{ij}(x_i w_i + x_j w_j)}{\sinh^3 d_{ij}},$$

$$dFM^{-1}\nabla I = 2\sum_{i=1}^3 m_i w_i x_i(2x_i^2 + 1) = 4\sum_{i=1}^3 m_i w_i x_i^3,$$

$$\lambda = \left[\sum_{1 \le i < j \le 3} m_i m_j \frac{2x_i x_j - \cosh d_{ij}(x_i^2 + x_j^2)}{\sinh^3 d_{ij}} \right] / \left[\sum_{i=1}^{3} m_i w_i^2 x_i^2 \right].$$

Now let $\mathbf{m} = (\sqrt{3}, m, 1)$*,* $\mathbf{q}_1 = (-1, \sqrt{2}), \mathbf{q}_2 = (0, 1), \mathbf{q}_3 = (\sqrt{2}, \sqrt{3})$*. So* $\mathbf{q} \in \mathcal{M}$*. At* $\mathbf{q}$*, direct computation leads to*

$$dFX = \frac{2(324m + 187\sqrt{3}m + 9\sqrt{3} + 15)}{(2 + \sqrt{3})^3(\sqrt{3} + 3)},$$

which is always positive. Thus $\mathcal{M}$ *is not invariant.*

The critical points of $U|_{S_c}$ are not isolated. Let $\mathbf{q}$ be an ordinary central configuration and ϕ an element of $SO(2) \times SO(2)$ or $SO(2)$. Then $\phi\mathbf{q}$ is also a central configuration. Thus it follows that the critical points of $U|_{S_c}$ are not isolated, but rather occur as manifolds of critical points. Similarly, these special central configurations are not isolated either. This fact suggests that we can further look at the central configurations as critical points of U subject to a quotient manifold. Note that both U and $(\mathbb{M}^3)^N$ are invariant under the isometry group, and the set S_c is invariant under the subgroup $SO(2) \times SO(2)$ or $SO(2)$. We thus have the following property.

Proposition 12.

1. *There is a one-to-one correspondence between the classes of central configurations on* $\mathbb{H}^2$ *and the critical points of the force function* $\hat{U}$ *induced by* U *on the quotient set* $S_c/SO(2)$*.*
2. *There is a one-to-one correspondence between the classes of ordinary central configurations in* $\mathbb{S}^3$ *and the critical points of the force function* $\hat{U}$ *induced by* U *on the quotient set* $S_c/(SO(2) \times SO(2))$*.*
3. *There is a one-to-one correspondence between the classes of special central configurations in* $\mathbb{S}^3$ *and the critical points of the force function* $\hat{U}$ *induced by* U *on the quotient set* $((\mathbb{S}^3)^N \setminus \Delta)/SO(4)$*.*

5.2 Existence and the Hessian

Using Proposition 11, we show that ordinary central configurations always exist for given masses $\mathbf{m}$. But this is not the case for special central configurations. We also discuss the computation of the Hessian of central configurations since they are critical points.

First we study the ordinary central configurations.

Theorem 12. *Assume that the masses* $\mathbf{m} \in \mathbb{R}_+^N$ *are in* $\mathbb{S}^3$ *or* $\mathbb{H}^2$*. Then there is at least one ordinary central configuration in* $\mathbb{S}^3$ *and at least one ordinary central configuration in* $\mathbb{H}^2$*.*

Proof. In $\mathbb{H}^2$, recall that S_c is the union of several open subsets of a $(2N-1)$-dimensional sphere. The boundary of each subset belongs to Δ, where U approaches $+\infty$. It follows that U attains a minimum at some non-singular configuration $\mathbf{q}$. This will be a critical point of U on S_c and hence an ordinary central configuration by Proposition 11.

In $\mathbb{S}^3$, we need to construct a connected component of S_c on whose boundary U approaches $+\infty$. Recall that there are two kinds of singularities, collision singularities and antipodal singularities, which are

$$\Delta^+ := \cup_{1\leq i<j\leq N}\{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = \mathbf{q}_j\} \setminus \cup_{1\leq i<j\leq N}\{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = -\mathbf{q}_j\},$$
$$\Delta^- := \cup_{1\leq i<j\leq N}\{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = -\mathbf{q}_j\} \setminus \cup_{1\leq i<j\leq N}\{\mathbf{q} \in (\mathbb{S}^3)^N : \mathbf{q}_i = \mathbf{q}_j\}.$$

U approaches $+\infty$ as the configuration approaches Δ^+, but approaches $-\infty$ as the configuration approaches Δ^-. Thus we need to construct a connected component of S_c whose boundary lies only in Δ^+.

As discussed above, $\mathbb{S}^2_{xyz}$ is an invariant submanifold of the gradient flow X. We thus confine the particles to $\mathbb{S}^2_{xyz}$ and order the masses such that $0 < m_1 \leq \cdots \leq m_N$,. Let $0 < c < m_1$. Then S_c is a smooth manifold. Let us further choose a configuration $\mathbf{q} \in S_c$ with all bodies lying near the North Pole $(0, 0, 1)$, which means that $z_i > 0$, $i = 1, ..., N$. Denote by $\mathcal{J}$ the connected component of the manifold S_c that contains the configuration $\mathbf{q}$. We claim that the boundary of $\mathcal{J}$ contains only points from Δ^+.

To prove this claim, we define the sets $\mathcal{U} = \{(x, y, z) \in \mathbb{S}^2_{xyz} \,|\, x^2 + y^2 < c/m_1,\ z > 0\}$, and $\mathcal{V} = \{(x, y, z) \in \mathbb{S}^2_{xyz} | x^2+y^2 < c/m_1,\ z \leq 0\}$. Since $I(\mathbf{q}) = \sum_{i=1}^N m_i(x_i^2+y_i^2) \geq m_1(x_i^2 + y_i^2)$, it follows that $x_i^2 + y_i^2 \leq c/m_1$, $i = 1, ..., N$, which means that for any configuration $\mathbf{q} \in \mathcal{J}$ each body lies either in $\mathcal{U}$ or in $\mathcal{V}$.

Let us now suppose that $\partial\mathcal{J} \cap \Delta^- \neq \emptyset$. Then there must exist a configuration $\bar{\mathbf{q}} = (\bar{\mathbf{q}}_1, \ldots, \bar{\mathbf{q}}_N) \in \mathcal{J}$ such that one body is in $\mathcal{U}$ and the another in $\mathcal{V}$, say, $\bar{\mathbf{q}}_1 \in \mathcal{U}$ and $\bar{\mathbf{q}}_2 \in \mathcal{V}$. Since $\mathcal{J}$ is connected, it is also path connected. Then there is a path in $\mathcal{J}$ connecting $\mathbf{q}$ and $\bar{\mathbf{q}}$, so there is a path that connects $\mathbf{q}_2 \in \mathcal{U}$ and $\bar{\mathbf{q}}_2 \in \mathcal{V}$. But this

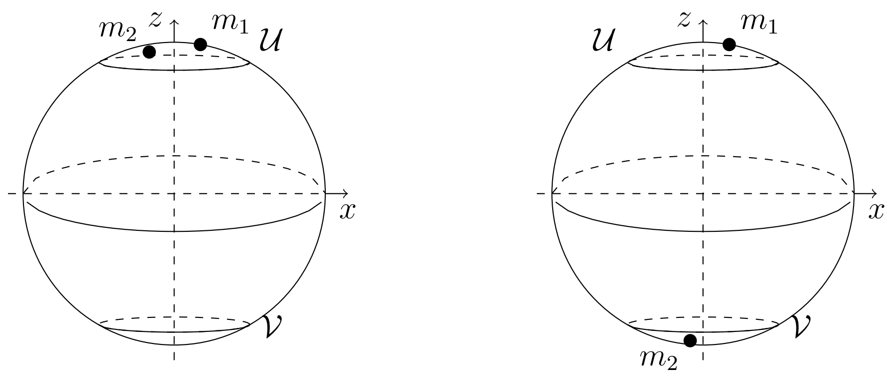

Figure 5.1: $\bar{\mathbf{q}}_1$, $\bar{\mathbf{q}}_2$ and $\mathbf{q}_1, \mathbf{q}_2$ on $\mathbb{S}^2_{xyz}$

is impossible since $\mathcal{U} \cap \mathcal{V} = \emptyset$. Thus $\mathcal{J}$ is a connected component of the manifold S_c whose boundary consists only of points from Δ^+.

Therefore $U \to +\infty$ as $\mathbf{q}$ approaches $\partial\mathcal{J}$. It follows that U attains a minimum at some configuration $\mathbf{q}$, which is then a critical point of U on S_c, hence an ordinary central configuration by Proposition 11. □

It is interesting to classify central configurations by their Morse index. Recall that if x is a critical point of a smooth function f on a manifold $\mathcal{M}$, there is a *Hessian* quadratic form on the tangent space $T_x\mathcal{M}$ that is given in local coordinates by the symmetric matrix of the second derivatives:

$$H(x)(\mathbf{v}) = \mathbf{v}^T D^2 V(x)\mathbf{v}.$$

The *Morse index* $\operatorname{ind}(x)$ is the maximum dimension of a subspace of $T_x\mathcal{M}$ on which $H(x)$ is negative-definite. The *nullity* is the dimension of

$$\ker H(x) = \{\mathbf{v} : H(x)(\mathbf{v}, \mathbf{u}) = 0 \text{ for all } \mathbf{u} \in T_x\mathcal{M}\},$$

where $H(x)(\mathbf{v}, \mathbf{u}) = \mathbf{v}^T D^2 V(x)\mathbf{u}$ is the symmetric bilinear form associated to $H(x)$. We are interested in the function $U|_{S_c}$ given by restricting the potential to the manifold S_c.

We may use the local coordinates of S_c, which is a $(dN-1)$-dimensional manifold for a d-dimensional central configuration. It is more convenient to use the coordinates of $\mathbb{S}^3$ ($\mathbb{H}^2$). Then the Hessian is given by a $dN \times dN$ matrix, also called $H(x)$, whose restriction to $T_{\mathbf{q}}S_c$ gives the correct values.

Lemma 4. *Let $\mathcal{M}$ be a smooth manifold, $\mathcal{N}$ be a submanifold of $\mathcal{M}$, and f_1 be a smooth function on $\mathcal{N}$. Assume that f is a smooth function on $\mathcal{M}$ and $f|_{\mathcal{N}} = f_1 + c$, where c is some constant, and that $x \in \mathcal{N}$ is a critical point of f and f_1. Denote the Hessian of f and the Hessian of f_1 by $H(x)$ and $H_1(x)$ respectively. Then $H(x)|_{T_x\mathcal{N}} = H_1(x)$.*

Proof. Let $(x_1, \cdots, x_n)$ be a local coordinate system of $\mathcal{N}$ near x. Extend this system to a local coordinate system of $\mathcal{M}$ near x, $(x_1, \cdots, x_n, y_1, \cdots, y_k)$. Since x is the critical point of f and f_1, we get

$$H(x) = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j + \frac{\partial^2 f}{\partial x^i \partial y^j} dx^i \otimes dy^j + \frac{\partial^2 f}{\partial y^i \partial y^j} dy^i \otimes dy^j,$$
$$H_1(x) = \frac{\partial^2 f_1}{\partial x^i \partial x^j} dx^i \otimes dx^j = \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \otimes dx^j,$$

where we used Einstein's summation. Now let $\mathbf{v} = a^i \frac{\partial}{\partial x^i} \in T_x\mathcal{N}$, we obtain $H(x)(\mathbf{v}) = H_1(x)(\mathbf{v})$ since $\frac{\partial^2 f}{\partial x^i \partial y^j} dx^i \otimes dy^j(\mathbf{v}) = \frac{\partial^2 f}{\partial y^i \partial y^j} dy^i \otimes dy^j(\mathbf{v}) = 0$. This remark completes the proof. □

Lemma 5. *Let $\mathbf{q}$ be a d-dimensional ordinary central configuration, and $I(\mathbf{q}) = c$. Let λ be the value of $\frac{\langle M^{-1}\nabla U, M^{-1}\nabla I\rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I\rangle}$ at $\mathbf{q}$. Then the Hessian of $U : S_c \to \mathbb{R}$ at the critical point $\mathbf{q}$ is given by $H(\mathbf{q})(\mathbf{v}) = \mathbf{v}^T H(\mathbf{q})\mathbf{v}$, where $H(\mathbf{q})$ is the $dN \times dN$ matrix*

$$H(\mathbf{q}) = D^2 U - \lambda D^2 I,$$

and $D^2 U$ and $D^2 I$ are the second derivatives matrices of U and I in some coordinates of $(\mathbb{S}^3)^N$ or $(\mathbb{H}^2)^N$.

Proof. We prove this result only in $\mathbb{S}^3$ since the proof for $\mathbb{H}^2$ is similar. Suppose that $\mathbf{q}$ is an ordinary central configuration in $\mathbb{S}^3$. Consider the manifold $(\mathbb{S}^3)^N \setminus \Delta$, the submanifold S_c, the smooth function $U - \lambda I : (\mathbb{S}^3)^N \setminus \Delta \to \mathbb{R}$, and the smooth function $U : S_c \to \mathbb{R}$. By Proposition 7 and Proposition 11, $\mathbf{q}$ is a critical point of the two functions. Restricted to the submanifold S_c, the two functions differ by a constant since $(U - \lambda I)|_{S_c} = U|_{S_c} - \lambda c$. Thus by the above lemma, we see that on $T_{\mathbf{q}} S_c$, the two Hessians are the same, which is $H(\mathbf{q}) = D^2 U - \lambda D^2 I$. This remark completes the proof. □

As noticed above, the critical points of $U|_{S_c}$ are not isolated, which implies that the ordinary central configuration are always degenerate as critical points for $d \geq 2$.

The following result describe the minimal degeneracy.

Proposition 13. *Let* $\mathbf{q} \in S_c$ *be an ordinary central configuration. Then the nullity of* $\mathbf{q}$ *as a critical point of* $U|_{S_c}$ *satisfies*

$$\text{null}(\mathbf{q}) \geq 1, \text{ in } \mathbb{H}^2, \quad \text{null}(\mathbf{q}) \geq 2, \text{ in } \mathbb{S}^3. \tag{5.1}$$

Proof. In $\mathbb{H}^2$, consider the curve of configurations $\mathbf{q}(t) = B_{\alpha,0}(t)\mathbf{q}$, which are also central configurations in S_c with the same value of λ. Thus we have the equation $\nabla_{\mathbf{q}_i} U(\mathbf{q}(t)) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}(t))$, $i = 1, ..., N$. Taking the derivative with respect to t at $t = 0$, we obtain

$$(D^2U - \lambda D^2 I)\dot{B}_{\alpha,0}(0)\mathbf{q} = 0.$$

Since $B_{\alpha,0}(t)\mathbf{q} \in S_c$, we get that $\dot{B}_{\alpha,0}(0)\mathbf{q} \in T_{\mathbf{q}} S_c$. Thus the nullity of the Hessian is at least one.

Similarly, in $\mathbb{S}^3$, we see that $\dot{A}_{\alpha,0}(0)\mathbf{q}$, $\dot{A}_{0,\beta}(0)\mathbf{q} \in T_{\mathbf{q}} S_c$ are in the kernel of $H(\mathbf{q})$. Thus the nullity of the Hessian is at least two. □

Now we study special central configurations in $\mathbb{S}^3$. Notice that by definition they are critical points of U. Thus the Hessian is easy to compute.

Definition 9. *N masses* $\mathbf{m} \in \mathbb{R}^N_+$ $(N \geq 2)$ *are said to be in* M_N *if* $m_1 + ... + m_N = 1$ *and there exists at least one special configuration for* $\mathbf{m}$.

Obviously, there are no special central configurations for two bodies. If so, then $\mathbf{F}_1 = m_1 m_2 \frac{\mathbf{q}_2 - \cos d_{12} \mathbf{q}_1}{\sin^3 d_{12}} = 0$, which implies that $\mathbf{q}_1 = \pm \mathbf{q}_2$.

Corollary 5. *M_2 is empty.*

Generally M_N is just a subset of $\{\mathbb{R}^N_+ | m_1 + ... + m_N = 1\}$. Now we turn to the Hessian.

Proposition 14. *Let* $\mathbf{q}$ *be a special central configuration in* $\mathbb{S}^3$ *of the masses* $\mathbf{m} \in M_N$. *Then the Hessian of* U *at the critical point* $\mathbf{q}$ *is given by* $H(\mathbf{q}) = D^2U(\mathbf{q})$, *where* D^2U *is the second derivatives matrix of* U *in some coordinates of* $\mathbb{S}^3$. *The nullity of* $\mathbf{q}$ *as a critical point of* $U : (\mathbb{S}^3)^N \setminus \Delta \to \mathbb{R}$ *satisfies*

$$\text{null}(\mathbf{q}) \geq 6. \tag{5.2}$$

Proof. We only need to prove the inequality (5.2). Notice that the symmetry of special central configurations is $SO(4)$, which is a six-dimensional Lie group. Let $\boldsymbol{\xi}_i$ $i = 1, ..., 6$, be the basis of $\mathfrak{so}(4)$, we see that each $\boldsymbol{\xi}_i \mathbf{q}$ is a null vector of $H(\mathbf{q})$. □

For ordinary central configurations and special central configurations, it is natural to call a critical point nondegenerate if its nullity is as small as possible, given the rotational symmetry.

Definition 10. *An ordinary central configuration or special central configuration is nondegenerate if the nullity of the corresponding critical point is as small as possible consistent with the rotational symmetry, i.e, equality holds in* (5.1) *and* (5.2)*.*

5.3 The Wintner-Smale Problem in Spaces of Constant Curvature

Recall that three equal masses on $\mathbb{S}^2_{xyz}$ possess a continuum of central configurations, see Example 4. Notice that these central configurations are on different S_c. In general, there is no obvious way to relate central configurations in S_{c_1} and central configurations in S_{c_2}. Thus we consider them belonging to different classes of central configurations. Notice that the existence proof of ordinary central configurations works for other constant values of I. Hence there always exist central configurations on S_c for c belonging to some open intervals. So we have the following obvious consequence.

Corollary 6. *Assume that the masses $m_1, \dots, m_N$ are in $\mathbb{S}^3$ or $\mathbb{H}^2$. Then for any positive values these masses take, the set of ordinary central configurations has the power of the continuum.*

Recall that the Wintner-Smale problem (Smale's 6th problem) asks whether for some given masses, $m_1, \dots, m_N > 0$, the number of classes of planar central configurations for the Newtonian N-body problem is finite or not [86]. If we extend the problem to the curved N-body problem in the following way: whether for some given masses, $m_1, \dots, m_N > 0$, the number of classes of central configurations for the curved N-body problem is finite or not, then this extension has an obvious and uninteresting answer. We modify the problem as follows for ordinary central configurations.

In the curved N-body problem, for given positive masses $m_1, \dots, m_N$ and all possible values of c, is the number of ordinary central configurations on S_c finite?

From now on, we will say that several masses possess a continuum of ordinary central configurations if the continuum of central configurations is on a certain set S_c. We will see in Chapter 7 that even for two equal masses, $m_1 = m_2 =: m$, there is a continuum of central configurations on S_m.

For special central configurations, we need to first study the mass set $M_N \in \mathbb{R}_+^N$. Thus we modify the problem as follows.

1. Characterize M_N.
2. In the curved N-body problem in $\mathbb{S}^3$, for given positive masses $\mathbf{m} \in M_N$, is the number of special central configurations finite?

Chapter 6

Central Configurations on $\mathbb{H}^2$

Recall that we denote by $\mathbb{H}^2$ the particular hyperbolic 2-sphere $\mathbb{H}^2_{xyw}$ and denote by $\mathbb{H}^1$ the particular hyperbola $\mathbb{H}^1_{xw}$. We give a few examples of central configurations on $\mathbb{H}^2$ and show that there is a neighbourhood of the singularity set where no central configurations exist. We then study the geodesic central configurations and generalize the celebrated Moulton's theorem of the Newtonian N-body problem. We also study their Hessian. Based on the exact count of the geodesic central configurations and their index, we give a lower bound for the number of central configurations on $\mathbb{S}_c$ by Morse inequality. Those familiar with the Newtonian N-body problem would notice that the above results are similar to the results on central configurations on $\mathbb{R}^2$. It is not surprising since $\mathbb{H}^2$ is homeomorphic to $\mathbb{R}^2$ and that two potentials on $\mathbb{R}^2$ and $\mathbb{H}^2$ are similar.

In the end, we focus on the non-geodesic central configurations of three bodies. Surprisingly, most of them are not equilateral triangles, which is different from the Newtonian N-body problem [90].

In this chapter, let $r_i = (x_i^2 + y_i^2)^{1/2}$.

6.1 Examples and the Extension of Shub's Lemma

We derive another form of the central configuration equation which would be useful later. After several examples, we show that there is a neighbourhood of the singularity set where no central configurations exist. In other words, the central configurations set on S_c is compact.

Proposition 15. *Consider the masses $m_1, \dots, m_N > 0$ on $\mathbb{H}^2$ at the configuration*

$\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_N)$, *where* $\mathbf{q}_i = (x_i, y_i, w_i)$. *If* $\mathbf{q}_i \neq (0,0,1)$, *then the i-th equation of* (3.5) *can be written as*

$$\sum_{j=1, j\neq i}^{N} \frac{m_j(x_i x_j + y_i y_j - r_i^2 \cosh d_{ij})}{\sinh^3 d_{ij}} = 2\lambda r_i^2 w_i^2, \quad \sum_{j=1, j\neq i}^{N} \frac{m_j(x_i y_j - x_j y_i)}{\sinh^3 d_{ij}} = 0. \tag{6.1}$$

Proof. Since $\mathbf{q}_i \neq (0,0,1)$, so $r_i = (x_i^2 + y_i^2)^{1/2} > 0$. Then the following three vectors

$$\mathbf{v}_{i1} = (x_i, y_i, 0), \ \mathbf{v}_{i2} = (-y_i, x_i, 0), \ \mathbf{v}_{i3} = (0, 0, w_i),$$

form an orthogonal basis of $T_{\mathbf{q}_i}\mathbb{R}^{2,1}$. Recall that

$$\nabla_{\mathbf{q}_i} U = \sum_{j=1, j\neq i}^{N} m_i m_j \frac{\mathbf{q}_j - \cosh d_{ij} \mathbf{q}_i}{\sinh^3 d_{ij}}, \quad \nabla_{\mathbf{q}_i} I = 2m_i(x_i w_i^2, y_i w_i^2, w_i r_i^2).$$

Then $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ is equivalent to $\nabla_{\mathbf{q}_i} U \cdot \mathbf{v}_{ik} = \lambda \nabla_{\mathbf{q}_i} I \cdot \mathbf{v}_{ik}$, $k = 1, 2, 3$. We obtain

$$\sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{\sinh^3 d_{ij}} (x_i x_j + y_i y_j - r_i^2 \cosh d_{ij}) = \lambda 2 m_i r_i^2 w_i^2,$$
$$\sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{\sinh^3 d_{ij}} (x_i y_j - y_i x_j) = 0,$$
$$\sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{\sinh^3 d_{ij}} (-w_i w_j + w_i^2 \cosh d_{ij}) = -\lambda 2 m_i r_i^2 w_i^2.$$

Adding the first and the third equation we obtain an identity. Thus the equation $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ is equivalent to (6.1). □

We give two examples of central configurations.

Example 6 (Lagrangian central configurations of three equal masses)**.** *Let three equal masses* $\mathbf{m} = (m, m, m)$ *be at*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3), \ \mathbf{q}_j = (x_j, y_j, w_j), \ j = 1, 2, 3,$$
$$x_j = r\cos\beta_j, \quad y_j = r\sin\beta_j, \quad w_j = \sqrt{1 + r^2}, \quad \beta_j = \frac{2\pi(j-1)}{3}, \ j = 1, 2, 3,$$

where r *could be any positive value, see Figure 6.1. By symmetry, we see that* $\nabla_{\mathbf{q}_i} U$ *is pointing towards the vertex* $A = (0, 0, 1)$, *so it is collinear with* $\nabla_{\mathbf{q}_i} \sinh^2 d(\mathbf{q}_i, A)$.

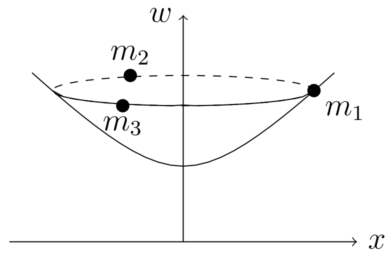

Figure 6.1: Lagrangian central configurations on $\mathbb{H}^2$

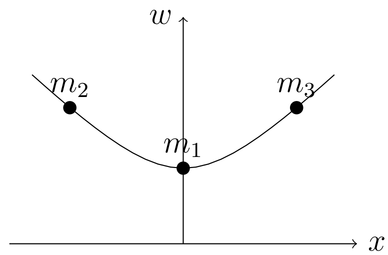

Figure 6.2: One geodesic central configurations on $\mathbb{H}^1$

Thus the central configuration equation (4.1), $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} m_i \sinh^2 d(\mathbf{q}_i, A)$, *is satisfied. Note that* $d_{12} = d_{13} = d_{23}$, *which is reminiscent of the 3-body central configuration in the Newtonian* N*-body problem found by Lagrange [90], so we will call them Lagrangian central configurations. Using the formula given in Proposition 8, we obtain*

$$\lambda = \frac{\langle M^{-1}\nabla U, M^{-1}\nabla I\rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I\rangle} = \frac{-3m}{2\sinh^3 d_{12}} = -\frac{m}{2\sqrt{3}r^3\left(1+\frac{3r^2}{4}\right)^{3/2}},$$

since $\cosh d_{12} = 1 + \frac{3r^2}{2}, \quad \sinh d_{12} = \sqrt{\cosh^2 d_{12} - 1}.$

Example 7 (Geodesic central configurations of three equal masses)**.** *Let three equal masses* $\mathbf{m} = (m, m, m)$ *be at*

$$\mathbf{q}_1 = (-r, w), \quad \mathbf{q}_2 = (0, 1), \quad \mathbf{q}_3 = (r, w),$$

with r *being any positive value and* $r^2 - w^2 = -1$, *see Figure 6.2. Let us check that the central configuration equation* (4.1): $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} m_i \sinh^2 d(\mathbf{q}_i, A)$, *is satisfied. For* $\mathbf{q}_2$, *the symmetry implies that* $\nabla_{\mathbf{q}_2} U = \mathbf{F}_2 = 0$. *Lemma 1 says that* $\nabla_{\mathbf{q}_2} I = 0$, *so* (4.1) *holds for* $i = 2$. *For* $\mathbf{q}_1$ *and* $\mathbf{q}_3$, *the symmetry implies that* $|\mathbf{F}_1| = |\mathbf{F}_3|$ *and they are collinear with* $\nabla_{\mathbf{q}_i} \sinh^2 d(\mathbf{q}_i, A)$. *Thus the central configuration equation*

$\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} m_i \sinh^2 d(\mathbf{q}_i, A)$ *holds also for* $i = 1, 3$.

Using the formula given in Proposition 8, we obtain

$$d_{12} = d_{23}, \quad r_2^2 = 0, \quad r_1^2 = r_3^2 = r^2,$$

$$x_1x_2 + y_1y_2 = x_2x_3 + y_2y_3 = 0, \quad x_1x_3 + y_1y_3 = -r^2.$$

$$\cosh d_{12} = w, \quad \sin^3 d_{12} = r^3, \quad \cosh d_{13} = w^2 + r^2, \quad \sinh^3 d_{13} = 8r^3w^3,$$

which yield

$$\lambda = -\frac{m}{2w^2}\left(\frac{\cosh d_{12}}{\sinh^3 d_{12}} + \frac{1 + \cosh d_{13}}{\sinh^3 d_{13}}\right) = -\frac{m}{2r^3}\left(\frac{1}{w} + \frac{1}{4w^3}\right).$$

6.1.1 The Extension of Shub's Lemma

Let us extend Shub's Lemma [80] to spaces of constant curvature. In its original form, this lemma shows that there are no central configurations near the singularity set Δ in S_c for given masses in the Newtonian N-body problem.

