

Analysis and Synthesis of Distributed Control Systems under Communication
Constraints

by

Yuanye Chen

B.Eng., Harbin Institute of Technology, 2010

M.Eng., Harbin Institute of Technology, 2012

A Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mechanical Engineering

© Yuanye Chen, 2017
University of Victoria

All rights reserved. This dissertation may not be reproduced in whole or in part, by
photocopying or other means, without the permission of the author.

Analysis and Synthesis of Distributed Control Systems under Communication
Constraints

by

Yuanye Chen

B.Eng., Harbin Institute of Technology, 2010

M.Eng., Harbin Institute of Technology, 2012

Supervisory Committee

Dr. Yang Shi, Supervisor
(Department of Mechanical Engineering)

Dr. Daniela Constantinescu, Departmental Member
(Department of Mechanical Engineering)

Dr. Kui Wu, Outside Member
(Department of Computer Science)

Supervisory Committee

Dr. Yang Shi, Supervisor
(Department of Mechanical Engineering)

Dr. Daniela Constantinescu, Departmental Member
(Department of Mechanical Engineering)

Dr. Kui Wu, Outside Member
(Department of Computer Science)

ABSTRACT

With the help of rapidly advancing communication technology, control systems are increasingly integrated via communication networks. Networked control systems (NCSs) bring significant advantages such as flexible and scalable structures, easy implementation and maintenance, and efficient resources distribution and allocation. NCSs empowers to finish some complicated tasks using some relatively simple systems in a collaborated manner. However, there are also some challenges and constraints subject to the imperfection of communication channels. In this thesis, the stabilization problems and the performance limitation problems of control systems subject to networked-induced constraints are studied.

Overall, the thesis mainly includes two parts: 1) Consensus and consensusability of multi-agent systems (MASs); 2) Delay margins of NCSs. Chapter 2 and Chapter 3 deal with the consensus problems of MASs, which aim to properly design the control protocols to ensure the state convergence of all the agents. Chapter 4 and Chapter 5 focus on the consensusability analysis, exploring how the dynamics of the agents and the networked induced constraints impact the overall systems for achieving consensus. Chapter 6 pays attention to the delay margins of discrete-time linear time-invariant (LTI) systems, studying how the dynamics of the plants limit the time delays that can be tolerated by LTI controllers.

In Chapter 2, the leader-following consensus problem of MASs with general linear dynamics and arbitrary switching topologies is considered. The MAS with arbitrary switching topologies is formulated as a switched system. Then the leader-following consensus problem is transformed to the stability problem of the corresponding switched system. A necessary and sufficient consensus condition is derived. The condition is also extended to MASs with time-varying delays.

In Chapter 3, the consensus problem of MASs with general linear dynamics is studied. Motivated by the multiple-input multiple-output (MIMO) communication technique, a general framework is considered in which different state variables are exchanged in different independent communication topologies. This novel framework could improve the control system design flexibility and potentially improve the system performance. Fully distributed consensus protocols are proposed and analyzed for the settings of fixed and switching multiple topologies. The protocols can be applied using only local information. And the control gains can be designed depending on the dynamics of the individual agent. By transforming the overall MASs into cascade systems, necessary and sufficient conditions are provided to guarantee the consensus under fixed and switching state-variables-dependent topologies, respectively.

Chapter 4 investigates the consensusability problem for MASs with time-varying delays. The bounded delays can be arbitrarily fast time-varying. The communication topology is assumed to be undirected and fixed. Considering general linear dynamics under average state protocols, the consensus problem is then transformed into a robust control problem. Sufficient frequency domain criteria are established in terms of small-gain theorem by analyzing the delay dependent gains for both continuous-time and discrete-time systems. The controller synthesis problems can be solved by applying the frequency domain design methods.

The consensusability problem of general linear MASs considering directed topologies are explored from a frequency domain perspective in Chapter 5. By investigating the properties of Laplacian spectra, a consensus criterion is established based on the stability of several complex weighted closed-loop systems. Furthermore, for single-input MASs, frequency domain consensusability criteria are proposed on the basis of the stability margins, which depend on the \mathcal{H}_∞ norm of the complementary sensitivity function determined by the agents' unstable poles. The corresponding design procedure is also developed.

Chapter 6 studies the delay margin problem of discrete-time LTI systems. For general LTI plants with multiple unstable poles and nonminimum phase zeros, we

employ analytic function interpolation and rational approximation techniques to derive bounds on delay margins. Readily computable and explicit lower bounds are found by computing the real eigenvalues of a constant matrix, and LTI controllers can be synthesized based on the \mathcal{H}_∞ control theory to achieve the bounds. The results can be also consistently extended to the case of systems with time-varying delays. For first-order unstable plants, we also obtain bounds achievable by proportional-integral-derivative (PID) controllers, which are of interest to PID control design and implementation. It is worth noting that unlike its continuous-time counterpart, the discrete-time delay margin problem being considered herein constitutes a simultaneous stabilization problem, which is known to be rather difficult. While previous work on the discrete-time delay margin led to negative results, the bounds developed in this chapter provide instead a guaranteed range of delays within which the delayed plants can be robustly stabilized, and in turn solve the special class of simultaneous stabilization problems in question.

Finally, in Chapter 7, the thesis is summarized and some future research topics are also presented.

Contents

Supervisory Committee	ii
Abstract	iii
Table of Contents	vi
List of Tables	ix
List of Figures	x
Acknowledgements	xii
Acronyms	xiii
1 Introduction	1
1.1 An Overview on Networked Control Systems	1
1.2 Multi-Agent Systems	4
1.2.1 An Overview on Multi-Agent Systems	4
1.2.2 Consensus of Multi-Agent Systems	5
1.2.3 Communication Constraints of Multi-Agent Systems	7
1.2.4 Methodologies	9
1.3 Motivations and Contributions	11
1.3.1 Consensus and Consensusability of Multi-Agent Systems	11
1.3.2 Delay Margins of Discrete-Time Systems	12
2 Leader-Following Consensus for Multi-Agent Systems with Switching Topologies and Time-Varying Delays	13
2.1 Introduction	13
2.2 Problem Formulation	15
2.3 The Switching Topology Case	17

2.4	The Switching Topologies and Time-Varying Delays Case	19
2.5	Simulation Examples	22
2.6	Conclusion	27
3	A Fully Distributed Approach for Consensus of Multi-Agent Systems under Multiple Communication Topologies	29
3.1	Introduction	29
3.2	Preliminaries and Problem Formulation	31
3.2.1	Basic Concepts from Graph Theory	31
3.2.2	Preliminary Results on Consensus	32
3.2.3	Problem Formulation	33
3.3	Main Results	34
3.3.1	The Multiple Fixed Communication Topologies Case	34
3.3.2	The Multiple Switching Communication Topologies Case	39
3.4	Simulation Examples	42
3.5	Conclusion	47
4	Consensus for Linear Multi-Agent Systems with Time-Varying Delays	48
4.1	Introduction	48
4.2	Problem Formulation	50
4.3	Main Results	51
4.4	Extensions to Discrete-Time Systems	58
4.5	Simulation Examples	62
4.6	Conclusion	67
5	Distributed Consensus of Linear Multi-Agent Systems: A Laplacian Spectra Based Method	69
5.1	Introduction	69
5.2	Problem Formulation	71
5.3	Laplacian Spectra Based Consensus of General Linear Agents	74
5.4	Consensusability of Single-Input Agents	77
5.5	Simulation Example	83
5.6	Conclusion	84

6	Simultaneous Stabilization of Discrete-Time Delay Systems: Bounds on Delay Margin	86
6.1	Introduction	86
6.2	Problem formulation	89
6.3	Systems with time-varying delays	103
6.4	Delay margin with PID controllers	108
6.5	Illustrative examples	115
6.6	Conclusion	119
7	Conclusions and Future Work	121
7.1	Summary of the Thesis	121
7.2	Future Work	122
7.2.1	Consensus of Heterogenous Multi-Agent Systems	122
7.2.2	Other Directions of Future Work	124
A	Publications	126
	Bibliography	128

List of Tables

Table 1.1 Classification and corresponding representative results on consensus.	7
---	---

List of Figures

Figure 1.1	General architecture of NCSs.	2
Figure 1.2	Research scheme of NCSs.	3
Figure 1.3	Classifications of MASs.	6
Figure 1.4	The research framework of MASs.	9
Figure 2.1	Possible communication topologies.	23
Figure 2.2	The distribution of the topologies.	24
Figure 2.3	The state trajectories.	24
Figure 2.4	The deviation trajectories between agents and the leader.	25
Figure 2.5	The distribution of the topologies.	26
Figure 2.6	The distribution of the time delays.	26
Figure 2.7	The state trajectories.	27
Figure 2.8	The deviation trajectories between agents and the leader.	27
Figure 3.1	The system block diagram considering fixed topologies.	36
Figure 3.2	The system block diagram considering switching topologies.	40
Figure 3.3	The block diagram of the modified cascade system.	41
Figure 3.4	The auxiliary state variable trajectories.	43
Figure 3.5	The state variable trajectories.	44
Figure 3.6	The switching signals describing the time-varying topologies.	45
Figure 3.7	The auxiliary state variable trajectories.	46
Figure 3.8	The state variable trajectories.	46
Figure 4.1	The block diagram of $G(s)$	54
Figure 4.2	Loop transformation of $G(s)$	55
Figure 4.3	Loop transformation of $G(s)$ with $\hat{\Lambda}$ and $\hat{\Delta}$	55
Figure 4.4	The block diagram for controller design of unstable $P(s)$	58
Figure 4.5	The block diagram of $G(z)$	60
Figure 4.6	Loop transformation of $G(z)$	60

Figure 4.7 Loop transformation of $G(z)$ with $\hat{\Lambda}$ and $\hat{\Delta}$	60
Figure 4.8 The communication topology in Example 4.1.	62
Figure 4.9 The singular value plot based on Theorem 4.1 in Example 4.1.	63
Figure 4.10 The singular value plot based on Theorem 4.2 in Example 4.1.	64
Figure 4.11 State trajectories of the system in Example 4.1.	64
Figure 4.12 Deviation trajectories of the system in Example 4.1.	65
Figure 4.13 The communication topology in Example 4.2.	65
Figure 4.14 The singular value plot based on Theorem 4.3 in Example 4.2.	66
Figure 4.15 The singular value plot based on Theorem 4.4 in Example 4.2.	66
Figure 4.16 State trajectories of the system in Example 4.2.	67
Figure 4.17 Deviation trajectories of the system in Example 4.2.	67
Figure 5.1 The region of Υ	73
Figure 5.2 The region of Ω and \mathbf{G}	79
Figure 5.3 The state trajectories.	84
Figure 5.4 The state deviation trajectories.	85
Figure 6.1 Standard feedback control structure.	89
Figure 6.2 Rational approximation for $\phi(\omega)$	92
Figure 6.3 Feedback control systems with time-varying delay.	104
Figure 6.4 Small-gain setup of systems with time-varying delay.	104
Figure 6.5 The frequency response of $\phi_\delta(\omega)$	106
Figure 6.6 Lower bounds on the delay margin of system (6.45).	116
Figure 6.7 Step response of system (6.45) with controller (6.46).	117
Figure 6.8 The time-varying delay.	117
Figure 6.9 Step response of system (6.47) with controller (6.49).	118
Figure 6.10 Step response of system (6.50) with controller (6.51).	119
Figure 6.11 Step response of system (6.50) with controller (6.51).	119
Figure 7.1 The heterogenous MASs.	123

ACKNOWLEDGEMENTS

First and foremost, I would like to express my gratitude to my supervisor Dr. Yang Shi for his continuous guidance and support. He has a broad vision on research, and he can always explore the cutting edge topics in his area. He is very patient, passionate, and always ready to providing insightful thoughts and suggestions on academic problems as well as career development. I also really appreciate some precious opportunities that were selflessly provided by Dr. Yang Shi.

I sincerely thank Dr. Jie Chen for his kind help and support during the visit in City University of Hong Kong. I am deeply impressed by his immense knowledge and meticulous attitude. His ideology, focusing on fundamental but profound research, motivates me to pursue better and higher achievements.

I am grateful to the committee members, Dr. Daniela Constantinescu, Dr. Kui Wu, Dr. Jason Gu for their constructive comments and suggestions.

I am appreciative of my friends and colleagues in the Applied Control and Information Processing Lab at the University of Victoria. I would like to express my thankfulness to Dr. Jian Wu, Dr. Huiping Li, Dr. Xiaotao Liu, Dr. Mingxi Liu, Bingxian Mu, Chao Shen, Yiming Zhao, Jicheng Chen, Kunwu Zhang, Xiang Sheng, Wei Chen, Lei Zuo and Henglai Wei for making our lab a joyful and productive place in which to work. And it is really lucky to get the chance to work in Dr. Chen's group in City University of Hong Kong. I also wish to thank Dr. Tian Qi, Dr. Dan Ma, Dr. Andong Liu, Dr. Fang Song, Dr. Haibao Chen, Dr. Fei Chen, Jianqi Chen, Shengquan He, Yuezun Lv, Adil Zulfikar, Patrick Deenen for the unforgettable time we have spent in Hong Kong.

I also gratefully acknowledge the financial support from the Chinese Scholarship Council (CSC), the Natural Sciences and Engineering Research Council of Canada (NSERC), the Hong Kong University Grants Committee (RGC) under Project CityU 11201514, CityU 111613, the Department of Mechanical Engineering and the Faculty of Graduate Studies (FGS) at the University of Victoria, the IEEE Control System Society (CSS), and Mr. Alfred Smith and Mrs. Mary Anderson Smith Scholarship.

Finally and most importantly, I would like to thank my parents for their persistent and unconditional love and support.

Acronyms

LTI	linear time-invariant
MAS	multi-agent system
MIMO	multiple-input multiple-output
NCS	networked control system
PID	proportional-integral-derivative
SISO	single-input single-output

Chapter 1

Introduction

In this chapter, the background and literature review of NCSs and MASs are introduced. The detailed motivations and contributions of this thesis are also presented.

1.1 An Overview on Networked Control Systems

NCSs are control systems in which controllers, sensors and actuators are spatially distributed. Information between different components of the systems can be exchanged over a packet-based network such as control area network (CAN), fieldbus, or more recently, the wired or wireless Ethernet. With the help of the networking, the control systems can be implemented on a large scale with high reliability and flexibility as well as low installation and maintenance cost. Consequently, NCSs have been widely applied in a variety of areas such as mobile sensor networks [1, 2, 3, 4], process control systems [5], teleoperation systems [6], and power systems [7].

Traditional control theory mainly studies the feedback control systems interconnected via ideal channels. In NCSs, the information exchanges via the imperfect channels among sensors, actuators and controllers, with the communication rate bounded by the channel capacity. A general architecture of NCSs is shown in Figure 1.1. Because of the imperfect channels, there are several distinct constraints for NCSs compared with conventional control systems.

- **Band-limited channels:** In reality, the communication network can only exchange a finite amount of information per unit time, which could pose significant limitations on the performance of NCSs. Some research effort has focused on

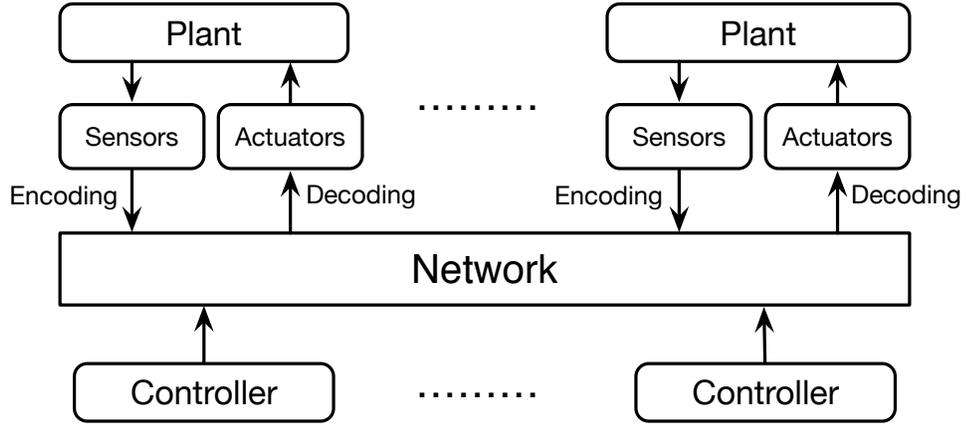


Figure 1.1: General architecture of NCSs.

the stabilization [8, 9] and performance limitations [10, 11] of a feedback system over finite capacity channels.

- **Time delays:** Since the data are exchanged over a network, there exist the network access delays and the transmission delays, both of which are usually called network-induced delays, depending on the network conditions. The time delays can be constant or time-varying, deterministic or stochastic, which usually degrade the stability and performance of the control systems. There are extensive results on the stabilization of time-delayed systems [12, 13, 14]. And recently some researchers have devoted their research interest to the delay margin problems [15, 16, 17, 18].
- **Packet losses:** Because of transmission errors and network traffic congestions, packet losses also pose an inevitable constraint that the data may be lost when transmitting over networks. Furthermore, long network-induced delays can cause the packet disordering, resulting in the dropout of old packets when the latest packet has received.