Recall that $S_c = \{\mathbf{q} \in (\mathbb{H}^2)^N \setminus \Delta | I(\mathbf{q}) = c\}$. Let X be a point in Δ, $X = (\mathbf{q}'_1, ..., \mathbf{q}'_{k_1}, \mathbf{q}'_{k_1+1}, ..., \mathbf{q}'_{k_2}, ..., \mathbf{q}'_*, ..., \mathbf{q}'_N)$, where $\mathbf{q}'_1 = ... = \mathbf{q}'_{k_1}$, $\mathbf{q}'_{k_1+1} = ... = \mathbf{q}'_{k_2}$, $\mathbf{q}'_* = ... = \mathbf{q}'_N$, $\mathbf{q}'_i = (x'_i, y'_i, w'_i)$. Assume that $\sum_{i=1}^N m_i(x_i'^2 + y_i'^2) = c$, i.e., X belongs to $I^{-1}(c)$. Our purpose is to show that there is some neighbourhood $\mathcal{U}$ of X in $I^{-1}(c)$, such that there are no central configurations in $S_c \cap \mathcal{U}$. We represent a point in such a neighbourhood of X by $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, ...\mathbf{q}_N)$,

$$\mathbf{q}_i = (x'_i + \delta_{i1}, y'_i + \delta_{i2}, w'_i + \delta_{i3}),$$

with $\mathbf{q}_i \in \mathbb{H}^2$, $\mathbf{q} \notin \Delta$, and $I(\mathbf{q}) = c$. The configuration X defines a partition of the bodies into *clusters*, where m_i, m_j are in the same cluster if there is an l such that $k_l < i < j \leq k_{l+1}$ or $* \leq i < j \leq N$.

We can assume that there are at least two clusters away from $(0, 0, 1)$. If there is no such cluster, then $\mathbf{q}'_1 = ... = \mathbf{q}'_N = (0, 0, 1)$. This contradicts with the fact that $\sum_{i=1}^N m_i(x_i'^2 + y_i'^2) = c$. If there is only one such cluster, say, $(x'_1, y'_1, w'_1) \neq (0, 0, 1)$, and $\mathbf{q}'_{k_1+1} = ... = \mathbf{q}'_N = (0, 0, 1)$, then equation (4.2), $\sum_{i=1}^N m_i x_i w_i = 0$, can't be satisfied for $\mathbf{q}$ sufficiently close to X, since

$$\sum_{i=1}^{N} m_i x_i w_i \approx \sum_{i=1}^{k_1} m_i x_i' w_i' + \sum_{i=k_1+1}^{N} m_i x_i' w_i' = x_1' w_1' (\sum_{i=1}^{k_1} m_i).$$

Thus $\mathbf{q}$ can't be a central configuration.

Proposition 16. *For fixed masses $m_1, ...m_N$ on $\mathbb{H}^2$, there is a neighbourhood of Δ in S_c that contains no central configurations.*

Proof. We need to show that the force function $U = \sum_{1 \leq i < j \leq N} m_i m_j \coth d_{ij}$ restricted to the $(2N-1)$-sphere $I^{-1}(c)$ has no critical points in a neighbourhood of $X \in \Delta \cap I^{-1}(c)$. Let X be the point as defined above. We have shown that we can assume that there are at least two clusters away from $(0,0,1)$. Thus, we may require $k_1 \geq 2$, $\mathbf{q}_N' \neq \mathbf{q}_1'$, and $\mathbf{q}_N' \neq (0,0,1)$.

We use (x, y) as the local coordinates of $\mathbb{H}^2$ and let $\bar{\mathbf{q}}_i = (x_i, y_i)$. We proceed as follows: It is easy to find that the differential of U is

$$dU = \sum_{i=1}^{N} \frac{\partial U}{\partial x_i} dx_i + \frac{\partial U}{\partial y_i} dy_i = \sum_{i=1}^{N} \left(\sum_{j=1, j\neq i}^{N} m_i m_j \frac{x_j - \frac{w_j}{w_i} x_i}{\sinh^3 d_{ij}} dx_i + m_i m_j \frac{y_j - \frac{w_j}{w_i} y_i}{\sinh^3 d_{ij}} dy_i \right).$$

For a point $\mathbf{q} \in S_c$ that approaches X, we will pick a bounded vector $\mathbf{v}(\mathbf{q}) = (\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_N) \in T_{\mathbf{q}} S_c$ such that

$$dU\mathbf{v} \to -\infty.$$

If this is done, then we can conclude that any point $\mathbf{q} \in S_c$ sufficiently close to X can't be a critical point of $U|_{S_c}$. Let $\mathbf{v}_i = (v_{i1}, v_{i2}) = v_{i1} \frac{\partial}{\partial x_i} + v_{i2} \frac{\partial}{\partial y_i}$. We do this by letting

1. $\mathbf{v}_i = w_i^2 \bar{\mathbf{q}}_i = w_i^2 (x_i, y_i), 1 \leq i \leq k_1,$
2. $\mathbf{v}_{k_1+1} = \mathbf{v}_{k_1+2} = ... = \mathbf{v}_{*-1} = 0,$
3. $\mathbf{v}_i = w_i \mathbf{v}_0, * \leq i \leq N,$

where $v_{01}(\sum_{i=*}^{N} m_i w_i x_i) + v_{02}(\sum_{i=*}^{N} m_i w_i y_i) = -\sum_{1}^{k_1} m_i w_i^2 r_i^2$. Note that this is a linear equation of (v_{01}, v_{02}) for any given $\mathbf{q}$ and that the coefficients have the property

$$(\sum_{i=*}^{N} m_i w_i x_i, \sum_{i=*}^{N} m_i w_i y_i) \approx (\sum_{i=*}^{N} m_i)(w_N' x_N', w_N' y_N') \neq (0,0),$$

for $\mathbf{q}$ sufficiently close to X. So we can always find such a $\mathbf{v}_0 = (v_{01}, v_{02})$. The vector $\mathbf{v}$ constructed in this way is bounded and it is in $T_{\mathbf{q}}S_c$, since

$$dI\mathbf{v} = \sum_{i=1}^{N} m_i(x_i v_{i1} + y_i v_{i2}) = \sum_{1}^{k_1} m_i w_i^2 r_i^2 + v_{01}(\sum_{i=*}^{N} m_i w_i x_i) + v_{02}(\sum_{i=*}^{N} m_i w_i y_i) = 0.$$

Let us show that $dU\mathbf{v} \to -\infty$ for $\mathbf{q}$ sufficiently close to X. Note that we can write $(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) = \sum_{j=1, j\neq i}^{N} \frac{m_i m_j}{\sinh^3 d_{ij}} (\bar{\mathbf{q}}_j - \frac{w_j}{w_i}\bar{\mathbf{q}}_i)$. Let $\cdot$ be the inner product in $\mathbb{R}^2$. Then

$$dU\mathbf{v} = \sum_{i=1}^{k_1}(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot \mathbf{v}_i + \sum_{i=k_1+1}^{*-1}(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot \mathbf{v}_i + \sum_{i=*}^{N}(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot \mathbf{v}_i.$$

The first sum goes to $-\infty$ when $\mathbf{q} \to X$. Explicitly, $\sum_{i=1}^{k_1}(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot \mathbf{v}_i$ is

$$\begin{aligned}
&\sum_{i=1}^{k_1}\sum_{j=1, j\neq i}^{k_1} \frac{m_i m_j}{\sinh^3 d_{ij}}(\bar{\mathbf{q}}_j - \frac{w_j}{w_i}\bar{\mathbf{q}}_i) \cdot \mathbf{v}_i + \sum_{i=1}^{k_1}\sum_{j=k_1+1}^{N} \frac{m_i m_j}{\sinh^3 d_{ij}}(\bar{\mathbf{q}}_j - \frac{w_j}{w_i}\bar{\mathbf{q}}_i) \cdot \mathbf{v}_i \\
=& \sum_{1\le i<j\le k_1} \frac{m_i m_j}{\sinh^3 d_{ij}} \left((\bar{\mathbf{q}}_j - \frac{w_j}{w_i}\bar{\mathbf{q}}_i) \cdot \mathbf{v}_i + (\bar{\mathbf{q}}_i - \frac{w_i}{w_j}\bar{\mathbf{q}}_j) \cdot \mathbf{v}_j \right) + O(1) \\
=& \sum_{1\le i<j\le k_1} \frac{m_i m_j}{\sinh^3 d_{ij}} \left((w_i^2 + w_j^2)\bar{\mathbf{q}}_i \cdot \bar{\mathbf{q}}_j - w_i w_j(\bar{\mathbf{q}}_i \cdot \bar{\mathbf{q}}_i + \bar{\mathbf{q}}_j \cdot \bar{\mathbf{q}}_j) \right) + O(1),
\end{aligned}$$

where $O(1)$ means a bounded term. Note that $(w_i^2 + w_j^2)\bar{\mathbf{q}}_i \cdot \bar{\mathbf{q}}_j - w_i w_j(\bar{\mathbf{q}}_i \cdot \bar{\mathbf{q}}_i + \bar{\mathbf{q}}_j \cdot \bar{\mathbf{q}}_j)$ is

$$\begin{aligned}
&(x_i x_j + y_i y_j)(w_i^2 + w_j^2) - w_i w_j (r_i^2 + r_j^2) \\
&= (w_i w_j - \cosh d_{ij})(w_i^2 + w_j^2) - w_i w_j (w_i^2 + w_j^2 - 2) \\
&= -\cosh d_{ij}(w_i^2 + w_j^2) + (w_i^2 + w_j^2) - (w_i^2 + w_j^2) + 2 w_i w_j \\
&= (w_i^2 + w_j^2)(1 - \cosh d_{ij}) - (w_i - w_j)^2 \le (w_i^2 + w_j^2)(1 - \cosh d_{ij}).
\end{aligned}$$

When $\mathbf{q}$ approaches X, d_{ij} approaches 0 for $1 \le i < j \le k_1$. Thus

$$\begin{aligned}
\sum_{i=1}^{k_1}(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}) \cdot \mathbf{v}_i &\le \sum_{1\le i<j\le k_1} \frac{m_i m_j}{\sinh^3 d_{ij}}(w_i^2 + w_j^2)(1 - \cosh d_{ij}) + O(1) \\
&\le \sum_{1\le i<j\le k_1} -\frac{m_i m_j}{\sinh^3 d_{ij}}(w_1^2 + w_2^2)(\frac{d_{12}^2}{2} + \frac{d_{12}^4}{4!} + ...) \to -\infty.
\end{aligned}$$

The second sum is obviously zero. The third sum is bounded. Explicitly, $\sum_{i=*}^{N}(\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i})\cdot$ $\mathbf{v}_i$ is

$$\sum_{i=*}^{N}\sum_{j=1}^{*-1}\frac{m_i m_j}{\sinh^3 d_{ij}}(\bar{\mathbf{q}}_j - \frac{w_j}{w_i}\bar{\mathbf{q}}_i)\cdot\mathbf{v}_i + \sum_{*\le i<j\le N}\frac{m_i m_j}{\sinh^3 d_{ij}}[(\bar{\mathbf{q}}_j - \frac{w_j}{w_i}\bar{\mathbf{q}}_i)\cdot\mathbf{v}_i + (\bar{\mathbf{q}}_i - \frac{w_i}{w_j}\bar{\mathbf{q}}_j)\cdot\mathbf{v}_j]$$

$$= O(1) + \sum_{*\le i<j\le N}\frac{m_i m_j}{\sinh^3 d_{ij}}\,(w_i\bar{\mathbf{q}}_j\cdot\mathbf{v}_0 - w_j\bar{\mathbf{q}}_i\cdot\mathbf{v}_0 + w_j\bar{\mathbf{q}}_i\cdot\mathbf{v}_0 - w_i\bar{\mathbf{q}}_j\cdot\mathbf{v}_0)$$

$$= O(1).$$

We have shown that $dU\mathbf{v} \to -\infty$ for $\mathbf{q}$ sufficiently close to X, where $\mathbf{v} \in T_{\mathbf{q}}S_c$ is bounded. Thus $\mathbf{q}$ can't be a critical point of $U|_{S_c}$ when $\mathbf{q}$ is sufficiently close to X, a remark that completes the proof. □

6.2 Geodesic Central Configurations

In this section, we study geodesic central configurations. In the Newtonian N-body problem, Moulton's theorem completely solves the Wintner-Smale problem in the 1-dimensional case [64]. By a similar approach, we obtain the exact number of geodesic central configurations on S_c. We also discuss the inverse problem of geodesic central configurations. Recall that we have shown that any geodesic central configuration in $\mathbb{H}^2$ is equivalent to some central configuration on $\mathbb{H}^1$.

Theorem 13. *Given masses $m_1, \ldots, m_N > 0$ on $\mathbb{H}^1$, for any $c > 0$, there are exactly $N!/2$ geodesic central configurations on S_c, one for each ordering of the masses along $\mathbb{H}^1$.*

Proof. We follow the idea used to prove Moulton's theorem of the Newtonian N-body problem, [1, 62]: first show that the manifold S_c contains $N!$ components, each homemorphic to an $(N-1)$-dimensional disk; then prove that the critical points of $U|_{S_c}$ are local minima on these disks; then show that there is just one minimum on each such disk.

To prove that each ordering of the N masses corresponds to an open $(N-1)$-disk of S_c, we again use the homomorphism $\pi : \mathbb{R}^1 \to \mathbb{H}^1$, $\pi(x) = (x, w)$. The map induces a homemorphism from $(\mathbb{R}^1)^N$ to $(\mathbb{H}^1)^N$, which we still denote by π. Thus the function $I : (\mathbb{H}^1)^N \to \mathbb{R}$ induces a function $\bar{I} : (\mathbb{R}^1)^N \to \mathbb{R}$ by $\bar{I}(\mathbf{q}') = I(\pi\mathbf{q}')$, where $\mathbf{q}'$ is a point in $(\mathbb{R}^1)^N$. It is easy to see that $\bar{I}^{-1}(c)$ is homeomorphic to $I^{-1}(c)$.

Define the singularity set in $(\mathbb{R}^1)^N$ as $\Delta = \cup_{1\le i<j\le N}\{\mathbf{q}' \in (\mathbb{R}^1)^N : x_i = x_j\}$. Then π is a homomorphism between the singularity set in $(\mathbb{H}^1)^N$ and the singularity set of $(\mathbb{R}^1)^N$. Therefore, the set $S_c = \{\mathbf{q} \in (\mathbb{H}^1)^N \setminus \Delta | I(\mathbf{q}) = c\}$ is homemorphic to the set $\{\mathbf{q}' \in (\mathbb{R}^1)^N \setminus \Delta' | \bar{I}(\mathbf{q}') = c\}$, which consists of $N!$ open $(N-1)$-disks [1, 62] (each corresponding to an ordering). Thus we prove that each ordering of the N masses corresponds to an open $(N-1)$-disk of S_c.

Let $\mathcal{V}$ be one of these open disks. Its boundary belongs to Δ, where U approaches $+\infty$. Thus we conclude that there is at least one minimum in $\mathcal{V}$. This minimum gives a central configuration. We now show that every critical point $\mathbf{q}$ of $U|_{\mathcal{V}}$ is a local minimum by studying the Hessian $H(\mathbf{q})$ on $T_{\mathbf{q}}S_c$. We use $(\sinh\theta, \cosh\theta)$ as the coordinates of $\mathbb{H}^1$ and assume that the disk $\mathcal{V}$ corresponds to the ordering $\theta_1 < \theta_2 < ... < \theta_N$. By Lemma 5, on $T_{\mathbf{q}}S_c$, the Hessian is given by the $N \times N$ matrix $D^2U(\mathbf{q}) - \lambda D^2 I(\mathbf{q})$, where λ is the constant in the equation of the central configuration equation and it is negative by Proposition 8. Then

$$U = \sum_{1\le i<j\le N} m_i m_j \coth(\theta_j - \theta_i), \quad I = \sum_{i=1}^N m_i(x_i^2 + y_i^2) = \sum_{i=1}^N m_i \sinh^2\theta_i.$$

Straightforward computations show that $H(\mathbf{q}) = D^2U(\mathbf{q}) - \lambda D^2 I(\mathbf{q})$ is

$$2\begin{bmatrix} \sum\limits_{j=1,j\neq 1}^N \frac{m_1 m_j \cosh d_{1j}}{\sinh^3 d_{1j}} & -\frac{m_1 m_2 \cosh d_{12}}{\sinh^3 d_{12}} & \cdots & -\frac{m_1 m_N \cosh d_{1N}}{\sinh^3 d_{1N}} \\ -\frac{m_2 m_1 \cosh d_{12}}{\sinh^3 d_{12}} & \sum\limits_{j=1,j\neq 2}^N \frac{m_2 m_j \cosh d_{2j}}{\sinh^3 d_{2j}} & \cdots & -\frac{m_2 m_N \cosh d_{2N}}{\sinh^3 d_{2N}} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{m_1 m_N \cosh d_{1N}}{\sinh^3 d_{1N}} & \cdots & \cdots & \sum\limits_{j=1,j\neq N}^N \frac{m_N m_j \cosh d_{Nj}}{\sinh^3 d_{Nj}} \end{bmatrix}$$

$$- 2\lambda \begin{bmatrix} m_1 \cosh 2\theta_1 & 0 & \cdots & 0 \\ 0 & m_2 \cosh 2\theta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & m_N \cosh 2\theta_N \end{bmatrix}.$$

For the first matrix, take any nonzero vector $\mathbf{v} = (v_1, \cdots, v_N)$, not necessarily in

$T_{\mathbf{q}}S_c$. We have

$$\mathbf{v}^T(D^2U)\mathbf{v} = \sum_{i=1}^{N}\sum_{j=1}^{N}(D^2U)_{ij}v_iv_j = 2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_im_j\cosh d_{ij}}{\sinh^3 d_{ij}}v_i^2$$

$$-2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_im_j\cosh d_{ij}}{\sinh^3 d_{ij}}v_iv_j = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_im_j\cosh d_{ij}}{\sinh^3 d_{ij}}(v_i - v_j)^2 \geq 0.$$

The equality holds only for $\mathbf{v} = k(1, 1, ..., 1) = k\sum_{i=1}^{N}\frac{\partial}{\partial\theta_i}$. The second matrix is always positive-definite, which makes the sum of the two matrices positive-definite. Therefore, the Hessian is positive-definite on the subspace $T_{\mathbf{q}}S_c$ and the critical point $\mathbf{q}$ is a local minimum of $U|_{\mathcal{V}}$.

Now we show that such a minimum of $U|_{\mathcal{V}}$ is unique. Assume that there are two such minima. Connect these two points with a continuous family of curves. As the two ends are local minima, there must be a local maximum on each curve. Then the minimum of all these maxima must be a saddle point of $U|_{\mathcal{V}}$, in contradiction with the positive-definiteness of the Hessian.

Note that a 180° rotation in the xy-plane changes the ordering, which means that we counted each case twice, so there are exactly $N!/2$ classes of geodesic central configurations, a remark that completes the proof. □

We use $(\sinh\theta, \cosh\theta)$ as the coordinates of $\mathbb{H}^1$. Now let us talk about the inverse problem: suppose λ and the N distinct points $\theta_1, \theta_2, ..., \theta_N$ are given, and consider the problem of determining $\mathbf{m} \in \mathbb{R}_+^N$ so that the geodesic central configuration exists. The following discussion is similar to the discussion in [64]. There will be no loss of generality by fixing the ordering such that $\theta_1 < ... < \theta_N$. Then

$$U = \sum_{1\leq i<j\leq N} m_im_j\coth(\theta_j - \theta_i), \quad I = \sum_{i=1}^{N} m_i(x_i^2 + y_i^2) = \sum_{i=1}^{N} m_i\sinh^2\theta_i.$$

Thus the central configuration is given by the equations

$$
\begin{array}{lllll}
0+\dfrac{m_2}{\sinh^2 d_{12}} & +...+\dfrac{m_{N-1}}{\sinh^2 d_{1,N-1}} & +\dfrac{m_N}{\sinh^2 d_{1N}} & = \lambda \sinh 2\theta_1, \\
-\dfrac{m_1}{\sinh^2 d_{12}}+0 & +...+\dfrac{m_{N-1}}{\sinh^2 d_{2,N-1}} & +\dfrac{m_N}{\sinh^2 d_{2N}} & = \lambda \sinh 2\theta_2, \\
\vdots & & \vdots & \\
-\dfrac{m_1}{\sinh^2 d_{N-1,1}}-\dfrac{m_2}{\sinh^2 d_{N-1,2}} & +...-0 & +\dfrac{m_N}{\sinh^2 d_{N-1,N}} & = \lambda \sinh 2\theta_{N-1}, \\
-\dfrac{m_1}{\sinh^2 d_{N1}}-\dfrac{m_2}{\sinh^2 d_{N2}} & +...-\dfrac{m_{N-1}}{\sinh^2 d_{N,N-1}} & +0 & = \lambda \sinh 2\theta_N.
\end{array}
$$

This is a linear system on the masses. The solution space of the linear system depends on the determinant of D, the coefficient matrix. Note that D is an anti-symmetric matrix.

When N is even, $\det D$ is the square of an associate Pfaffian [70], and in general is not zero. Therefore, generally, for given $\lambda, \theta_1, ..., \theta_N$, there always exists a unique solution of masses in $\mathbb{R}^N$. However, these masses are not necessarily all positive.

When N is odd, then $\det D = 0$. But the minors of the the main diagonal entries are anti-symmetric matrices of even order, thus generally they are non-zero. So the rank of the coefficient matrix is generally $N-1$, which implies that the vector on the right hand side $(\sinh 2\theta_1, ..., \sinh 2\theta_N)$ has to satisfy one linear relation. Recall that for ordinary central configurations we have (4.2), which is

$$\sum_{i=1}^N m_i x_i w_i = \frac{1}{2}\sum_{i=1}^N m_i \sinh 2\theta_i = 0.$$

Adding this equation to the system, we obtain a homogeneous linear system of $(\lambda, m_1, ..., m_N)$ with the coefficient matrix

$$
E := \begin{bmatrix}
0 & \sinh 2\theta_1 & \sinh 2\theta_2 & ... & \sinh 2\theta_N \\
-\sinh 2\theta_1 & 0 & \frac{1}{\sinh^2 d_{12}} & ... & \frac{1}{\sinh^2 d_{1N}} \\
-\sinh 2\theta_2 & -\frac{1}{\sinh^2 d_{12}} & 0 & ... & \frac{1}{\sinh^2 d_{2N}} \\
\vdots & & & & \vdots \\
-\sinh 2\theta_N & \frac{1}{\sinh^2 d_{N1}} & \frac{1}{\sinh^2 d_{N2}} & ... & \frac{1}{\sinh^2 d_{N,N-1}}
\end{bmatrix},
$$

which is anti-symmetric and of even order. Thus the determinant, the square of the associate Pfaffian, must be zero such that the system has solutions in $\mathbb{R}^{N+1}$. The Pfaffian is a polynomial on entries of E, which is determined by $\theta_1, \theta_2, ..., \theta_N$. Thus when N is odd, the configuration itself has to satisfy one equation for the existence of a geodesic central configuration.

6.3 A Lower Bound for the Number of Central Configurations by Morse Theory

While it is generally not easy to study the index of central configurations analytically, it turns out that each geodesic central configuration of N bodies has index $N-2$. Then we describe how to use Morse theory to prove the existence of central configurations. This is a generalization of the result of Palmore [68] in the planar Newtonian N-body problem.

Now we study the Hessian of the $N!/2$ geodesic central configurations on $S_c = \{\mathbf{q} \in (\mathbb{H}^2)^N \setminus \Delta | I(\mathbf{q}) = c\}$. For our purpose, we use the coordinate system of $\mathbb{H}^2$,

$$(x, y, w) = (\sinh\theta, \cosh\theta\sinh\varphi, \cosh\theta\cosh\varphi),\ \ \theta, \varphi \in \mathbb{R}.$$

Then $\mathbb{H}^1$ corresponds to $\varphi = 0$. Restricted to $\varphi = 0$, this system is identical with the one we used in the proof of the previous theorem, especially, $d_{ij} = |\theta_i - \theta_j|$. This system gives a homemorphism between $\mathbb{R}^2 = (\theta, \varphi)$ and $\mathbb{H}^2$. Then

$$U = \sum_{1\leq i<j\leq N} m_i m_j \coth d_{ij}, \quad I = \sum_{i=1}^{N} m_i(x_i^2 + y_i^2) = \sum_{i=1}^{N} m_i(\sinh^2\theta_i + \cosh^2\theta_i \sinh^2\varphi_i).$$

Order the coordinates as $(\theta_1, ..., \theta_N, \varphi_1, ..., \varphi_N)$. For a geodesic central configuration $(\theta_1, ..., \theta_N, 0, ..., 0)$, direct computations lead to

$$H(\mathbf{q}) = D^2U - \lambda D^2 I = \begin{bmatrix} \frac{\partial^2 U}{\partial\theta_i\partial\theta_j} & 0 \\ 0 & \frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j} \end{bmatrix} - \lambda \begin{bmatrix} \frac{\partial^2 I}{\partial\theta_i\partial\theta_j} & 0 \\ 0 & \frac{\partial^2 I}{\partial\varphi_i\partial\varphi_j} \end{bmatrix}.$$

Thus it is enough to study the upper left block $H_\theta := [\frac{\partial^2 U}{\partial\theta_i\partial\theta_j} - \lambda\frac{\partial^2 I}{\partial\theta_i\partial\theta_j}]_{N\times N}$ and the lower right block $H_\varphi : [\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j} - \lambda\frac{\partial^2 I}{\partial\varphi_i\partial\varphi_j}]_{N\times N}$.

Obviously, the upper left block H_θ is the same as the matrix obtained in the proof of the previous theorem. We have proved that this matrix is positive-definite. More precisely, this block acts on $T_{\mathbf{q}}(\mathbb{H}^1)^N$, which is spanned by the N vectors $\frac{\partial}{\partial\theta_i}$, ..., $\frac{\partial}{\partial\theta_N}$. In this N-dimensional space, there is a 1-dimensional subspace that is not in $T_{\mathbf{q}}S_c$. It is generated by

$$\nabla I = \sum_{i=1}^{N} m_i \sinh 2\theta_i \frac{\partial}{\partial\theta_i},$$

which is actually orthogonal to S_c. Thus we see that there is an $(N-1)$-dimensional subspace (in the geodesic directions) of $T_{\mathbf{q}}S_c$ on which $H(\mathbf{q})$ is positive-definite.

The lower right block acts on the complementary subspace spanned by the N vectors $\frac{\partial}{\partial\varphi_i}$, ..., $\frac{\partial}{\partial\varphi_N}$. At the geodesic central configuration $\mathbf{q}$, it is easy to see that $dI\frac{\partial}{\partial\varphi_i} = 0$ for each i since $\varphi_i = 0$. Thus the N-dimensional subspace belongs to $T_{\mathbf{q}}S_c$. Explicitly, this block is

$$H_\varphi = \begin{bmatrix} \sum_{j=1, j\neq 1}^{N} \frac{-m_1 m_j \cosh\theta_1 \cosh\theta_j}{\sinh^3 d_{1j}} & \frac{m_1 m_2 \cosh\theta_1 \cosh\theta_2}{\sinh^3 d_{12}} & \cdots & \frac{m_1 m_N \cosh\theta_1 \cosh\theta_N}{\sinh^3 d_{1N}} \\ \frac{m_2 m_1 \cosh\theta_1 \cosh\theta_2}{\sinh^3 d_{12}} & \sum_{j=1, j\neq 2}^{N} \frac{-m_2 m_j \cosh\theta_2 \cosh\theta_j}{\sinh^3 d_{2j}} & \cdots & \frac{m_2 m_N \cosh\theta_2 \cosh\theta_N}{\sinh^3 d_{2N}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{m_1 m_N \cosh\theta_1 \cosh\theta_N}{\sinh^3 d_{1N}} & \cdots & \cdots & \sum_{j=1, j\neq N}^{N} \frac{-m_N m_j \cosh\theta_N \cosh\theta_j}{\sinh^3 d_{Nj}} \end{bmatrix}$$
$$- 2\lambda \begin{bmatrix} m_1 \cosh^2\theta_1 & 0 & \cdots & 0 \\ 0 & m_2 \cosh^2\theta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & m_N \cosh^2\theta_N \end{bmatrix}.$$

First, notice that there is a null vector of $H(\mathbf{q})$ in this subspace. Proposition 13 shows, by the $SO(2)$ symmetry, that there is at least one null vector for the Hessian of any central configuration on $\mathbb{H}^2$. In the xyw-coordinates, obviously, the null vector is

$$\mathbf{v} = \sum_{i=1}^{N} -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} = \sum_{i=1}^{N} x_i \frac{\partial}{\partial y_i}.$$

Expressed in the θ, φ coordinates, it is in the subspace spanned by the N vectors $\frac{\partial}{\partial\varphi_i}$, ..., $\frac{\partial}{\partial\varphi_N}$, and

$$\mathbf{v} = \sum_{i=1}^{N} -\frac{\sinh\theta_i}{\cosh\theta_i} \frac{\partial}{\partial\varphi_i} = -\left(0, ...0, \frac{\sinh\theta_1}{\cosh\theta_1}, ..., \frac{\sinh\theta_N}{\cosh\theta_N}\right). \tag{6.2}$$

Thus $\mathbf{v}$ is a null vector of H_φ.

We will need the following inequalities on distance.

Proposition 17. *On $\mathbb{H}^1 = (\sinh\theta, \cosh\theta)$, for N distinct points with $\theta_1 < \theta_2 < \cdots < \theta_N$, we have the following inequalities:*

1. *if $k < i < j$, then $\frac{1}{\sinh^3(\theta_j-\theta_k)\cosh\theta_j} < \frac{1}{\sinh^3(\theta_i-\theta_k)\cosh\theta_i}$;*
2. *if $i < j < k$, then $\frac{1}{\sinh^3(\theta_k-\theta_j)\cosh\theta_j} > \frac{1}{\sinh^3(\theta_k-\theta_i)\cosh\theta_i}$.*

Proof. If $k < i < j$, what we need to show is

$$\sinh^3(\theta_j - \theta_k)\cosh\theta_j - \sinh^3(\theta_i - \theta_k)\cosh\theta_i > 0.$$

View this as a function of θ_j, i.e.,

$$f(x) = \sinh^3(x - \theta_k)\cosh x - \sinh^3(\theta_i - \theta_k)\cosh\theta_i,\ x > \theta_i.$$

Then $f(\theta_i) = 0$, and

$$\begin{aligned} f'(x) &= \sinh^3(x-\theta_k)\sinh x + 3\sinh^2(x-\theta_k)\cosh(x-\theta_k)\cosh x \\ &= \sinh^2(x-\theta_k)\left(\sinh(x-\theta_k)\sinh x + 3\cosh(x-\theta_k)\cosh x\right) \\ &= \sinh^2(x-\theta_k)\cosh(2x-\theta_k) + 2\sinh^2(x-\theta_k)\cosh(x-\theta_k)\cosh x > 0. \end{aligned}$$

So for $k < i < j$, we always have $\frac{1}{\sinh^3(\theta_j-\theta_k)\cosh\theta_j} < \frac{1}{\sinh^3(\theta_j-\theta_k)\cosh\theta_i}$. The proof of the other inequality is similar. □

The following theorem extends the result on the indices of geodesic central configurations in $\mathbb{R}^3$ [52]. As done in [67], the essential idea of the proof is due to Conley.

Theorem 14. *Every geodesic central configuration $\mathbf{q}$ on $\mathbb{H}^2$ is nondegenerate with* $\mathrm{null}(\mathbf{q}) = 1$ *and* $\mathrm{ind}(\mathbf{q}) = N-2$*. In the geodesic tangent directions, which are $(N-1)$-dimensional, $H(\mathbf{q})$ is positive definite, while in the normal directions it is positive on a 1-dimensional subspace, zero on another 1-dimensional subspace, negative-definite on the rest $(N-2)$-dimensional subspaces.*

Proof. We first simplify the form of the matrix H_φ. Introduce the following three

$N \times N$ matrices:

$$
\begin{aligned}
C &:= \operatorname{diag}\{\cosh\theta_1, \cdots, \cosh\theta_N\},\\
M &:= \operatorname{diag}\{m_1, \cdots, m_N\},\\
A &:= \begin{bmatrix}
\sum_{j=1, j\neq 1}^{N} \frac{-m_j \cosh\theta_j}{cosh\theta_1 \sinh^3 d_{1j}} & \frac{m_2}{\sinh^3 d_{12}} & \cdots & \frac{m_N}{\sinh^3 d_{1N}}\\
\frac{m_1}{\sinh^3 d_{12}} & \sum_{j=1, j\neq 2}^{N} \frac{-m_j \cosh\theta_j}{\cosh\theta_2 \sinh^3 d_{2j}} & \cdots & \frac{m_N}{\sinh^3 d_{2N}}\\
\cdots & \cdots & \cdots & \cdots\\
\frac{m_1}{\sinh^3 d_{1N}} & \cdots & \cdots & \sum_{j=1, j\neq N}^{N} \frac{-m_j \cosh\theta_j}{\cosh\theta_N \sinh^3 d_{Nj}}
\end{bmatrix}.
\end{aligned}
$$

Then it is easy to check that $H_\varphi = CM(A - 2\lambda)C$. Thus to study the eigenvalues of H_φ is equivalent to studying the eigenvalues of $A - 2\lambda$. Precisely, note that $M^{-\frac{1}{2}}$ and $C^{-\frac{1}{2}}$ are well defined. Then H_φ is congruent to $G_1 := (C^{-\frac{1}{2}})^T H_\varphi C^{-\frac{1}{2}}$, which is similar to $C^{\frac{1}{2}} G_1 C^{-\frac{1}{2}} = H_\varphi C^{-1} = CM(A - 2\lambda)$. Similarly, we can get rid of CM. By Sylvester's law of inertia [37], we have

$$n_0(H_\varphi) = n_0(A - 2\lambda), \quad n_-(H_\varphi) = n_-(A - 2\lambda), \; n_+(H_\varphi) = n_+(A - 2\lambda),$$

where $n_0(*)$ is the number of zero eigenvalues of matrix $*$, $n_-(*)$ the number of negative eigenvalues, and $n_+(*)$ the number of positive eigenvalues.

To study the eigenvalues of $A - 2\lambda$, it is enough to study the eigenvalues of A and compare them with the negative number 2λ. First, notice that there are two obvious eigenvectors of A:

$$
\begin{aligned}
\mathbf{v}_1 &= (\cosh\theta_1, \cdots, \cosh\theta_N), & A\mathbf{v}_1 &= 0\mathbf{v}_1,\\
\mathbf{v}_2 &= (\sinh\theta_1, \cdots, \sinh\theta_N), & A\mathbf{v}_2 &= 2\lambda\mathbf{v}_2.
\end{aligned}
$$

The first vector can be obtained by inspecting the matrix A. The second vector $\mathbf{v}_2$ equals $-C\mathbf{v}$, where $\mathbf{v} = -(\frac{\sinh\theta_1}{\cosh\theta_1}, ..., \frac{\sinh\theta_N}{\cosh\theta_N})$ is the null vector of $\mathbb{H}_\varphi$, see (6.2). Since $H_\varphi \mathbf{v} = CM(A - 2\lambda)C\mathbf{v} = 0$, we have

$$AC\mathbf{v} = 2\lambda C\mathbf{v}, \; \Rightarrow A\mathbf{v}_2 = 2\lambda\mathbf{v}_2.$$

Now we employ the idea of Conley to show that all other eigenvalues of A are smaller than 2λ. The idea is to consider the linear system in $\mathbb{R}^N$:

$$\dot{\mathbf{u}} = A\mathbf{u}, \mathbf{u} = (u_1, \cdots, u_N) \in \mathbb{R}^N.$$

Conley observed that to show that all other eigenvalues of A are smaller than 2λ is equivalent to showing that, in the flow on $\mathbb{R}^N$, the line determined by $\mathbf{v}_2$ is an attractor. It is enough to find a "cone", K, around $\mathbf{v}_2$ that is carried strictly inside itself by the flow (except for the origin).

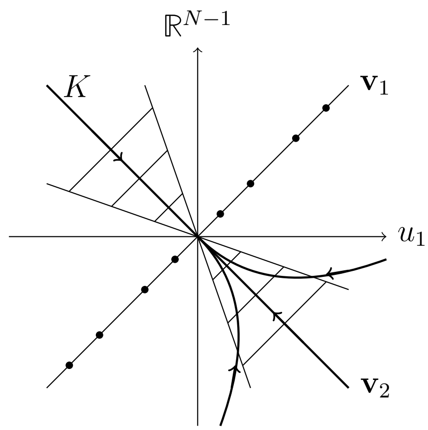

Figure 6.3: The linear flow in $\mathbb{R}^N$.

Suppose that the ordering of the geodesic central configuration $\mathbf{q}$ is $\theta_1 < \theta_2 < \cdots < \theta_N$. Then let

$$K = \left\{ \mathbf{u} \in \mathbb{R}^N | \sum_{i=1}^{N} m_i \cosh\theta_i u_i = 0, \frac{u_1}{\cosh\theta_1} \leq \frac{u_2}{\cosh\theta_2} \leq \cdots \leq \frac{u_N}{\cosh\theta_N} \right\}.$$

Endow $\mathbb{R}^N$ with the metric by the matrix M. Then the cone K is in the $(N-1)$-dimensional subspace perpendicular to $\mathbf{v}_1$. Note that $\mathbf{v}_2 \in K$. First, by equation (4.2), we have $0 = \sum_{i=1}^N m_i x_i w_i = \sum_{i=1}^N m_i \sinh\theta_i \cosh\theta_i$. Second, since $\tanh\theta$ is an increasing function, we have $\frac{\sinh\theta_1}{\cosh\theta_1} \leq \frac{\sinh\theta_2}{\cosh\theta_2} \leq \cdots \leq \frac{\sinh\theta_N}{\cosh\theta_N}$. The boundary ∂K consists of points for which one or more equalities hold. However, except for the origin, at least one inequality must hold (otherwise $\mathbf{u} = k(\cosh\theta_1, \cdots, \cosh\theta_N) = k\mathbf{v}_1$).

Consider a boundary point with

$$\frac{u_1}{\cosh\theta_1} \leq \cdots \frac{u_i}{\cosh\theta_i} = \cdots = \frac{u_j}{\cosh\theta_j} \leq \cdots \leq \frac{u_N}{\cosh\theta_N}.$$

To prove that at this point the flow is pointing inwards, see Figure 6.3, we need to show $\frac{\dot{u}_j}{\cosh\theta_j} - \frac{\dot{u}_i}{\cosh\theta_i} > 0$. Direct computation shows that $\frac{\dot{u}_j}{\cosh\theta_j} - \frac{\dot{u}_i}{\cosh\theta_i}$ is

$$\sum_{k=1,k\neq j}^{N} \frac{m_k}{\cosh\theta_j \sinh^3 d_{kj}}\left(u_k - \frac{u_j\cosh\theta_k}{\cosh\theta_j}\right) - \sum_{k=1,k\neq i}^{N} \frac{m_k}{\cosh\theta_i \sinh^3 d_{ki}}\left(u_k - \frac{u_i\cosh\theta_k}{\cosh\theta_i}\right)$$
$$= \sum_{k=1,k\neq i,j}^{N} m_k\left(\frac{u_k}{\sinh^3 d_{kj}\cosh\theta_j} - \frac{u_j\cosh\theta_k}{\cosh^2\theta_j} - \frac{u_k}{\sinh^3 d_{ki}\cosh\theta_i} + \frac{u_i\cosh\theta_k}{\cosh^2\theta_i}\right)$$
$$+ \frac{m_i}{\sinh^3 d_{ij}\cosh\theta_j}\left(u_i - \frac{u_j\cosh\theta_i}{\cosh\theta_j}\right) - \frac{m_j}{\sinh^3 d_{ij}\cosh\theta_i}\left(u_j - \frac{u_i\cosh\theta_j}{\cosh\theta_i}\right).$$

Since $\frac{u_i}{\cosh\theta_i} = \frac{u_j}{\cosh\theta_j}$, the last two terms are zero, and the first part can be nicely written as

$$\sum_{k=1,k\neq i,j}^{N} m_k\left(u_k - \frac{u_i\cosh\theta_k}{\cosh\theta_i}\right)\left(\frac{1}{\sinh^3 d_{kj}\cosh\theta_j} - \frac{1}{\sinh^3 d_{ki}\cosh\theta_i}\right).$$

Every term in this sum is non-negative:

1. If $k < i$, then $u_k - \frac{u_i\cosh\theta_k}{\cosh\theta_i} \leq 0$ and $\frac{1}{\sinh^3 d_{kj}\cosh\theta_j} - \frac{1}{\sinh^3 d_{ki}\cosh\theta_i} < 0$ by Proposition 17.

2. If $i \leq k \leq j$, then $u_k - \frac{u_i\cosh\theta_k}{\cosh\theta_i} = 0$.

3. If $i < k$, then $u_k - \frac{u_i\cosh\theta_k}{\cosh\theta_i} \geq 0$ and $\frac{1}{\sinh^3 d_{kj}\cosh\theta_j} - \frac{1}{\sinh^3 d_{ki}\cosh\theta_i} > 0$ by Proposition 17.

Moreover, at least one term is strictly positive since at least one inequality in the definition of the cone must hold. Thus we have proved $\frac{\dot{u}_j}{\cosh\theta_j} - \frac{\dot{u}_i}{\cosh\theta_i} > 0$ on ∂K and the boundary point moves into the interior of the cone as required. This proves that all the other eigenvalues of A are smaller than 2λ, thus the N eigenvalues of $A - 2\lambda$ are $-2\lambda > 0, 0, \lambda_3 < 0, \cdots, \lambda_N < 0$. Therefore we get

$$n_0(H_\varphi) = n_0(A-2\lambda) = 1,\ \ n_-(H_\varphi) = n_-(A-2\lambda) = N-2,\ n_+(H_\varphi) = n_+(A-2\lambda) = 1.$$

This remark completes the proof. □

We can now extend the result of Smale [85] and Palmore [68] on the number of central configurations in the Newtonian N-body problem. We will make use of Morse theory and assume that, for generic masses, the central configurations has nullity 1, the minimum value compatible with the symmetry, see Proposition 13.

Recall that a central configuration on $\mathbb{H}^2$ is a critical point of $U : S_c \to \mathbb{R}$ and that $S_c = \{\mathbf{q} \in (\mathbb{H}^2)^N \setminus \Delta | I(\mathbf{q}) = c\}$. The group $SO(2)$ acts freely on S_c, which reduces U as a smooth function on the quotient manifold

$$\mathcal{M} = (S_c)/SO(2).$$

A *Morse function* is such a function that all its critical points are nondegenerate. Thus assuming that the all central configurations for generic masses are nondegenerate is the same as assuming that all critical points of $U : \mathcal{M} \to \mathbb{R}$ are nondegenerate, that is, $U : \mathcal{M} \to \mathbb{R}$ is a Morse function. Recall that the critical points of $U : \mathcal{M} \to \mathbb{R}$ correspond to classes of central configurations in a 1-1 manner. Thus the counting of central configurations is the same as the counting of critical points of U restricted to the quotient manifold.

The Morse inequality relates the indices of critical points of a Morse function on a smooth manifold to the topology of the manifold. The inequality is most easily expressed in terms of polynomial generating functions. Define a *Morse polynomial*

$$M(t) = \sum_k \gamma_k t^k, \quad \gamma_k = \text{ number of critical points of index } k$$

and the *Poincaré polynomial* $P(t) = \sum_k \beta_k t^k$, where β_k is the k-th Betti number of the manifold. By the Betti numbers, we mean the ranks of the homology groups $H_k(\mathcal{M}, \mathbb{R})$ with real coefficients. Then the Morse inequalities can be written as

$$M(t) = P(t) + (1 + t)R(t),$$

where $R(t)$ is some polynomial with non-negative integer coefficients [43]. Thus the Poincaré polynomial can be used to get an estimate of the number of critical points.

The above Morse inequality holds for a compact manifold. The manifold we are interested is S_c, a non-compact manifold. Recall that Proposition 16 shows that the critical point set of $U|_{S_c}$ is compact and that, near the boundary of S_c, U approaches $+\infty$. Thus we can restrict to a compact set of the form $\mathcal{K} = \{\mathbf{q} \in S_c : U(\mathbf{q}) \leq U_0\}$ for some sufficiently large U_0. Therefore, Morse inequality applies.

The following two proofs are almost the same as in the Newtonian N-body problem, but we reproduce them here for completeness. For more details, we refer the readers to the nice book [52].

Proposition 18. *For the curved N-body problem in $\mathbb{H}^2$, the Poincaré polynomial of $S_c/SO(2)$ is*

$$P(t) = (1+2t)...(1+(N-1)t).$$

Proof. Consider again the homemorphism $\pi : \mathbb{R}^2 \to \mathbb{H}^2$, $\pi(x, y) = (x, y, w)$. The map induces a homemorphism from $(\mathbb{R}^2)^N$ to $(\mathbb{H}^2)^N$, which we still denote by π. Thus the function $I : (\mathbb{H}^2)^N \to \mathbb{R}$ induces a function $\bar{I} : (\mathbb{R}^2)^N \to \mathbb{R}$ by $\bar{I}(\mathbf{q}') = I(\pi\mathbf{q}')$, where $\mathbf{q}'$ is a point in $(\mathbb{R}^2)^N$. Note that π is also a homemorphism between the singularity set Δ in $(\mathbb{H}^2)^N$ and $\Delta' = \cup_{1\le i<j\le N}\{\mathbf{q}' \in (\mathbb{R}^2)^N\,;\, \mathbf{q}'_i = \mathbf{q}'_j\}$. Then $S_c \simeq \bar{I}^{-1}(c) \setminus \Delta'$. Also π commutes with the $SO(2)$ action on $\mathbb{R}^2$. Thus we obtain

$$S_c/SO(2) = \mathcal{M} \simeq (\bar{I}^{-1}(c) \setminus \Delta')/SO(2).$$

Note that $(\mathbb{R}^2)^N \setminus\ \Delta' \simeq \mathbb{R}_+ \times (I^{-1}(c) \setminus \Delta')$. By induction [52], we can find that the Poincaré polynomial of $(\mathbb{R}^2)^N \setminus \Delta'$ is

$$(1+t)(1+2t)...(1+(N-1)t).$$

Now Künneth's theorem from algebraic topology shows that the Poincaré polynomial of a product space is the product of the Poincaré polynomials of the factors. Then the Poincaré polynomial of $\bar{I}^{-1}(c) \setminus \Delta'$ is the same as the Poincaré polynomial of $(\mathbb{R}^2)^N \setminus \Delta'$ since the Poincaré polynomial of $\mathbb{R}_+$ is 1.

There is a global cross-section to the $SO(2)$ action consisting of all the points in $\bar{I}^{-1}(c) \setminus \Delta'$, where the vector $\mathbf{q}'_1$ to $\mathbf{q}'_2$ is the direction of the positive first-coordinate axis [52]. By this, there is a homemorphism

$$[(\bar{I}^{-1}(c) \setminus \Delta')/SO(2)] \times SO(2) \simeq \bar{I}^{-1}(c) \setminus \Delta'.$$

Since the Poincaré polynomial of $SO(2) \simeq S^1$ is $(1 + t)$, we see that the Poincaré polynomial of $\mathcal{M}$ is

$$P(t) = (1+2t)...(1+(N-1)t)$$

by Künneth's theorem. This remark completes the proof. □

Proposition 19. *Suppose that all of the central configurations are nondegenerate for a certain choice of masses in the curved N-body problem in $\mathbb{H}^2$. Then on S_c $(c > 0)$,* *there are at least*

$$\frac{(3N-4)(N-1)!}{2}$$

central configurations, of which at least

$$\frac{(2N-4)(N-1)!}{2}$$

are non-geodesic.

Proof. Obviously, the compact subset $\mathcal{K} = \{\mathbf{q} \in S_c : U(\mathbf{q}) \leq U_0\}$ is homotopic to S_c, where U_0 is sufficiently large. Thus the Poincaré polynomial of $\mathcal{K}/SO(2)$ is $P(t) = (1+2t)...(1+(N-1)t)$. We have assumed that U is a Morse function on $\mathcal{K}/SO(2)$. We further assume that its Morse polynomial is

$$M(t) = \sum_{k=0}^{2N-2} \gamma_k t^k, \quad \gamma_k = \text{ number of critical points of index } k.$$

Thus the Morse inequality implies that there is some $R(t)$ with non-negative integer coefficients such that $M(t) = P(t) + (1+t)R(t)$. The simplest estimate is obtained by setting $t = 1$,

$$\sum_{k=0}^{2N-2} \gamma_k \geq P(1) \geq N!/2,$$

which predicts the existence of the $N!/2$ geodesic central configurations. Theorem 14 shows that the index of each geodesic central configuration in $\mathcal{K}/SO(2)$ is $N-2$. Then γ_{N-2} is at least $N!/2$. On the other hand, the coefficient of t^{N-2} in $P(t)$ is $2 \cdot 3...(N-1) = (N-1)!$.

Let $R(t) = \sum_k r_k t^k$. Then the coefficient of t^{N-2} in $(1+t)R(t)$ is $r_{N-2} + r_{N-3}$. So

$$r_{N-2} + r_{N-3} + (N-1)! \geq N!/2.$$

Setting $t = 1$ in the Morse inequality now gives

$$\sum_{k=0}^{2N-2} \gamma_k \geq N!/2 + 2(r_{N-2} + r_{N-3}) \geq 3N!/2 - 2(N-1)! = \frac{(3N-4)(N-1)!}{2}.$$

Subtracting $N!/2$ gives the non-geodesic estimate. This remark completes the proof. □

6.4 Central Configurations in the 3-Body Problem

In this section, we study the 3-body non-geodesic central configurations. We obtain one necessary condition, which implies that all equilateral triangle central configurations must have equal masses.

When $N = 3$, Proposition 19 gives that the lower bound of non-geodesic central configurations is 2.

Corollary 7. *For generic three masses on $\mathbb{H}^2$, there are at least two non-geodesic central configurations on S_c.*

Lemma 6. *Consider three masses $\mathbf{m} \in \mathbb{R}^3_+$ on $\mathbb{H}^2$ at $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\mathbf{q}_i = (x_i, y_i, w_i)$. Suppose that $\mathbf{q}$ is a non-geodesic central configuration. Then there is some $k \neq 0$ such that*

$$(w_1, w_2, w_3) = k(\sinh^3 d_{23}, \sinh^3 d_{13}, \sinh^3 d_{12}). \tag{6.3}$$

Proof. Let $\bar{\mathbf{q}}_i = (x_i, y_i)$. Then the equations (4.2) $\sum_{i=1}^3 m_i w_i x_i = \sum_{i=1}^3 m_i w_i y_i = 0$ can be written as

$$\sum_{i=1}^{3} m_i w_i \bar{\mathbf{q}}_i = 0.$$

This can be viewed as a linear relation of the three vectors $\bar{\mathbf{q}}_1$, $\bar{\mathbf{q}}_2$ and $\bar{\mathbf{q}}_3$. Then $\text{rank}(\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2, \bar{\mathbf{q}}_3) = 2$, since it is greater than one. If the rank is one, consider the 2-dimensional plane $V := \{(x, y, w) \in \mathbb{H}^2 | yx_1 = xy_1\}$. Then all three particles are on V, so the configuration is on a geodesic.

Thus no particle could be at the vertex $(0, 0, 1)$ and the second equation of (6.1) applies, which can be written as

$$0 = \sum_{j=1, j\neq i}^{3} \frac{m_j \det[\bar{\mathbf{q}}_j, \bar{\mathbf{q}}_i]}{\sinh^3 d_{ij}} = \det\left[\sum_{j=1, j\neq i}^{3} \frac{m_j \bar{\mathbf{q}}_j}{\sinh^3 d_{ij}}, \bar{\mathbf{q}}_i\right].$$

Then we obtain three more linear relations of $\bar{\mathbf{q}}_1$, $\bar{\mathbf{q}}_2$ and $\bar{\mathbf{q}}_3$: $\sum_{j=1,j\neq i}^3 \dfrac{m_j \bar{\mathbf{q}}_j}{\sinh^3 d_{ij}} + \theta_i \bar{\mathbf{q}}_i = 0$ for $i = 1, 2, 3$. Since $\text{rank}(\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2, \bar{\mathbf{q}}_3) = 2$, all linear relations are the same up to a constant. Consider the first two relations,

$$m_1 w_1 \bar{\mathbf{q}}_1 + m_2 w_2 \bar{\mathbf{q}}_2 + m_3 w_3 \bar{\mathbf{q}}_3 = 0, \quad \theta_1 \bar{\mathbf{q}}_1 + \frac{m_2 \bar{\mathbf{q}}_2}{\sinh^3 d_{12}} + \frac{m_3 \bar{\mathbf{q}}_3}{\sinh^3 d_{13}} = 0.$$

We obtain

$$\frac{w_2}{1/\sinh^3 d_{12}} = \frac{w_3}{1/\sinh^3 d_{13}}, \text{ or } \frac{w_2}{\sinh^3 d_{13}} = \frac{w_3}{\sinh^3 d_{12}}.$$

Similarly, we get $\frac{w_1}{\sinh^3 d_{23}} = \frac{w_3}{\sinh^3 d_{12}}$. So we have

$$\frac{w_1}{\sinh^3 d_{23}} = \frac{w_2}{\sinh^3 d_{13}} = \frac{w_3}{\sinh^3 d_{12}},$$

which implies that (w_1, w_2, w_3) and $(\sinh^3 d_{23}, \sinh^3 d_{13}, \sinh^3 d_{12})$ are collinear, as desired. □

With this necessary condition, we show that the equilateral triangle central configurations exist only for three equal masses. This result has been partially obtained in [29], where Florin Diacu and S. Popa showed that an equilateral triangle negative elliptic relative equilibrium must have three equal masses. By Corollary 1, we can conclude that equilateral triangle negative elliptic-hyperbolic relative equilibria and equilateral triangle negative hyperbolic relative equilibria must have three equal masses. Hence we state this result in the context of central configurations.