Figure 1.2 shows the research scheme of NCSs. For tradition NCSs, our research interest lies in the constraints of time delays. Unlike most of the existing results on the stabilization problems of time-delay systems, in this thesis, we focus our research effort on a performance limitation problem, named delay margin, which is motivated by the idea of gain and phase margins in robust control theory. The definition of delay margin may be first proposed as an open problem in [19]: For a fixed finite-dimensional LTI plant, is there an upper bound on the uncertain delay that can be

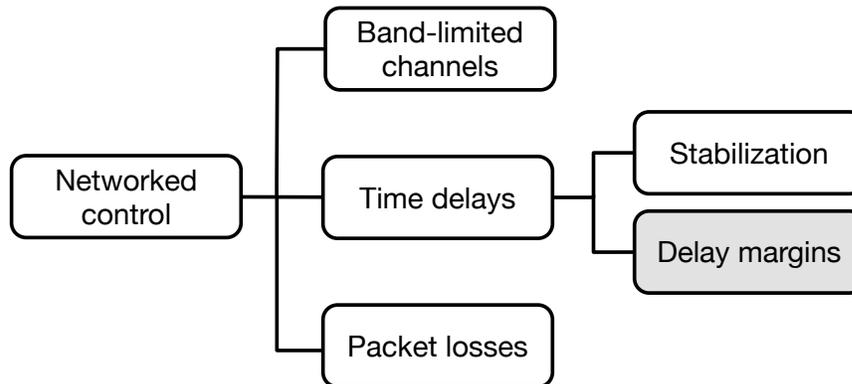


Figure 1.2: Research scheme of NCSs.

tolerated by an LTI stabilizing controller? Most of the research interest has been attracted to the continuous-time systems [15, 17, 18]. In [15], by substituting the delay with a proper designed all-pass transfer function in frequency domain, the upper bounds on delay margins are derived based on the phase analysis considering the single-input single-output (SISO) plants for the cases with one real unstable pole, with a pair of complex unstable poles, and with one real unstable pole and one real nonminimum phase zero, respectively. In [17], the authors follow a similar idea from [15] and extend the upper bounds on delay margins to the cases of SISO plants with 2 and 3 different real unstable poles, respectively. In [18], both SISO and MIMO plants are taken into consideration. The authors propose a novel rational approximation method for the time delays, which bridges the delay margin problems to some \mathcal{H}_∞ control problems. Then analytical interpolation method is applied to derive the lower bounds on delay margins for plants containing multiply different unstable poles and nonminimum phase zeros. For continuous-time systems, if a controller can stabilize a plant, the closed-loop system can also tolerate a sufficient small delay following the continuity, which is a great advantage for analysis. This indicates the delay margins for continuous-time systems are always great than 0. When it comes to discrete-time systems, the delay margin problems turn out to be a special class of simultaneous stabilization problems, which are more challenging. It is only proved in [16] that the delay margin of a discrete-time system is zero whenever the plant contains a negative real unstable pole by investigating the simultaneous stabilization of the original plant and the one-step delayed plant.

1.2 Multi-Agent Systems

1.2.1 An Overview on Multi-Agent Systems

In recent years, the rapid development of the communication technology not only gives power to link the components within a control system, but also makes it possible to connect a group of simple autonomous systems, usually called an MAS, via a communication network. With the help of the information exchange, the MASs can fulfill complex tasks cooperatively, though each agent could only handle simple jobs. MASs amazingly bring many advantages compared to the conventional control systems, like flexibility, robustness, and cost efficiency. The application of MASs distributes in a variety of areas, including sensor networks [2, 3, 4], smart grids [20, 21, 22], and vehicle platoons [23].

Since the MASs are related to a group of agents, the system dynamics of each agent essentially become research concerns. And it is observed that the dynamics have a closed relationship with the convergence rate and the final state of the overall system [24]. Extensively results have been carried out for agents with relative simple dynamics, like 1-order and 2-order integrator dynamics [25, 26], which are some featured but simplified cases for general linear dynamics. The research on MASs mainly focusing on the following dynamics.

- **General linear dynamics:** General linear dynamics are very representative and have been widely applied. The study on general linear dynamics [27, 28] extends the study of the MASs with low-order linear dynamics to a larger category of applications, since the well-investigated 1-order and 2-order integrator dynamics are special cases of general linear dynamics.
- **Nonholonomic mobile robots:** Nonholonomic constraints play an important role in the study of mobile robot systems. The mobile robot systems are underactuated since the number of control input is less than that of the states. This inherent property brings additional challenges for the cooperative control [29, 30].
- **Rigid bodies:** The rigid bodies dynamics, like those in [31], represent a large class of mechanical systems, like robotic arms. And the convergence analysis is usually based on the property of the matrix with skew-symmetric structure.

- **General complex networks:** The study of general complex networks usually focuses on a special class of 1-order state-dependent nonlinear dynamics, like those in [32], which is a nonlinear extension of 1-order integrator dynamics.
- **Nonlinear oscillators:** Nonlinear oscillators are usually used to model the physical process of diffusion. The dynamics are usually described by the Kuramoto equation, like those in [33].

Most of the existing results are based on the assumption that all the agents in the MASs share the same dynamics, usually called homogenous MASs. There also exist some results on the heterogenous MASs, which indicates that the dynamics of each agent could be different [34, 35, 36].

Furthermore, the research results of MASs can be also characterized in the following directions.

- **Consensus and consensus like problems (synchronization, rendezvous):** Consensus means that the group of agents reach a common states asymptotically only using local information. This is a prerequisite for MASs to achieve more complicated tasks and has been extensively studied.
- **Formation and formation like problems (flocking):** Formation indicates the group behaviour of all the agents forming a designed geometrical configuration with only local information exchange. The formation could be achieved leaderless using the methods based on the state-transition matrices [37] or the Lyapunov function [38]. The formation could also be fulfilled tracking a leader as a reference, which is usually more challenging [39, 40].
- **Distributed estimation:** Distributed estimation is usually needed because of the absence of global information in applications. The scheme has been widely applied in sensor network [2, 41].

Figure 1.3 provides the classification of MASs from some different perspectives. In this thesis, we mainly focus our research effort on the consensus of homogenous general linear MASs.

1.2.2 Consensus of Multi-Agent Systems

When MASs with a cooperative scheme are adopted to fulfill some complicated tasks, the agents need to interact with their neighbours over a communication network to

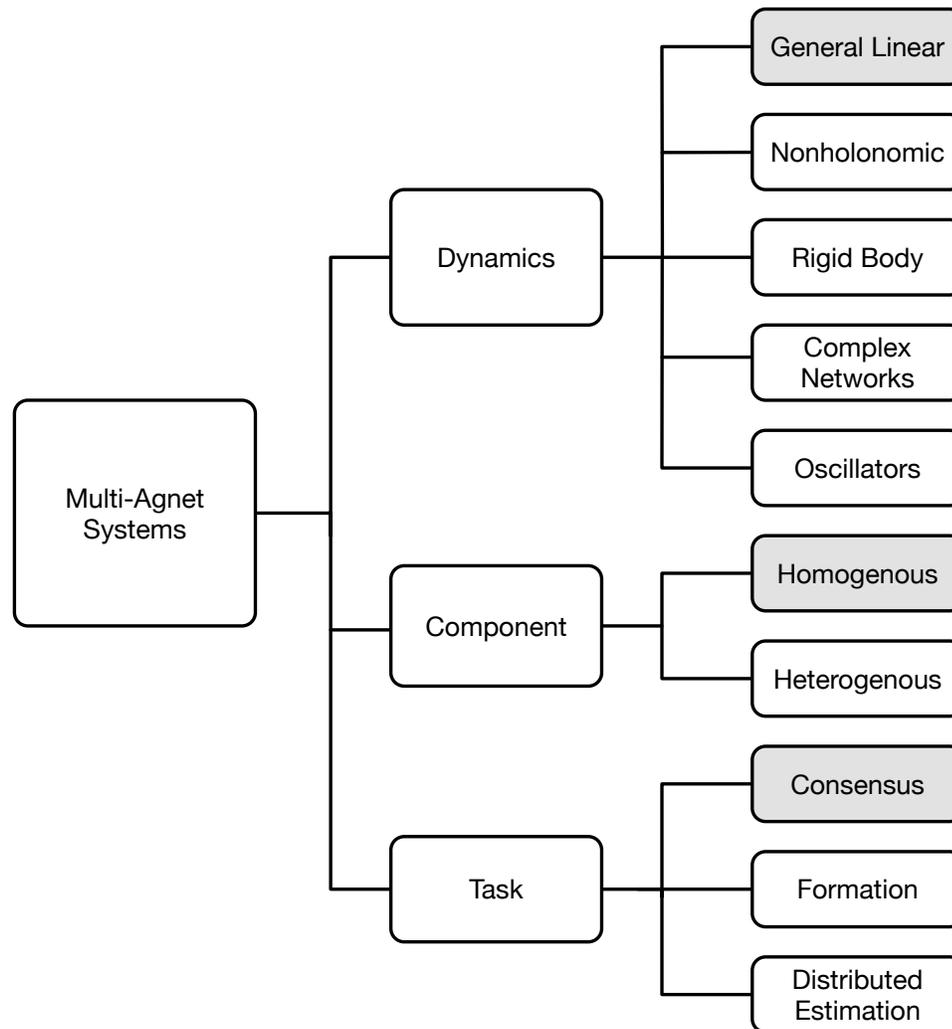


Figure 1.3: Classifications of MASs.

reach a common state. This problem is usually called consensus, which lays foundations for other cooperative control problems including formation, flocking and swarming.

Consensus is a problem with a long history in the decision making area. The study of consensus has been paid growing attentions since the publish of [42]. By model the undirected communication topology as an graph, the consensus phenomenon in [43] is explained theoretically under the assumption that the communication topology is jointly connected. Later, in [44], consensus conditions are proposed when the communication topology is directed. The average based consensus can be achieved if the communication topology keeps balanced and strongly connected. While in [45], consensus conditions are proved if the communication topology has a jointly directed

spanning tree. Similar results are also shown in [46].

Several emerging directions have been investigated very recently. The first direction is the research on more general systems, like high-order integrator systems and general linear systems, etc. In [47], consensus problem of general linear dynamics is studied, and sufficient conditions are proposed. In [27], necessary and sufficient like consensus conditions are established for general linear dynamics. Later, the results in [27] are extended in [48] by introducing dynamic controllers, which improves the system performance.

Another direction is the research on heterogenous MASs. This problem is initially discussed for relative simple dynamics. In [34, 35, 36], the heterogenous consensus problems are studied for 2-order dynamics. In [49, 50], more general systems are further explored. Especially in [49], a necessary and sufficient consensus condition is proved for SISO linear dynamics.

In addition to the aforementioned literatures, consensus of MASs has been studied in a variety aspects including dynamics, topology, time framework, etc. Some main categories and corresponding representative papers are shown in Table 1.1.

Table 1.1: Classification and corresponding representative results on consensus.

1-order:	[25], [24], [51]
2-order:	[52], [26], [53]
High-order:	[54], [55], [56]
Linear:	[57], [28], [58]
Nonlinear:	[59], [60], [61]
Heterogenous:	[62], [63]
Homogenous:	[24], [27], [61]
Continuous-time:	[44], [59], [64]
Discrete-time:	[65], [27], [48]
Sampled-data:	[66], [67], [68]
Fixed topology:	[69], [70], [71]
Switching topology:	[42], [46], [72]
Leaderless:	[46], [73]
Leader-following:	[64], [72]

1.2.3 Communication Constraints of Multi-Agent Systems

Since the information of agents is exchanged over a communication network, a variety of network-induced constraints are inevitable. Some representative networked

constraints for MASs are listed below.

- **Communication topology:** The communication topology for an MAS, which could be fixed [69, 70, 71] or switching [42, 46, 72], directed [45, 74] or undirected [27, 48], plays an significant role on the system performance [24]. For undirected topology case, the algebra connectivity is usually closely related to the rate of convergence. However, for directed topology case, since the Laplacian matrix is no longer symmetric, it is more challenging for the performance and convergence analysis, which still deserves in-depth exploration. There is also some recent research interest considering the state-variable dependent topologies. This problem could be brought because the state variables are not updated via the switching communication topologies simultaneously. For example, in [75, 76, 77, 78], the position information and the velocity information are exchanged in two different topologies, respectively.
- **Time delays:** Delay effect is an important issue on consensus, since the information of agents will be inherently delayed when transmitted via a network. Another source of delay is related to the computation time and execution time of each agent. Consensus with time delays has been extensively studied in existing literatures [79, 80, 81, 58, 56, 82]. Most of the existing works have been focused on agents of simple dynamics with a constant delay. Consensus conditions for more general dynamics and time-varying delays are still needed to be further investigated.
- **Sampling:** In traditional digital control systems, signals are usually periodically sampled and updated with the same rate synchronically. However, in MASs, the data sampling and data updating may be asynchronous [83, 80, 66]. In addition, considering the bandwidth constraints, non-uniform sampling [84], event-trigger scheme [85, 86, 87], or multirate sampling may be adopted to reduce the communication loads. To the best knowledge of the author, multirate sampling consensus is still remaining as an open problem.
- **Packet dropouts:** When transported in a network, the data packets may be dropped because of network traffic congestions and limited network reliability. In this situation, the agent cannot received the data from its neighbours. Hence it is challenging to handle the consensus problems with data missing. Related results can be found in literatures like [88] and [89].

- **Quantization errors:** Rather than exchange with the exact values, the data transferred in networks is usually rounded off and represented with finite bits. Because of this, there exists a difference between the real data and transmitted data, which may affect the system in terms of performance and stability. This factor has been also mainly studied considering simple dynamics [90, 91, 27].

A research framework of MASs considering communication constraints is shown in Figure 1.4. Because of the mentioned communication constraints, the system performance might be degraded. Even worse, the stability could be destroyed. Therefore, it is of great importance and challenge to design controllers to guarantee the desired performance under these constraints. This is one of the motivations in this research thesis.

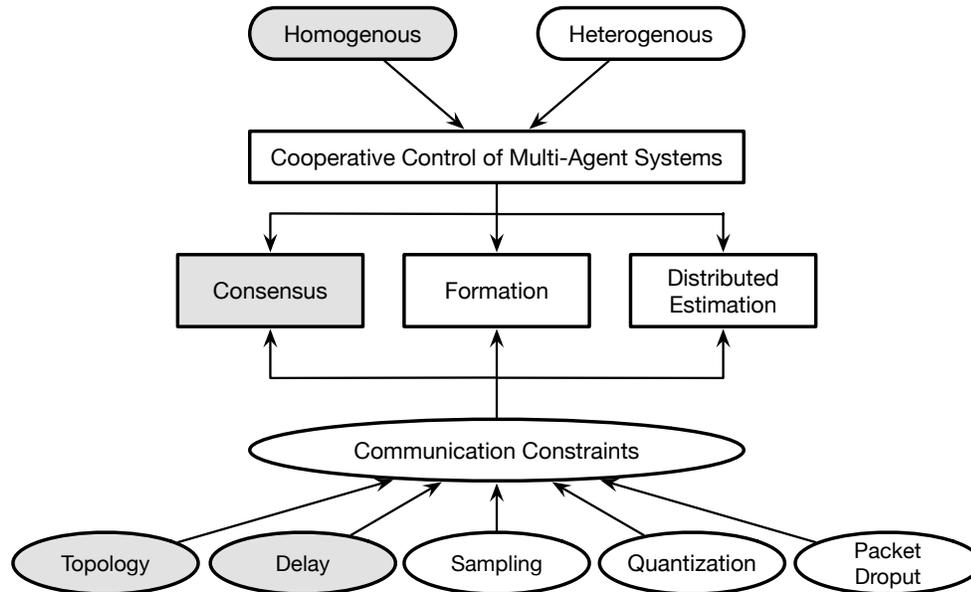


Figure 1.4: The research framework of MASs.

1.2.4 Methodologies

The following three methods are widely used to deal with the consensus problem of MASs.

- **Lyapunov stability theory based methods:** The basic idea of Lyapunov stability theory based methods is to transform the original multi-agent dynamics to an associate error dynamics. Then the consensus condition can be evaluated

by the properly constructed Lyapunov function. This kind of methods not only can be applied to LTI systems or linear systems, but also can be used to deal with time-varying systems [44, 72] or nonlinear systems[61]. In addition, Lyapunov stability theory based methods are compatible to many advanced control schemes, like adaptive control [28], and receding horizon control [92]. However, only sufficient conditions can be obtained and the conditions may be very conservative.

- **State-transition matrices based methods:** When applying this kind of methods, the state-transition matrices of MASs are usually transformed into stochastic matrices [42]. Based on the production convergence property on an infinite sequence of stochastic matrix, consensus can be ensured. Usually, the final state also can be calculated based on the initial state. The state-transition matrices based methods can also deal with the consensus problem with stable error dynamics since the infinite sequence of stable state-transition matrices will converge to 0. Necessary and sufficient consensus conditions can be established. However, these methods can be only applied under the discrete-time systems.
- **Frequency domain methods:** Frequency domain methods are also very powerful to solve the consensus problems and analyze the consensusability. By some proper transformations, consensus problems can be bridged to the stabilization problems of error dynamics. And the frequency domain methods like Nyquist criterion [44], stability margin optimization [58], and pole analysis [27, 48], can be employed. Especially in [48, 58], consensus protocols with dynamics controllers are introduced to improve the consensus performance. Necessary and sufficient or necessary and sufficient like conditions can be derived.

In this thesis, Lyapunov stability theory based methods are adopted to achieve some preliminary results, which may tend to be conservative. Frequency domain based methods or state-transition matrix based methods will be further employed for more sound and less conservative results.

1.3 Motivations and Contributions

1.3.1 Consensus and Consensusability of Multi-Agent Systems

Motivated by the aforementioned networked constraints, one of our main concerns in this thesis is to propose appropriate control schemes and develop conditions that guarantee consensus of MASs subject to these constraints. This can be considered as the stabilization problem in the feedback control theory, which aims to ensuring the stability of the feedback systems with proper design controllers.