Corollary 8. *For a central configuration formed by three bodies on $\mathbb{H}^2$, by saying that they are on $w = c$ is equivalent to saying that they form an equilateral triangle. For these central configurations, the three masses must be equal.*

Proof. The first claim is clear from (6.3). For these central configurations, the two conditions that $w_1 = w_2 = w_3$ and that $d_{12} = d_{13} = d_{23}$ imply that their position is given by

$$\mathbf{q}_i = (r\cos(\beta_i + \alpha), r\sin(\beta_i + \alpha), w), \quad \beta_i = \frac{2i\pi}{3}, \quad i = 1, 2, 3.$$

Thus the three vectors $\bar{\mathbf{q}}_1$, $\bar{\mathbf{q}}_2$ and $\bar{\mathbf{q}}_3$ have another linear relation: $\bar{\mathbf{q}}_1 + \bar{\mathbf{q}}_2 + \bar{\mathbf{q}}_3 = 0$. Compared with the first relation derived above, $\sum_{i=1}^{3} m_i w_i \bar{\mathbf{q}}_i = 0$, we obtain $m_1 = m_2 = m_3$, as desired. □

Chapter 7

Ordinary Central Configurations in $\mathbb{S}^3$

In this chapter, we study the ordinary central configurations in $\mathbb{S}^3$. As mentioned in Chapter 5, the topology of the set S_c is much more complex than in $\mathbb{H}^2$, which makes the study harder. Thus we concentrate only on the simple cases: geodesic central configurations and $\mathbb{S}^2$ central configurations of three bodies. Recall that they can be found on $\mathbb{S}^1_{xz}$ and $\mathbb{S}^2_{xyz}$ by Theorem 11 and Corollary 4. Unless otherwise stated, in this chapter, we use $\mathbb{S}^1$ and $\mathbb{S}^2$ for $\mathbb{S}^1_{xz}$ and $\mathbb{S}^2_{xyz}$ respectively.

We first give several examples, then turn to geodesic central configurations of two and three masses. We find that Moulton's theorem cannot be generalized directly. Surprisingly, there is a continuum of central configurations on some S_c for two equal masses. For three masses, we study the inverse problem: for a given configuration on $\mathbb{S}^1$, find positive masses such that a geodesic central configuration exists. In the end, we study the 3-body non-geodesic central configurations. We obtain one necessary condition which shows that all equilateral triangle central configurations must have equal masses. This condition also leads to the existence of another class of central configurations, which are easy to describe.

In this chapter, let $r_i = (x_i^2 + y_i^2)^{1/2}$ and $\rho_i = (z_i^2 + w_i^2)^{1/2}$.

7.1 Examples

One class of the following examples are $\mathbb{S}^3$ central configurations, which contrast with what happens in $\mathbb{H}^3$. It also shows the complexity of classifying central configurations in $\mathbb{S}^3$.

Proposition 20. *Consider the masses* $m_1, \ldots, m_N > 0$ *on* $\mathbb{S}^3$ *at the configuration* $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$*, where* $\mathbf{q}_i = (x_i, y_i, w_i)$*. If* $\mathbf{q}_i = (x_i, y_i, z_i, w_i) \notin \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$*, then the* i*-th equation of* (3.5) *can be written as*

$$\begin{cases} \sum_{j=1, j\neq i}^N \dfrac{m_j(x_i x_j + y_i y_j - r_i^2 \cos d_{ij})}{\sin^3 d_{ij}} = 2\lambda r_i^2 \rho_i^2 \\ \sum_{j=1, j\neq i}^N \dfrac{m_j(x_i y_j - x_j y_i)}{\sin^3 d_{ij}} = 0 \\ \sum_{j=1, j\neq i}^N \dfrac{m_j(z_i w_j - z_j w_i)}{\sin^3 d_{ij}} = 0. \end{cases} \tag{7.1}$$

Proof. Since $\mathbf{q}_i \notin \mathbb{S}^1_{xy} \cup \mathbb{S}^1_{zw}$, the following four vectors

$$\mathbf{v}_{i1} = (x_i, y_i, 0, 0),\ \mathbf{v}_{i2} = (-y_i, x_i, 0, 0),\ \mathbf{v}_{i3} = (0, 0, z_i, w_i),\ \mathbf{v}_{i4} = (0, 0, -w_i, z_i),$$

form an orthogonal basis of $T_{\mathbf{q}_i}\mathbb{R}^4$. Recall that

$$\nabla_{\mathbf{q}_i} U = \sum_{j=1, j\neq i}^N m_i m_j \frac{\mathbf{q}_j - \cos d_{ij}\mathbf{q}_i}{\sin^3 d_{ij}}, \quad \nabla_{\mathbf{q}_i} I = 2m_i(x_i\rho_i^2, y_i\rho_i^2, -z_i r_i^2, -w_i r_i^2).$$

That $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ is equivalent to that $\nabla_{\mathbf{q}_i} U \cdot \mathbf{v}_{ik} = \lambda \nabla_{\mathbf{q}_i} I \cdot \mathbf{v}_{ik}, k = 1, 2, 3, 4$. We obtain

$$\sum_{j=1, j\neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}}(x_i x_j + y_i y_j - r_i^2 \cos d_{ij}) = \lambda 2 m_i r_i^2 \rho_i^2,$$

$$\sum_{j=1, j\neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}}(x_i y_j - y_i x_j) = 0,$$

$$\sum_{j=1, j\neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}}(z_i z_j + w_i w_j - \rho_i^2 \cos d_{ij}) = -\lambda 2 m_i r_i^2 \rho_i^2,$$

$$\sum_{j=1, j\neq i}^N \frac{m_i m_j}{\sin^3 d_{ij}}(z_i w_j - w_i z_j) = 0.$$

Adding the first and the third equation we obtain an identity. Thus the equation $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ is equivalent to (7.1). □

Example 8 (Lagrangian central configurations of three equal masses)**.** *Let three equal* *masses* $\mathbf{m} = (m, m, m)$ *on* $\mathbb{S}^2$ *be at*

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3),\ \ \mathbf{q}_j = (x_j, y_j, z_j),\ \ j = 1, 2, 3,$$
$$x_j = r\cos\beta_j, \quad y_j = r\sin\beta_j, \quad z_j = \sqrt{1-r^2}, \quad \beta_j = \frac{2\pi(j-1)}{3},$$

see Figure 7.1. This example has appeared in Chapter 4, where we used it as an

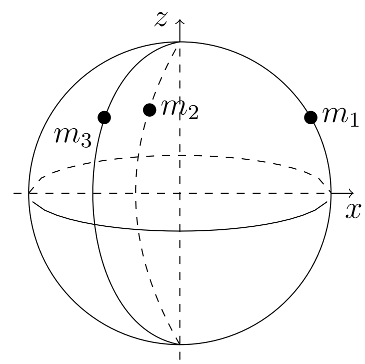

Figure 7.1: Lagrangian central configurations on $\mathbb{S}^2_{xyz}$

illustration of central configuration counting. We proved there that it is a central configuration. Using the formula given in Proposition 8, we obtain

$$\lambda = \frac{\langle M^{-1}\nabla U, M^{-1}\nabla I\rangle}{\langle M^{-1}\nabla I, M^{-1}\nabla I\rangle} = \frac{-3m}{2\sin^3 d_{12}} = -\frac{m}{2\sqrt{3}r^3\left(1-\frac{3r^2}{4}\right)^{3/2}},$$

since $\cos d_{12} = 1 - \frac{3r^2}{2}$, $\sin d_{12} = \sqrt{1-\cos^2 d_{12}}$.

Example 9 (Geodesic central configurations of three equal masses)**.** *Let three equal masses* $\mathbf{m} = (m, m, m)$ *on* $\mathbb{S}^1$ *be at*

$$\mathbf{q}_1 = (-x, z), \quad \mathbf{q}_2 = (0, 1), \quad \mathbf{q}_3 = (x, z),$$

where $z \neq 0, \pm 1$, *see Figure 7.2. Let* N *be the north pole* $(0, 1)$. *Let us check that the central configuration equation* (4.1), $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} m_i \sin^2 d(\mathbf{q}_i, N)$, *is satisfied. For* $\mathbf{q}_2$, *the symmetry implies that* $\nabla_{\mathbf{q}_2} U = \mathbf{F}_2 = 0$. *Lemma 1 says that* $\nabla_{\mathbf{q}_2} I = 0$, *so* (4.1) *holds for* $i = 2$. *For* $\mathbf{q}_1$ *and* $\mathbf{q}_3$, *the symmetry implies that* $|\mathbf{F}_1| = |\mathbf{F}_3|$ *and they are collinear with* $\nabla_{\mathbf{q}_i} \sin^2 d(\mathbf{q}_i, N)$. *Thus the central configuration equation* $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} m_i \sin^2 d(\mathbf{q}_i, N)$ *holds also for* $i = 1$ *and* 3.

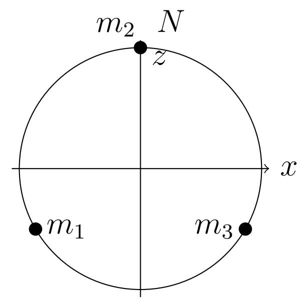

Figure 7.2: One geodesic central configurations on $\mathbb{S}^1$

Using the formula given in Proposition 8, we obtain

$$d_{12} = d_{23}, \quad r_2^2 = 0, \quad r_1^2 = r_3^2 = x^2,$$

$$x_1x_2 + y_1y_2 = x_2x_3 + y_2y_3 = 0, \quad x_1x_3 + y_1y_3 = -x^2.$$

$$\cos d_{12} = z, \quad \sin^3 d_{12} = |x|^3, \quad \cos d_{13} = z^2 - r^2, \quad \sin^3 d_{13} = 8|x|^3|z|^3,$$

which yield

$$\lambda = -\frac{m}{2z^2}\left(\frac{\cos d_{12}}{\sin^3 d_{12}} + \frac{1+\cos d_{13}}{\sin^3 d_{13}}\right) = -\frac{m}{2|x|^3}\left(\frac{1}{z} + \frac{1}{4|z|^3}\right).$$

Recall that there is no 3-dimensional central configurations in $\mathbb{H}^3$, see Theorem 10. In $\mathbb{S}^3$, we will construct a 2-parameter family of 3-dimensional ordinary central configurations of five masses. This example has been published in [92]. Suppose that the masses are $m_1 = m_2 = m, m_3 = m_4 = m_5 = 1$, and their positions are given by

$$\begin{array}{llll}
x_1 = 0, & y_1 = 0, & z_1 = \cos\theta, & w_1 = \sin\theta, \\
x_2 = 0, & y_2 = 0, & z_2 = \cos\theta, & w_2 = -\sin\theta, \\
x_3 = r, & y_3 = 0, & z_3 = c, & w_3 = 0, \\
x_4 = r\cos\frac{2\pi}{3}, & y_4 = r\sin\frac{2\pi}{3}, & z_4 = c, & w_4 = 0, \\
x_5 = r\cos\frac{4\pi}{3}, & y_5 = r\sin\frac{4\pi}{3}, & z_5 = c, & w_5 = 0,
\end{array}$$

where $c \in (-1,1) \setminus \{0\}$, $r > 0$, $r^2 + c^2 = 1$ and $\theta \in (0,\pi) \setminus \{\frac{\pi}{2}\}$. Such configurations depend on two parameters, c and θ, and we denote them by $\mathbf{q}(c,\theta)$. It is easy to see that these configurations are not confined to any 2-sphere. In Figure 7.3, we illustrate such a configuration in a $\mathbb{R}^3$ hyperplane by the stereographic projection of $\mathbb{S}^3$ from $(0,0,1,0)$ onto the corresponding equatorial $\mathbb{R}^3$ hyperplane, i.e.,

$$\bar{x} = \frac{x}{1-z}, \quad \bar{y} = \frac{y}{1-z}, \quad \bar{w} = \frac{w}{1-z}.$$

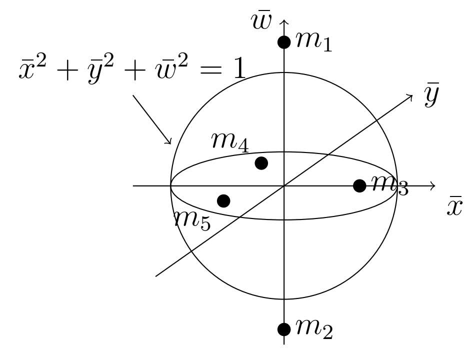

Figure 7.3: A configuration $\mathbf{q}(c,\theta)$ with $(c,\theta) \in (-1,0) \times (0, \frac{\pi}{2})$.

This example is harder to verify, thus we state it in a proposition.

Proposition 21. *For any $(c,\theta) \in (-1,0) \times (0, \frac{\pi}{2})$ and $(c,\theta) \in (0,1) \times (\frac{\pi}{2}, \pi)$ the configurations $\mathbf{q}(c,\theta)$ constructed above are central configurations if*

$$m = -\frac{3c|\sin^3 2\theta|}{2\cos\theta(1-c^2\cos^2\theta)^{3/2}}. \tag{7.2}$$

Generally, they are ordinary central configurations.

Proof. We check that the central configuration equations $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, i = 1, ..., 5,$ are satisfied. The function U can be written as $U = U_1 + U_2$, where

$$U_1 = \cot d_{34} + \cot d_{45} + \cot d_{35}, \; U_2 = m^2 \cot d_{12} + m \sum_{i=3}^{5} (\cot d_{1i} + \cot d_{2i}).$$

Note that the three equal masses m_3, m_4, and m_5 form an ordinary central configuration themselves, i.e., $\nabla_{\mathbf{q}_i} U_1 = \lambda_1 \nabla_{\mathbf{q}_i} I$, for $i = 3,4,5$, $\lambda_1 = \frac{-3}{2\sin^3 d_{34}}$, see Example 8. Note that $\nabla_{\mathbf{q}_1} I = \nabla_{\mathbf{q}_2} I = 0$ by Lemma 1. Thus $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I$ is satisfied if and only if there is some constant λ_2 such that

$$\nabla_{\mathbf{q}_i} U_2 = \lambda_2 \nabla_{\mathbf{q}_i} I, i = 3,4,5, \quad \text{and} \quad \mathbf{F}_1 = \mathbf{F}_2 = 0.$$

By symmetry, we only need to check $\nabla_{\mathbf{q}_3} U_2 = \lambda_2 \nabla_{\mathbf{q}_3} I$, and $\mathbf{F}_1 = 0$.

Note that $d_{13} = d_{23} = d_{14} = d_{24} = d_{15} = d_{25}$, $d_{34} = d_{45} = d_{35}$, and

$$\cos d_{12} = \cos 2\theta,\ \cos d_{13} = c\cos\theta, \quad \cos d_{34} = \frac{3}{2}c^2 - \frac{1}{2}.$$

Some straightforward computation shows

$$\begin{aligned}
\nabla_{\mathbf{q}_3} U_2 = \mathbf{F}_{31} + \mathbf{F}_{32} &= \frac{m(\mathbf{q}_1 - \cos d_{13}\mathbf{q}_3)}{\sin^3 d_{13}} + \frac{m(\mathbf{q}_2 - \cos d_{23}\mathbf{q}_3)}{\sin^3 d_{23}} \\
&= \frac{m}{\sin^3 d_{13}}(\mathbf{q}_1 + \mathbf{q}_2 - 2\cos d_{13}\mathbf{q}_3) = \frac{m}{\sin^3 d_{13}}((0,0,2\cos\theta,0) - 2c\cos\theta(r,0,c,0)) \\
&= \frac{-2mr\cos\theta}{\sin^3 d_{13}}(c,0,-r,0)
\end{aligned}$$

Recall that $\nabla_{\mathbf{q}_3} I = (x_3\rho_3^2, y_3\rho_3^2, -z_3r_3^2, -w_3r_3^2) = 2rc(c,0,-r,0)$. Thus we can write that

$$\nabla_{\mathbf{q}_3} U_2 = \lambda_2 \nabla_{\mathbf{q}_3} I,\ \lambda_2 = \frac{-m\cos\theta}{c\sin^3 d_{13}}.$$

By direct computation, we obtain

$$\begin{aligned}
\mathbf{F}_1 = \mathbf{F}_{12} + \sum_{j=3}^{5} \mathbf{F}_{1j} &= \frac{m^2}{|\sin^3 2\theta|}(\mathbf{q}_2 - \cos 2\theta \mathbf{q}_1) + \sum_{i=3}^{5} \frac{m}{\sin^3 d_{13}}(\mathbf{q}_i - \cos d_{13}\mathbf{q}_1) \\
&= \frac{m^2}{|\sin^3 2\theta|}(\mathbf{q}_2 - \cos 2\theta\mathbf{q}_1) + \frac{m}{\sin^3 d_{13}}(\sum_{i=3}^{5}\mathbf{q}_i - 3c\cos\theta\mathbf{q}_1) \\
&= m\sin\theta\left(\frac{2m\cos\theta}{|\sin^3 2\theta|} + \frac{3c}{\sin^3 d_{13}}\right)(0,0,\sin\theta,-\cos\theta).
\end{aligned}$$

Thus $\mathbf{F}_1 = 0$ if and only if $m = -\frac{3c|\sin^3 2\theta|}{2\cos\theta(1-c^2\cos^2\theta)^{3/2}}$. Since we need positive masses, $c\cos\theta$ needs to be negative.

We have thus obtained a 2-parameter family of central configurations $\mathbf{q}(c,\theta)$ for any $(c,\theta) \in (-1,0)\times(0,\frac{\pi}{2})$, and $(c,\theta)\in(0,1)\times(\frac{\pi}{2},\pi)$. The central configuration equations $\nabla_{\mathbf{q}_i} U = \lambda(c,\theta)\nabla_{\mathbf{q}_i} I, i = 1,...,5$, are satisfied, and the constant is

$$\begin{aligned}
\lambda(c,\theta) = \lambda_1 + \lambda_2 &= \frac{-3}{2\sin^3 d_{34}} - \frac{m\cos\theta}{c\sin^3 d_{13}} = \frac{-3}{2\sin^3 d_{34}} + \frac{3|\sin^3 2\theta|}{2\sin^6 d_{13}} \\
&= \frac{3}{2}\left(\frac{-8}{3\sqrt{3}(1+3c^2)^{3/2}(1-c^2)^{3/2}} + \frac{|\sin^3 2\theta|}{(1-c^2\cos^2\theta)^3}\right),
\end{aligned}$$

which is zero on a 1-dimensional manifold. Factually, the set $\lambda = 0$ is homeomorphic to two open intervals. Thus generally, $\mathbf{q}(c, \theta)$ are ordinary central configurations. This remark completes the proof. $\square$

Moreover, if $(c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})$, then the masses m_3, m_4, m_5 are contained in the unit ball, $\bar{x}^2 + \bar{y}^2 + \bar{w}^2 \leq 1$, and the masses m_1, m_2 are outside it, see Figure 7.3. This happens because

$$\bar{w}_1 = \frac{w_1}{1 - z_1} = \frac{\sin \theta}{1 - \cos \theta} > 1, \ \bar{x}_3^2 + \bar{y}_3^2 = (\frac{x_3}{1 - z_3})^2 + (\frac{y_3}{1 - z_3})^2 = \frac{1 + c}{(1 - c)} < 1.$$

Similarly, if $(c, \theta) \in (0, 1) \times (\frac{\pi}{2}, \pi)$, then masses m_3, m_4, m_5 are outside the unit ball, but the masses m_1, m_2 are inside the ball.

7.2 Geodesic Central Configurations of Two and Three Bodies

In this section, we focus on geodesic central configurations. For $N = 2$, We find the number of geodesic central configurations on S_c for $c \in (0, m_1 + m_2)$ (when c equals one of the endpoints, we get special central configurations). For $N = 3$, We find all possible shapes that allow positive masses to form central configurations.

Notice that the symmetry group of ordinary central configurations $SO(2) \times SO(2)$ contains two elements

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Restricted on $\mathbb{S}^1_{xz}$, they are $(x, z) \to (-x, z)$ and $(x, z) \to (x, -z)$. Thus we can always assume that one particle, say m_1, belongs to a particular quadrant.

For $N = 2$, we use the coordinates $(x, z) = (-\sin \theta, \cos \theta)$, where θ is measured from $(0, 1)$ anti-clockwise, see Figure 7.4. Assume that $\mathbf{q_1}$ is in the second quadrant, but not at the endpoints. Otherwise, $\nabla_{\mathbf{q_1}} I = 0$ by Lemma 1, but $\nabla_{\mathbf{q_1}} U$ is not zero since there are only two bodies, which implies that there will be no central configurations. Thus let $\mathbf{q_1} = (-\sin \theta_1, \cos \theta_1)$, $\quad \mathbf{q_2} = (-\sin \theta_2, \cos \theta_2)$, $\quad 0 \leq \theta_1 < \theta_2 \leq 2\pi, \theta_1 \in (0, \frac{\pi}{2})$. Then

$$U(\mathbf{q}) = m_1 m_2 \cot d_{12} \ \text{ and } \ I(\mathbf{q}) = m_1 \sin^2 \theta_1 + m_2 \sin^2 \theta_2,$$

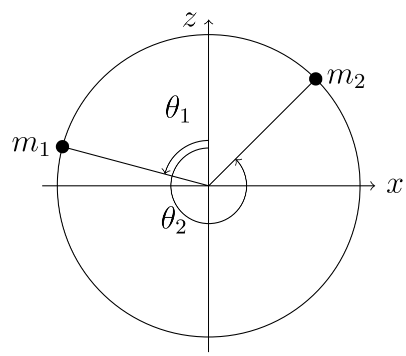

Figure 7.4: A configuration of two masses on $\mathbb{S}^1$

where $d_{12} = \min\{\theta_2 - \theta_1, 2\pi - (\theta_2 - \theta_1)\}$.

Theorem 15. *Consider two masses m_1 and m_2 on $\mathbb{S}^1$ with positions $\mathbf{q}_1$ and $\mathbf{q}_2$ as above. Then these bodies can form a central configuration if and only if*

$$m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0. \tag{7.3}$$

The number of central configurations depends on the size $I(\mathbf{q}) = c$ and it is given in the table below, where $\bar{m} := m_1 + m_2$. The table on the left is for $m_1 < m_2$, whereas the table on the right is for the $m_1 = m_2 =: m$.

size: $I(\mathbf{q}) = c$	*number*
$c \in (0, m_1)$	*2*
$c \in [m_1, m_2]$	*0*
$c \in (m_2, \bar{m})$	*2*

size: $I(\mathbf{q}) = c$	*number*
$c \in (0, m)$	*2*
$c = m$	∞
$c \in (m, \bar{m})$	*2*

When the masses are equal and $c = m$, all of the ∞ central configurations are degenerate and the set they form has the power of the continuum.

Proof. The central configuration equation $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, i = 1, 2$, is given by $\frac{\partial U}{\partial \theta_1} = \lambda \frac{\partial I}{\partial \theta_1}$ and $\frac{\partial U}{\partial \theta_2} = \lambda \frac{\partial I}{\partial \theta_2}$. Explicitly, they are

$$\frac{\pm m_1 m_2}{\sin^2(\theta_2 - \theta_1)} = \lambda m_1 \sin 2\theta_1 \quad \text{and} \quad \frac{\mp m_1 m_2}{\sin^2(\theta_2 - \theta_1)} = \lambda m_2 \sin 2\theta_2,$$

where the signs depend on whether d_{12} equals $\theta_2 - \theta_1$ or $2\pi - (\theta_2 - \theta_1)$. From these equations we obtain the condition

$$m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0.$$

Figure 7.5: The graphs of $\sin^2\theta_2 = \frac{c(m_1-c)}{m_2(\bar{m}-2c)}$ for $m_1 < m_2$ (left) and $m_1 = m_2 =: m$ (right) in coordinates $(c, \sin^2\theta_2)$.

This relationship implies that $\theta_2 \in \left(\frac{1}{2}\pi, \pi\right)$ or $\theta_2 \in \left(\frac{3}{2}\pi, 2\pi\right)$.

To find the number of central configurations on S_c, we solve the system

$$\begin{cases} m_1 \sin^2\theta_1 + m_2 \sin^2\theta_2 = c \\ m_1 \sin 2\theta_1 + m_2 \sin 2\theta_2 = 0, \end{cases}$$

and obtain

$$\sin^2\theta_2 = \frac{c(m_1-c)}{m_2(\bar{m}-2c)} \quad \text{and} \quad \sin^2\theta_1 = \frac{c(m_2-c)}{m_1(\bar{m}-2c)}.$$

Notice that $\sin 2\theta_i \neq 0$, so let

$$0 < \frac{c(m_1-c)}{m_2(\bar{m}-2c)} < 1, \quad 0 < \frac{c(m_2-c)}{m_1(\bar{m}-2c)} < 1.$$

We are then led to

$$0 < c < m_1, \quad m_2 < c < \bar{m},$$

a fact that can also be seen in the graphs of Figure 7.5, where a typical function of the form $\frac{c(m_1-c)}{m_2(\bar{m}-2c)}$ is represented for $m_1 < m_2$, on the left, and $m_1 = m_2$, on the right.

Thus having c in this range, we can solve for $\theta_i, i = 1, 2$. Using the fact that $\theta_1 \in \left(0, \frac{\pi}{2}\right)$ and $\theta_2 \in \left(\frac{1}{2}\pi, \pi\right) \cup \left(\frac{3}{2}\pi, 2\pi\right)$, we see that there are exactly two central configurations for each c: $(\theta_1, \theta_2) \in \left(0, \frac{\pi}{2}\right) \times \left(\frac{1}{2}\pi, \pi\right)$ and $(\theta_1, \theta_2+\pi) \in \left(0, \frac{\pi}{2}\right) \times \left(\frac{3}{2}\pi, 2\pi\right)$. If $m_1 = m_2 = m$ and $I = m$, the system reduces to

$$\sin^2\theta_1 + \sin^2\theta_2 = 1, \ \sin 2\theta_1 + \sin 2\theta_2 = 0,$$

The two equations are not independent. The solution is a 1-dimensional manifold: $\{\theta_1 \in (0, \pi/2), \ \theta_2 = \theta_1 + \pi/2 \ \text{ or } \ \theta_2 = \theta_1 + 3\pi/2\}$. Thus there is a continuum of central configurations on S_m. Moreover, S_m is this 1-dimensional manifold, on which $U = m^2 \cot(\frac{\pi}{2}) = 0$. As critical points of $U|_{S_m}$, the nullity is bigger than the dimension of the symmetry, which is zero in this case.

Thus all central configurations on S_m are degenerate. This remark completes the proof. □

Thus we see that Moulton's theorem can not be generalized to $\mathbb{S}^1$ even for $N = 2$. For higher N, it is more difficult to find the number of central configurations on S_c. We discuss the inverse problem: suppose that λ and the N distinct points $\theta_1, \theta_2, ..., \theta_N$ are given, and consider the problem of determining $\mathbf{m} \in \mathbb{R}^N_+$ so that the geodesic central configurations exist [64]. Similar to the discussion in Chapter 6, there is a unique solution of the masses for even N, though some m_i may be negative. For odd N, there are solutions of the masses if the shape satisfies one equation (to make the Pfaffian of a certain matrix zero). We study this in detail for $N = 3$. This result is the main part of [91].

We return to the usual angular coordinates of $\mathbb{S}^1$, $(x, z) = (\cos\theta, \sin\theta)$. As discussed at the beginning of this section, we assume that $\mathbf{q}_1$ belongs to the first quadrant $\theta_1 \in [0, \frac{\pi}{2}]$. Furthermore, recall the transform in $SO(4)$ that maps a central configuration on $\mathbb{S}^2_{xyz}$ to a central configuration on $\mathbb{S}^2_{xzw}$,

$$(x, y, z, w) \to (z, w, x, y).$$

Restricted on $\mathbb{S}^1$, it is $(x, z) \to (z, x)$, the reflection about the straight line $z = x$. Since we are concerned only with the shapes, after this reflection $\theta_1 = 0$ becomes $\theta_1 = \frac{\pi}{2}$, so we assume that $\theta_1 \in [0, \frac{\pi}{2})$. We consider the triangle formed by the three positions in the xz-plane. The triangle may be an acute triangle or an obtuse triangle, since the right triangle implies antipodal singularity.

Acute Triangles: We first consider the acute triangular shapes:

$$\mathbf{q}_1 = (\cos\theta, \sin\theta),\ \mathbf{q}_2 = (\cos(\theta+\alpha), \sin(\theta+\alpha)),\ \mathbf{q}_2 = (\cos(\theta+\alpha+\beta), \sin(\theta+\alpha+\beta)),$$

where $\theta \in [0, \frac{\pi}{2}), \alpha \in (0, \pi), \beta \in (0, \pi), \alpha + \beta \in (\pi, 2\pi)$. We have $d_{12} = \alpha$, $d_{23} = \beta$, $d_{13} = 2\pi - \alpha - \beta$, see Figure 7.6. Then

$$U = m_1 m_2 \cot\alpha + m_2 m_3 \cot\beta - m_1 m_3 \cot(\alpha + \beta),$$

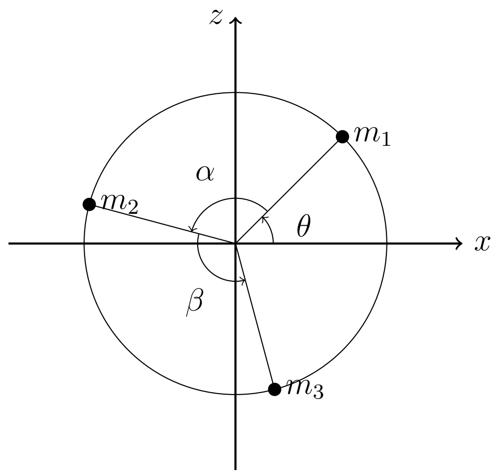

Figure 7.6: An acute triangle configuration on $\mathbb{S}^1$

$$I = m_1 \cos^2\theta + m_2 \cos^2(\theta+\alpha) + m_3 \cos^2(\theta+\alpha+\beta).$$

Theorem 16. *Consider an acute geodesic configuration determined by* θ, α, β *as above. Then the configuration admits a solution of masses in* $\mathbb{R}^3_+$ *if and only if*

$$\frac{\sin 2\theta}{\sin^2\beta} + \frac{\sin 2(\theta+\alpha)}{\sin^2(\alpha+\beta)} + \frac{\sin 2(\theta+\alpha+\beta)}{\sin^2\alpha} = 0. \tag{7.4}$$

Proof. The central configuration equation $\nabla_{\mathbf{q}_i} U = \lambda \nabla_{\mathbf{q}_i} I, i = 1,2,3,$ is given by $\frac{\partial U}{\partial\theta} = \lambda\frac{\partial I}{\partial\theta}$, $\frac{\partial U}{\partial\alpha} = \lambda\frac{\partial I}{\partial\alpha}$, and $\frac{\partial U}{\partial\beta} = \lambda\frac{\partial I}{\partial\beta}$. Explicitly, we have

$$\begin{cases} m_1 \sin 2\theta + m_2 \sin 2(\theta+\alpha) + m_3 \sin 2(\theta+\alpha+\beta) = 0, \\ \frac{m_1 m_2}{\sin^2\alpha} - \frac{m_1 m_3}{\sin^2(\alpha+\beta)} = \lambda m_2 \sin 2(\theta+\alpha) + \lambda m_3 \sin 2(\theta+\alpha+\beta), \\ \frac{m_2 m_3}{\sin^2\beta} - \frac{m_1 m_3}{\sin^2(\alpha+\beta)} = \lambda m_3 \sin 2(\theta+\alpha+\beta). \end{cases}$$

Using the first equation, (which is (4.2) $\sum_{i=1}^3 m_i x_i z_i = 0$), the last two equations can be solved by

$$m_1 = (-\lambda \sin 2(\theta+\alpha+\beta) + \frac{m_2}{\sin^2\beta}) \sin^2(\alpha+\beta), \quad m_3 = (\lambda \sin 2\theta + \frac{m_2}{\sin^2\alpha}) \sin^2(\alpha+\beta).$$

Substituting these expressions back to the first equation, direct computation leads to

(7.4). Thus the system is equivalent to

$$\begin{cases} \frac{\sin 2\theta}{\sin^2\beta} + \frac{\sin 2(\theta+\alpha)}{\sin^2(\alpha+\beta)} + \frac{\sin 2(\theta+\alpha+\beta)}{\sin^2\alpha} = 0, \\ m_1 = (-\lambda \sin 2(\theta+\alpha+\beta) + \frac{m_2}{\sin^2\beta}) \sin^2(\alpha+\beta), \\ m_3 = (\lambda \sin 2\theta + \frac{m_2}{\sin^2\alpha}) \sin^2(\alpha+\beta). \end{cases}$$

Note that for fixed $\lambda \neq 0$ (the case when $\lambda = 0$ are the special central configurations) we can choose m_2 so large such that m_1, m_3 are positive. This remark completes the proof. □

Thus to find acute triangle geodesic central configurations is to solve (7.4), which can be written as $A \sin 2\theta = -B \cos 2\theta$, where

$$A = \frac{1}{\sin^2\beta} + \frac{\cos 2\alpha}{\sin^2(\alpha+\beta)} + \frac{\cos 2(\alpha+\beta)}{\sin^2\alpha}, \ B = \frac{\sin 2(\alpha+\beta)}{\sin^2\alpha} + \frac{\sin 2\alpha}{\sin^2(\alpha+\beta)}.$$

Given (α, β), we discuss the number of possible values of θ to have the equation satisfied. If $AB \neq 0$, then there is a unique $\theta \in [0, \frac{\pi}{2})$, $\tan 2\theta = -B/A$. If $A = 0, B \neq 0$, then there also is a unique θ, $\theta = \frac{\pi}{4}$.

Corollary 9. *For any (α, β) with $A^2 + B^2 \neq 0$, there is a unique $\theta \in [0, \frac{\pi}{2})$ that satisfies* (7.4).

If $A \neq 0, B = 0$, then there also is a unique θ, $\theta = 0$. When $A = B = 0$, (7.4) is satisfied for any $\theta \in [0, \frac{\pi}{2})$. Generally, it is not easy to solve the equation $A = B = 0$. Let us consider one special case, when the acute triangle is isosceles, $d_{12} = d_{13}$, i.e., $\alpha = 2\pi - \alpha - \beta$.

Proposition 22. *Assuming $d_{12} = d_{13} \neq \frac{2\pi}{3}$, then $A \neq 0$, $B = 0$, and $\theta = 0$. In this case, $m_2 = m_3$. If $d_{12} = d_{13} = \frac{2\pi}{3}$, any $\theta \in [0, \frac{\pi}{2})$ satisfies* (7.4)*. Then we can make* $\mathbf{m}$ *run over a 3-dimensional submanifold of $\mathbb{R}^3_+$ by properly choosing m_2, λ and θ.*

Proof. If $\alpha = 2\pi - \alpha - \beta$, then $\alpha + \beta = 2\pi - \alpha, \beta = 2\pi - 2\alpha$ and (7.4) becomes

$$\begin{aligned} 0 &= \frac{\sin 2\theta}{\sin^2\beta} + \frac{\sin 2(\theta+\alpha) + \sin 2(\theta-\alpha)}{\sin^2\alpha} = \frac{\sin 2\theta}{\sin^2 2\alpha} + 2\frac{\sin 2\theta \cos 2\alpha}{\sin^2\alpha} \\ &= \frac{\sin 2\theta}{\sin^2 2\alpha}\left(1 + 8\cos^2\alpha \cos 2\alpha\right). \end{aligned}$$

Solving for $1 + 8\cos^2\alpha \cos 2\alpha = 0$, we get $\alpha = \frac{2\pi}{3}$ since in this case we need $\alpha > \frac{\pi}{2}$.

Thus if $\alpha \neq \frac{2\pi}{3}$, then θ must equal 0. By the equation $m_3 = (\lambda \sin 2\theta + \frac{m_2}{\sin^2 \alpha}) \sin^2(\alpha + \beta)$, we see that $m_2 = m_3$.

If $\alpha = \beta = \frac{2\pi}{3}$, then any $\theta \in [0, \frac{\pi}{2})$ satisfies (7.4). In this case, direct computation leads to

$$(m_1, m_2, m_3) = (-3/4\lambda \sin(2\theta + \frac{2\pi}{3}) + m_2, m_2, 3/4\lambda \sin 2\theta + m_2).$$

The three components are not dependent, thus the masses form a 3-dimensional submanifold of $\mathbb{R}^3_+$ with coordinates (m_2, λ, θ). This remark completes the proof. □

Further analysis shows that the only solution of $A = B = 0$ is the equilateral triangle. It is interesting to notice that the masses can run over a 3-dimensional submanifold of $\mathbb{R}^3_+$. In other words, the highly symmetric equilateral triangle central configuration can be formed by distinct masses. This fact is reminiscent of the Lagrangian central configurations of the Newtonian 3-body problem, where any three masses can form an equilateral triangle central configuration.

Obtuse Triangles: We now consider the obtuse triangular shapes. Again, assume that $\theta_1 \in [0, \frac{\pi}{2})$. Then θ_2, θ_3 belong to $(\theta_1, \theta_1 + \pi)$ or $(\theta_1 + \pi, \theta_1 + 2\pi)$. By the transform $(x, z) \to (z, x)$, one type becomes the other type. Since we are concerned only with the shapes, we assume that θ_2, θ_3 belong to $(\theta_1, \theta_1 + \pi)$. Also, note that $\theta_1 \neq 0$. If $\theta_1 = 0$, Lemma 1 implies that $\nabla_{\mathbf{q}_1} I = 0$. Then $\nabla_{\mathbf{q}_1} U = \mathbf{F}_{12} + \mathbf{F}_{13} = 0$, which can never be satisfied since both $\mathbf{F}_{12}$ and $\mathbf{F}_{13}$ are pointing upwards. An obtuse triangular shape is of the form

$$\mathbf{q}_1 = (\cos\theta, \sin\theta),\ \mathbf{q}_2 = (\cos(\theta+\alpha), \sin(\theta+\alpha)),\ \mathbf{q}_2 = (\cos(\theta+\alpha+\beta), \sin(\theta+\alpha+\beta)).$$

There are further restrictions. Note that $\nabla_{\mathbf{q}_1} U$ is pointing upwards. Comparing this fact with Figure 4.1, we see that $\lambda < 0$. Note that $\nabla_{\mathbf{q}_3} U = \mathbf{F}_{31} + \mathbf{F}_{32}$ is also pointing upwards. If $\mathbf{q}_3$ is in the third quadrant, then the equation $\nabla_{\mathbf{q}_3} U = \lambda \nabla_{\mathbf{q}_3} I$ can only be satisfied for positive λ, see Figure 4.1. This contradiction shows that it is necessary to require $\mathbf{q}_3$ to be in the second quadrant. Thus we further assume that $\theta \in (0, \frac{\pi}{2}), \alpha, \beta \in (0, \pi), \alpha + \beta + \theta < \pi$, see Figure 7.7. We have $d_{12} = \alpha$, $d_{23} = \beta$, $d_{13} = \alpha + \beta$. Then

$$U = m_1 m_2 \cot\alpha + m_2 m_3 \cot\beta + m_1 m_3 \cot(\alpha + \beta),$$

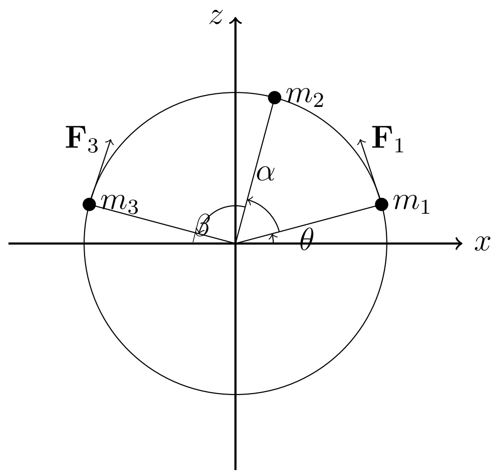

Figure 7.7: An obtuse triangle configuration on $\mathbb{S}^1$

$$I = m_1 \cos^2\theta + m_2\cos^2(\theta+\alpha) + m_3\cos^2(\theta+\alpha+\beta).$$

Theorem 17. *Consider an obtuse geodesic configuration determined by θ, α, β as above. Then the configuration admits a solution of masses in $\mathbb{R}^3_+$ if and only if*

$$\frac{\sin 2\theta}{\sin^2\beta} - \frac{\sin 2(\theta+\alpha)}{\sin^2(\alpha+\beta)} + \frac{\sin 2(\theta+\alpha+\beta)}{\sin^2\alpha} = 0. \tag{7.5}$$

Proof. Similarly the central configuration equations are

$$\begin{cases} m_1\sin 2\theta + m_2\sin 2(\theta+\alpha) + m_3\sin 2(\theta+\alpha+\beta) = 0, \\ \frac{m_1m_2}{\sin^2\alpha} + \frac{m_1m_3}{\sin^2(\alpha+\beta)} = \lambda m_2\sin 2(\theta+\alpha) + \lambda m_3 \sin 2(\theta+\alpha+\beta), \\ \frac{m_2m_3}{\sin^2\beta} + \frac{m_1m_3}{\sin^2(\alpha+\beta)} = \lambda m_3\sin 2(\theta+\alpha+\beta). \end{cases}$$

Using the first equation, (which is (4.2) $\sum_{i=1}^3 m_ix_iz_i = 0$), the last two equations can be solved by

$$m_1 = (\lambda\sin 2(\theta+\alpha+\beta) - \frac{m_2}{\sin^2\beta})\sin^2(\alpha+\beta), \quad m_3 = (-\lambda\sin 2\theta - \frac{m_2}{\sin^2\alpha})\sin^2(\alpha+\beta).$$

Substituting these expressions back to the first equation, direct computation leads to

(7.5). Thus the system is equivalent to

$$\begin{cases} \frac{\sin 2\theta}{\sin^2\beta} - \frac{\sin 2(\theta+\alpha)}{\sin^2(\alpha+\beta)} + \frac{\sin 2(\theta+\alpha+\beta)}{\sin^2\alpha} = 0, \\ m_1 = (\lambda \sin 2(\theta+\alpha+\beta) - \frac{m_2}{\sin^2\beta}) \sin^2(\alpha+\beta), \\ m_3 = (-\lambda \sin 2\theta - \frac{m_2}{\sin^2\alpha}) \sin^2(\alpha+\beta). \end{cases}$$

Note that $\sin 2\theta > 0$, $\sin 2(\theta+\alpha+\beta) < 0$. Thus to get positive masses, λ has to be negative. For fixed $\lambda < 0$ we can choose m_2 so small such that m_1, m_3 are positive. This remark completes the proof. □

Thus to find obtuse triangular geodesic central configurations we must solve (7.5), which can be written as $A \sin 2\theta = B \cos 2\theta$, where

$$A = \frac{1}{\sin^2\beta} - \frac{\cos 2\alpha}{\sin^2(\alpha+\beta)} + \frac{\cos 2(\alpha+\beta)}{\sin^2\alpha},\ B = -\frac{\sin 2(\alpha+\beta)}{\sin^2\alpha} + \frac{\sin 2\alpha}{\sin^2(\alpha+\beta)}.$$

Given (α, β), we further discuss the number of values of θ to have the equation satisfied. If $AB \neq 0$, then there is a unique $\theta \in (0, \frac{\pi}{2})$, $\tan 2\theta = B/A$.

Corollary 10. *For any (α, β) with $AB \neq 0$, there is a unique $\theta \in (0, \frac{\pi}{2})$ that satisfies (7.4).*

If $A = 0, B \neq 0$, then there also is a unique θ, $\theta = \frac{\pi}{4}$. When $A \neq 0, B = 0$, there is no solution. Generally, such pairs of (α, β) exist and they form a 1-dimensional manifold in the $\alpha\beta$-plane. When $A = B = 0$, (7.5) is satisfied for any $\theta \in (0, \frac{\pi}{2})$ that keeps the shape constraint to $\alpha + \beta + \theta < \pi$. Generally, such pairs of (α, β) exist and are finite in the $\alpha\beta$-plane. It is not easy to solve the equation $A = B = 0$. Let us consider one special case, when the obtuse triangle is isosceles $d_{12} = d_{23}$, i.e., $\alpha = \beta$.

Proposition 23. *Assuming $d_{12} = d_{23} \neq \bar{\beta}$, where $\cos^2 \bar{\beta} = \frac{1+\sqrt{3}}{4}$, then $\theta + \alpha = \frac{\pi}{2}$. In this case, $m_1 = m_3$. If $d_{12} = d_{23} = \bar{\beta}$, any $\theta \in (0, \frac{\pi}{2})$ with $2\bar{\beta} + \theta < \pi$ satisfies (7.5). In this case, we can let $\mathbf{m}$ run over a 3-dimensional submanifold of $\mathbb{R}^3_+$ by properly choosing m_2, λ and θ.*

Proof. If $\alpha = \beta$, then (7.5) becomes

$$\begin{aligned} 0 &= \frac{\sin(2\theta + 2\beta - 2\beta) + \sin(2\theta + 2\beta + 2\beta)}{\sin^2 \beta} - \frac{\sin 2(\theta + \beta)}{\sin^2 2\beta} \\ &= \frac{\sin(2\theta + 2\beta)\cos 2\beta}{\sin^2 \beta} - \frac{\sin 2(\theta + \beta)}{\sin^2 2\beta} \\ &= \frac{\sin(2\theta + 2\beta)}{\sin^2 2\beta}\left(4\cos^2 \beta \cos 2\beta - 1\right). \end{aligned}$$

Solving for $4\cos^2 \beta \cos 2\beta - 1 = 0$, we get $\cos^2 \beta = \frac{1+\sqrt{3}}{4}$. There is only one solution, since in this case we need $2\bar{\beta} + \theta < \pi$.

Thus if $\beta \neq \bar{\beta}$, then $\theta + \alpha$ must equal $\frac{\pi}{2}$. Then $\mathbf{F}_2 = 0$. Since $d_{12} = d_{23}$, we get $m_1 = m_3$.

If $\alpha = \beta = \bar{\beta}$, then any $\theta \in (0, \frac{\pi}{2})$ with $2\bar{\beta} + \theta < \pi$ satisfies (7.5). Similar to the case of acute triangle shapes, the masses form a 3-dimensional submanifold of $\mathbb{R}^3_+$ with local coordinates (m_2, λ, θ). This remark completes the proof. □

7.3 Central Configurations in the 3-Body Problem

In this section, we study the 3-body non-geodesic central configurations. We obtain a necessary condition, which shows that all equilateral triangle central configurations must have equal masses. This condition also helps to find all 3-body central configurations on $z = c$.

Similar to Lemma 6, we have the following result:

Lemma 7. *Consider three masses $\mathbf{m} \in \mathbb{R}^3_+$ on $\mathbb{S}^2$ at $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\mathbf{q}_i = (x_i, y_i, z_i)$. Suppose that $\mathbf{q}$ is a non-geodesic ordinary central configuration. Then there is some $k \neq 0$ such that*

$$(z_1, z_2, z_3) = k(\sin^3 d_{23}, \sin^3 d_{13}, \sin^3 d_{12}). \tag{7.6}$$

With this necessary condition, we show that the equilateral triangle central configurations occur only for three equal masses. This result has been obtained in [29], where Florin Diacu and Sergiu Popa showed that equilateral triangle elliptic relative equilibria and equilateral triangle elliptic-elliptic relative equilibria in $\mathbb{S}^3$ exist only for three equal masses.

Corollary 11. *For a non-geodesic ordinary central configuration of three bodies on $\mathbb{S}^2$, if they form an equilateral triangle, then $z_1 = z_2 = z_3$ and the three masses must* *be equal.*

Corollary 12. *Let* $\mathbf{q}$ *be a non-geodesic ordinary central configuration of three bodies on* $\mathbb{S}^2$. *Then* z_1, z_2, z_3 *are all in* $(-1, 0)$ *or all in* $(0, 1)$.

Proof. Equation (7.6) implies that we only need to show that they can not be zero and ± 1. If $z_1 = \pm 1$, then $x_1 = y_1 = 0$, z_2 and z_3 are not ± 1 or zero by Lemma 7. Then (4.2), $0 = \sum_{i=1}^{3} m_i x_i z_i = \sum_{i=1}^{3} m_i y_i z_i$ gives $m_2 x_2 z_2 + m_3 x_3 z_3 = 0$ and $m_2 y_2 z_2 + m_3 y_3 z_3 = 0$. It implies $x_2 y_3 = x_3 y_2$. Thus the three bodies are on the same 2-dimensional plane $x = ky$ for some constant k. Then $\mathbf{q}$ is a geodesic configuration. This is a contradiction. If $z_1 = 0$, then $z_2 = z_3 = 0$. Thus the configuration lies on $\mathbb{S}^1_{xy}$, which is a special central configuration by Proposition 6. This is also a contradiction, a remark that completes the proof. □

Now we study the central configurations on $z = c$. We can assume that $m_3 = 1$. In this case, (7.6) implies that $\sin d_{12} = \sin d_{13} = \sin d_{23}$. In $\mathbb{H}^2$, the corresponding equation $\sinh d_{12} = \sinh d_{13} = \sinh d_{23}$ implies the only central configurations on $w = c$ are the Lagrangian central configurations. The trigonometry function is not monotonic, which implies that there are more central configurations on $z = c$ on $\mathbb{S}^2$. Note that the mutual distances could be

$$d_{ij} = a \text{ or } \ \pi - a, \quad \text{for some } a \in (0, \pi).$$

If they are all equal, then we get the Lagrangian central configuration, see Example 8. Now let us assume that only two of them are equal. Without lose of generality, assume that $d_{12} = d_{13}, d_{23} = \pi - d_{12} \neq d_{12}$. More precisely, let the configuration be $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, $\quad \mathbf{q}_i = (x_i, y_i, z_i)$, $\quad i = 1, 2, 3$,

$$\begin{array}{lll} x_1 = \sin\theta, & y_1 = 0, & z_1 = \cos\theta, \\ x_2 = \sin\theta\cos\varphi, & y_2 = \sin\theta\sin\varphi, & z_2 = \cos\theta, \\ x_3 = \sin\theta\cos\varphi, & y_3 = -\sin\theta\sin\varphi, & z_3 = \cos\theta, \end{array}$$

where $\varphi \neq \frac{2\pi}{3}$, $\theta \in (0, \pi/2)$. It turns out that there are central configurations of this kind.

Proposition 24. *Consider three masses* m_1, m_2, m_3 *on* $\mathbb{S}^2$ *with positions* $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$

as above. Then these bodies form an ordinary central configuration if and only if

$$(m_1, m_2, m_3) = (-2\cos\varphi, 1, 1), \tag{7.7}$$
$$\cos^2\theta = 1 + \frac{2}{(\cos\varphi - 1)(2\cos\varphi + 3)}, \tag{7.8}$$

where $\varphi \in (\frac{\pi}{2}, \frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi)$.

Proof. Since no particle is on $S^1_{xy} \cup S^1_{zw}$, $\mathbf{q}$ is a central configuration if (7.1) is satisfied for $i = 1, 2, 3$. Since $w_i = 0$, there are six equations, which can be written as two linear systems of the masses. We first prove that the conditions (7.7) and (7.8) are necessary. Then we show that they are sufficient.

Substituting the configuration into (7.1), we obtain the two linear systems

$$\begin{bmatrix} 0 & \sin\varphi & -\sin\varphi \\ -\sin\varphi & 0 & -2\cos\varphi\sin\varphi \\ \sin\varphi & 2\cos\varphi\sin\varphi & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{7.9}$$

and

$$\begin{bmatrix} 0 & 1-\cos d_{12} & 1-\cos d_{12} \\ 1-\cos d_{12} & 0 & 1+\cos d_{12} \\ 1-\cos d_{12} & 1+\cos d_{12} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = -2\lambda\sin^2\theta\sin^3 d_{12} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{7.10}$$

It is easy to see that the solution of system (7.9) is (7.7), $(m_1, m_2, m_3) = (-2\cos\varphi, 1, 1)$, since we have assumed $m_3 = 1$. Since the masses must be positive, we get $\varphi \in (\frac{\pi}{2}, \pi)$, and then $\varphi \in (\frac{\pi}{2}, \frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi)$. Since we have assumed that $d_{23} + d_{12} = \pi$, we have

$$\begin{aligned} 0 &= \cos d_{23} + \cos d_{12} = \mathbf{q}_1 \cdot \mathbf{q}_2 + \mathbf{q}_2 \cdot \mathbf{q}_3 \\ &= \sin^2\theta\cos\varphi + \cos^2\theta + \sin^2\theta(\cos^2\varphi - \sin^2\varphi) + \cos^2\theta \\ &= 2\cos^2\theta + \sin^2\theta(\cos\varphi + \cos^2\varphi - \sin^2\varphi), \end{aligned}$$

thus we obtain (7.8),

$$\cos^2\theta = 1 + \frac{2}{(\cos\varphi - 1)(2\cos\varphi + 3)}, \ \sin^2\theta = \frac{-2}{(\cos\varphi - 1)(2\cos\varphi + 3)}.$$

We then show that the necessary conditions, (7.7) and (7.8), are sufficient, i.e., system (7.10) is solved by $(-2\cos\varphi, 1, 1)$ when (7.8) holds. Let $\mathbf{m} = (-2\cos\varphi, 1, 1)$. Then system (7.10) has two equations:

$$1 - \cos d_{12} = -\lambda \sin^2\theta \sin^3 d_{12}, \quad 2(1+\cos\varphi)(1-\cos d_{12}) = 1 + \cos d_{12}.$$

Note that $\cos d_{12} = 1 - \sin^2\theta(1-\cos\varphi)$. The first equation is solved by letting $\lambda = -\frac{1-\cos\varphi}{\sin^3 d_{12}} = -\frac{m_1+m_2+m_3}{2\sin^3 d_{12}}$. The second equation is $2\cos^2\theta + \sin^2\theta(2\cos^2\varphi - 1 + \cos\varphi) = 0$, which is the same as the equation derived from $0 = \cos d_{23} + \cos d_{12}$, thus equivalent to (7.8). Thus the second linear system is solved by $(-2\cos\varphi, 1, 1)$. These configurations satisfying (7.7) and (7.8) are central configurations with

$$\lambda = -\frac{m_1+m_2+m_3}{2\sin^3 d_{12}}.$$

This remark completes the proof. □

The existence of these central configurations was also shown in [32] with a different method. Our approach is advantageous since the exact configurations and the relationship between the masses and the value of c can be given. Using (7.8), we obtain the following theorem.

Theorem 18. *There are two types of ordinary central configurations for three bodies on $z = c \in (0,1)$:*

1. *ordinary central configurations for $\mathbf{m} = (1,1,1)$ at the vertices of a equilateral triangle, namely, the Lagrangian central configuration, which occurs for $c \in (0,1)$;*
2. *ordinary central configurations for $\mathbf{m} = (m_1, 1, 1)$ $m_1 \in (0,2) \setminus \{1\}$ at the vertices of an isosceles triangle as stated in Proposition 24, and the value of c is $\sqrt{1 + \frac{2}{(-\frac{m_1}{2}-1)(-m_1+3)}}$ in the range of $(0, 3/5] \setminus \{\sqrt{7/15}\}$.*

When $c = \sqrt{7/15}$, we get $m_1 = 1$. The central configurations is the Lagrangian central configuration. The maximal value of c is for $m_1 = 1/2$. It is interesting to notice that for one triple of masses, there is only one value of c that allows central configurations on $z = c$, and that these central configurations do not exist for values of c higher than $3/5$.