The leader-following consensus problems considering the constraint of time delays have been studied in [93, 94]. However, the consensus conditions established are only applicable to second-order dynamics. In Chapter 2, we focus our research effort on general linear dynamics. A novel consensus analysis scheme is proposed which can model switching topologies and time-varying delays on account of switched systems in a unified way. Under this scheme, we cast the leader-following consensus problems into augmented switched control problems. In light of the theory on switched control systems, necessary and sufficient conditions are derived depending on the state-transition matrices of subsystems depending on different topologies and different time-varying delays.

In recent years, multiple-input multiple-output (MIMO) communication technique has been rapidly developed and widely applied especially for wireless communication [95, 96, 97, 98]. An MIMO channel includes different SISO subchannels, which could significantly improve the communication capacity. Inspired by this, Chapter 3 studies the consensus problem considering the MIMO communication channels, which can be described as multiple state-variables-dependent communication topologies. The overall closed-loop system can be modelled as a cascade system. And we establish the necessary and sufficient consensus conditions following some first-order consensus results and some stability conditions.

Another branch of our research is the consensusability problem, focusing on existence and synthesis of distributed controllers for achieving consensus [48, 27]. This problem explores the intrinsic of feedback that can neither be overcome nor circumvent regardless how the controller may be design [99], which is characterized as a fundamental performance limitation.

Note that, in [48, 27], consensusability problems are studied without considering the networked induced constraints. In Chapter 4, we take the time-varying delays

into consideration, and suppose the communication topology is undirected like that in [48, 27]. We isolate the time-varying delay part in the delayed MAS and analyze the corresponding input-to-output gain. Then small-gain theorem plays an important role on the convergence analysis. We prove that a delayed MAS is consensusable if the \mathcal{H}_∞ norm of the complementary sensitivity function is less than the delay dependent gain.

In real applications, directed communication topology is more general and practical. However, it is more challenging for analysis since the corresponding Laplacian matrix is no longer symmetric. In Chapter 5, we investigate the consensusability problems under directed communication topology. By fully utilizing the properties of Laplacian spectra, we cast the complex spectra as the gain and phase margins of the systems, which are two commonly used performance indices in robust control theory. The consensusability conditions are derived based on the robust analysis of the closed-loop systems. Furthermore, a systematical controller synthesis method is also developed.

1.3.2 Delay Margins of Discrete-Time Systems

Inspired by the existing results on delay margins, we focus our research effort on the delay margins of discrete-time systems. In Chapter 6, the bounds on delay margins for discrete-time LTI systems with multiple unstable poles and nonminimum phase zeros are studied. Since the time delay is an infinite-order process in frequency domain, which is challenging to be handled directly, we first adopt the rational approximation technique to get a finite-order approximation for the delay. Based on this, the delay margin problems can be solved as \mathcal{H}_∞ performance limitation problems. By employing the analytical interpolation method, explicit lower bounds can be calculated as the real eigenvalues of a constant matrix. In addition, with the approximations depending on the characters of the delays, we can also determine the bounds considering the time-varying delays. Furthermore, a more practical case of first-order unstable plant using PID control is also studied using a direct phase analysis method.

Chapter 2

Leader-Following Consensus for Multi-Agent Systems with Switching Topologies and Time-Varying Delays

2.1 Introduction

In recent years, increasing research interest has been focusing on cooperative control of MASs. An MAS consists of a group of autonomous agents which are connected by a communication network. Even though each agent can only solve relatively simple tasks, the overall system can fulfill complex tasks in a cooperative way with information exchanging among different agents. Compared to the conventional centralized control systems, MASs bears many advantages, such as economics, speed, reliability, and scalability. The application of MASs can be found in a variety of areas, such as formation control [100, 101, 102], flocking [38, 40], rendezvous [103], and sensor networks [104, 1].

A fundamental problem for MASs is to control the agents to reach a common state based on local information from their neighbours, which is usually called consensus. The consensus problem has been studied considering relative simple dynamics: Vicsek model [43, 105], first-order dynamics [46, 24] and second-order dynamics [105, 42, 44, 46, 69, 52, 106, 107, 108, 109]. Particularly, in [105], the authors propose the consensus condition of Vicsek model with asynchronous agent clocks. Furthermore, in [46], the authors demonstrate necessary and sufficient conditions for first-order dynamics MASs if the union of the directed communication topologies has a spanning tree

frequently enough. Besides, in [52], the authors illustrate necessary and sufficient conditions for second-order dynamics MASs under a fixed directed communication topology with a spanning tree. In recent years, more results are proposed for high-order dynamics [55, 56], general linear dynamics [27, 48] and nonlinear dynamics [47, 27, 48, 110]. In [27], single input or single output linear dynamics are studied in frequency domain, and some necessary and sufficient like results are demonstrated under fixed communication topology. In [48], the authors extend the results in [27] by adding a dynamic filter, which improves the performance and increases the design flexibility. In [110], a distributed receding horizon control scheme is proposed to solve the consensus problem considering nonlinear dynamics.

A practical topic is consensus of a group of agents with a leader, where the dynamics of the leader is independent of the other agents. Hence the leader is followed by other agents when reaching consensus. This is usually called a leader-following consensus problem, which keeps gaining a lot of research attention. Because the agents are connected via a network, the information exchange will be delayed inherently, and the communication topologies may switch because of communication constraints. Such kind of network-induced factors may significantly deteriorate the performance, or even destroy the stability. Many researchers study the leader-following consensus problems considering network-induced factors [111, 71, 112, 72, 113, 93, 94]. In [111, 71, 112], leader-following consensus problems are studied with time-varying delays considering second-order dynamics or general linear dynamics. In [72, 113], leader-following consensus problems are investigated under switching topologies with general linear dynamics. In [93, 94], both delays and switching topologies are studied for second-order systems.

In this chapter, the leader-following consensus problems with switching topologies and time delays are studied in a general scenario. Instead of considering second-order dynamics or integral dynamics, general linear dynamics are investigated. Furthermore, time-varying delays are considered instead of constant delays. In order to tackle the arbitrary switching topologies, a novel idea is proposed to model the MAS as a switched system. Consequently, the leader-following consensus problem can be studied in the framework of switched systems. A necessary and sufficient leader-following consensus condition with arbitrary switching topologies is proposed in light of theoretical results on switched systems. Then by transforming the systems with time-varying delays to switched systems with arbitrary switching delays, a necessary and sufficient leader-following consensus condition with arbitrary switching topolo-

gies and time-varying delays is established. The main contribution of this work is two-fold:

- A switched system model for leader-following MAS is presented, which bridges the leader-following consensus problems to the stability problems of switched systems.
- A necessary and sufficient condition is established for leader-following consensus problems with arbitrary switching topologies and time-varying delays.

The rest of this chapter is organized as follows. The problem formulation and preliminaries are proposed in Section 2.2. The leader-following consensus problems under switching topologies, without delays and with time-varying delays, are investigated in Section 2.3 and Section 2.4, respectively. Numerical examples are shown in Section 2.5. Finally the chapter is concluded in Section 2.6.

Notation in this chapter: The space of real number is represented by \mathbb{R} . $\|\cdot\|$ denotes 1-norm or infinity norm of a matrix. $\|\cdot\|_2$ denotes 2-norm of a matrix or a vector. $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. $\text{diag}\{a_1, \dots, a_n\}$ represents the diagonal matrix whose diagonal entries are a_1, \dots, a_n . \otimes represents Kronecker product. T represents the transpose of a matrix. \cup represents the union of sets. I stands for the identity matrix.

2.2 Problem Formulation

Consider an MAS which consists of N agents and a leader. Each agent can be modelled as the following general linear dynamics

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \quad x_i(0) = x_{i0}, \quad (2.1)$$

where $i \in \mathcal{N} = \{1, 2, \dots, N\}$. $x_i(k) \in \mathbb{R}^n$ is the state of agent i , $u_i(k) \in \mathbb{R}^m$ is the input of agent i , which is generated only based on the local information from its neighbours, and x_{i0} is the initial state of agent i . The dynamics of the leader, labeled as $i = 0$, are

$$x_0(k+1) = Ax_0(k), \quad x_0(0) = x_{00}. \quad (2.2)$$

Similarly, $x_0(k) \in \mathbb{R}^n$ is the state of the leader, and x_{00} is the initial state of the leader.

The leader and the N agents are connected via a communication topology represented by a graph. The communication among the N agents can be described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the index set, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the edge set and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix. The element $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, which means there is information exchange from agents j to i , and $a_{ij} = 0$ otherwise. Assume that there is no edge from an agent to itself, which indicates $a_{ii} = 0$. For undirected topology, it is well known that the adjacency matrix is symmetric. A graph \mathcal{G} is called connected if there exists a path between any two agent i, j . Let $\mathcal{N}_i = \{j \mid (i, j) \in \mathcal{E}\}$ represent the neighbourhood of the i th agent, and $\deg_i = \sum_{j=1}^N a_{ij}$ represent the degree of the i th agent. Denote $\mathcal{D} := \text{diag}\{\deg_1, \deg_2, \dots, \deg_N\}$. Then the Laplacian matrix of graph \mathcal{G} is $\mathcal{L}_{\mathcal{G}} := \mathcal{D} - \mathcal{A}$, which is symmetric and positive semi-definite for undirected graphs. By representing the leader as vertex 0, we can get an extended graph $\bar{\mathcal{G}}$, which includes \mathcal{G} , index 0, and the communication topology between the leader and its neighbours.

The following assumptions are supposed to be satisfied throughout the chapter.

Assumption 2.1. *The topology is undirected and connected at any time instant k .*

Assumption 2.2. *The topology can switch arbitrarily.*

Noticing the topology is switching, $\bar{\mathcal{G}}_{\sigma(k)}$ is adopted to describe the time-varying topology, where $\sigma(k)$ is a switching signal whose value equals the index of the graph at each time instant k . Suppose topology can arbitrarily switch among M possible topologies, and we consider possible graphs set $\{\bar{\mathcal{G}}_p\} \subseteq \{\bar{\mathcal{G}} : \bar{\mathcal{G}} \text{ is connected}\}$, where p is the index of the graph. Denote $p \in \mathcal{P}$, $\{\bar{\mathcal{G}} : \bar{\mathcal{G}} \text{ is connected}\}$ is finite if N is finite, therefore \mathcal{P} is finite if N is finite. Denote $\mathcal{P} = \{1, 2, \dots, M\}$, then the switching signal $\sigma(k) \in \mathcal{P}$.

Under the proposed communication topology, our aim is to apply the control input u_i , which is only based on local information, to ensure that all the N agents will follow the leader. This can be precisely described by the condition in (2.3).

$$\lim_{k \rightarrow \infty} \|x_i(k) - x_0(k)\|_2 = 0, \quad \forall i \in \mathcal{N}, \quad (2.3)$$

for any finite initial condition $x_i(0)$, $i = 0, 1, \dots, N$.

The following lemma for switched systems will be useful for the leader-following consensus analysis and it is adopted from [114].

Lemma 2.1. *A switched linear system*

$$x(k+1) = \Phi_{\sigma(k)}x(k), \quad \Phi_{\sigma(k)} \in \{\Phi_1, \Phi_2, \dots, \Phi_M\},$$

is globally asymptotically stable under arbitrary switching if and only if there exists a finite number c such that

$$\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_c}\| < 1, \quad \forall \Phi_{i_j} \in \{\Phi_1, \Phi_2, \dots, \Phi_M\}, \quad (2.4)$$

for all $j = 1, 2, \dots, c$.

2.3 The Switching Topology Case

In this section, we focus on the leader-following consensus problem when the communication topology can switch arbitrarily. We adopt the following local consensus protocol

$$u_i(k) = K \left(\sum_{j=1}^N a_{ij}(k)(x_j(k) - x_i(k)) \right) + K d_i(k)(x_0(k) - x_i(k)), \quad (2.5)$$

where $K \in \mathbb{R}^{m \times n}$ is a static matrix, and $d_i(k)$ denotes the adjacency between the leader and agent i . $d_i(k) > 0$ if the leader and agent i is connected, and $d_i(k) = 0$ otherwise. Then we have the main result of this section.

Theorem 2.1. *Consider the leader-following MAS in (2.1)-(2.2). Suppose Assumptions 2.1 and 2.2 are satisfied. All the agents can follow the leader under the consensus protocol in (2.5) if and only if there exists a finite number c , such that*

$$\|\Psi_{i_1}\Psi_{i_2}\cdots\Psi_{i_c}\| < 1, \quad \forall \Psi_{i_j} \in \{\Psi_{\sigma(k)}\}, \quad j = 1, \dots, c,$$

where

$$\begin{aligned}\Psi_{\sigma(k)} &= \hat{A} - ((\mathcal{L}_{\sigma(k)} + D_{\sigma(k)}) \otimes \hat{B}), \\ \hat{A} &= I_N \otimes A, \\ \hat{B} &= BK, \\ D_{\sigma(k)} &= \text{diag}\{d_1(k), d_2(k), \dots, d_N(k)\}.\end{aligned}$$

Proof. Denote the state error between the leader and agent i as

$$\bar{x}_i(k) := x_i(k) - x_0(k). \quad (2.6)$$

Then the closed-loop error dynamics can be written as

$$\begin{aligned}\bar{x}_i(k) &= A\bar{x}_i(k) + Bu_i(k) \\ &= A\bar{x}_i(k) + BK \sum_{j=1}^N a_{ij}(k)(\bar{x}_j(k) - \bar{x}_i(k)) - BKd_i(k)\bar{x}_i(k).\end{aligned}$$

Denote vector $\bar{X}(k) := [\bar{x}_1^T(k), \bar{x}_2^T(k), \dots, \bar{x}_N^T(k)]^T$, therefore the closed-loop error dynamics of the leader-following MAS can be described by

$$\begin{aligned}\bar{X}(k+1) &= (\hat{A} - ((\mathcal{L}_{\sigma(k)} + D_{\sigma(k)}) \otimes \hat{B}))\bar{X}(k) \\ &= \Psi_{\sigma(k)}\bar{X}(k).\end{aligned} \quad (2.7)$$

Therefore, the dynamics in (2.7) is a switched system. And the stability of system in (2.7) is equivalent to the leader-following consensus of the original MAS. By applying Lemma 2.1 to the system in (2.7), Theorem 2.1 is proved. \square

Remark 2.1. *When implementing Theorem 2.1, it is required to test the norms of all combinations $\Psi_{i_1}\Psi_{i_2}\dots\Psi_{i_c}$. Denote the number of elements in $\{\sigma(t)\}$ as M . Obviously, the number of the combinations is M^c . This means the computational complex increases exponentially with respect to c . Therefore the leader-following consensus condition could be hard to test when the required c is very large.*

Corollary 2.1. *Consider the leader-following MAS in (2.1)-(2.2). Suppose Assumptions 2.1 and 2.2 are satisfied. All the agents can follow the leader under consensus protocol in (2.5) if $\forall \Psi_{\sigma(k)}$, such that $\|\Psi_{\sigma(k)}\|_F < 1$.*

Proof. If $\|\Psi_{\sigma(k)}\|_F < 1$, there exists a finite number c , such that

$$\sqrt{nN}\|\Psi_{i_1}\|_F\|\Psi_{i_2}\|_F\cdots\|\Psi_{i_c}\|_F < 1, \forall \Psi_{i_j} \in \{\Psi_{\sigma(k)}\}. \quad (2.8)$$

And by the properties of the matrix norm, we have

$$\begin{aligned} \sqrt{nN}\|\Psi_{i_1}\|_F\|\Psi_{i_2}\|_F\cdots\|\Psi_{i_c}\|_F &\geq \sqrt{nN}\|\Psi_{i_1}\Psi_{i_2}\cdots\Psi_{i_c}\|_F \\ &\geq \sqrt{nN}\|\Psi_{i_1}\Psi_{i_2}\cdots\Psi_{i_c}\|_2 \\ &\geq \|\Psi_{i_1}\Psi_{i_2}\cdots\Psi_{i_c}\|. \end{aligned}$$

Hence there exists a finite number c such that

$$\|\Psi_{i_1}\Psi_{i_2}\cdots\Psi_{i_c}\| < 1.$$

□

Remark 2.2. *Corollary 2.1 tends to be conservative especially when the dimension of $\Psi_{\sigma(k)}$ is large. However, the computational complexity is independent of c . In this regard, Corollary 2.1 is preferable when the dimension of $\Psi_{\sigma(k)}$ is not too large.*

2.4 The Switching Topologies and Time-Varying Delays Case

In this section, we extend the result in the previous section to the case when there exist time-varying communication delays. Hence the consensus protocol is delay-dependent, which indicates

$$\begin{aligned} u_i(k) = K\left(\sum_{j=1}^N a_{ij}(k)(x_j(k - \tau(k))) - x_i(k - \tau(k))\right) \\ + Kd_i(k)(x_0(k - \tau(k)) - x_i(k - \tau(k))). \end{aligned} \quad (2.9)$$

And the delay term $\tau(k)$ is supposed to satisfy the following assumptions.

Assumption 2.3. *All the information exchange is subject to the same time delay $\tau(k) \in [0, \tau_{\max}]$, where τ_{\max} is a finite constant.*

Assumption 2.4. *The time delay $\tau(k)$ changes arbitrarily.*

The main result in this section is presented in the following theorem.