Chapter 8

Special Central Configurations in $\mathbb{S}^3$

In this chapter, we study special central configurations. They are quite different from the ordinary central configurations. The main difference comes from the fact that equation (3.6) is a linear combination of the position vectors only. We first give some interesting examples of special central configurations and show that M_N is not empty for $N \geq 3$. The concept of special central configurations extends naturally to $\mathbb{S}^n$ for $n \geq 1$. Following the idea of Dziobek, we discuss these configuration where N masses span an $(N-2)$-sphere. In the end, we return to special central configurations in $\mathbb{S}^3$ and focus on M_3.

8.1 Examples and the Mass Set M_N

Recall that special central configurations in $\mathbb{S}^3$ are critical points of U, i.e., $\nabla_{\mathbf{q}_i} U = \mathbf{F}_i = 0$, $1 \leq i \leq N$, and the central configuration equation (3.5) can be written as

$$\sum_{j\neq i, j=1}^{N} \frac{m_j \mathbf{q}_j}{\sin^3 d_{ij}} + \theta_i \mathbf{q}_i = 0,\ 1 \leq i \leq N, \tag{8.1}$$

where θ_i is some constant. If there is such a constant, it must be $-\sum_{j\neq i,j=1}^{N} \frac{m_j \cos d_{ij}}{\sin^3 d_{ij}}$, see Proposition 5.

Since special central configurations are invariant under the action of the $SO(4)$ group, unless otherwise stated we place geodesic configurations on $\mathbb{S}^1_{xy}$ and denote $\mathbb{S}^1_{xy}$ by $\mathbb{S}^1$, place $\mathbb{S}^2$ configurations on $\mathbb{S}^2_{xyz}$ and denote $\mathbb{S}^2_{xyz}$ by $\mathbb{S}^2$.

Example 10 ($2k+1$ equal masses on $\mathbb{S}^1$)**.** *Place $2k+1$ equal masses $\mathbf{m}=(m,...,m)$ on $\mathbb{S}^1$ at the vertices of the regular $(2k+1)$-gon. That is*

$$\mathbf{q}_i = \left(\cos\frac{2i\pi}{2k+1}, \sin\frac{2i\pi}{2k+1}\right), \ i = 1, ..., 2k+1.$$

By symmetry, it is enough to check that (8.1) holds for $i = 2k+1$. Group the other $2k$ masses in k pairs $(j, 2k+1-j)$, $j = 1, ..., k$. For each pair, using $d_{2k+1,j} = d_{2k+1,2k+1-j}$, the sum of $\frac{m_j\mathbf{q}_j}{\sin^3 d_{2k+1,j}}$ and $\frac{m_{2k+1-j}\mathbf{q}_{2k+1-j}}{\sin^3 d_{2k+1,2k+1-j}}$ is

$$\frac{m}{\sin^3 d_{2k+1,j}}(\mathbf{q}_j+\mathbf{q}_{2k+1-j}) = 2\frac{m}{\sin^3 d_{2k+1,j}}\left(\cos\frac{2j\pi}{2k+1}, 0\right),$$

a vector collinear with $\mathbf{q}_{2k+1} = (1,0)$. Thus it is easy to see that (8.1) holds for $i = 2k+1$. Therefore the configuration is a special central configuration.

Example 11 (4 equal masses $\mathbf{m} = (m, ..., m)$ on $\mathbb{S}^2$ and 5 equal masses in $\mathbb{S}^3$, [19, 25])**.** *Place 4 equal masses on $\mathbb{S}^2$ at the vertices of the regular tetrahedron. By symmetry, it is enough to check that (8.1) holds for $i = 1$. Using $d_{ij} = d_{kl}$, we find that $\sum_{i=2}^4 \frac{m_i\mathbf{q}_i}{\sin^3 d_{i1}} = \frac{m}{\sin^3 d_{i1}}(\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) = -\frac{m}{\sin^3 d_{i1}}\mathbf{q}_1$, since $\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4 = 0$. Thus (8.1) is satisfied for $i = 1$. So this configuration is a special central configuration.*

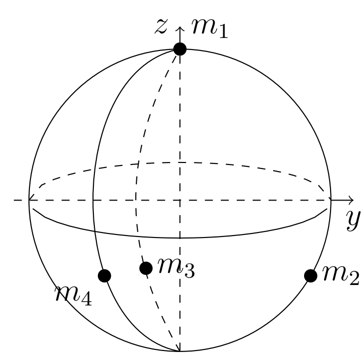

Figure 8.1: The regular tetrahedron special central configuration

Similarly, place 5 equal masses in $\mathbb{S}^3$ at the vertices of the regular pentatope. Then this configuration is a special central configuration.

Example 12 (6 masses and $2k_1+1+2k_2+1 (k_1, k_2 \geq 1)$ masses on two complementary circles in $\mathbb{S}^3$)**.** *Let V be a 2-dimensional linear space in $\mathbb{R}^4$ and $V^\perp$ be its orthogonal complement. Then the circles: $V \cap \mathbb{S}^3$ and $V^\perp \cap \mathbb{S}^3$ are called* complementary*. For* *example,* $\mathbb{S}^1_{xy}$ *and* $\mathbb{S}^1_{zw}$ *are complementary. One interesting property about complementary circles is that the distance between two points on complementary circles is independent of their positions. Indeed, let* $\mathbf{q}_1 \in V \cap \mathbb{S}^3$ *and* $\mathbf{q}_2 \in V^\perp \cap \mathbb{S}^3$*, then*

$$\cos d_{12} = \mathbf{q}_1 \cdot \mathbf{q}_2 = 0,\ \ d_{12} = \frac{\pi}{2}.$$

Using this remarkable geometry fact, Florin Diacu constructed the following interesting special central configuration [19, 25]. Let $\mathbf{m} = (m, m, m, \bar{m}, \bar{m}, \bar{m})$*. Place three masses* m *at the vertices of an equilateral triangle on* $\mathbb{S}^1_{xy}$*, and place three masses* $\bar{m}$ *at the vertices of an equilateral triangle on* $\mathbb{S}^1_{zw}$*. We need to check that* (8.1) *holds for* $i = 1, 4$*. For* $i = 1$*,*

$$\sum_{i=2}^{6} \frac{m_i \mathbf{q}_i}{\sin^3 d_{i1}} = \frac{m}{\sin^3 d_{21}}(\mathbf{q}_2 + \mathbf{q}_3) + \bar{m}(\mathbf{q}_4 + \mathbf{q}_5 + \mathbf{q}_6).$$

The first term on the right hand side is collinear with $\mathbf{q}_1$ *since* $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0$*, and the second term is zero. Thus* (8.1) *holds for* $i = 1$*. Similarly, it also holds for* $i = 4$*. Therefore the configuration is a special central configuration.*

This construction can be generalized to $2k_1+1+2k_2+1$ *masses* $\mathbf{m} = (m, ..., m, \bar{m}, ..., \bar{m})$*. Place the first* $2k_1 + 1$ *masses* m *at the vertices of a regular* $(2k_1 + 1)$*-gon on* $\mathbb{S}^1_{xy}$*, and place the last* $2k_2 + 1$ *masses* $\bar{m}$ *at the vertices of a regular* $(2k_2 + 1)$*-gon on* $\mathbb{S}^1_{zw}$*. For* $i = 1$*,*

$$\sum_{i=2}^{2k_1+2k_2+2} \frac{m_i \mathbf{q}_i}{\sin^3 d_{i1}} = \sum_{i=2}^{2k_1+1} \frac{m}{\sin^3 d_{i1}} \mathbf{q}_i + \bar{m} \sum_{i=2k_1+2}^{2k_1+2k_2+2} \mathbf{q}_i.$$

The first term on the right hand side is collinear with $\mathbf{q}_1$*, by Example 10, and the second term is zero. Thus this configuration is a special central configuration.*

Theorem 19. M_N *is not empty for* $N \geq 3$*.*

Proof. Examples 10 show that M_N is not empty for $N = 4k+1, 4k+3$. Examples 12 show that M_N is not empty for $N = 4k + 2 = 2k + 1 + 2k + 1$. For $N = 4k$, if $k = 1$, Examples 11 show that M_4 is not empty; if $k \geq 2$, let $k = k_1 + k_2$ with $k_1, k_2 \geq 1$. Since

$$4k + 2 = 2k_1 + 1 + 2k_2 + 1,$$

Examples 12 show that M_N is not empty for $N = 4k$. This remark completes the proof. □

We can check that those special central configurations for all equal masses are nondegenerate. Thus by the implicit function theorem, the set M_N has subsets homeomorphic to $\mathbb{R}^{N-1}$.

8.2 Dziobek Special Central Configurations

In this section, we consider special central configurations in $\mathbb{S}^n$ for $n \geq 1$. Following the idea of Dziobek [35] and Albouy [2], we discuss these configurations where N masses span an $(N-2)$-sphere. We obtain a criterion in terms of mutual distances and the volumes formed by the position vectors. We also manage to separate the equations of the criterion further into two sets of equations, the S-equations and the M-equations.

In this section only, let $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$, $\mathbf{q}_i \in \mathbb{R}^{n+1}, \mathbf{q}_i \cdot \mathbf{q}_i = 1$, where $n \geq 1$. Define $U : (\mathbb{S}^n)^N \setminus \Delta \to \mathbb{R}$, $U(\mathbf{q}) = \sum_{1 \leq i < j \leq N} m_i m_j \cot d_{ij}$, where $\cos d_{ij} = \mathbf{q}_i \cdot \mathbf{q}_j$.

Definition 11. *Consider the masses $m_1, \ldots, m_N$ in $\mathbb{S}^n$. Then a configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, $i = 1, ..., N$, is called a special central configuration if it is a critical point of U, i.e.*

$$\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \mathbf{F}_i = 0, \ i = 1, ..., N. \tag{8.2}$$

Obviously, equation (8.1) still holds in this case. Similarly to Theorem 9, we have:

Theorem 20. *There is no special central configuration in any closed hemisphere of $\mathbb{S}^n$ (i.e. a hemisphere that contains its boundary), as long as at least one body does not lie on the boundary.*

In the Newtonian N-body problem, a central configuration of N bodies span at most an $(N-1)$-dimensional affine plane. Those that span an $(N-2)$-dimensional affine plane are called Dziobek central configurations [2, 3, 35]. Note that (8.1) implies that the N position vectors are always dependent for a special central configuration $\mathbf{q}$ of N bodies, $1 \leq \text{rank}(\mathbf{q}_1, ..., \mathbf{q}_N) \leq N-1$. When they span an $(N-1)$-plane, they span an $(N-2)$-sphere. Thus it is natural to have the following definition.

Definition 12. *A Dziobek configuration is a configuration of N bodies that span $\mathbb{R}^{N-1}$, or equivalently, span $\mathbb{S}^{N-2}$.*

Let $\{\mathbf{q}_1, \cdots, \mathbf{q}_N\}$ be a collection of vectors in $\mathbb{R}^{N-1}$. Assume the rank of these N vectors is $N-1$. Consider the $(N-1) \times N$ matrix:

$$X = [\mathbf{q}_1, \cdots, \mathbf{q}_N].$$

Since the rank of X is $N-1$, $\dim \ker X = 1$. There is a nice formula for the kernel. Let X_k be the $(N-1)\times(N-1)$ matrix obtained from X by deleting the k-th column and let $|X_k|$ denote its determinant.

Lemma 8. *Let*

$$\Delta = (\Delta_1, ...\Delta_N) = (|X_1|, -|X_2|, ..., (-1)^{k+1}|X_k|, ...). \tag{8.3}$$

Then Δ is the base of $\ker X$. In other words, $\Delta \neq 0$ and

$$\Delta_1 \mathbf{q}_1 + \cdots + \Delta_N \mathbf{q}_N = 0. \tag{8.4}$$

Proof. Assume that $\Delta_N = (-1)^{N+1}|X_N| \neq 0$. Consider the linear system in $\mathbb{R}^{N-1}$, $X_N x = \mathbf{q}_N$. By Crammer's rule, the solution is

$$\begin{aligned} x_k &= \frac{\det(\mathbf{q}_1, \cdots, \mathbf{q}_{k-1}, \mathbf{q}_N, \mathbf{q}_{k+1}, \cdots, \mathbf{q}_{N-1})}{\det(\mathbf{q}_1, \cdots, \mathbf{q}_{N-1})} \\ &= (-1)^{N-k-1} \frac{\det(\mathbf{q}_1, \cdots, \mathbf{q}_{k-1}, \mathbf{q}_{k+1}, \cdots, \mathbf{q}_{N-1}, \mathbf{q}_N)}{\det(\mathbf{q}_1, \cdots, \mathbf{q}_{N-1})} = \frac{(-1)^{N-k-1}|X_k|}{|X_N|} = \frac{-\Delta_k}{\Delta_N}. \end{aligned}$$

Then (8.4) follows. □

For convenience, let $S_{ij} = \frac{1}{\sin^3 d_{ij}}$ for $i \neq j$. Then (8.1) becomes

$$\sum_{j \neq i} m_j S_{ij} \mathbf{q}_j + \theta_i \mathbf{q}_i = 0, \ 1 \leq i \leq N.$$

There are $N(N-1)$ scalar equations. As what happens in the Newtonian N-body problem [3, 61], they are equivalent to the following $N(N-1)/2$ scalar equations for Dziobek special central configurations.

Theorem 21. *Let $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$ be a Dziobek configuration in $\mathbb{S}^{N-2}$. Let $S_{ij} = \frac{1}{\sin^3 d_{ij}}, i \neq j$ and Δ be given by (8.3). Then $\mathbf{q}$ is a Dziobek special central configuration* *if and only if there is a real number $k \neq 0$ such that*

$$m_i m_j S_{ij} = k\Delta_i\Delta_j \text{ for any } i \neq j. \tag{8.5}$$

Proof. Since $\mathbf{q}$ is a Dziobek special central configuration, then we have equation (8.1). That is, for each $j = 1, ..., N$, the vector

$$(m_1 S_{1j}, m_2 S_{2j}, ..., \theta_j, ..., m_N S_{Nj})$$

is a solution to the equation $x_1\mathbf{q}_1 + ... + x_N\mathbf{q}_N = 0$. By Lemma 8, this vector is collinear with $(\Delta_1, ..., \Delta_N)$. Note that $\Delta_j \neq 0$ for all j. If $\Delta_1 = 0$, then $\mathbf{q}_2, ..., \mathbf{q}_N$ span $\mathbb{R}^{N-2}$, or $\mathbb{S}^{N-3}$. Then the configuration lies in a closed hemisphere of $\mathbb{S}^{N-2}$, a conclusion which contradicts with Theorem 20. Similarly, $\theta_j \neq 0$ for all j. Let $k_j = \frac{\theta_j}{\Delta_j}$. Then

$$k_j\Delta_i = m_i S_{ij}.$$

Since $S_{ij} = S_{ji}$, the vector $(k_1, ..., k_N)$ is a multiple of $(\Delta_1/m_1, ..., \Delta_N/m_N)$. So we get (8.5) for some real number $k \neq 0$. On the other hand, if $\mathbf{q}$ satisfies (8.5), then it is easy to check that (8.1) are satisfied. This remark completes the proof. □

Corollary 13. *Let $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$ be a Dziobek central configuration in $\mathbb{S}^{N-2}$. If $\sum_{i=1}^N m_i\mathbf{q}_i = 0$, then $\mathbf{q}$ is formed by N equal masses located at the vertices of a regular $(N-1)$-simplex.*

Proof. By Lemma 8, the mass vector $\mathbf{m}$ is a multiple of $(\Delta_1, ..., \Delta_N)$. Then equations of (8.5) are $S_{ij} = \bar{k}$ for all i, j, where $\bar{k}$ is some constant. Thus the configuration is a regular $(N-1)$-simplex, which implies that $\Delta_1 = \Delta_2 = ... = \Delta_N$, i.e., $m_1 = m_2 = ...m_N$. This remark completes the proof. □

Eliminating the constant k, we get $\frac{N(N-1)}{2} - 1$ equations from (8.5). Multiplying two of the equations gives

$$S_{ij}S_{kl} = S_{il}S_{kj} = S_{ik}S_{jl}$$

for any four indices $i, j, k, l \in \{1, ..., N\}$. Thus we can obtain $2 \times C_N^4$ mass-independent constraints on the shapes. Obviously, some of these constraints are redundant. We are going to find a proper set of constraints on the shapes.

Denote by M-equations the following $N-1$ equations:

$$m_2 = \frac{S_{1N}\Delta_2}{S_{12}\Delta_N}m_N,\ m_3 = \frac{S_{1N}\Delta_3}{S_{13}\Delta_N}m_N,\ ..., m_{N-1} = \frac{S_{1N}\Delta_{N-1}}{S_{1,N-1}\Delta_N}m_N, m_1 = \frac{S_{2N}\Delta_1}{S_{12}\Delta_N}m_N.$$

Denote by S-equations the following $\frac{N(N-3)}{2}$ equations:

$$\begin{aligned} \frac{S_{k-1,k-2}}{S_{k+1,k-2}} &= \frac{S_{k-1,k}}{S_{k+1,k}}, && k=3,...,N-1,\\ \frac{S_{k-1,k+j}}{S_{k,k+j}} &= \frac{S_{k-1,k+j+1}}{S_{k,k+j+1}}, && k=2,...,N-2,\ j=2,...,N-k-1. \end{aligned}$$

We are going to show the following result.

Theorem 22. *The equations of* (8.5) $\Leftrightarrow$ *the M-equations* $\cup$ *the S-equations.*

It is easy to derive the the M-equations and the S-equations from the equations of (8.5). To show the other direction, we place the $\frac{N(N-1)}{2}$ equations of (8.5) into a matrix,

$$A := \begin{bmatrix} 0=0 & \frac{m_1m_2}{\Delta_1\Delta_2}=\frac{k}{S_{12}} & \frac{m_1m_3}{\Delta_1\Delta_3}=\frac{k}{S_{13}} & \frac{m_1m_4}{\Delta_1\Delta_4}=\frac{k}{S_{14}} & \cdots & \frac{m_1m_N}{\Delta_1\Delta_N}=\frac{k}{S_{1N}}\\ 0=0 & 0=0 & \frac{m_2m_3}{\Delta_2\Delta_3}=\frac{k}{S_{23}} & \frac{m_2m_4}{\Delta_2\Delta_4}=\frac{k}{S_{24}} & \cdots & \frac{m_2m_N}{\Delta_2\Delta_N}=\frac{k}{S_{2N}}\\ 0=0 & 0=0 & 0=0 & \frac{m_3m_4}{\Delta_3\Delta_4}=\frac{k}{S_{34}} & \cdots & \frac{m_3m_N}{\Delta_3\Delta_N}=\frac{k}{S_{3N}}\\ \vdots & \vdots & \vdots & \vdots & ... & \vdots\\ 0=0 & 0=0 & 0=0 & 0=0 & \cdots & \frac{m_{N-1}m_N}{\Delta_{N-1}\Delta_N}=\frac{k}{S_{N-1,N}}\\ 0=0 & 0=0 & 0=0 & 0=0 & \cdots & 0=0 \end{bmatrix}.$$

Denote by A_{ij} the ij-th element of the matrix A, and denote by A_i the set of the nontrivial equations of the i-th row of A. We first prove the following result.

Lemma 9 (Evelyn's Lemma)**.** *The S-equations* $\cup A_1 \cup A_{23} \Rightarrow A_1 \cup A_2 \cup ... \cup A_{N-1}$.

Proof. Denote by $\mathcal{J}$ the S-equations. We will prove it in the following way,

$$\mathcal{J} \cup A_1 \cup A_{23} \Rightarrow \mathcal{J} \cup A_1 \cup A_2 \Rightarrow \mathcal{J} \cup A_1 \cup A_2 \cup A_3 \Rightarrow ... \Rightarrow \mathcal{J} \cup A_1 \cup ... \cup A_{N-1}.$$

We begin with the derivation of A_2. First, A_{24} can be derived from A_{13}, A_{14}, A_{23}, and one equation of $\mathcal{J}$, $\frac{S_{13}}{S_{23}} = \frac{S_{14}}{S_{24}}$. Indeed, the four equations are

$$\frac{m_1m_3}{\Delta_1\Delta_3} = \frac{k}{S_{13}},\ \frac{m_1m_4}{\Delta_1\Delta_4} = \frac{k}{S_{14}},\ \frac{m_2m_3}{\Delta_2\Delta_3} = \frac{k}{S_{23}},\ \frac{S_{13}}{S_{23}} = \frac{S_{14}}{S_{24}}.$$

Using them, we obtain

$$\frac{\Delta_3}{\Delta_4} = \frac{m_3 S_{13}}{m_4 S_{14}} = \frac{m_3 S_{23}}{m_4 S_{24}} = \frac{k \Delta_2 \Delta_3}{m_2 m_4 S_{24}},$$

which is A_{24}. Similarly, A_{25} can be derived from A_{14}, A_{15}, A_{24} and $\frac{S_{14}}{S_{24}} = \frac{S_{15}}{S_{25}}$. Actually, the equation $A_{2,k+1}$ can be derived from $A_{1k}, A_{1,k+1}, A_{2k}$ and $\frac{S_{1k}}{S_{2k}} = \frac{S_{1,k+1}}{S_{2,k+1}}$. That is, $\mathcal{J} \cup A_1 \cup A_{23} \Rightarrow \mathcal{J} \cup A_1 \cup A_2$.

Now we derive A_3. First, A_{34} can be derived from A_{12}, A_{14}, A_{23} and $\frac{S_{21}}{S_{41}} = \frac{S_{23}}{S_{43}}$. Indeed, the four equations are

$$\frac{m_1 m_2}{\Delta_1 \Delta_2} = \frac{k}{S_{12}}, \ \frac{m_1 m_4}{\Delta_1 \Delta_4} = \frac{k}{S_{14}}, \ \frac{m_2 m_3}{\Delta_2 \Delta_3} = \frac{k}{S_{23}}, \ \frac{S_{21}}{S_{41}} = \frac{S_{23}}{S_{43}}.$$

Using them, we obtain

$$\frac{\Delta_2}{\Delta_4} = \frac{m_2 S_{12}}{m_4 S_{14}} = \frac{m_2 S_{23}}{m_4 S_{34}} = \frac{k \Delta_2 \Delta_3}{m_3 m_4 S_{34}},$$

which is A_{34}. Then, $A_{3,k+1}$ can be derived from $A_{2k}, A_{2,k+1}, A_{3k}$ and $\frac{S_{2k}}{S_{3k}} = \frac{S_{2,k+1}}{S_{3,k+1}}$. That is, $\mathcal{J} \cup A_1 \cup A_2 \Rightarrow \mathcal{J} \cup A_1 \cup A_2 \cup A_3$.

If we have $A_1 \cup ... \cup A_{k-1}$, we can obtain A_k. First, the element $A_{k,k+1}$ can be derived from $A_{k-2,k-1}, A_{k-2,k+1}$, $A_{k-1,k}$ and one equation of $\mathcal{J}$ $\frac{S_{k-1,k-2}}{S_{k+1,k-2}} = \frac{S_{k-1,k}}{S_{k+1,k}}$. The other equations $A_{k,k+j}$ $(j \geq 2)$, can be derived from $A_{k-1,k+j-1}, A_{k-1,k+j}$, $A_{k,k+j-1}$ and one equation of $\mathcal{J}$ $\frac{S_{k-1,k+j-1}}{S_{k,k+j-1}} = \frac{S_{k-1,k+j}}{S_{k,k+j}}$.

This procedure can be carried out until we obtain A_{N-1}. That is, $\mathcal{J} \cup A_1 \cup A_{23} \Rightarrow A_1 \cup ... \cup A_{N-1}$. This remark completes the proof. □

proof of Theorem 22. Denote by $\mathcal{K}$ the M-equations. "The equations of (8.5) $\Rightarrow$ $\mathcal{K} \cup \mathcal{J}$": It is obvious that the equations of (8.5) imply $\mathcal{J}$. By A_{1j} and A_{1N}, we obtain the first $N-1$ equations of $\mathcal{K}$. The last one $m_1 = \frac{S_{2N}\Delta_1}{S_{12}\Delta_N} m_N$ can be derived from A_{23}, A_{1N} and $\frac{S_{13}}{S_{23}} = \frac{S_{1N}}{S_{2N}}$, one equation which can be derived from $\mathcal{J}$.

"The equations of (8.5) $\Leftarrow \mathcal{K} \cup \mathcal{J}$": We first show that

$$\mathcal{K} \cup \mathcal{J} \Rightarrow A_1 \cup A_{23}.$$

Let $k = \frac{m_1 m_N S_{1N}}{\Delta_1 \Delta_N}$, then A_{1N} is true. Then the first $N-1$ equations of $\mathcal{K}$ imply $A_{12}, ..., A_{1N}$. It is easy to verify that the last equation of $\mathcal{K}$ and $\frac{S_{13}}{S_{23}} = \frac{S_{1N}}{S_{2N}}$, one equation which can be derived from $\mathcal{J}$, imply A_{23}. Thus we have

$$\mathcal{K} \cup \mathcal{J} \Rightarrow \mathcal{J} \cup A_1 \cup A_{23} \Rightarrow A_1 \cup ... \cup A_{N-1},$$

where we have used Lemma 9. This remark completes the proof. □

Thus to build a Dziobek special central configuration, we first need to find the shape, the solutions of the S-equations, and then the shape determines the masses by the M-equations.

The M-equations imply that Δ_i $(i \geq 2)$ is of the same sign as Δ_1, which is quite different form what happens in the Newtonian N-body problem [2, 3, 61]. This is equivalent to the condition proposed in Theorem 20: the configuration $\mathbf{q}$ does not lie in a closed hemisphere. In other words, there is some k such that $\Delta_1\Delta_k = (-1)^{k+1}|X_1||X_k| < 0$ if and only if $\mathbf{q}$ lies in a closed hemisphere. For example, let $\Delta_1\Delta_2 = -|X_1||X_2| < 0$. Consider the subspace V spanned by $(\mathbf{q}_3, ..., \mathbf{q}_N)$. Its dimension should be $N-2$. If not, then $\Delta_1 = \Delta_2 = 0$ and (8.5) can't hold. Thus there is a vector $\mathbf{u}$ such that the orthogonal complement of V is $V^{\perp} = t\mathbf{u}$. Then assume

$$\mathbf{q}_1 = t_1\mathbf{u} + \sum_{j=3}^{N} t_{j1}\mathbf{q}_j, \quad \mathbf{q}_2 = t_2\mathbf{u} + \sum_{j=3}^{N} t_{j2}\mathbf{q}_j,$$

and we obtain $|X_1| = t_1|\mathbf{u}, \mathbf{q}_3, ..., \mathbf{q}_N|$ and $|X_2| = t_2|\mathbf{u}, \mathbf{q}_3, ..., \mathbf{q}_N|$. Thus $|X_1||X_2| > 0$ implies that $t_1t_2 > 0$. Assume that t_1 and t_2 are both positive. Consider the coordinates system of $\mathbb{R}^{N-1}$ based on the basis $\{\mathbf{u}, \mathbf{q}_3, .., \mathbf{q}_N\}$. The configuration is given by $\mathbf{q}_1 = (t_1 > 0, t_{31}...t_{N1})$, $\mathbf{q}_2 = (t_2 > 0, t_{32}...t_{N2})$, $\mathbf{q}_3 = (0, 1, 0, ...0)$,...,$\mathbf{q}_N = (0, 0, 0, ...1)$, which implies that $\mathbf{q}$ lies in a closed hemisphere.

Back to special central configurations in $\mathbb{S}^3$, Theorem 22 has the following three immediate applications.

Proposition 25 $(N = 3, \mathbb{S}^1)$**.** *Consider three masses $\mathbf{m} \in \mathbb{R}^3_+$ on $\mathbb{S}^1$ at a nonsingular configuration $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$. Then $\mathbf{q}$ is a Dziobek special central configuration if and only if $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ are not all in the same semicircle and the mass vector is*

$$\mathbf{m} = \left(\frac{\sin^3 d_{12}\Delta_1}{\sin^3 d_{23}\Delta_3}, \frac{\sin^3 d_{12}\Delta_2}{\sin^3 d_{13}\Delta_3}, 1 \right) m_3.$$

Obviously, the equilateral triangle special central configuration on $\mathbb{S}^1$ of three equal masses presented in Example 10 satisfies all of these conditions. The shape

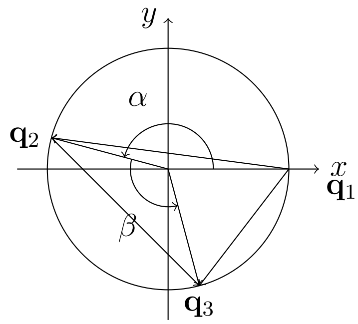

Figure 8.2: An acute triangle special central configuration

constraint is mild here. For a Dziobek special central configuration in the ordering $0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi$, where $\mathbf{q}_i = (\cos\varphi_i, \sin\varphi_i)$, the constraints are $\varphi_2 - \varphi_1 < \pi$, $\varphi_3 - \varphi_1 < \pi$ and $2\pi - \varphi_3 < \pi$. That implies, $\angle\mathbf{q}_3 = 1/2(\varphi_2 - \varphi_1) < \frac{\pi}{2}$ and similarly $\angle\mathbf{q}_1 < \frac{\pi}{2}$, $\angle\mathbf{q}_1 < \frac{\pi}{2}$. In other words, the particles form an acute triangle, see Figure 8.2. Let $d_{12} = \alpha, d_{23} = \beta$. Then $d_{13} = 2\pi - (\alpha + \beta)$ and

$$0 < \alpha < \pi, \quad 0 < \beta < \pi, \quad \pi < \alpha + \beta < 2\pi. \tag{8.6}$$

Notice that on $\mathbb{S}^1$, we have $\Delta_1 = |\mathbf{q}_2, \mathbf{q}_3| = \sin d_{23} = \sin\beta$, $\Delta_2 = -|\mathbf{q}_1, \mathbf{q}_3| = \sin d_{13} = \sin(\alpha+\beta)$, $\Delta_3 = |\mathbf{q}_1, \mathbf{q}_2| = \sin d_{12} = \sin\alpha$. Thus the masses satisfy

$$\frac{m_2}{\sin^2\alpha} = \frac{m_3}{\sin^2(\alpha+\beta)}, \qquad \frac{m_1}{\sin^2\alpha} = \frac{m_3}{\sin^2\beta}, \qquad \frac{m_2}{\sin^2\beta} = \frac{m_1}{\sin^2(\alpha+\beta)}. \tag{8.7}$$

These central configurations have been found by several different methods [19, 32, 91]. We will refer to them as *acute triangle special central configurations*. In Chapter 9, we will study the stability of the associated relative equilibria.

Proposition 26 ($N = 4$, $\mathbb{S}^2$)**.** *Consider four masses $\mathbf{m} \in \mathbb{R}^4_+$ on $\mathbb{S}^2$ at a nonsingular configuration $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$. Then $\mathbf{q}$ is a Dziobek special central configuration if and only if $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ are not all in the same hemisphere;*

$$\sin d_{12} \sin d_{34} = \sin d_{13} \sin d_{24} = \sin d_{14} \sin d_{23}$$

are satisfied and the mass vector is

$$\mathbf{m} = \left(\frac{\sin^3 d_{12}\Delta_1}{\sin^3 d_{24}\Delta_4}, \frac{\sin^3 d_{12}\Delta_2}{\sin^3 d_{14}\Delta_4}, \frac{\sin^3 d_{13}\Delta_3}{\sin^3 d_{14}\Delta_4}, 1 \right) m_4.$$

Obviously, the regular tetrahedron special central configuration on $\mathbb{S}^2$ of four equal masses presented in Example 11 satisfies all of these conditions.

Proposition 27 ($N = 5$, $\mathbb{S}^3$)**.** *Consider five masses $\mathbf{m} \in \mathbb{R}^5_+$ in $\mathbb{S}^3$ at a nonsingular configuration $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5)$. Then $\mathbf{q}$ is a Dziobek special central configuration if and only if $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5$ are not all in the same hemisphere; the five constraints*

$$\begin{aligned} &\sin d_{13} \sin d_{24} = \sin d_{14} \sin d_{23}, && \sin d_{13} \sin d_{25} = \sin d_{15} \sin d_{23}, \\ &\sin d_{24} \sin d_{35} = \sin d_{34} \sin d_{25}, && \sin d_{12} \sin d_{34} = \sin d_{14} \sin d_{23}, \\ &\sin d_{23} \sin d_{45} = \sin d_{34} \sin d_{25} \end{aligned}$$

are satisfied and the mass vector is

$$\mathbf{m} = \left(\frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{25} \Delta_5}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{15} \Delta_5}, \frac{\sin^3 d_{13} \Delta_3}{\sin^3 d_{15} \Delta_5}, \frac{\sin^3 d_{14} \Delta_4}{\sin^3 d_{15} \Delta_5}, 1 \right) m_5.$$

Obviously, the regular pentatope special central configuration in $\mathbb{S}^3$ of five equal masses presented in Example 11 satisfies all of these conditions.

8.3 Special Central Configurations in the 3- and 4-Body Problem

We show that all special central configurations of three bodies are Dziobek special central configurations. We have found all such special central configurations in the last section, which enable us to find the mass set M_3 [33].

Proposition 28. *Every special central configuration of three bodies in $\mathbb{S}^3$ occurs on a great circle. Every special central configuration of four bodies in $\mathbb{S}^3$ occurs on a great 2-sphere.*

Proof. We only prove this for $N = 3$. By the equations (8.1), $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ are linearly dependent. Thus they must be on a great circle. □

Thus all special central configurations of three bodies are Dziobek special central configurations. They are acute triangle special central configurations on $\mathbb{S}^1$. Let $\mathbf{q}_i = (\cos \varphi_i, \sin \varphi_i)$. If they are ordered as $0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi$, then one such

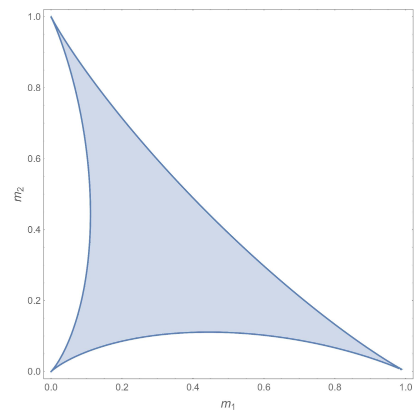

Figure 8.3: M_3 projected onto m_1m_2 plane

configurations is determined by a point (α, β) in the open region $\mathcal{U}$ in $\mathbb{R}^2$,

$$0 < \alpha < \pi, \quad 0 < \beta < \pi, \quad \pi < \alpha + \beta < 2\pi.$$

where $\alpha = d_{12}$ and $\beta = d_{23}$, see Figure 8.2. By (8.7), we have $\frac{m_2}{\sin^2 \alpha} = \frac{m_3}{\sin^2(\alpha+\beta)}$, $\frac{m_1}{\sin^2 \alpha} = \frac{m_3}{\sin^2 \beta}$, $\frac{m_2}{\sin^2 \beta} = \frac{m_1}{\sin^2(\alpha+\beta)}$, i.e., each point (α, β) determines one unique mass vector $\mathbf{m} = m_3 \left(\frac{\sin^2 \alpha}{\sin^2 \beta}, \frac{\sin^2 \alpha}{\sin^2(\alpha+\beta)}, 1 \right)$. Thus (8.7) determines a map,

$$\chi : \mathcal{U} \to (0,1) \times (0,1),$$
$$(\alpha, \beta) \mapsto (m_1, m_2) = \frac{1}{\tau} \left(\frac{\sin^2 \alpha}{\sin^2 \beta}, \frac{\sin^2 \alpha}{\sin^2(\alpha + \beta)} \right),$$

where $\tau = \frac{\sin^2 \alpha}{\sin^2 \beta} + \frac{\sin^2 \alpha}{\sin^2(\alpha+\beta)} + 1$.

Thus to find M_3, the set in $\mathbb{R}^3_+$ with $\sum_{i=1}^3 m_i = 1$ that possess special central configurations, is to find the image of the map χ.

Theorem 23.

$$M_3 = \{\mathbf{m} \in \mathbb{R}^3_+ | m_1 + m_2 + m_3 = 1, \; m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2 - 2m_1 m_2 m_3 < 0\},$$

see Figure 8.4*. For any* $\mathbf{m} \in M_3$*, there are two special central configurations, one for each ordering of the masses on* $\mathbb{S}^1$*.*

Proof. Our approach is to find the inverse map $(\alpha, \beta) = \chi^{-1}(m_1, m_2)$. Then inequalities (8.6) would yield inequalities for (m_1, m_2), which would give $\mathcal{V} := \chi(\mathcal{U})$.

By (8.7), we set $u = \sqrt{\frac{m_3}{m_1}} = \frac{\sin\beta}{\sin\alpha}$, $v = \sqrt{\frac{m_3}{m_2}} = -\frac{\sin(\alpha+\beta)}{\sin\alpha}$. Then

$$-v = \cos\beta + \frac{\sin\beta}{\sin\alpha}\cos\alpha = \pm\sqrt{1-u^2\sin^2\alpha} + u\cos\alpha,$$

$$(-u\cos\alpha - v)^2 = u^2\cos^2\alpha + 2uv\cos\alpha + v^2 = 1 - u^2\sin^2\alpha,$$

and we obtain

$$\begin{aligned} \cos\alpha &= \frac{1-u^2-v^2}{2uv} = \frac{m_1m_2 - m_3(m_1+m_2)}{2m_3\sqrt{m_1m_2}} \\ \cos\beta &= -v - u\cos\alpha = \frac{u^2-v^2-1}{2v} = \frac{m_3(m_2-m_1)-m_1m_2}{2m_1\sqrt{m_2m_3}}. \end{aligned} \tag{8.8}$$

Both $\cos\alpha$ and $\cos\beta$ are injective on $\mathcal{U}$, so the inverse map χ^{-1} is

$$\alpha = \cos^{-1}\frac{m_1m_2 - m_3(m_1+m_2)}{2m_3\sqrt{m_1m_2}}, \quad \beta = \cos^{-1}\frac{m_3(m_2-m_1)-m_1m_2}{2m_1\sqrt{m_2m_3}}.$$

To find $\chi(\mathcal{U})$, note that condition (8.6) is equivalent to

$$\cos^2\alpha < 1, \quad \cos^2\beta < 1, \quad \sin(\alpha+\beta) < 0.$$

Substituting (8.8) into the above inequalities and using the normalization condition $m_1 + m_2 + m_3 = 1$, we find that the three inequalities lead to the same relationship:

$$m_1^2m_2^2 + m_2^2m_3^2 + m_1^2m_3^2 - 2m_1m_2m_3 < 0,$$

which is the image of the map χ, and it gives M_3.

Since the inverse map exists, for one mass vector in M_3, and the ordering $0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi$, there is one and only one special central configuration. For the second ordering, $0 = \varphi_1 < \varphi_3 < \varphi_2 < 2\pi$, reflecting it about the y-axis changes the ordering to $0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi$, thus there is also one and only one special central configuration for this ordering. This remark completes the proof. □

Corollary 14. *For each acute isosceles triangle special central configuration, if the masses m_1 and m_3 are at the vertices of the base, then $m_1 = m_3$ and $m_1 < 4m_2$. Reciprocally, if $m_1 = m_3$ and $m_1 < 4m_2$, then the only kind of special central configuration these three masses can form is an acute isosceles triangle.*

Proof. Let the special central configuration be an acute isosceles triangle, for instance, $\alpha = \beta$, then by (8.7), we have $m_1 = m_3$. Consequently

$$m_1^2 m_2^2 + m_2^2 m_3^2 + m_1^2 m_3^2 - 2m_1 m_2 m_3 = m_1^2 \left(m_1^2 + 2m_2^2 - 2m_2\right) < 0.$$

Therefore $m_1^2 < 2m_2(1 - m_2) = 2m_2 \cdot 2m_1$, which implies that $m_1 < 4m_2$.

Reciprocally, if $m_1 = m_3$ and $m_1 < 4m_2$, then the bodies can form an acute isosceles triangle special central configuration by the first part of this corollary. They can only form such configuration by Theorem 23. □

Chapter 9

Stability of the Associated Relative Equilibria

In this chapter, we study the stability of the relative equilibria associated with the acute triangle special central configuration. The result has been published in [33]. These special central configurations remain special central configurations after any rotation in $SO(4)$. We assume that they are on $\mathbb{S}^1_{xy} \subset \mathbb{S}^2_{xyz}$. In this chapter, we use $\mathbb{S}^1$ for $\mathbb{S}^1_{xy}$ and $\mathbb{S}^2$ for $\mathbb{S}^2_{xyz}$. The associated relative equilibria are on $\mathbb{S}^1$. We first show that these orbits are Lyapunov stable on $\mathbb{S}^1$. Then we study their linear stability on $\mathbb{S}^2$. We find that their linear stability depends on the angular velocity ω. More precisely, the associated relative equilibria are linearly stable if and only if the angular velocity is greater than a certain value determined by the configuration.

9.1 The Setup

Recall that the curved N-body problem is a Hamiltonian system. For our purpose, we confine the system to $\mathbb{S}^2$. Using the spherical coordinates $\mathbf{q}_i = (x_i, y_i, z_i) = (\sin\theta_i \cos\varphi_i, \sin\theta_i \sin\varphi_i, \cos\theta_i)$, the kinetic energy has the form

$$T(\varphi_1, \theta_1, \dots, \varphi_N, \theta_N) = \sum_{i=1}^{N} \frac{m_i}{2} \left(\dot{\theta}_i^2 + \sin^2\theta_i \dot{\varphi}_i^2 \right),$$

and the conjugate momenta are given by

$$p_{\theta_i} = m_i \dot{\theta}_i, \quad p_{\varphi_i} = m_i \sin^2\theta_i \dot{\varphi}_i, \ i = 1, \dots, N.$$

The force function is unchanged,

$$U(\varphi_1, \theta_1, \cdots, \varphi_N, \theta_N) = \sum_{1 \le i < j \le N} m_i m_j \cot d_{ij},$$

and the Hamiltonian is $H = T - U$. The symplectic form is $w = d\left(\sum_i p_{\theta_i} d\theta_i + p_{\varphi_i} d\varphi_i\right)$.

Recall that all the special central configurations of three bodies, the acute triangle special central configurations, have been found in Chapter 8, see Figure 9.1. Let $\mathbf{q}_i = (\cos\varphi_i, \sin\varphi_i, 0)$. If they are ordered as $0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi$, then one such configuration is determined by a point $(\alpha = \varphi_2 - \varphi_1, \beta = \varphi_3 - \varphi_2)$ in the open region $\mathcal{U}$ in $\mathbb{R}^2$,

$$0 < \alpha < \pi, \quad 0 < \beta < \pi, \quad \pi < \alpha + \beta < 2\pi.$$

For each such configuration, there are three masses that form a special central configuration, see (8.7), i.e.,

$$\frac{m_2}{\sin^2\alpha} = \frac{m_3}{\sin^2(\alpha+\beta)}, \; \frac{m_1}{\sin^2\alpha} = \frac{m_3}{\sin^2\beta}, \; \frac{m_2}{\sin^2\beta} = \frac{m_1}{\sin^2(\alpha+\beta)}.$$

By Corollary 1, there is a 2-parameter family of relative equilibria associated to each acute triangle special central configuration: $A_{\omega,\eta}(t)\mathbf{q}$ with $\omega, \eta \in \mathbb{R}$. Since $z_i = w_i = 0$, the rotation in the zw space does not affect $\mathbf{q}$. Thus there is only a one-parameter family of relative equilibria: $A_{\omega,0}(t)\mathbf{q}$ with $\omega \in \mathbb{R}$. We are going to study their stability on $\mathbb{S}^1$ and $\mathbb{S}^2$. We will call the relative equilibria with $\omega = 0$ *fixed-point solutions*. In the spherical coordinates, these solutions have the form:

$$\theta_i(t) = \frac{\pi}{2}, \quad \varphi_i(t) = \varphi_i + \omega t, \quad p_{\theta_i} = 0, \quad p_{\varphi_i} = m_i\omega, \quad i = 1, 2, 3.$$

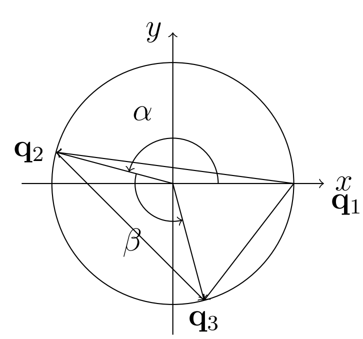

Figure 9.1: An acute triangle special central configuration

9.2 Reduction and Stability on $\mathbb{S}^1$

It is easy to see that if the initial velocity of each particle is confined to the tangent space $T(\mathbb{S}^1)$, then the motion takes place in $T^*(\mathbb{S}^1)^3$ forever, where $T^*(\mathbb{S}^1)$ is the cotangent bundle of $\mathbb{S}^1$. In other words, $T^*(\mathbb{S}^1)^3$ is an invariant manifold of the equations of motion. In this section, we are going to study the stability of the relative equilibria associated with all acute triangle special central configurations on this invariant manifold. We first perform the reduction of the 3-body problem on $T^*(\mathbb{S}^1)^3$ and then prove that all these solutions are Lyapunov stable.

Confined to $T^*(\mathbb{S}^1)^3$, the Hamiltonian system takes the form

$$H(\varphi_i, p_{\varphi_i}) = \sum_i \frac{1}{2}\frac{p_{\varphi_i}^2}{m_i} - \sum_{j\neq i} m_i m_j \cot d_{ij}, \;\; w = d\left(\sum_i p_{\varphi_i} d\varphi_i\right).$$

It is easy to see that in these coordinates, the angular momentum integral, see (2.2), is

$$\omega_{xy} = \sum_{i=1}^{N} m_i(\dot{x}_i y_i - x_i \dot{y}_i) = p_{\varphi_1} + p_{\varphi_2} + p_{\varphi_3}.$$

We further restrict the study of the stability of relative equilibria to the quotient space S_p' of $T^*(\mathbb{S}^1)^3$. Define

$$S_p := \{(\mathbf{q}, \mathbf{p}) \in T^*(\mathbb{S}^1)^3 : \omega_{xy} = p\},$$

and let S_p' be the quotient space under the Lie group $SO(2)$. For all values of p, these spaces are always smooth manifolds and have dimension 4. We can apply this procedure to the fixed-point solutions as well. The advantage of this approach is to eliminate the drift caused by the $SO(2)$ symmetry on the bodies. Indeed, by symmetry, if we perturb a fixed-point solution with some initial velocity $\dot{\varphi}_i = \epsilon$, then we obtain a relative equilibrium with angular velocity ϵ. We can therefore introduce the definition below.

Definition 13. *A relative equilibrium associated with an acute triangle special central configuration is Lyapunov stable on $\mathbb{S}^1$ if the corresponding rest point of the reduced Hamiltonian system on the quotient space S_p' is Lyapunov stable.*

Thus we need to explicitly find the quotient manifold S_p' and the *reduced Hamiltonian H_p* (see [82] for a theoretical approach to this kind of problem). The angular momentum here behaves like the linear momentum of the Newtonian N-body problem. Thus to perform the reduction, we introduce a type of Jacobi coordinates [57]:

$$\tilde{\varphi} = \frac{1}{\bar{m}}(m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3), \quad \phi_1 = \varphi_2 - \varphi_1, \quad \phi_2 = \varphi_3 - \nu_1\varphi_1 - \nu_2\varphi_2,$$

where

$$\bar{m} = m_1 + m_2 + m_3, \quad \nu_1 = \frac{m_1}{m_1 + m_2}, \quad \nu_2 = \frac{m_2}{m_1 + m_2}.$$

The corresponding conjugate momenta are then given by

$$p_{\tilde{\varphi}} = \bar{m}\dot{\tilde{\varphi}}, \quad p_{\phi_1} = \nu_3\dot{\phi}_1, \quad p_{\phi_2} = \nu_4\dot{\phi}_2,$$

where $\nu_3 = \frac{m_1 m_2}{m_1 + m_2}$ and $\nu_4 = \frac{(m_1+m_2)m_3}{\bar{m}}$. It is easy to verify that

$$p_{\tilde{\varphi}}d\tilde{\varphi} + p_{\phi_1}d\phi_1 + p_{\phi_2}d\phi_2 = p_{\varphi_1}d\varphi_1 + p_{\varphi_2}d\varphi_2 + p_{\varphi_3}d\varphi_3$$

and that the Hamiltonian is

$$H = \frac{1}{2}\left(\frac{p_{\tilde{\varphi}}^2}{\bar{m}} + \frac{p_{\phi_1}^2}{\nu_3} + \frac{p_{\phi_2}^2}{\nu_4}\right) - U(\phi_1, \phi_2),$$

where $U(\phi_1, \phi_2)$ is the force function in the new variables. Note that $\varphi_2 - \varphi_1 = \phi_1 = \alpha$, and

$$\beta = \varphi_3 - \varphi_2 = \phi_2 + \nu_1\varphi_1 + \nu_2\varphi_2 - \varphi_2 = \phi_2 - \nu_1\phi_1.$$

Recall that $d_{12} = \alpha, d_{23} = \beta, d_{13} = 2\pi - (\alpha + \beta)$. Thus

$$\begin{aligned} U(\phi_1, \phi_2) &= m_1 m_2 \cot\alpha + m_2 m_3 \cot\beta - m_1 m_3 \cot(\alpha + \beta) \\ &= m_1 m_2 \cot\phi_1 + m_2 m_3 \cot(\phi_2 - \nu_1\phi_1) - m_1 m_3 \cot(\phi_2 + \nu_2\phi_1). \end{aligned}$$

We now study the stability of the fixed-point solution. In this case $\dot{\varphi}_1 = \dot{\varphi}_2 = \dot{\varphi}_3 = 0$, we have $\bar{m}\dot{\tilde{\varphi}} = 0$, and $\omega_{xy} = 0$. So the quotient manifold is S_0'. On this quotient manifold, $p_{\tilde{\varphi}} = 0$ and we can set $\tilde{\varphi} = 0$ since we identify all points that differ by a rotation. Thus we use $\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}$ as the coordinates of the 4-dimensional manifold S_0'. In these coordinates, we have the reduced Hamiltonian function

$$H_0 = \frac{1}{2}\left(\frac{p_{\phi_1}^2}{\nu_3} + \frac{p_{\phi_2}^2}{\nu_4}\right) - U(\phi_1, \phi_2).$$

Suppose $\mathbf{q}$ is a special central configuration on $\mathbb{S}^1$ of three masses m_1, m_2, m_3. Their positions are given by

$$\varphi_1 = 0,\ \varphi_2 = \alpha,\quad \varphi_3 = \alpha + \beta,$$

(see Figure 9.1). Then the fixed-point solution $(0, \alpha, \alpha+\beta, 0, 0, 0)$ we want to study in $T^*(\mathbb{S}^1)^3$ becomes a rest point of the reduced Hamiltonian system on S_0':

$$\phi_1 = \alpha, \quad \phi_2 = \beta + \nu_1\alpha, \quad p_{\phi_1} = 0, \quad p_{\phi_2} = 0.$$

Let us denote this rest point by X. Our goal is to study the stability of X.

Theorem 24. *Every acute triangle fixed-point solution is Lyapunov stable on* $\mathbb{S}^1$*.*

Proof. We will show that the rest point X is a local minimum of the Hamiltonian H_0. Since the Hamiltonian is preserved during any motion, we can then conclude that the rest point X is stable. The rest point X is obviously a minimum of the kinetic energy. We will show that it is also a local maximum of U by studying its Hessian matrix.

We use the variables (α, β) instead of (ϕ_1, ϕ_2). Then

$$U(\alpha, \beta) = m_1 m_2 \cot\alpha + m_2 m_3 \cot\beta - m_1 m_3 \cot(\alpha+\beta),$$

and the Hessian matrix $H(\alpha, \beta)$ of U at (α, β) has the form

$$H(\alpha,\beta) = 2\begin{bmatrix} \frac{m_1 m_2 \cos\alpha}{\sin^3\alpha} - \frac{m_1 m_3 \cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} & -\frac{m_1 m_3 \cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} \\ -\frac{m_1 m_3 \cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} & \frac{m_2 m_3 \cos\beta}{\sin^3\beta} - \frac{m_1 m_3 \cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} \end{bmatrix}.$$

Recall that equations (8.7) are

$$\frac{m_2}{\sin^2\alpha} = \frac{m_3}{\sin^2(\alpha+\beta)}, \quad \frac{m_2}{\sin^2\beta} = \frac{m_1}{\sin^2(\alpha+\beta)}, \quad \frac{m_1}{\sin^2\alpha} = \frac{m_3}{\sin^2\beta}.$$

By direct computation, we find that the diagonal entries of $\frac{1}{2}H(\alpha,\beta)$ are

$$\frac{m_1 m_2 \cos\alpha}{\sin^3\alpha} - \frac{m_1 m_3 \cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} = \frac{m_1 m_2}{\sin^2\alpha}\frac{\sin\beta}{\sin(\alpha+\beta)\sin\alpha} < 0,$$
$$\frac{m_2 m_3 \cos\beta}{\sin^3\beta} - \frac{m_1 m_3 \cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} = \frac{m_1 m_3}{\sin^2(\alpha+\beta)}\frac{\sin\alpha}{\sin(\alpha+\beta)\sin\beta} < 0.$$

Hence the trace is negative. Here we used the fact that (α, β) satisfies the inequalities

(8.6): $0 < \alpha < \pi$, $0 < \beta < \pi$, $\pi < \alpha + \beta < 2\pi$. Thus $\sin\alpha > 0$, $\sin\beta > 0$, and $\sin(\alpha+\beta) < 0$. Further straightforward computations show that the determinant of $\frac{1}{2}H(\alpha,\beta)$ is

$$\begin{aligned}
&\frac{m_1m_2\cos\alpha}{\sin^3\alpha}\frac{m_2m_3\cos\beta}{\sin^3\beta} - \left(\frac{m_1m_2\cos\alpha}{\sin^3\alpha} + \frac{m_2m_3\cos\beta}{\sin^3\beta}\right)\frac{m_1m_3\cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} \\
&= \frac{m_1m_3m_2^2\cos\alpha\cos\beta}{\sin^3\alpha\sin^3\beta} - \frac{m_1m_3}{\sin^2(\alpha+\beta)}\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}\frac{m_1m_3\cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} \\
&= \frac{m_1m_3m_2^2}{\sin^3\alpha\sin^3\beta}(\cos\alpha\cos\beta - \cos(\alpha+\beta)) = \frac{m_1m_3m_2^2}{\sin^2\alpha\sin^2\beta} > 0.
\end{aligned}$$

These two facts imply that the two eigenvalues of $H(\alpha,\beta)$ are both negative. Then the special central configuration is a local maximum of U. We can thus conclude that the rest point $X = (\alpha, \beta + \nu_1\alpha, 0, 0)$ is a local minimum of $H = T - U$. This remark completes the proof. □

We further study the stability of the other associated relative equilibria $A_{\omega,0}(t)\mathbf{q}$ with $\omega \neq 0$. In this case $\dot\varphi_1 = \dot\varphi_2 = \dot\varphi_3 = \omega$, we have $\bar m\dot{\tilde\varphi} = \bar m\omega$, and $\omega_{xy} = \bar m\omega$. So the quotient manifold is $S'_{\bar m\omega}$. On this quotient manifold, $p_{\tilde\varphi} = \bar m\omega$ and we can set $\tilde\varphi = 0$ since we identify all points that differ by a rotation. Thus we use $\phi_1, \phi_2, p_{\phi_1}, p_{\phi_2}$ as the coordinates of the 4-dimensional manifold $S'_{\bar m\omega}$. Under these coordinates, we have the reduced Hamiltonian

$$H_{\bar m\omega} = \frac{1}{2}\left(\frac{p_{\phi_1}^2}{\nu_3} + \frac{p_{\phi_2}^2}{\nu_4}\right) - U(\phi_1,\phi_2) + \frac{\bar m\omega^2}{2}.$$

Then the relative equilibrium $A_{\omega,0}(t)\mathbf{q}$ in $T^*(\mathbb{S}^1)^3$ becomes a rest point in $S'_{\bar m\omega}$,

$$\phi_1 = \alpha,\ \phi_2 = \beta + \nu_1\alpha,\ p_{\phi_1} = 0,\ p_{\phi_2} = 0.$$

Let us denote this rest point by X_1. Note that H_0 and $H_{\bar m\omega}$ are the same up to a constant. Then we can conclude that X_1 is also a local minimum of $H_{\bar m\omega}$. So we have proved the following result.