Theorem 2.2. *Consider the leader-following MAS in (2.1)-(2.2). Suppose Assumptions 2.1-2.4 are satisfied. All the agents can follow the leader under consensus protocol in (2.9) if and only if there exists a finite number c , such that*

$$\|\Psi_{i_1}\Psi_{i_2}\cdots\Psi_{i_c}\| < 1, \quad \forall \Psi_{i_j} \in \{\Psi_{\sigma(k)\tau(k)}\}, \quad j = 1, \dots, c,$$

where

$$\begin{aligned} \Psi_{\sigma(k)0} &= \begin{bmatrix} \hat{A} + H_{\sigma(k)} & & & & & \\ & I & 0 & & & \\ & & I & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & I & 0 \end{bmatrix}, \\ \Psi_{\sigma(k)1} &= \begin{bmatrix} \hat{A} & H_{\sigma(k)} & & & & \\ I & 0 & & & & \\ & I & 0 & & & \\ & & & \ddots & \ddots & \\ & & & & I & 0 \end{bmatrix}, \\ \Psi_{\sigma(k)2} &= \begin{bmatrix} \hat{A} & H_{\sigma(k)} & & & & \\ I & 0 & & & & \\ & I & 0 & & & \\ & & & \ddots & \ddots & \\ & & & & I & 0 \end{bmatrix}, \\ &\vdots \\ \Psi_{\sigma(k)\tau_{\max}} &= \begin{bmatrix} \hat{A} & & & & H_{\sigma(k)} & \\ I & 0 & & & & \\ & I & 0 & & & \\ & & & \ddots & \ddots & \\ & & & & I & 0 \end{bmatrix}, \\ H_{\sigma(k)} &= -((\mathcal{L}_{\sigma(k)} + D_{\sigma(k)}) \otimes \hat{B}). \end{aligned}$$

Proof. Denote the state error between the leader and agent i as

$$\bar{x}_i(k) = x_i(k) - x_0(k).$$

Therefore the closed-loop error dynamics can be written as

$$\begin{aligned} \bar{x}_i(k) &= A\bar{x}_i(k) + Bu_i(k) \\ &= A\bar{x}_i(k) \\ &\quad + BK \sum_{j=1}^N a_{ij}(k)(\bar{x}_j(k - \tau(k)) - \bar{x}_i(k - \tau(k))) - BKd_i(k)\bar{x}_i(k - \tau(k)). \end{aligned}$$

Since $\tau(k) \in [0, \tau_{\max}]$, and $\tau(k)$ can change arbitrarily. We can take $\tau(k)$ as a switching signal. Denote vector $\bar{X}(k) := [\bar{x}_1^T(k), \bar{x}_2^T(k), \dots, \bar{x}_N^T(k)]^T$. By augmenting the state, the closed-loop error dynamics of the MAS can be described by a switched system, which contains two independent switching signals $\sigma(k)$ and $\tau(k)$.

When $\tau(k) = 0$,

$$\begin{bmatrix} \bar{X}(k+1) \\ \bar{X}(k) \\ \vdots \\ \bar{X}(k - \tau_{\max} + 1) \end{bmatrix} = \Psi_{\sigma(k)0} \begin{bmatrix} \bar{X}(k) \\ \bar{X}(k-1) \\ \vdots \\ \bar{X}(k - \tau_{\max}) \end{bmatrix}.$$

When $\tau(k) = 1$,

$$\begin{bmatrix} \bar{X}(k+1) \\ \bar{X}(k) \\ \vdots \\ \bar{X}(k - \tau_{\max} + 1) \end{bmatrix} = \Psi_{\sigma(k)1} \begin{bmatrix} \bar{X}(k) \\ \bar{X}(k-1) \\ \vdots \\ \bar{X}(k - \tau_{\max}) \end{bmatrix}.$$

\vdots

When $\tau(k) = \tau_{\max}$,

$$\begin{bmatrix} \bar{X}(k+1) \\ \bar{X}(k) \\ \vdots \\ \bar{X}(k - \tau_{\max} + 1) \end{bmatrix} = \Psi_{\sigma(k)\tau_{\max}} \begin{bmatrix} \bar{X}(k) \\ \bar{X}(k-1) \\ \vdots \\ \bar{X}(k - \tau_{\max}) \end{bmatrix}.$$

Therefore the system matrix of the closed-loop error dynamics is switching within the set $\{\Psi_{\sigma(k)\tau(k)}\} = \{\Psi_{\sigma(k)0}\} \cup \{\Psi_{\sigma(k)1}\} \cup \dots \cup \{\Psi_{\sigma(k)\tau_{\max}}\}$. And the system can be regarded as a switched system

$$\begin{bmatrix} \bar{X}(k+1) \\ \bar{X}(k) \\ \vdots \\ \bar{X}(k - \tau_{\max} + 1) \end{bmatrix} = \Psi_{\sigma(k)\tau(k)} \begin{bmatrix} \bar{X}(k) \\ \bar{X}(k-1) \\ \vdots \\ \bar{X}(k - \tau_{\max}) \end{bmatrix}. \quad (2.10)$$

The stability of the switched system in (2.10) is equivalent to the leader-following consensus of the original MAS. Hence, with the help of Lemma 2.1, Theorem 2.2 is proved. \square

2.5 Simulation Examples

In this section, two examples are presented to validate the theoretical results. In the first example, we consider the case when the communication topologies are switching. In the second example, we consider the case with switching topologies as well as time-varying delays.

The possible topologies for the two numerical examples are shown as Figure 2.1. Let $a_{ij}(k) = 1$ if agent i and agent j are connected, otherwise $a_{ij}(k) = 0$. Similarly, Let $d_i(k) = 1$ if agent i is connected to the leader, otherwise, $d_i(k) = 0$.

Example 2.1. Consider an MAS which includes four agents and one leader with the dynamics $A = 1.02$, $B = 1$, and the control gain $K = 0.21$. Hence $\{\Psi_{\sigma(k)}\} =$

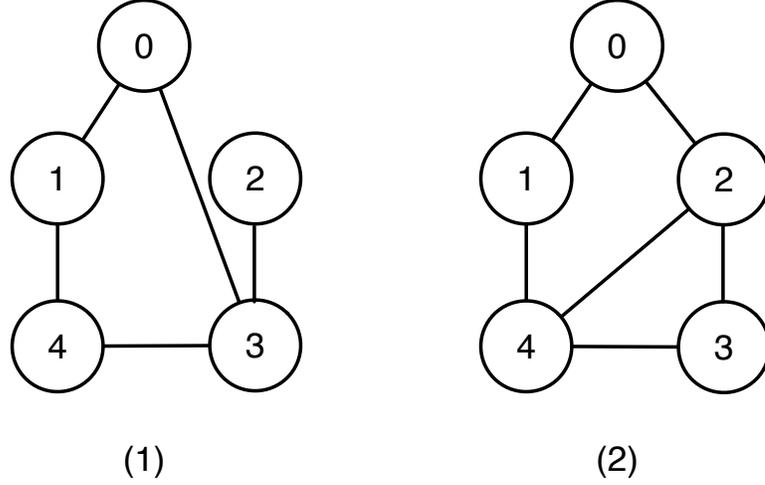


Figure 2.1: Possible communication topologies.

$\{\Psi_1, \Psi_2\}$, where

$$\Psi_1 = \begin{bmatrix} 0.6 & 0 & 0 & 0.21 \\ 0 & 0.81 & 0.21 & 0 \\ 0 & 0.21 & 0.39 & 0.21 \\ 0.21 & 0 & 0.21 & 0.6 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} 0.6 & 0 & 0 & 0.21 \\ 0 & 0.39 & 0.21 & 0.21 \\ 0 & 0.21 & 0.6 & 0.21 \\ 0.21 & 0.21 & 0.21 & 0.39 \end{bmatrix}.$$

By testing the matrix norms, it is found that

$$\|\Psi_{i_1} \Psi_{i_2} \cdots \Psi_{i_{15}}\|_\infty < 1, \quad \forall \Psi_{i_j} \in \{\Psi_{\sigma(k)}\},$$

where $j = 1, 2, \dots, 15$.

The initial condition for the simulations are $x_0(0) = 0.1$, $x_1(0) = 1.4$, $x_2(0) = -1.3$, $x_3(0) = -0.8$, and $x_4(0) = 0.9$. The distribution of the topologies in the simulation is generated randomly, which is shown as Figure 2.2. The state trajectories and deviation trajectories between the agents and the leader are shown in Figure 2.3 and Figure 2.4, respectively. We can see that the leader-following consensus is achieved asymptotically.

Example 2.2. Consider an MAS with the same dynamics and control gain as in Example 2.1. But the information exchange is subject to a time-varying delay $\tau(k)$,

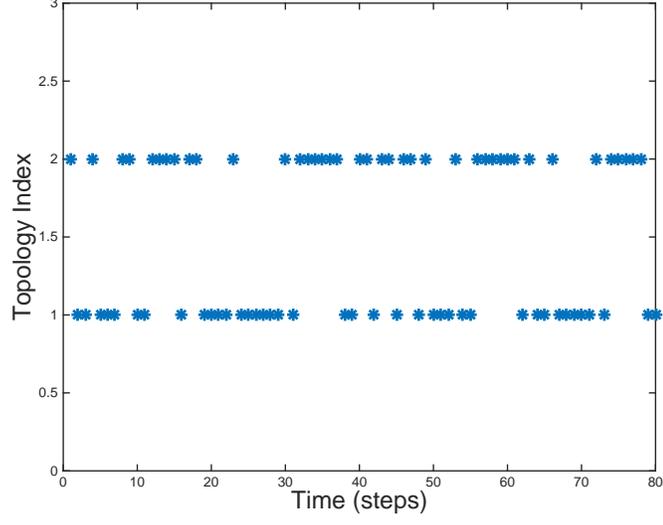


Figure 2.2: The distribution of the topologies.

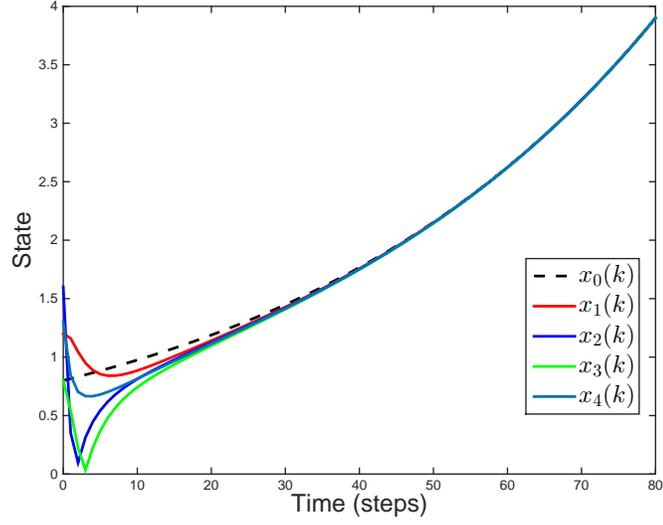


Figure 2.3: The state trajectories.

which is bounded by $\tau_{\max} = 1$. Therefore $\{\Psi_{\sigma(k)\tau(k)}\} = \{\Psi_{10}, \Psi_{11}, \Psi_{20}, \Psi_{21}\}$, where

$$\Psi_{10} = \begin{bmatrix} \hat{A} + H_1 & 0 \\ I & 0 \end{bmatrix}, \quad \Psi_{11} = \begin{bmatrix} \hat{A} & H_1 \\ I & 0 \end{bmatrix},$$

$$\Psi_{20} = \begin{bmatrix} \hat{A} + H_2 & 0 \\ I & 0 \end{bmatrix}, \quad \Psi_{21} = \begin{bmatrix} \hat{A} & H_2 \\ I & 0 \end{bmatrix},$$

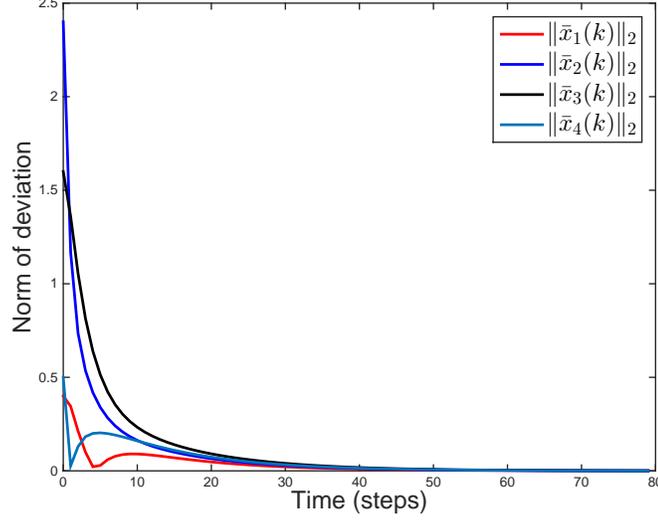


Figure 2.4: The deviation trajectories between agents and the leader.

$$\hat{A} = \begin{bmatrix} 1.02 & 0 & 0 & 0 \\ 0 & 1.02 & 0 & 0 \\ 0 & 0 & 1.02 & 0 \\ 0 & 0 & 0 & 1.02 \end{bmatrix},$$

and

$$H_1 = \begin{bmatrix} -0.42 & 0 & 0 & 0.21 \\ 0 & -0.21 & 0.21 & 0 \\ 0 & 0.21 & -0.63 & 0.21 \\ 0.21 & 0 & 0.21 & -0.42 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.42 & 0 & 0 & 0.21 \\ 0 & -0.63 & 0.21 & 0.21 \\ 0 & 0.21 & -0.42 & 0.21 \\ 0.21 & 0.21 & 0.21 & -0.63 \end{bmatrix}.$$

We can test that

$$\|\Psi_{i_1} \Psi_{i_2} \cdots \Psi_{i_{25}}\|_\infty < 1, \quad \forall \Psi_{i_j} \in \{\Psi_{\sigma(k)\tau(k)}\},$$

where $j = 1, 2, \dots, 25$.

The initial conditions for the simulation are $x_0(0) = 0.9$, $x_1(0) = -2.5$, $x_2(0) = 3$, $x_3(0) = 2.1$, and $x_4(0) = 1.7$. The distributions of the topologies and delays in the simulation are generated randomly, which are shown as Figure 2.5 and Figure 2.6,

respectively. The state trajectories and deviation trajectories between the agents and the leader are shown in Figure 2.7 and Figure 2.8, respectively. The leader-following consensus is achieved asymptotically regardless of the switching topologies and time delays.

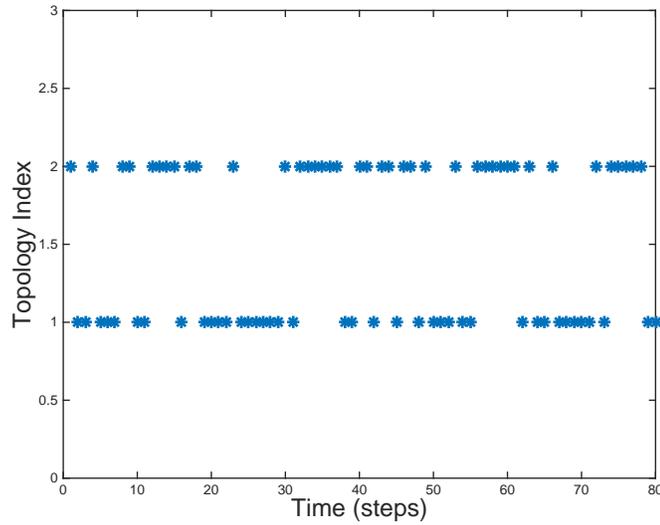


Figure 2.5: The distribution of the topologies.

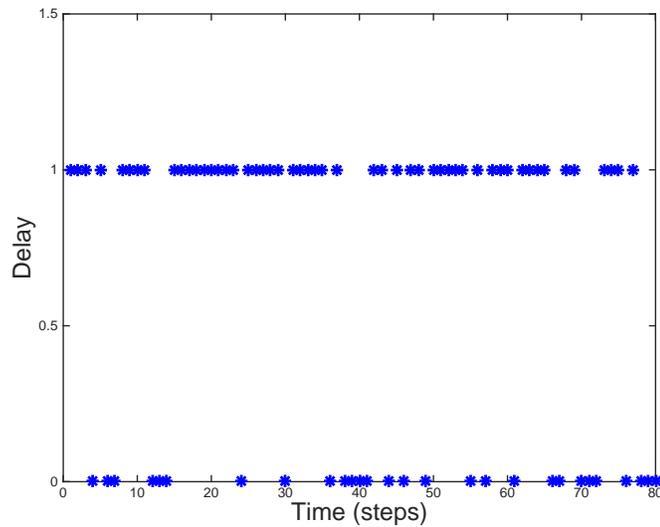


Figure 2.6: The distribution of the time delays.

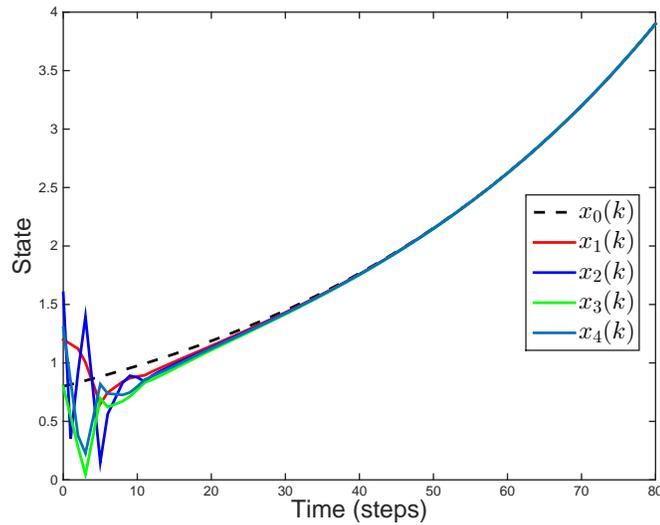


Figure 2.7: The state trajectories.