Theorem 25. *Every relative equilibrium associated to an acute triangle special central configuration is Lyapunov stable on* $\mathbb{S}^1$.

9.3 Stability on $\mathbb{S}^2$

In this section we study the linear stability of the these solutions on $\mathbb{S}^2$. Unlike in the previous case, their stability depends on the angular velocity ω. We first introduce rotating coordinates to treat a general relative equilibrium on $\mathbb{S}^2$ as a rest point, and obtain the linearized system $\dot{v} = L(v - X_\omega)$. We then compute L for relative equilibria associated with special central configurations on the equator. As in the Newtonian N-body problem, we study the stability of the rest points on a proper subspace [60, 62]. We show that these solutions are linearly stable if and only if ω^2 is greater than a critical value.

So consider a general relative equilibrium on $\mathbb{S}^2$ with angular velocity ω and introduce the rotating coordinates

$$\overline{\theta}_i = \theta_i,\ \ \dot{\overline{\theta}}_i = \dot{\theta}_i,\ \ \overline{\varphi}_i = \varphi_i - \omega t,\ \ \dot{\overline{\varphi}}_i = \dot{\varphi}_i - \omega,\ \ \overline{p}_{\theta_i} = p_{\theta_i},\ \ \overline{p}_{\varphi_i} = p_{\varphi_i}, i = 1, ..., N.$$

In these new coordinates, the original Hamiltonian system

$$\begin{aligned} \dot{\theta}_i &= \frac{\partial H}{\partial p_{\theta_i}} = \frac{p_{\theta_i}}{m_i}, & \dot{p}_{\theta_i} &= -\frac{\partial H}{\partial \theta_i} = \frac{p_{\varphi_i}^2 \cos\theta_i}{m_i \sin^3\theta_i} + \frac{\partial U}{\partial \theta_i}, \\ \dot{\varphi}_i &= \frac{\partial H}{\partial p_{\varphi_i}} = \frac{p_{\varphi_i}}{m_i \sin^2\theta_i}, & \dot{p}_{\varphi_i} &= -\frac{\partial H}{\partial \varphi_i} = \frac{\partial U}{\partial \varphi_i},\ \ i = 1, ..., N, \end{aligned}$$

becomes

$$\begin{aligned} \dot{\overline{\theta}}_i &= \frac{\overline{p}_{\theta_i}}{m_i}, & \dot{\overline{p}}_{\theta_i} &= \frac{\overline{p}_{\varphi_i}^2 \cos\overline{\theta}_i}{m_i \sin^3\overline{\theta}_i} + \frac{\partial U}{\partial \overline{\theta}_i}, \\ \dot{\overline{\varphi}}_i &= \frac{\overline{p}_{\varphi_i}}{m_i \sin^2\overline{\theta}_i} - \omega, & \dot{\overline{p}}_{\varphi_i} &= \frac{\partial U}{\partial \overline{\varphi}_i},\ \ i = 1, ..., N. \end{aligned}$$

This system is Hamiltonian with

$$H = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{\overline{p}_{\theta_i}^2}{m_i} + \frac{\overline{p}_{\varphi_i}^2}{m_i \sin^2\overline{\theta}_i} - 2\overline{p}_{\varphi_i}\omega\right) - U$$

and symplectic form $w = d\left(\sum_{i=1}^{n} \overline{p}_{\theta_i} d\overline{\theta}_i + \overline{p}_{\varphi_i} d\overline{\varphi}_i\right)$.

We will use $\theta_i, \varphi_i, p_{\theta_i}, p_{\varphi_i}$ instead of $\overline{\theta}_i, \overline{\varphi}_i, \overline{p}_{\theta_i}\overline{p}_{\varphi_i}$ if no further confusion arises. Denote by X_ω the rest point in the new Hamiltonian system that corresponds to a relative equilibrium $A_{\omega,0}(t)\mathbf{q}$. Then X_ω is

$$\theta_i(t)=\theta_i(0),\quad \varphi_i(t)=\varphi_i(0),\quad p_{\theta_i}=0,\quad p_{\varphi_i}=\omega m_i\sin^2\theta_i,$$

and we are going to study the stability of X_ω for the linearized system

$$\dot{v}=L(v-X_\omega),\quad v=(\theta_1,..,\theta_N,\varphi_1,..,\varphi_N,p_{\theta_1},..,p_{\theta_N},p_{\varphi_1},..,p_{\varphi_N}).$$

By straightforward computation, we obtain .

$$L=\begin{bmatrix} 0 & 0 & M^{-1} & 0\\ K & 0 & 0 & M^{-1}C^{-1}\\ \frac{\partial^2 U}{\partial\theta_i\partial\theta_j}+R & \frac{\partial^2 U}{\partial\theta_i\partial\varphi_j} & 0 & -K^T\\ \frac{\partial^2 U}{\partial\varphi_i\partial\theta_j} & \frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j} & 0 & 0\end{bmatrix},\quad R=\begin{bmatrix} \frac{-p_{\varphi_1}^2(1+2\cos^2\theta_1)}{m_1\sin^4\theta_1} & \cdots & 0\\ 0 & \frac{-p_{\varphi_2}^2(1+2\cos^2\theta_2)}{m_2\sin^4\theta_2} & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \frac{-p_{\varphi_N}^2(1+2\cos^2\theta_N)}{m_N\sin^4\theta_N}\end{bmatrix},$$

$$K=\begin{bmatrix} \frac{-2p_{\varphi_1}\cos\theta_1}{m_1\sin^3\theta_1} & \cdots & 0\\ 0 & \frac{-2p_{\varphi_2}\cos\theta_2}{m_2\sin^3\theta_2} & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \frac{-2p_{\varphi_N}\cos\theta_N}{m_N\sin^3\theta_N}\end{bmatrix},\quad M^{-1}=\begin{bmatrix} \frac{1}{m_1} & \cdots & 0\\ 0 & \frac{1}{m_2} & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \frac{1}{m_N}\end{bmatrix},\quad C^{-1}=\begin{bmatrix} \frac{1}{\sin^2\theta_1} & \cdots & 0\\ 0 & \frac{1}{\sin^2\theta_2} & 0\\ \vdots & \cdots & \vdots\\ 0 & \cdots & \frac{1}{\sin^2\theta_N}\end{bmatrix},$$

where L is a $4N\times 4N$ matrix, R, K, M^{-1}, and C^{-1} are $N\times N$ matrices.

It is generally difficult to find the normal form of L. However, for relative equilibria associated to a special central configuration on the equator, things are easier. Direct computation leads to the following result.

Lemma 10. *For special central configurations on the equator,* $L=\begin{bmatrix} 0 & 0 & M^{-1} & 0\\ 0 & 0 & 0 & M^{-1}\\ H_1(\omega) & 0 & 0 & 0\\ 0 & H_2 & 0 & 0\end{bmatrix}$, *where* $H_1(\omega)=\left[\frac{\partial^2 U}{\partial\theta_i\partial\theta_j}\right]-\omega^2 M$ *and* $H_2=\left[\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j}\right]$. *The elements of* $H_1(0)=\left[\frac{\partial^2 U}{\partial\theta_i\partial\theta_j}\right]$

and H_2 are

$$[H_1(0)]_{ij} = \frac{m_i m_j}{\sin^3 d_{ij}}, \qquad [H_1(0)]_{ii} = -\sum_{j\neq i, j=1}^{n} [H_1(0)]_{ij}\cos d_{ij},$$

$$[H_2]_{ij} = \frac{-2m_i m_j \cos d_{ij}}{\sin^3 d_{ij}}, \qquad [H_2]_{ii} = -\sum_{j\neq i, j=1}^{n} [H_2]_{ij}.$$

We use the notations $H_1(\omega)$ and H_2 to indicate that the lower left $2N \times 2N$ block is related to the Hessian of U at the special central configuration on $\mathbb{S}^2$. Especially, H_2 is just the Hessian of U at the special central configuration on $\mathbb{S}^1$. We can find the normal form of L by finding the normal forms of $H_1(\omega)M^{-1}$ and H_2M^{-1}.

Lemma 11. *$H_1(\omega)M^{-1}$ and H_2M^{-1} are diagonalizable. If $\mathbf{u} \in \mathbb{C}^N$ is an eigenvector of $H_1(\omega)M^{-1}$ (H_2M^{-1}) with eigenvalue $\lambda \neq 0$, then there exists a 2-dimensional invariant subspace of L in $\mathbb{C}^{4N}$ on which L is $\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix}$. If $\mathbf{u} \in \mathbb{C}^N$ is an eigenvector of $H_1(\omega)M^{-1}$ (H_2M^{-1}) with eigenvalue 0, then there exists a 2-dimensional invariant subspace of L in $\mathbb{C}^{4N}$ on which L is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.*

Proof. It is enough to prove this for $H_1(\omega)M^{-1}$. Note $H_1(\omega)$ is symmetric, thus $H_1(\omega)M^{-1}$ is symmetric with respect to the inner product $\langle \mathbf{v}, \mathbf{u}\rangle = \mathbf{v}^T M^{-1}\mathbf{u}$ so it is diagonalizable with respect to some M^{-1} orthogonal basis. Now suppose $H_1(\omega)M^{-1}\mathbf{u} = \lambda\mathbf{u}$, $\lambda \neq 0$. Then .

$$\begin{bmatrix} 0 & 0 & M^{-1} & 0 \\ 0 & 0 & 0 & M^{-1} \\ H_1(\omega) & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{M^{-1}\mathbf{u}}{\sqrt{\lambda}} & \frac{M^{-1}\mathbf{u}}{-\sqrt{\lambda}} \\ 0 & 0 \\ \mathbf{u} & \mathbf{u} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M^{-1}\mathbf{u} & M^{-1}\mathbf{u} \\ 0 & 0 \\ \sqrt{\lambda}\mathbf{u} & -\sqrt{\lambda}\mathbf{u} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{M^{-1}\mathbf{u}}{\sqrt{\lambda}} & \frac{M^{-1}\mathbf{u}}{-\sqrt{\lambda}} \\ 0 & 0 \\ \mathbf{u} & \mathbf{u} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix}.$$

Similarly, if $H_1(\omega)M^{-1}\mathbf{u} = 0$, then

$$\begin{bmatrix} 0 & 0 & M^{-1} & 0 \\ 0 & 0 & 0 & M^{-1} \\ H_1(\omega) & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} M^{-1}\mathbf{u} & 0 \\ 0 & 0 \\ 0 & \mathbf{u} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}\mathbf{u} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M^{-1}\mathbf{u} & 0 \\ 0 & 0 \\ 0 & \mathbf{u} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This remark completes the proof. □

Normally, a rest point X_ω is called linearly stable if X_ω is a stable rest point of the linearized system. However, as for relative equilibria of the curved N-body problem, the symmetries and integrals of the problem make it impossible to satisfy this condition.

Recall that a special central configuration on $\mathbb{S}^2$ remains a special central configuration on $\mathbb{S}^2$ after any rotation in $SO(3)$, which is a 3-dimensional Lie group. Let $\boldsymbol{\xi}_i$ $i=1,2,3$, be the basis of $\mathfrak{s}o(3)$, we see that each $\boldsymbol{\xi}_i\mathbf{q}$ is a null vector of the Hessian $\begin{bmatrix} H_1(0) & 0 \\ 0 & H_2 \end{bmatrix}$.

Lemma 12. *Let $H_1(0)$ and H_2 be the matrices defined in Lemma 10. Then*

$$H_1(0)\mathbf{v}_1 = 0, \quad H_1(0)\mathbf{v}_2 = 0, \quad H_2\mathbf{v}_3 = 0,$$

where

$$\mathbf{v}_1 = (y_1, y_2, ..., y_N), \ \mathbf{v}_2 = (x_1, x_2, ..., x_N), \ \mathbf{v}_3 = (1, 1, ..., 1). \tag{9.1}$$

Proof. By Proposition 14, there are at least three null vectors for the Hessian of a special central configuration on $\mathbb{S}^2$, $\boldsymbol{\xi}_i\mathbf{q}$, $i=1,2,3$, where $\boldsymbol{\xi}_i$ is the basis of $\mathfrak{so}(3)$. Recall that the basis of $\mathfrak{so}(3)$ [41] is given by

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which correspond to the three rotations (around the three axes) in $SO(3)$. In xyz-coordinates, $J_x\mathbf{q}_i = (0, -z_i, y_i)$, $J_y\mathbf{q}_i = (-z_i, 0, x_i)$, $J_z\mathbf{q}_i = (-y_i, x_i, 0)$. Thus for special central configurations on $\mathbb{S}^1$, $(z_i = 0)$, the three null vectors are

$$\mathbf{v}_1 = (0, 0, y_1, 0, 0, y_2, ..., 0, 0, y_N), \quad \mathbf{v}_2 = (0, 0, x_1, 0, 0, x_2, ..., 0, 0, x_N),$$
$$\mathbf{v}_3 = (-y_1, x_1, 0, -y_2, x_2, 0, ..., -y_N, x_N, 0).$$

Transforming them into the spherical coordinates, they are

$$\mathbf{v}_1 = y_1\frac{\partial}{\partial\theta_1} + y_2\frac{\partial}{\partial\theta_2} + ... + y_N\frac{\partial}{\partial\theta_N}, \quad \mathbf{v}_2 = x_1\frac{\partial}{\partial\theta_1} + x_2\frac{\partial}{\partial\theta_2} + ... + x_N\frac{\partial}{\partial\theta_N},$$
$$\mathbf{v}_3 = \frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2} + ... + \frac{\partial}{\partial\varphi_N}.$$

Thus we obtain $H_1(0)\mathbf{v}_1 = 0$, $H_1(0)\mathbf{v}_2 = 0$, and $H_2\mathbf{v}_3 = 0$. This remark completes the proof. □

Now we consider the stability of the fixed-point solutions, i.e, $\omega = 0$. In this case, $H_1(\omega)$ has at least two null vectors and H_2 has at least one null vector. Consider the 6-dimensional subspace E_1 of $\mathbb{C}^{4N}$ spanned by the vectors

$$\begin{bmatrix} \mathbf{v}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ M\mathbf{v}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{v}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ M\mathbf{v}_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{v}_3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ M\mathbf{v}_3 \end{bmatrix}. \tag{9.2}$$

Then Lemma 11 implies that E_1 is an invariant subspace for L. The matrix of $L|_{E_1}$ in this basis is

$$L|_{E_1} = \text{diag}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Though all eigenvalues on E_1 are 0, there are three nontrivial Jordan blocks, a fact which implies that the rest point is not linearly stable in the conventional sense. This instability is trivial as a natural effect of the symmetry of this Hamiltonian system. Indeed, we can perturb a fixed-point solution into a relative equilibrium by any rotation in $SO(3)$. Then the angular positions of these orbits drift away from each other, a property mathematically reflected by the nontrivial Jordan blocks, as remarked in [60].

It is traditional in celestial mechanics to view the drifts in this subspace as harmless. Indeed, they can be eliminated by fixing the angular momentum and passing to a quotient manifold under the action of the rotational symmetry group. Thus it is reasonable to formulate a definition of linear stability based on the behaviour of L in a complementary subspace [60]. To define such a subspace, it is necessary to introduce the *skew inner product* of two complex vectors $\mathbf{v}, \mathbf{u} \in \mathbb{C}^{4N}$:

$$\Omega(\mathbf{v}, \mathbf{u}) = \mathbf{v}^T J\mathbf{u}, \quad J_{4N\times 4N} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Using the fact that $L = JS$, $S^T = S$, where S is the Hessian matrix of the Hamiltonian at X_0, we obtain that

$$\Omega(\mathbf{v}, L\mathbf{u}) = -\Omega(L\mathbf{v}, \mathbf{u}).$$

With the help of this property it is easy to show that the skew-orthogonal complement of an invariant subspace of L is again invariant. Indeed, let E denote the skew orthogonal complement in $\mathbb{C}^{4N}$ of E_1, that is,

$$E = \{\mathbf{v} \in \mathbb{C}^{4N} : \Omega(\mathbf{v}, \mathbf{u}) = 0 \text{ for all } \mathbf{u} \in E_1\}.$$

Then E is an L invariant subspace of dimension $4N - 6$.

Definition 14. *A fixed-point solutions X_0 associated with a special central configuration on $\mathbb{S}^1$ is called* linearly stable *if X_0 is a stable rest point of the linearized equation restricted to E.*

For $N = 3$, i.e., the relative equilibria associated with acute triangle special central configurations, we have $E = \mathbb{C}^6$, and we can find the normal form of $L|_E$.

Theorem 26. *For each acute triangle fixed-point solution, $L|_E$ is diagonalizable and in a properly chosen basis,*

$$L|_E = diag\Big\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|}\Big\},$$

where $0, 0, \lambda_1 > 0$ are the eigenvalues of $H_1(0)M^{-1}$, and $0, \lambda_2 < 0, \lambda_3 < 0$ are the eigenvalues of $H_2 M^{-1}$. All acute triangle fixed-point solutions on the equator are unstable on $\mathbb{S}^2$.

Proof. We first find the eigenvalues of $H_1(0)M^{-1}$ and H_2M^{-1}. Recall that $d_{12} = \alpha, d_{23} = \beta, d_{13} = 2\pi - (\alpha + \beta)$. Using α, β and Lemma 10, we obtain that

$$H_1(0)M^{-1} = \begin{bmatrix} \frac{-m_2 \cos\alpha}{\sin^3\alpha} + \frac{m_3\cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} & \frac{m_1}{\sin^3\alpha} & \frac{-m_1}{\sin^3(\alpha+\beta)} \\ \frac{m_2}{\sin^3\alpha} & \frac{-m_1\cos\alpha}{\sin^3\alpha} + \frac{-m_3\cos\beta}{\sin^3\beta} & \frac{m_2}{\sin^3\beta} \\ \frac{-m_3}{\sin^3(\alpha+\beta)} & \frac{m_3}{\sin^3\beta} & \frac{m_1\cos(\alpha+\beta)}{\sin^3(\alpha+\beta)} + \frac{-m_2\cos\beta}{\sin^3\beta} \end{bmatrix}.$$

Lemma 12 implies that $M\mathbf{v}_1$ and $M\mathbf{v}_2$ are two null vectors of $H_1(0)M^{-1}$. Thus the third eigenvalue λ_1 equals the trace of the matrix. Using the same trick as in the proof of Theorem 24, we find that the second diagonal entry is

$$\frac{-m_1\cos\alpha}{\sin^3\alpha} + \frac{-m_3\cos\beta}{\sin^3\beta} = -\frac{m_3}{\sin^2\beta}\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} > 0.$$

The first one and the third one are just opposite to diagonal entries of the matrix $H(\alpha, \beta)$ in the proof of Theorem 24, so they are positive. Hence $\lambda_1 > 0$.

Lemma 12 also implies that H_2M^{-1} has one null vector $M\mathbf{v}_3$. Note that the proof of Theorem 24 implies that acute triangle special central configurations are local maxima of U on $(\mathbb{S}^1)^3 \setminus \Delta$. Thus the two other eigenvalues of $H_2 = \left[\frac{\partial^2 U}{\partial\varphi_i\partial\varphi_j}\right]$ are both negative. Note that $M^{-\frac{1}{2}}$ is well defined. Then H_2 is congruent to $H_2' := (M^{-\frac{1}{2}})^T H_2 M^{-\frac{1}{2}}$, which is similar to $M^{\frac{1}{2}}H_2'M^{-\frac{1}{2}} = H_2M^{-1}$. By Sylvester's law of inertia [37], we have

$$n_-(H_2M^{-1}) = n_-(H_2) = 2, \quad n_0(H_2M^{-1}) = n_0(H_2) = 1,$$

where $n_0(A)$ is the number of zero eigenvalues and $n_-(A)$ is the number of negative eigenvalues of matrix A. This proves the eigenvalues of $H_1(0)M^{-1}$ are $0, 0, \lambda_1 > 0$, and the eigenvalues of H_2M^{-1} are $0, \lambda_2 < 0, \lambda_3 < 0$.

By Lemma 11, we see that L on $\mathbb{C}^{12}$ is similar to

$$\mathrm{diag}\left\{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \sqrt{\lambda_1}, -\sqrt{\lambda_1}, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|}\right\}.$$

Recall that the normal form of $L|_{E_1}$ is given by the first three nontrivial Jordan blocks. We thus obtain that $L|_E$ is similar to

$$\mathrm{diag}\left\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|}\right\}.$$

Then the positive eigenvalue $\sqrt{\lambda_1}$ indicates that all acute triangle fixed-point solutions are unstable on $\mathbb{S}^2$, a remark that completes the proof. □

Now we study the linear stability of other associated relative equilibria $\omega \neq 0$. Recall that the Hessian part is now $\begin{bmatrix}H_1(0) - \omega^2 M & 0\\ 0 & H_2\end{bmatrix}$. Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ do not correspond to the Jordan blocks of L any more. Define E_2 as the 2-dimensional subspace spanned by the last two vectors of (9.2). Then L has one Jordan block on E_2. By the same reason, it is reasonable to define stability based on the behaviour of the system on the complementary space, that is

$$\tilde{E} = \{\mathbf{v} \in \mathbb{C}^{4N} : \Omega(\mathbf{v}, \mathbf{u}) = 0 \text{ for all } \mathbf{u} \in E_2\}.$$

Then $\tilde{E}$ is an L invariant subspace of dimension $4N - 2$.

Definition 15. *A relative equilibrium X_ω $(\omega \neq 0)$ associated with a special central* *configuration on the equator is called* linearly stable *if X_ω is a stable rest point of the linearized equation restricted to $\tilde{E}$.*

Theorem 27. *Let X_ω ($\omega \neq 0$) be a relative equilibrium associated with an acute triangle special central configuration on the equator. Then it is unstable on $\mathbb{S}^2$ if and only if $0 < \omega^2 \leq \lambda_1$, and it is linearly stable if and only if $\lambda_1 < \omega^2$, where*

$$\lambda_1 = -\frac{m_2}{\sin^2\alpha}\frac{\sin\beta}{\sin(\alpha+\beta)\sin\alpha} - \frac{m_3}{\sin^2\beta}\frac{\sin\alpha}{\sin(\alpha+\beta)\sin\beta} - \frac{m_3}{\sin^2\beta}\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}.$$

Proof. By Lemma 11 and the proof of the above theorem, we only need to find the eigenvalues of $H_1(0)M^{-1} - \omega^2$. Note that $H_1(0)M^{-1}$ has three eigenvectors, with two eigenvalues being zero. Thus there exists an invertible matrix P such that $H_1(0)M^{-1} = P\text{diag}\{\lambda_1, 0, 0\}P^{-1}$. Therefore $H_1(0)M^{-1} - \omega^2 = P\text{diag}\{\lambda_1 - \omega^2, -\omega^2, -\omega^2\}P^{-1}$. By Lemma 11, we obtain that $L|_{\tilde{E}}$ is similar to

$$\text{diag}\Big\{\sqrt{\lambda_1-\omega^2}, -\sqrt{\lambda_1-\omega^2}, i\omega, -i\omega, i\omega, -i\omega, i\sqrt{|\lambda_2|}, -i\sqrt{|\lambda_2|}, i\sqrt{|\lambda_3|}, -i\sqrt{|\lambda_3|}\Big\}.$$

Thus the relative equilibrium is unstable on $\mathbb{S}^2$ if and only if $0 < \omega^2 \leq \lambda_1$, and it is linearly stable if and only if $\lambda_1 < \omega^2$. Direct computation leads to the value of λ_1. This remark completes the proof. □

It is interesting to compare the special central configuration on $\mathbb{S}^1$ and the collinear central configurations of the Newtonian N-body problem. The special central configuration of three particles are local maxima of the potential restricted on the equator, whereas the collinear central configurations are local minima of the potential restricted on a line. For each collinear configuration of the Newtonian N-body problem, there is only one angular velocity that produces a circular motion, whereas any angular velocity leads to circular motion in our case. Additionally, it is very interesting to notice that the stability depends on the velocity.

Chapter 10

Conclusions

To summarise, we find a natural way to define central configurations of the curved N-body problem. We separate them into two categories, the ordinary central configurations and the special central configurations. By characterizing ordinary central configurations as the critical points of the force function restricted to some submanifold, we show the existence of them. We also propose a problem which is analogous to Smale's 6-th problem for the Newtonian N-body problem. In $\mathbb{H}^2$, we obtain many results which are similar to the results on $\mathbb{R}^2$. In $\mathbb{S}^3$, we first study the geodesic ordinary central configurations for $N = 2, 3$, and the 3-body ordinary non-geodesic central configurations. Then we study special central configurations. We extend the definition of special central configuration to higher dimensional spheres. The Dziobek special central configurations are studied. We also study the stability of some relative equilibria associated to central configurations.

As it happens in the Euclidean case, the central configurations of the curved N-body problem are far from easy. Fixing N masses, to find all central configurations is to find all real solutions of $3N$ scalar equations in $3N$ unknowns, which is the main subject of algebraic geometry. To make things worse, the terms involved are usually not algebraic of the unknowns. For instance, d_{ij} is not algebraic in $\mathbf{q}_i, \mathbf{q}_j$. Central configurations of 3-body on $\mathbb{R}^2$ are settled long time ago, but curved 3-body central configurations is a very "non-linear" problem, as indicated in Chapter 7. As it always happens, problems on surfaces are harder than analogous problems in linear spaces.

There are several further directions. I am trying to solve the ordinary central configuration equations for three masses and the special central configuration equations for four masses. As mentioned earlier, the results of central configurations on $\mathbb{H}^2$ are similar to the ones on $\mathbb{R}^2$. I am trying to find the exact relationship between them. I am also planning to study the linear stability of relative equilibrium with Conley-Zehnder index theory.

These study of central configurations are only at the beginning. It is an object to study, and it is also a new tool. The properties of them to be found in future may shed more light on the equations of motion that govern this mathematical model.

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