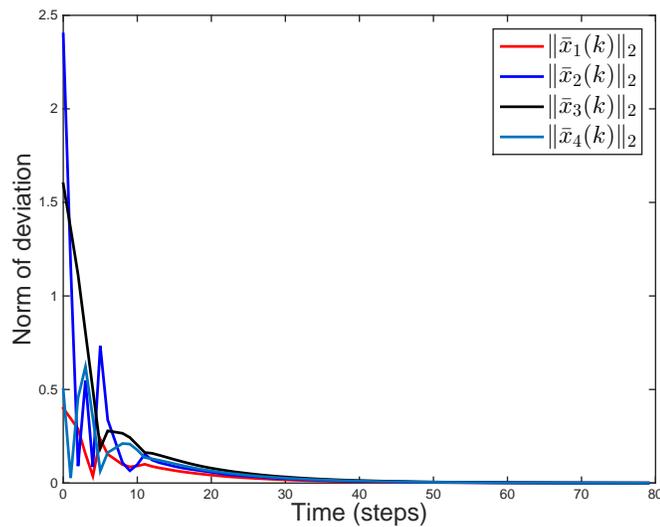


Figure 2.8: The deviation trajectories between agents and the leader.

2.6 Conclusion

In this chapter, the leader-following consensus problems for linear MASs with switching topologies and time-varying delays are studied from a switched system perspective. By modelling the multi-agent system as a switched system, the leader-following consensus problem is equivalent to the stability problem of the switched system. In this

scheme, norm based necessary and sufficient leader-following consensus conditions are established. The effectiveness of the obtained theoretical results are finally verified by numerical simulations.

Chapter 3

A Fully Distributed Approach for Consensus of Multi-Agent Systems under Multiple Communication Topologies

3.1 Introduction

The past years have witnessed the rapid increase of interest in the cooperative control of MASs among a variety of areas, such as sensor networks [2, 3, 4], autonomous vehicles [39, 115, 100], etc. An MAS is composed of multiply interacting autonomous agents. A key feature of MAS is the overall system can cooperatively fulfill complex assignments, which are difficult or impossible for an individual agent to solve. MASs also benefit the system implementation for their flexibility, reliability, and efficiency.

Most of the research interest on MASs have been focused on the fundamental problem: How to ensure the agents to reach an agreement only using local information. This is usually called consensus. The study of consensus has become prosperous since the work in [42]. Agents governed by relative simple dynamics, such as single integrator and double integrator dynamics [25, 24, 26, 108, 53], have been firstly investigated. Very recently, attentions have been gradually paid on the study of agents characterized by more general dynamics, like general linear dynamics [116, 70, 27, 48, 117, 92], heterogenous dynamics [118], etc.

It is worth pointing out that the majority of existing results are derived based

on the assumption that all the state variables of each agent are exchanged via the same communication topology. In recent years, the MIMO technology, which could significantly improve the communication capacity, has been promisingly developed. It has been widely applied in wireless communication [95, 96, 97, 98]. When introduced to networked control systems, an important feature of MIMO communication is that the number of the SISO subchannels is usually greater than the number of control inputs [119]. Hence the subchannels corresponding to different control inputs could be allocated as shown in [120]. That is to say, some higher capacity or more reliable subchannels are used for more important inputs while lower capacity or less reliable subchannels are used for less important inputs, which could give an additional freedom for system design. This motivates us to study the communication of MASs following the same idea that dominant state variables are exchanged via more reliable subchannels, while other state variables are just exchanged via less reliable subchannels. Due to the different reliabilities, the information loss could lead to different communication topologies for different state variables. In fact, if we only exchange part of the state information for some agents, the communication cost can be reduced. Considering the above-mentioned factors, some preliminary studies have been conducted in [75, 76, 77, 78] with independent position and velocity communication topologies. In these previous work, only agents with second-order dynamics are studied. And the design of controllers requires some global information related to the entire network.

Inspired by these previous results, in this chapter, we study the distributed consensus problem of general linear dynamics under multiple communication topologies, which transmitting different state variables. Both fixed and switching topologies cases are investigated. Our topology assumption on connectivity only requires that each of the topologies contains or jointly contains a spanning tree, which is very mild. The proposed distributed consensus protocols are based on the information exchange of auxiliary state. This scheme is similar to those in [121, 122]. In light of the proposed consensus control protocols, the consensus problem can be cast as a cascade system consisting of two subsystems. The convergence analysis is conducted based on the stability of each subsystem. The main contribution of this work is twofold:

- A novel multiple interconnection topologies scheme is proposed for MASs. Based on this scheme, necessary and sufficient consensus conditions for general linear dynamics are established.

- The controllers are designed and implemented in a fully distributed manner under a mild topology connection requirement.

The rest of this chapter is organized as follows. In Section 3.2, some relevant fundamental knowledge and the problem formulation are provided. In Section 3.3, the distributed consensus problems under fixed and switching multiple topologies are investigated. And the theoretical results are validated by the numerical examples in Section 3.4. Finally, the chapter is concluded in Section 3.5.

3.2 Preliminaries and Problem Formulation

In this section, some fundamental knowledge from graph theory and preliminary consensus results are introduced. Then the problem studied in this chapter is formulated.

3.2.1 Basic Concepts from Graph Theory

In the study of MASs, the communication topology among agents is commonly depicted by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. $\mathcal{V} = \{1, 2, \dots, N\}$ denotes the index set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the weighted adjacency matrix, where $i, j \in \mathcal{V}$. An edge of the graph \mathcal{G} is represented by an ordered pair of nodes $(j, i) \in \mathcal{E}$, which indicates the edge pointing from node j to node i . The edge (j, i) is usually called an incoming edge of node i and an outgoing edge of node j .

A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \dots$. A directed graph is strongly connected if there exists a directed path from each node to any other node. A directed graph contains a spanning tree if and only if there exists at least one node with a directed path to all the other nodes.

For a directed graph \mathcal{G} , the Laplacian matrix $\mathcal{L}_{\mathcal{G}} = [l_{ij}]$ is defined by

$$l_{ij} = \begin{cases} \sum_{k=1}^N a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases}.$$

Since the Laplacian matrix $\mathcal{L}_{\mathcal{G}}$ has zero row sums, 0 is an eigenvalue of $\mathcal{L}_{\mathcal{G}}$ with the associated eigenvector $\mathbf{1}_N$, where $\mathbf{1}_N$ represents the N dimensional vector of ones. This indicates that $\mathcal{L}_{\mathcal{G}}\mathbf{1}_N = \mathbf{0}$, where $\mathbf{0}$ represents the matrix or vector of zeros with proper dimensions.

The graph describing the communication topology can be time-varying. We adopt the notion of dwell time to handle this problem. Suppose there exists a finite lower bound on the time between the topology switching, which can be described as

$$\begin{aligned}\mathcal{A}(t) &= \mathcal{A}(t_{i'}), \\ \mathcal{L}_{\mathcal{G}}(t) &= \mathcal{L}_{\mathcal{G}}(t_{i'}),\end{aligned}$$

when $t \in [t_{i'}, t_{i'} + \tau)$. Here $i' = 1, 2, \dots$. $\tau > 0$ is the dwell time, and t_0, t_1, \dots is an infinite time sequence satisfying $t_{i'+1} - t_{i'} = \tau$. The union of a group of graphs is a graph whose vertex set and edge set are the union of the vertex sets and edge sets of all the graphs in this group. If there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[t_{j'}, t_{j'+1})$, $j' = 1, 2, \dots$, starting at $t_1 = 0$, the union of the directed graphs in each interval contains a spanning tree, we call the switching graph jointly containing a spanning tree.

3.2.2 Preliminary Results on Consensus

Consider N first-order integrator agents are connected by a fixed directed graph \mathcal{G} . Each agent updates its state $\xi_i \in \mathbb{R}^n$ with the following consensus protocol

$$\dot{\xi}_i(t) = \sum_{j=1}^n a_{ij}(\xi_j(t) - \xi_i(t)), \quad i = 1, 2, \dots, N, \quad (3.1)$$

where $a_{ij} \in \mathbb{R}$ are the adjacency matrix's elements of graph \mathcal{G} . Let $\mathcal{L}_{\mathcal{G}}$ denote the Laplacian matrix of graph \mathcal{G} , and let $\xi_i(0)$ denote the initial state of agent i . We have the following consensus result from [123].

Lemma 3.1. *The multi-agent system consisting of N first-order integrator dynamics under the consensus protocol in (3.1) achieves consensus asymptotically if and only if the directed graph \mathcal{G} has a directed spanning tree. In particular, $\xi_i(t) \rightarrow \sum_{i=1}^N v_i \xi_i(0)$, as $t \rightarrow \infty$, where $v = [v_1, v_2, \dots, v_N]^T \geq 0$, $\mathbf{1}_N^T v = 1$, and $\mathcal{L}_{\mathcal{G}}^T v = \mathbf{0}$.*

If the first-order MAS is connected by a single switching topology, the consensus protocol in (3.1) can be modified as

$$\dot{\xi}_i(t) = \sum_{j=1}^n a_{ij}(t)(\xi_j(t) - \xi_i(t)), \quad i = 1, 2, \dots, N, \quad (3.2)$$

where a_{ij} is time-varying. And the following lemma is proved in [123].

Lemma 3.2. *The system of N first-order integrator agents under the consensus protocol in (3.2) achieves consensus asymptotically if the switching directed graph $\mathcal{G}(t)$ jointly contains a spanning tree.*

Lemma 3.1 provides a necessary and sufficient consensus condition for first-order integrator MASs under single fixed topology, whereas Lemma 3.2 provides a sufficient consensus condition for first-order integrator MASs under single switching topology. In Lemma 3.1 and Lemma 3.2, for the i th agent, it requires the state of itself $\xi_i(t)$ and those of its neighbours $\xi_j(t)$, where $j \in \mathcal{N}_i$. Hence the result is fully distributed. Moreover, in fixed topology case, with specified Laplacian matrix and the initial conditions, the final consensus equilibrium point can also be calculated.

3.2.3 Problem Formulation

This chapter studies the consensus problem for a group of N general linear agents. The dynamics of each agent are

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad (3.3)$$

where $x_i(t) \in \mathbb{R}^n$, and $u_i(t) \in \mathbb{R}^m$. In order to make the problem solvable, we introduce the following assumption on the dynamics.

Assumption 3.1. *The pair (A, B) of the system in (3.3) is stabilizable.*

Suppose each agent can interact with some other agents. Different from the commonly adopted interaction framework that all the state variables are exchanged through the same topology, in our work, we consider a more general case, in which the information exchange can be implemented via several different and independent channels. Different channels are assumed to transmit different state variables. We use the directed graphs $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$ to represent these channels. And we make the following assumption for the graphs.

Assumption 3.2. *If the topologies are fixed, each of the graphs \mathcal{G}_i , $i = 1, 2, \dots, k$, contains a spanning tree. If the topologies are switching, the graph $\mathcal{G}_i(t)$, $i = 1, 2, \dots, k$, jointly contains a spanning tree.*

Under the proposed communication topologies, the objective is to design a distributed control input $u_i(t)$, which is only based on local information, to ensure all the agents to achieve consensus. This can be described mathematically as

$$\lim_{t \rightarrow \infty} x_i(t) = x_\infty, \quad \forall i = 1, 2, \dots, N,$$

where x_∞ is a constant vector with finite elements.

3.3 Main Results

In this section, fully distributed control strategies are introduced for MASs under fixed and switching multiple communication topologies to achieve consensus. Theoretical analysis on the convergence of the proposed control strategies is also demonstrated.

3.3.1 The Multiple Fixed Communication Topologies Case

This section studies the consensus problem of N agents with the dynamics in (3.3) exchanging information via different fixed communication topologies. The topologies are represented by $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$. That is to say, different communication topologies transmit different state variables. Considering the multiple topologies, it is very challenging to adopt the average based consensus protocols, like those in [70, 116], because the closed-loop error dynamics of the overall system contain multiple terms of different Laplacian matrices, which cannot be diagonalized simultaneously. Hence we adopt the following consensus protocol, inspired by the results in [118, 121]. Such consensus protocol turns out to be effective for the case considering the multiple state-variables-dependent topologies:

$$\begin{cases} \dot{\eta}_i^l = \sum_{j=1}^N a_{ij}^l (\eta_j^l - \eta_i^l), \\ \eta_i = [\eta_i^{1T}, \eta_i^{2T}, \dots, \eta_i^{kT}]^T, \\ u_i = K(x_i - \eta_i), \end{cases} \quad (3.4)$$

where $\eta_i^l \in \mathbb{R}^{n_l}$, $l = 1, 2, \dots, k$, is the auxiliary state corresponding to agent i transmitting in topology l , a_{ij}^l is the elements of the adjacency matrix of graph \mathcal{G}_l , $\sum_{l=1}^k n_l = n$, and $K \in \mathbb{R}^{m \times n}$ is the control gain to be designed.

It can be seen that the consensus protocol for agent i in (3.4) is developed with the auxiliary state of itself and the auxiliary state η_j^l from its neighbours of k different

topologies. Therefore, the consensus protocols can be applied in a fully distributed way. In addition, only relative information $\eta_j^l - \eta_i^l$ is exchanged between agents, so the absolute reference frame is not required.

Denote

$$\begin{aligned} x &:= [x_1^T, x_2^T, \dots, x_N^T]^T, \\ \eta &:= [\eta_1^T, \eta_2^T, \dots, \eta_N^T]^T. \end{aligned}$$

By applying the distributed consensus protocol in (3.4), the closed-loop system dynamics can be represented as

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} I_N \otimes (A + BK) & I_N \otimes BK \\ \mathbf{0} & -\tilde{\mathcal{L}} \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix}, \quad (3.5)$$

where

$$\begin{aligned} \tilde{\mathcal{L}} &= \sum_{l=1}^k \mathcal{L}_{\mathcal{G}_l} \otimes M_l, \\ M_l &= \begin{bmatrix} M_{l1} & & & \\ & M_{l2} & & \\ & & \ddots & \\ & & & M_{lk} \end{bmatrix}, \\ M_{lp} &= \begin{cases} \mathbf{0} & l \neq p \\ I_{n_p} & l = p \end{cases}, \end{aligned}$$

for $p = 1, 2, \dots, k$, I_N denotes the identity matrix of size N , and $\mathcal{L}_{\mathcal{G}_l}$ denotes the weighted Laplacian matrix of graph \mathcal{G}_l . The system in (3.5) can be considered as an autonomous system of state η and a system of state x with the input η . For any finite initial values of state x and state η , the consensus can be fulfilled by two steps:

1. Synchronizing the auxiliary state η_i with local information exchange;
2. Controlling the state x_i based on the synchronized auxiliary η_i to reach consensus.

Then the main result for consensus with fixed communication topologies is presented here.

Theorem 3.1. *Suppose Assumption 3.1 and Assumption 3.2 are satisfied. The MAS with dynamics in (3.3) under the control protocol in (3.4) can achieve consensus if and only if $A + BK$ is Hurwitz. In particular,*

$$x_\infty = \lim_{t \rightarrow \infty} x_i(t) = -(A + BK)^{-1} BK \eta_\infty,$$

where

$$\begin{aligned} \eta_\infty &= [\eta_\infty^{1\text{T}}, \eta_\infty^{2\text{T}}, \dots, \eta_\infty^{k\text{T}}]^\text{T}, \\ \eta_\infty^l &= \lim_{t \rightarrow \infty} \eta_i^l(t) = \sum_{i=1}^N v_i^l \eta_i^l(0), \\ v^l &= [v_1^l, v_2^l, \dots, v_n^l]^\text{T} \geq 0, \end{aligned}$$

with $\mathbf{1}_N^\text{T} v^l = 1$, and $\mathcal{L}_{\mathcal{G}_l}^\text{T} v^l = \mathbf{0}$.

Proof. Denote

$$A_1 := -\tilde{\mathcal{L}}, \quad (3.6)$$

$$A_2 := I_N \otimes (A + BK), \quad (3.7)$$

$$B_2 := I_N \otimes BK. \quad (3.8)$$

Then the closed-loop system in (3.5) can be studied as a cascade system. Figure 3.1 shows the block diagram of the cascade system. The subsystem S_1 is an autonomous system of the state η , and the subsystem S_2 is a system of the state x and the external input η .

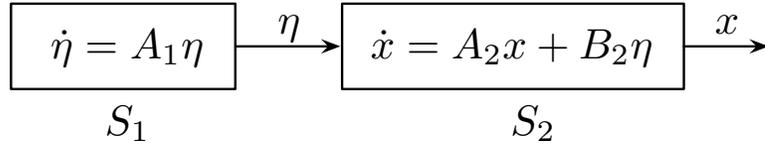


Figure 3.1: The system block diagram considering fixed topologies.

The subsystem S_1 can be considered as k independent first-order MASs with average consensus protocol. Suppose Assumption 3.1 is satisfied, that is to say, the topology \mathcal{G}_l contains a spanning tree. With the help of Lemma 3.1, all of the k

first-order MASs reach consensus asymptotically. The final state of η_i^l is

$$\eta_\infty^l = \lim_{t \rightarrow \infty} \eta_i^l(t) = \sum_{i=1}^N v_i \eta_i^l(0), \quad (3.9)$$

where $v^l = [v_1^l, v_2^l, \dots, v_n^l]^T \geq 0$, $\mathbf{1}_N^T v^l = 1$, and $\mathcal{L}_{\mathcal{G}_l}^T v^l = \mathbf{0}$, which indicates v^l is the left eigenvector of $\mathcal{L}_{\mathcal{G}_l}$ corresponding to the zero eigenvalue. Denote

$$\eta_\infty = \left[\eta_\infty^{1^T}, \eta_\infty^{2^T}, \dots, \eta_\infty^{k^T} \right]^T, \quad (3.10)$$

then we have

$$\lim_{t \rightarrow \infty} \eta_i(t) = \eta_\infty.$$

Since the state η_i converges to η_∞ asymptotically. The overall closed-loop system is stable if and only if the subsystem S_2 is stable. This indicates the system matrix A_2 needs to be Hurwitz, or $A + BK$ should be Hurwitz.

If $A + BK$ is Hurwitz, the final value of $x(t)$ can be calculated by the final value theorem

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s(sI - A_2)^{-1} B_2 (sI - A_1)^{-1}. \quad (3.11)$$

Because the subsystem S_1 reaches consensus asymptotically, and the state η_i converges to η_∞ , we have

$$\lim_{t \rightarrow \infty} \eta(t) = \lim_{s \rightarrow 0} s(sI - A_1)^{-1} = [\eta_\infty^T, \eta_\infty^T, \dots, \eta_\infty^T]^T. \quad (3.12)$$

With (3.11) and (3.12), we have

$$\begin{aligned} \lim_{t \rightarrow \infty} x(t) &= \lim_{s \rightarrow 0} (sI - A_2)^{-1} B_2 \lim_{s \rightarrow 0} s(sI - A_1)^{-1} \\ &= \lim_{s \rightarrow 0} (sI - A_2)^{-1} B_2 [\eta_\infty^T, \eta_\infty^T, \dots, \eta_\infty^T]^T. \end{aligned} \quad (3.13)$$

Substituting (3.7) and (3.8) into (3.13), we can derive

$$x_\infty = \lim_{t \rightarrow \infty} x_i(t) = -(A + BK)^{-1} BK \eta_\infty, \quad (3.14)$$

which indicates that $x_i(t)$ asymptotically converges to x_∞ . This implies that the MAS

reaches consensus asymptotically. And from (3.9), (3.10) and (3.14), we can see the consensus trajectory depends on the initial conditions. \square

Remark 3.1. *Theorem 3.1 shows the consensus problem can be solved with the control protocol in (3.4) by designing a control gain K such that $A+BK$ is Hurwitz. The control gain synthesis problem merely depends on the dynamics of each agent, regardless of the global information. Hence the control protocol in (3.4) can be designed and implemented distributedly.*

Theorem 3.1 gives a necessary and sufficient consensus conditions for general MASs. When $k = 1$, which indicates that all the state can be exchanged via a single topology, the problem reduces to the single communication topology problem. In such a case, the control protocol in (3.4) can be simplified as

$$\begin{cases} \dot{\eta}_i = \sum_{j=1}^N a_{ij}^1 (\eta_j - \eta_i), \\ u_i = K(x_i - \eta_i), \end{cases} \quad (3.15)$$

and

$$A_1 = -\mathcal{L}_{\mathcal{G}_1} \otimes I_n.$$

Then we can present the result for consensus of general linear multi-agent under single communication topology.

Corollary 3.1. *Suppose Assumption 3.1 and Assumption 3.2 are satisfied. The MAS with dynamics in (3.3) under the control protocol in (3.15) achieves consensus if and only if $A + BK$ is Hurwitz. In particular,*

$$x_\infty = \lim_{t \rightarrow \infty} x_i(t) = -(A + BK)^{-1} BK \eta_\infty,$$

where

$$\begin{aligned} \eta_\infty &= \lim_{t \rightarrow \infty} \eta_i(t) = \sum_{i=1}^N v_i \eta_i(0), \\ v^1 &= [v_1^1, v_2^1, \dots, v_n^1]^T \geq 0, \end{aligned}$$

with $\mathbf{1}_N^T v^1 = 1$, and $\mathcal{L}_{\mathcal{G}_1}^T v^1 = \mathbf{0}$.

3.3.2 The Multiple Switching Communication Topologies Case

This section studies the consensus problem of N agents with the dynamics in (3.3). However, the agents exchange information via different switching communication topologies represented by $\mathcal{G}_1(t), \mathcal{G}_2(t), \dots, \mathcal{G}_k(t)$. Each communication topology can only transmit part of the state variables. The following control protocol is adopted for the general linear multi-agents under multiple switching communication topologies.

$$\begin{cases} \dot{\eta}_i^l = \sum_{j=1}^N a_{ij}^l(t)(\eta_j^l - \eta_i^l), \\ \eta_i = [\eta_i^{1T}, \eta_i^{2T}, \dots, \eta_i^{kT}]^T, \\ u_i = K(x_i - \eta_i), \end{cases} \quad (3.16)$$

where $\eta_i^l \in \mathbb{R}^{n_l}$ is auxiliary state corresponding to agent i transmitting in topology l , $a_{ij}^l(t)$ is the elements of the adjacency matrix for graph $\mathcal{G}_l(t)$, and $K \in \mathbb{R}^{m \times n}$ is the control gain to be designed. In control protocol (3.16), for agent i , it requires the information of itself and η_j^l from all its neighbours in k different topologies. This indicates the control protocol can still be applied in a fully distributed way.

Under distributed control protocol in (3.16), the closed-loop system dynamics of this MASs can be represented by

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} I_N \otimes (A + BK) & I_N \otimes BK \\ \mathbf{0} & -\tilde{\mathcal{L}}(t) \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix}, \quad (3.17)$$

where

$$\begin{aligned} \tilde{\mathcal{L}}(t) &= \sum_{l=1}^k \mathcal{L}_{\mathcal{G}_l(t)} \otimes M_l, \\ M_l &= \begin{bmatrix} M_{l1} & & & \\ & M_{l2} & & \\ & & \ddots & \\ & & & M_{lk} \end{bmatrix}, \\ M_{lp} &= \begin{cases} \mathbf{0} & l \neq p \\ I_{n_p} & l = p \end{cases}, \end{aligned}$$

for $p = 1, 2, \dots, k$.

Then the main result for consensus with multiple switching communication topologies can be established.

Theorem 3.2. *Suppose Assumption 3.1 and Assumption 3.2 are satisfied. The MAS with dynamics in (3.3) under the control protocol in (3.16) achieves consensus if and only if $A + BK$ is Hurwitz.*

Proof. Denote

$$\tilde{A}_1 = -\tilde{\mathcal{L}}(t),$$

Then the closed-loop system in (3.17) can be studied as a cascade system, shown in Figure 3.2. The subsystem S'_1 is an autonomous system of the state η , and the subsystem S_2 is a system of the state x and the external input η .

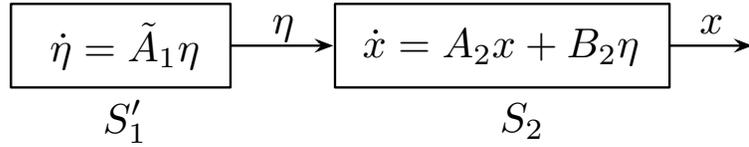


Figure 3.2: The system block diagram considering switching topologies.

The subsystem S'_1 can be considered as k different independent first-order MASs with average consensus protocol under switching topologies. Suppose Assumption 3.2 is satisfied, that is to say, the topology $\mathcal{G}_l(t)$, $l = 1, 2, \dots, k$, jointly contains a spanning tree. With the help of Lemma 3.2, all of the k first-order MASs reach consensus asymptotically. Denote the final state of η_i^l as

$$\eta_\infty^l = \lim_{t \rightarrow \infty} \eta_i^l(t).$$

And let

$$\eta_\infty = \left[\eta_\infty^{1\text{T}}, \eta_\infty^{2\text{T}}, \dots, \eta_\infty^{k\text{T}} \right]^{\text{T}},$$

then we have

$$\lim_{t \rightarrow \infty} \eta_i(t) = \eta_\infty,$$

which shows that the subsystem S'_1 reaches consensus asymptotically. Therefore the overall closed-loop system is stable if and only if the subsystem S_2 is stable. This requires the system matrix A_2 to be Hurwitz, or $A + BK$ to be Hurwitz.

If $A + BK$ is Hurwitz, define

$$\begin{aligned}\hat{x} &:= [\hat{x}_1^T, \hat{x}_2^T, \dots, \hat{x}_N^T]^T, \\ \hat{x}_i &:= x_i + (A + BK)^{-1}BK\eta_\infty, \\ \delta &:= [\delta_1^T, \delta_2^T, \dots, \delta_N^T]^T, \\ \delta_i &:= \eta_i - \eta_\infty.\end{aligned}$$

Then the system in Figure 3.2 can be transformed to the system in Figure 3.3. We can see that the external input for S'_2 asymptotically converges to 0. Since $A + BK$ is Hurwitz, \hat{x} converges to 0 asymptotically. Equivalently,

$$x_\infty = \lim_{t \rightarrow \infty} x_i(t) = -(A + BK)^{-1}BK\eta_\infty.$$

Hence the MAS reaches consensus asymptotically. \square

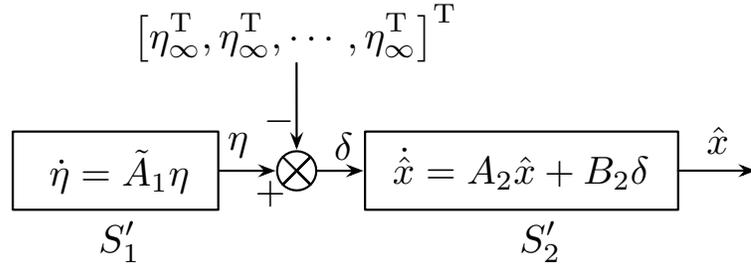


Figure 3.3: The block diagram of the modified cascade system.

Remark 3.2. *Theorem 3.2 shows that the consensus can be fulfilled with the consensus protocol in (3.16) depending on a control gain K such that $A + BK$ is Hurwitz. Namely, in the multiple switching interaction topologies case, the control gain synthesis problem is still based on the dynamics of each agent, which implies that only local information is needed. Hence the consensus protocol in (3.16) is in a distributed manner on both design and implementation.*

Theorem 3.2 gives necessary and sufficient consensus conditions for general MASs under multiple switching topologies. We can also derive the result under single switching communication topology. In this case, the consensus protocol in (3.16) can be

simplified as

$$\begin{cases} \dot{\eta}_i = \sum_{j=1}^N a_{ij}^1(t)(\eta_j - \eta_i), \\ u_i = K(x_i - \eta_i), \end{cases} \quad (3.18)$$

and

$$A_1 = -\mathcal{L}_{\mathcal{G}_1(t)} \otimes I_n.$$

Then the result is summarized below.

Corollary 3.2. *Suppose Assumption 3.1 and Assumption 3.2 are satisfied. The MAS with dynamics in (3.3) under the consensus protocol in (3.18) achieves consensus if and only if $A + BK$ is Hurwitz.*

3.4 Simulation Examples

In this section, two examples are illustrated to validate the theoretical results. The first example considers the case when the communication topologies are fixed. And the second example considers the case when the communication topologies are switching.

Example 3.1. *Consider an MAS consisting of 5 agents. The system matrices for each agent are defined as follows.*

$$A = \begin{bmatrix} -5.3758 & 4.2278 & 7.1702 \\ 3.3786 & 7.1897 & 4.3821 \\ 2.3868 & 6.6645 & 8.8498 \end{bmatrix},$$

$$B = \begin{bmatrix} -2.6677 \\ -8.2524 \\ -12.6030 \end{bmatrix}.$$

And it can be verified that the matrix A is unstable, but the matrix pair (A, B) is stabilizable. The control gain

$$K = \begin{bmatrix} -2.6677 & -8.2524 & -12.6030 \end{bmatrix}$$

is adopted to ensure $A + BK$ is Hurwitz.

Assuming that the first and the second auxiliary state variables are exchanged via graph \mathcal{G}_1 , and the third auxiliary state variable is exchanged via graph \mathcal{G}_2 . Define the Laplacian matrices corresponding to the communication topologies to be

$$\mathcal{L}_{\mathcal{G}_1} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{G}_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

It can be checked that both graph \mathcal{G}_1 and graph \mathcal{G}_2 contain a spanning tree, respectively.

In this simulation, each initial state variable is generated randomly from the set $[-5, 5]$. Under the consensus protocol proposed in (3.4), the auxiliary state and system state trajectories for each agent are shown in Figure 3.4 and Figure 3.5, respectively. It can be observed that the states of different agents converge to a same vector asymptotically, which means the system reaches consensus.

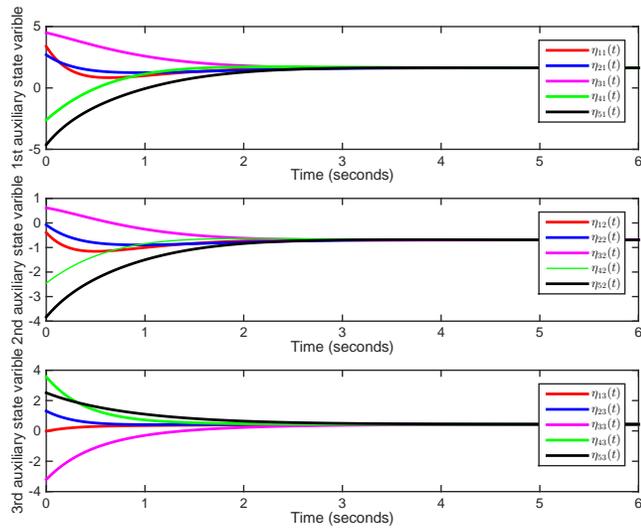


Figure 3.4: The auxiliary state variable trajectories.

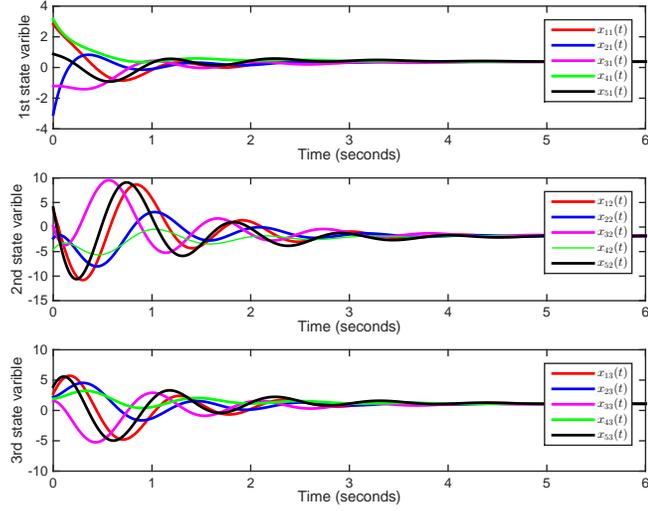


Figure 3.5: The state variable trajectories.

Example 3.2. Consider an MAS consisting of 5 agents. The system matrices for each agent are defined as follows.

$$A = \begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}.$$

And it can be verified that the matrix A is unstable, but the matrix pair (A, B) is stabilizable. The control gain

$$K = \begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}$$

is adopted to ensure $A + BK$ is Hurwitz. Assuming that the first auxiliary state variable is exchanged via graph $\mathcal{G}_1(t)$, and the second auxiliary state variable is exchanged via graph $\mathcal{G}_2(t)$. Suppose $\mathcal{G}_1(t)$ can switch between graph \mathcal{G}_{11} and graph \mathcal{G}_{12} , and $\mathcal{G}_2(t)$ can switch between graph \mathcal{G}_{21} and graph \mathcal{G}_{22} . The corresponding Laplacian matrices

are

$$\mathcal{L}_{\mathcal{G}_{11}} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathcal{L}_{\mathcal{G}_{12}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{G}_{21}} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}_{\mathcal{G}_{22}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The state of the switching topologies can be represented by switching signals, which are shown in Figure 3.6. The state 1 represents $\mathcal{G}_1(t) = \mathcal{G}_{11}$, or $\mathcal{G}_2(t) = \mathcal{G}_{21}$. And the state 2 represents $\mathcal{G}_1(t) = \mathcal{G}_{12}$, or $\mathcal{G}_2(t) = \mathcal{G}_{22}$. It can be verified that the graph $\mathcal{G}_1(t)$ and the graph $\mathcal{G}_2(t)$ are jointly containing a spanning tree.

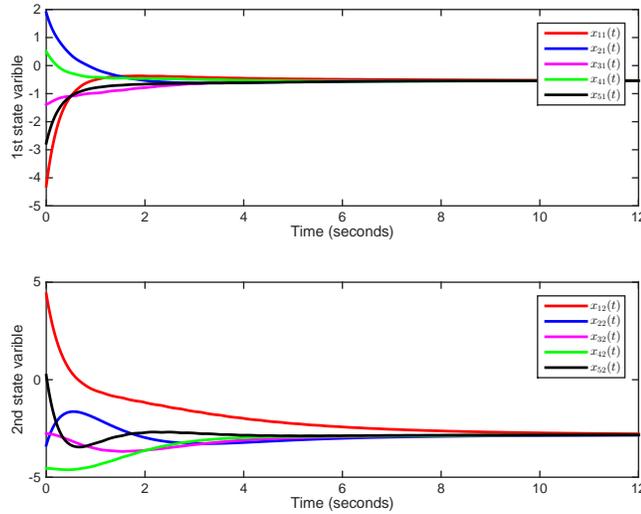


Figure 3.6: The switching signals describing the time-varying topologies.

In this simulation, each initial state variable is generated randomly from the set $[-5, 5]$. Under the consensus protocol proposed in (3.16), the auxiliary state trajectories and the system state trajectories for each agent are shown in Figure 3.7 and Figure 3.8, respectively. It is observed that the auxiliary state and system state of

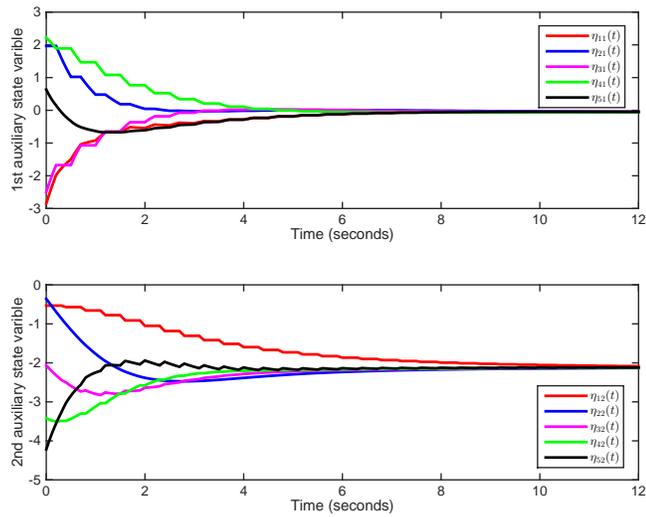


Figure 3.7: The auxiliary state variable trajectories.

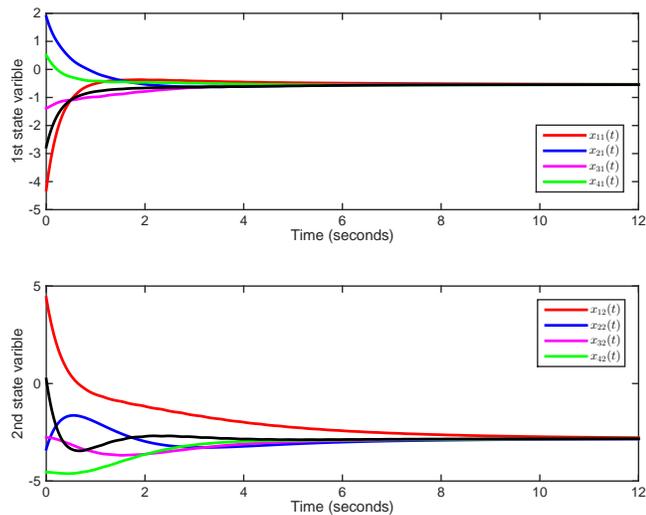


Figure 3.8: The state variable trajectories.

different agents converge asymptotically, implying that the system reaches consensus. The consensus can be achieved for any switching signal that ensuring the graphs jointly containing a spanning tree. We also simulate the system with other switching signals using different dwell time, and the consensus can still be fulfilled.

3.5 Conclusion

This chapter demonstrates a fully distributed method to deal with the consensus problem for a general linear MAS. A more practical interaction scheme has been considered: The information is exchanged in different state-variables-dependent communication topologies. The consensus protocols, which can be designed and implemented in a fully distributed manner, are proposed for systems under fixed topologies and switching topologies, respectively. Under the mild connectivity assumption that each of the topologies contains or jointly contains a spanning tree, necessary and sufficient conditions are established for both fixed topologies and switching topologies cases.

Chapter 4

Consensus for Linear Multi-Agent Systems with Time-Varying Delays

4.1 Introduction

In recent years, there has been increasing research interest in distributed cooperative of MASs, which finds broad applications in many areas such as formation control [100], flocking [40], synchronization [124], stochastic event capturing [125, 126], and distributed sensor networks [1]. When MASs with a cooperative scheme are adopted to fulfill some complicated tasks, the agents need to interact over a communication network to reach a common state; this is usually called consensus.

Consensus problems have attracted considerable attentions in the past few years. Noticeable literatures can be found on consensus problems of first-order dynamics, second-order dynamics, and simple Lipschitz nonlinear dynamics, to name a few [44, 46, 69, 52, 106, 107, 82, 127, 128]. Some recent results are proposed for consensus problems of general linear dynamics agents. In [47], the consensus problem of agents modelled by general continuous-time linear dynamics is investigated under fixed directed topologies. In [27], the consensusability for multiple agents with general discrete-time linear dynamics is analyzed under fixed directed and undirected topologies. The result in [27] is further improved in [48] by introducing a dynamic filter.

Delay effect is an important issue on consensus problems, since the information of agents will be inherently delayed when being transmitted via a network. Another source of delay is related to the computation time and execution time of each agent.

Because time delays can deteriorate the performance and even destroy the stability of a system, the time delay effect has become an attractive topic in networked control area [129, 130, 131]. Consensus of MASs with time delays has been extensively studied in existing literatures, e.g. [24, 79, 80, 81, 132, 133, 82, 134, 135]. Most of the existing works have focused on agents of simple dynamics, e.g. first-order or second-order integrator dynamics. Recently, consensusability of MASs with single-input linear dynamics is studied in [58], and a necessary and sufficient like condition are proposed for systems with constant delays. The consensus problem of general linear dynamics with large delays is considered in [56], but the dynamics of the agents is assumed not be exponentially unstable. In [82], the problem of consensus of linear MASs with time-varying delays is studied, and it is required that the eigenvalues of the system matrices must be on the imaginary axis.

In this chapter, the agent dynamics are assumed to be general linear systems. The delays are time-varying. The communication topology is undirected and fixed. Based on the average state feedback control protocols proposed, which is similar to those in [48] and [58], the consensus problems can be recast as robust control problems. In order to handle the dynamic controllers of the protocols, we solve the problems using frequency domain methods. By analyzing the delay dependent gains, and in light of small-gain theorem, sufficient frequency domain consensus criteria for both continuous-time and discrete-time systems are established. The criteria can be conveniently verified and implemented since they are based on the norms of complementary sensitivity functions, which are commonly used performance indices in frequency domain analysis.

The main contribution of this work is three-fold:

- The general linear dynamics agents are considered, characterizing a larger class of systems compared with the single-integrator or second-order dynamics. Essentially, the proposed consensus criteria can be applied to agents with unstable dynamics.
- The time-varying delays are considered, which are assumed to be bounded but can be arbitrarily fast time-varying. This characterizes the delayed systems more generally and practically.
- The frequency domain criteria have been established for both continuous-time and discrete-time systems.

The rest of this chapter is organized as follows. The problem formulation is proposed in Section 4.2. The consensus problems of continuous-time and discrete-time cases are solved in Section 4.3 and Section 4.4, respectively. Numerical examples are demonstrated in Section 4.5. Finally, the chapter is concluded in Section 4.6.

4.2 Problem Formulation

Consider a set of continuous-time agents with dynamics described by state-space models

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad (4.1)$$

where $i \in \mathcal{N} = \{1, 2, \dots, N\}$. $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and input of the i th agent, respectively. Therefore the transfer matrix of the i th agent is

$$P(s) = (sI - A)^{-1}B.$$

The set of N agents are connected over a communication topology represented by an undirected graph. The graph can be specified as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the index set of N agents, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}]$ with nonnegative elements is the adjacency matrix of the graph. The element $a_{ij} > 0$ represents there is information exchange between agent i and agent j , and $a_{ij} = 0$ otherwise, where $i, j \in \mathcal{V}$. It is assumed that there is no information flow from an agent to itself, which implies that $a_{ii} = 0$. For undirected topology, it is well known that the adjacency matrix is symmetric. A graph \mathcal{G} is called connected if there exists a path between any two agents $i, j \in \mathcal{V}$. Let $\mathcal{N}_i = \{j \mid i, j \in \mathcal{E}\}$ represent the neighbourhood of the i th agent, and $\deg_i = \sum_{j=1}^N a_{ij}$ represent the degree of the i th agent. Denote $\mathcal{D} := \text{diag}\{\deg_1, \deg_2, \dots, \deg_N\}$. Then the Laplacian matrix of graph \mathcal{G} is $\mathcal{L}_{\mathcal{G}} := \mathcal{D} - \mathcal{A}$, which is symmetric and positive semi-definite. The eigenvalues of $\mathcal{L}_{\mathcal{G}}$, denoted by λ_i , are nonnegative, and can be arranged in non-decreasing order as

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N.$$

Moreover, it is assumed that the graph is connected, which indicates $\lambda_2 > 0$.

Consider a local consensus protocol including a dynamic controller:

$$u_i(t) = \int_0^t K(t-l) \left(\sum_{j=1}^N a_{ij}(x_j(l - \tau_i(l))) - x_i(l - \tau_i(l)) \right) dl, \quad (4.2)$$

where $K(t)$ is the impulse response of the dynamic feedback controller. Let $K(s)$ represent the corresponding transfer matrix. The input of each agent is subject to time-varying delays $\tau_i(t) \in [0, \tau_{max}]$. The continuous-time MAS in (4.1) over a connected undirected graph \mathcal{G} is said to be consensusable under protocol in (4.2) if for any finite $x_{i0}, \forall i \in \mathcal{V}$, the condition in (4.3) holds,

$$\lim_{t \rightarrow \infty} \|x_j(t) - x_i(t)\|_2 = 0, \quad \forall i, j \in \mathcal{V}, \quad (4.3)$$

where $\|\cdot\|_2$ denotes the Euclidean norm of a vector. Hence a controller $K(s)$ is to be designed such that the states of the agents converge asymptotically regardless of the time-varying delays.

4.3 Main Results

Let $x_{iK}(t) \in \mathbb{R}^{n_K}$ be the state vector associated with the controller for the i th agent

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K,$$

then the input for the controller is $v_i(t) = \sum_{j=1}^N a_{ij}(x_j(t - \tau_i(t)) - x_i(t - \tau_i(t)))$. Denote the augmented state $X_i(t) := [x_i^T(t), x_{iK}^T(t)]^T$. Combining (4.1) with (4.2), the augmented state-space model can be transformed as

$$\begin{aligned} \dot{X}_i(t) &= \hat{A}X_i(t) + \hat{B}v_i(t), \\ v_i(t) &= [I, \mathbf{0}] \sum_{j=1}^N a_{ij}(X_j(t - \tau_i(t)) - X_i(t - \tau_i(t))), \end{aligned}$$

where

$$\hat{A} = \begin{bmatrix} A & BC_K \\ \mathbf{0} & A_K \end{bmatrix}, \quad \text{and} \quad \hat{B} = \begin{bmatrix} BD_K \\ B_K \end{bmatrix}.$$

The following lemma is useful and it is adopted from [136].

Lemma 4.1. *Let $\tilde{\Delta}$ denote the time-varying delay operator of $\tau(t) \in [0, \tau_{max}]$ in time*

domain. The corresponding time-varying delay can be considered as a perturbation in frequency domain, which is denoted by Δ . And the operator $\Delta_F := (\Delta - 1) \circ \frac{1}{s}$ has an l_2 induced gain $\|\Delta_F\|_\infty$ bounded by τ_{max} , where \circ is the composition symbol.

Our main result is given in the following theorem.

Theorem 4.1. *The continuous-time MAS in (4.1) with an undirected and connected topology is consensusable under the state feedback protocol in (4.2) if there exists properly designed controller $K(s)$ such that*

$$\|T_i(s)\|_\infty < \frac{1}{\tau_{max}|s|}, \quad i = 2, 3, \dots, N,$$

holds for $s = j\omega$, $\forall \omega \in [0, +\infty)$, where

$$T_i(s) = \lambda_i P(s) K(s) (I + \lambda_i P(s) K(s))^{-1},$$

$$j = \sqrt{-1}.$$

The proof of Theorem 4.1 is based on Lemma 4.1 and small-gain theorem. The basic idea is to transform the consensus problem into a set of linear control problems. Then small-gain theorem is applied to ensure the stability subject to time delays and connectivity constraints.

Proof. Denote the augmented vector

$$X(t) := [X_1^T(t), X_2^T(t), \dots, X_N^T(t)]^T,$$

therefore the dynamics for $X(t)$ can be described by

$$\dot{X}(t) = (I \otimes \hat{A})X(t) - (\tilde{\Delta} \circ \mathcal{L}_G \otimes \hat{B}[I \ \mathbf{0}])X(t),$$

$$\tilde{\Delta} = \text{diag}\{\tilde{\Delta}_1, \tilde{\Delta}_2, \dots, \tilde{\Delta}_N\},$$

where $\tilde{\Delta}_i$ is the delay operator corresponding to $\tau_i(t)$. Denote the average state of all the agents by

$$\bar{X}(t) := \frac{1}{N} \sum_{i=1}^N X_i(t) = \frac{1}{N} (\mathbf{1}^T \otimes I) X(t),$$

where $\mathbf{1}$ is a compatible dimension vector with all entries to be one. Then the deviation of each agent from the average state is

$$\delta_i(t) := X_i(t) - \bar{X}(t).$$

Let $\delta(t) = [\delta_1^T(t), \delta_2^T(t), \dots, \delta_N^T(t)]^T$, and it leads to

$$\dot{\delta}(t) = (I \otimes \hat{A})\delta(t) - (\tilde{\Delta} \circ \mathcal{L}_{\mathcal{G}} \otimes \hat{B}[I \ \mathbf{0}])\delta(t). \quad (4.4)$$

Based on the definition of consensus in (4.3), the system achieves consensus if and only if

$$\lim_{t \rightarrow \infty} \|(I \otimes [I \ \mathbf{0}])\delta(t)\|_2 = 0.$$

$\mathcal{L}_{\mathcal{G}}$ is symmetric, hence there exists a unitary matrix $\mathcal{U} = [\frac{1}{\sqrt{N}}, \phi_2, \dots, \phi_N]$ such that

$$\Lambda = \mathcal{U}^T \mathcal{L}_{\mathcal{G}} \mathcal{U} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}.$$

Denote

$$\begin{aligned} \tilde{\delta}(t) &:= (\mathcal{U} \otimes I)^T \delta(t) = [\tilde{\delta}_1^T(t), \tilde{\delta}_2^T(t), \dots, \tilde{\delta}_N^T(t)], \\ \tilde{\Delta}' &:= \mathcal{U}^T \tilde{\Delta} \mathcal{U}. \end{aligned}$$

With the property of Kronecker product, the dynamics in (4.4) can be transformed as

$$\dot{\tilde{\delta}}(t) = (I \otimes \hat{A})\tilde{\delta}(t) - (\tilde{\Delta}' \circ \Lambda \otimes \hat{B}[I \ \mathbf{0}])\tilde{\delta}(t). \quad (4.5)$$

Set the initial value of the controller $x_{iK}(0) = 0$, then

$$\begin{aligned} \tilde{\delta}_1(t) &= \left(\frac{\mathbf{1}^T}{\sqrt{N}}\right)\delta(t) = \sum_{i=1}^N \delta_i(t) = 0, \\ \tilde{\delta}_i(0) &= (\phi_i^T \otimes I)\delta(0) = [\omega_i^T \ \mathbf{0}^T]^T, \\ \omega &= [\omega_1^T, \omega_2^T, \dots, \omega_N^T]^T, \quad i = 1, 2, \dots, N, \end{aligned}$$

where $\omega_i \neq 0$ because the initial states of the agents $X(0) \neq 0$. Since $\mathcal{U} \otimes I$ is

nonsingular, the condition $\lim_{t \rightarrow \infty} \|(I \otimes [I \ \mathbf{0}])\delta(t)\|_2 = 0$ is equivalent to $\lim_{t \rightarrow \infty} \|(I \otimes [I \ \mathbf{0}])\tilde{\delta}(t)\|_2 = 0$, which means $\lim_{t \rightarrow \infty} \|(I \otimes [I \ \mathbf{0}])\tilde{\delta}_i(t)\|_2 = 0$, for $i = 2, 3, \dots, N$. In light of (4.5), taking the Laplace transform gives rise to

$$(I \otimes [I \ \mathbf{0}])\tilde{\delta}(s) = (I \otimes [I \ \mathbf{0}])(sI - I \otimes \hat{A} + \Delta' \circ \Lambda \otimes \hat{B}[I \ \mathbf{0}])^{-1}\tilde{\delta}(0). \quad (4.6)$$

Submitting \hat{A} , \hat{B} and \hat{K} , (4.6) can be simplified as

$$(I \otimes [I \ \mathbf{0}])\tilde{\delta}(s) = (sI - I \otimes A + \Delta' \circ \Lambda \otimes BK(s))^{-1}\omega.$$

Therefore, the MAS can reach consensus if and only if the transfer matrix

$$G(s) = (sI - I \otimes A + \Delta' \circ \Lambda \otimes BK(s))^{-1}$$

is asymptotically stable. Figure 4.1 shows the closed-loop block diagram of $G(s)$. By loop transformation, the system shown in Figure 4.1 is equivalent to the system in Figure 4.2.

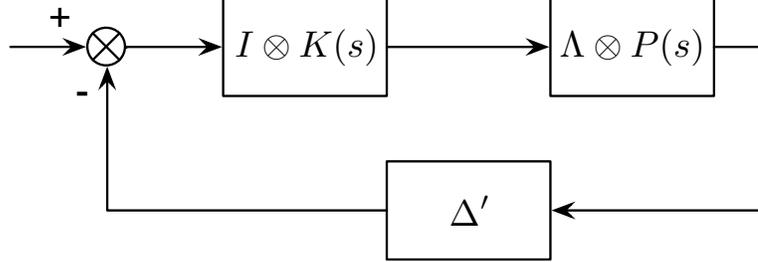


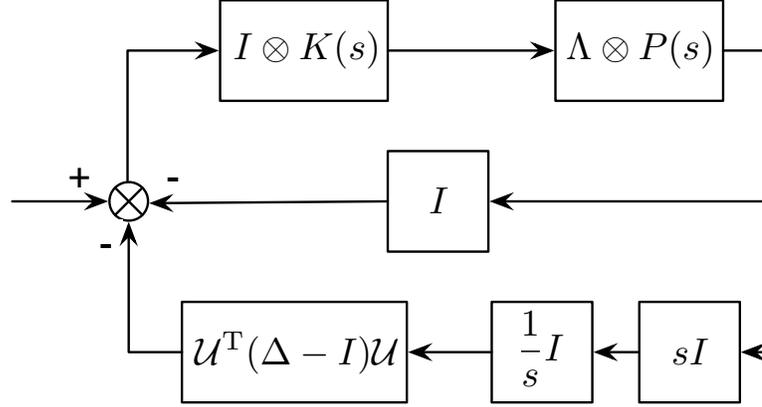
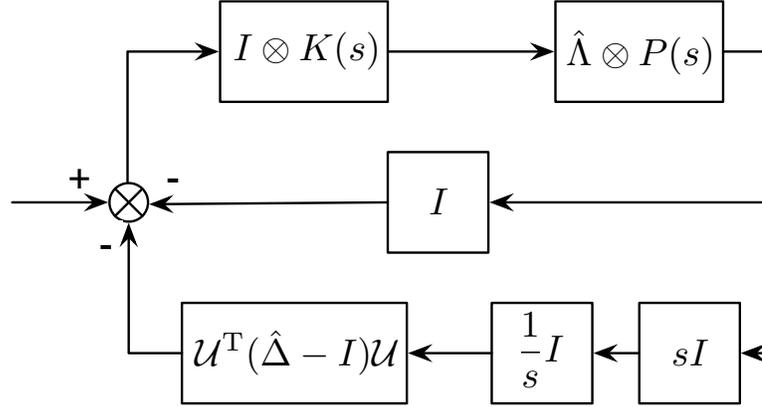
Figure 4.1: The block diagram of $G(s)$.

Suppose (A, B) is stabilizable, and the closed-loop transfer matrices of $\Lambda \otimes P(s)$ and $I \otimes K(s)$ are stable. By small-gain theorem, the system is consensusable if

$$\|sT(s)\|_\infty \left\| \mathcal{U}^T(\Delta - I)\mathcal{U} \circ \frac{1}{s}I \right\|_\infty < 1, \quad (4.7)$$

where

$$T(s) = \Lambda \otimes P(s)K(s)(I + \Lambda \otimes P(s)K(s))^{-1}.$$

Figure 4.2: Loop transformation of $G(s)$.Figure 4.3: Loop transformation of $G(s)$ with $\hat{\Lambda}$ and $\hat{\Delta}$.

Equivalently, we can rewrite (4.7) as

$$\|sT_i(s)\|_\infty \left\| \mathcal{U}^T(\Delta - I)\mathcal{U} \circ \frac{1}{s}I \right\|_\infty < 1, \quad (4.8)$$

where $i = 1, 2, \dots, N$. Since unitary transformation can preserve norms,

$$\begin{aligned} \left\| \mathcal{U}^T(\Delta - I)\mathcal{U} \circ \frac{1}{s}I \right\|_\infty &= \left\| (\Delta - I) \circ \frac{1}{s}I \right\|_\infty \\ &= \max_{i=1,2,\dots,N} \left(\left\| (\Delta_i - 1) \circ \frac{1}{s} \right\|_\infty \right). \end{aligned}$$

With the help of Lemma 4.1, we have

$$\max_{i=1,2,\dots,N} \left(\left\| (\Delta_i - 1) \circ \frac{1}{s} \right\|_{\infty} \right) < \tau_{\max}. \quad (4.9)$$

Combining (4.8), (4.9) and $\lambda_1 = 0$, Theorem 4.1 can be proved. \square

Theorem 4.1 provides a sufficient consensus criterion for the MAS in (4.1) with control protocol in (4.2). For any given $K(s)$, we can verify if the overall MAS can reach consensus by checking $\|T_i(s)\|_{\infty}$.

Theorem 4.1 can be considered as a gain margin problem, and a result only depending on λ_2 and λ_N can be also obtained.

Theorem 4.2. *The continuous-time MAS in (4.1) with an undirected and connected topology is consensusable under the state feedback protocol in (4.2) if there exists properly designed controller $K(s)$ such that*

$$\|T_2(s)\|_{\infty} < \frac{1}{k_{\max}\tau_{\max}|s| + k_{\max} - 1}$$

holds for $s = j\omega$, $\forall \omega \in [0, +\infty)$, where

$$T_2(s) := \lambda_2 P(s) K(s) (I + \lambda_2 P(s) K(s))^{-1},$$

and $k_{\max} = \lambda_N / \lambda_2$.

Proof. By loop transformation, the system in Figure 4.2 is equivalent to the one in Figure 4.3, where

$$\begin{aligned} \hat{\Lambda} &= \text{diag}\{\lambda_1, \lambda_2, \lambda_2, \dots, \lambda_2\}, \\ \hat{\Delta} &= \text{diag}\{\Delta_1, k_2\Delta_2, k_3\Delta_3, \dots, k_n\Delta_N\}, \\ k_i &= \lambda_i / \lambda_2. \end{aligned}$$

By Lemma 4.1,

$$\begin{aligned} \left\| \mathcal{U}^T(\hat{\Delta} - I)\mathcal{U} \circ \frac{1}{s}I \right\|_{\infty} &= \max \left(\left\| (k_i\Delta_i - 1) \circ \frac{1}{s} \right\|_{\infty} \right) \\ &\leq k_i\tau_{max} + \frac{k_i - 1}{|s|} \\ &\leq k_{max}\tau_{max} + \frac{k_{max} - 1}{|s|}, \end{aligned}$$

where $i = 2, 3, \dots, N$. In light of small-gain theorem, the system can be consensusable if

$$\|T_2(s)\|_{\infty} \left\| \mathcal{U}^T(\hat{\Delta} - I)\mathcal{U} \circ \frac{1}{s}I \right\|_{\infty} < 1,$$

hence Theorem 4.2 can be proved. \square

Theorem 4.2 tends to be conservative especially when $\{\lambda_i\}_{i=2}^N$ are sparsely distributed in $[\lambda_2, \lambda_N]$; but it implies that λ_2 and λ_N are important factors of a graph and can be easily estimated compared to the distribution of $\{\lambda_i\}_{i=2}^N$ [137]. In this regard, Theorem 4.2 is preferable to Theorem 4.1.

The controller $K(s)$ can be designed directly by loop shaping method based on Theorem 4.2. Without loss of generality, we consider the case: $P(s)$ is unstable. If the system in (4.1) is stabilizable, there exists a controller $K_1(s)$ such that the closed-loop system $\hat{P}(s)$, consisting of $\lambda_2 P(s)$ and $K_1(s)$, is stable. The internal stabilizing controllers $K_2(s)$ for $\hat{P}(s)$ can be denoted as the following form [138],

$$K_2(s) = Q(s)(I - \hat{P}(s)Q(s))^{-1}.$$

The sensitivity function for $\hat{P}(s)$ is

$$\begin{aligned} \hat{S}(s) &= (I + \hat{P}(s)K_2(s))^{-1} \\ &= (I + \hat{P}(s)Q(s))(I - \hat{P}(s)Q(s))^{-1})^{-1}. \end{aligned}$$

By matrix inversion lemma,

$$\hat{S}(s) = I - \hat{P}(s)Q(s).$$

The complementary sensitivity function

$$\hat{T}(s) = I - \hat{S}(s) = \hat{P}(s)Q(s).$$

Suppose there is a nominal complementary sensitivity function $\hat{T}_0(s)$ satisfying Theorem 4.2, then

$$Q(s) = \hat{P}^{-1}(s)\hat{T}_0(s).$$

Hence $K_2(s)$ can be calculated by $Q(s)$. If $P(s)$ is stable, we can apply the loop shaping method by directly taking $P(s)$ as $\hat{P}(s)$.

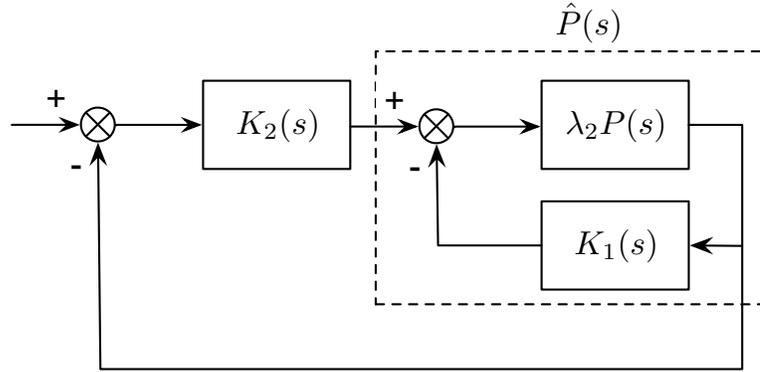


Figure 4.4: The block diagram for controller design of unstable $P(s)$.

4.4 Extensions to Discrete-Time Systems

Consider a set of N discrete-time agents with dynamics described by state-space models

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \quad x_i(0) = x_{i0}, \quad (4.10)$$

where $i \in \mathcal{V} = \{1, 2, \dots, N\}$. $x_i(k) \in \mathbb{R}^n$, and $u_i(k) \in \mathbb{R}^m$ are the state and input of the i th agent, respectively. Hence the transfer matrix of the i th agent is

$$P(z) = (zI - A)^{-1}B.$$

Suppose the discrete-time system is under the same topologies as the continuous-time case. Consider a local consensus protocol based on dynamic controller subject

to time-varying delays

$$u_i(k) = \sum_{l=0}^k K(k-l) \left(\sum_{j=1}^N a_{ij} (x_j(l - \tau_i(l))) - x_i(l - \tau_i(l)) \right). \quad (4.11)$$

The delay $\tau_i(k) \in \{1, 2, \dots, \tau_{max}\}$. $K(l)$ denotes the impulse response of the dynamic feedback controller. Let $K(z)$ represent the corresponding transfer matrix. The discrete-time MAS in (4.10) with a connected undirected graph \mathcal{G} is said to be consensusable under protocol in (4.11) if for any finite $x_{i0}, \forall i \in \mathcal{V}$

$$\lim_{k \rightarrow \infty} \|x_j(k) - x_i(k)\|_2 = 0, \quad \forall i, j \in \mathcal{V}.$$

Hence the controller $K(z)$ is to be designed such that the states of the agents converge asymptotically regardless of the time delays.

The following lemma is useful and it is adopted from [136].

Lemma 4.2. *Let $\tilde{\Delta}$ denote the time-varying delay operator of $\tau(k) \in \{1, 2, \dots, \tau_{max}\}$ in time domain. The corresponding time-varying delay can be considered as a perturbation in frequency domain, which is denoted by Δ . And the operator $\Delta_F := (\Delta - 1) \circ \frac{1}{s}$ has an l_2 induced gain $\|\Delta_F\|_\infty$ bounded by τ_{max} .*

Our main result is given in the following theorem.

Theorem 4.3. *The discrete-time MAS in (4.10) with an undirected and connected topology is consensusable under the state feedback protocol in (4.11) if there exists properly designed controller $K(z)$ such that*

$$\|T_i(z)\|_\infty < \frac{|z|}{\tau_{max}|z-1|}, \quad i = 2, 3, \dots, N,$$

holds for $z = e^{j\omega}, \forall \omega \in [0, +\infty)$, where

$$T_i(z) := \lambda_i P(z) K(z) (I + \lambda_i P(z) K(z))^{-1}.$$

Proof. Similar to the continuous-time case, the MAS can reach consensus if and only if the transfer matrix

$$G(z) = (zI - I \otimes A + \Delta' \circ \Lambda B K(z))^{-1}$$

is asymptotically stable. Figure 4.5 shows the closed-loop block diagram of $G(z)$. By loop transformation, the system in Figure 4.5 is equivalent to the system in Figure 4.6.

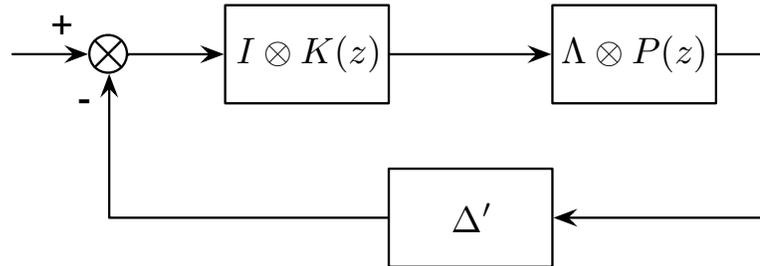


Figure 4.5: The block diagram of $G(z)$.

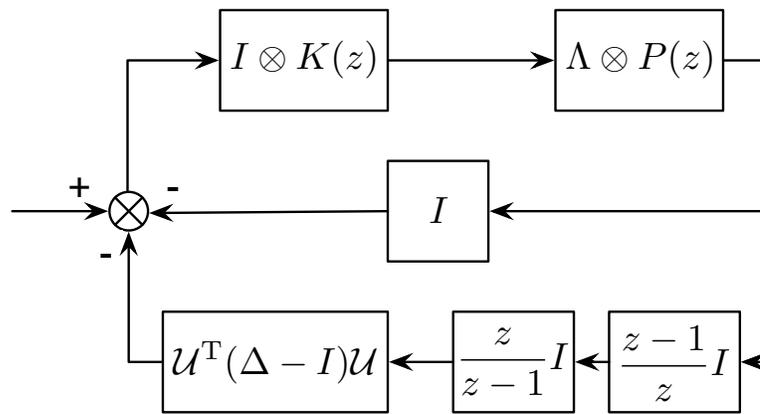


Figure 4.6: Loop transformation of $G(z)$.

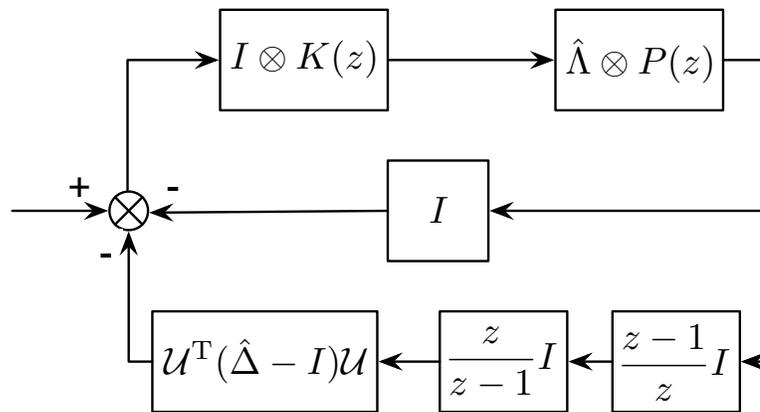


Figure 4.7: Loop transformation of $G(z)$ with $\hat{\Lambda}$ and $\hat{\Delta}$.

Suppose (A, B) is stabilizable, and the closed-loop transfer matrices of $\lambda_i P(z)$ and $K(z)$ are stable. Based on small-gain theorem, the system is consensusable if

$$\left\| \frac{z-1}{z} T_i(z) \right\|_{\infty} \max_{i=1,2,\dots,N} \left(\left\| \frac{z}{z-1} (\Delta_i - 1) \circ I \right\|_{\infty} \right) < 1. \quad (4.12)$$

Combining (4.12) with Lemma 4.2, Theorem 4.3 can be proved. \square

Similar to the continuous-time case, the result only depending on λ_2 and λ_N can be also obtained.

Theorem 4.4. *The discrete-time MAS in (4.10) with an undirected and connected topology is consensusable under the state feedback protocol in (4.11) if there exists properly designed controller $K(z)$ such that*

$$\|T_2(z)\|_{\infty} < \frac{1}{k_{max} \tau_{max} \left| \frac{z-1}{z} \right| + k_{max} - 1}$$

holds for $z = e^{j\omega}$, $\forall \omega \in [0, +\infty)$, where

$$T_2(z) := \lambda_2 P(z) K(z) (I + \lambda_2 P(z) K(z))^{-1},$$

and $k_{max} = \lambda_N / \lambda_2$.

Proof. By loop transformation, the system in Figure 4.6 is equivalent to the system in Figure 4.7. By Lemma 4.2,

$$\begin{aligned} \max_{i=2,3,\dots,N} \left(\left\| \frac{z}{z-1} (k_i \Delta_i - 1) \circ I \right\|_{\infty} \right) &\leq k_i \tau_{max} + (k_i - 1) \frac{|z|}{|z-1|} \\ &\leq k_{max} \tau_{max} + (k_{max} - 1) \frac{|z|}{|z-1|}. \end{aligned}$$

Similarly, with small-gain theorem, Theorem 4.4 can be proved. \square

Remark 4.1. *Since the discrete-time system results are easier for application on physical plants, we derived the discrete-time results with similar methods for the continuous-time case. Compared to Theorem 4.1 and Theorem 4.2, Theorem 4.3 and Theorem 4.4 are more friendly for digital implementation.*

Remark 4.2. *In our results, the closed-loop systems are transformed before employing small-gain theorem. The upper bounds are functions of s or z , which are relative large*

especially for low frequencies. This implies the results are not very conservative since the low-frequency performance is more important for application.

Remark 4.3. *The developed results depend on the nonzero eigenvalues of the Laplacian matrix, which can be determined or approximated by existing results on graph theory [139]. Take the ring graph for example. If a ring graph of N nodes with the weight of each edge equaling to 1, the nonzero Laplacian eigenvalues are $2 - 2 \cos \frac{2\pi k}{N}$, where $k \in \mathbb{Z}$, and $1 \leq k \leq \frac{N}{2}$. Hence $\lambda_2 = 2 - 2 \cos \frac{2\pi}{N}$, and $\lambda_N \leq 4$. Based on Theorem 4.2 or Theorem 4.4, we can establish the consensus conditions with the determined $\lambda_2 = 2 - 2 \cos \frac{2\pi}{N}$ and the approximated $\bar{\lambda}_N = 4$. For large-scale networks, if the involved criteria are satisfied based on the determined or approximated Laplacian matrix eigenvalues, the system can still reach consensus.*

4.5 Simulation Examples

In this section, some illustrative examples are presented to show the validity of the theoretical results.

Example 4.1. *Suppose the system consists of four agents. Figure 4.8 shows the communication topology of the system. The eigenvalues of the Laplacian matrix are $\lambda_1 = 0$, $\lambda_2 = 0.5858$, $\lambda_3 = 2$, and $\lambda_4 = 3.4142$. The system matrices of the agents are*

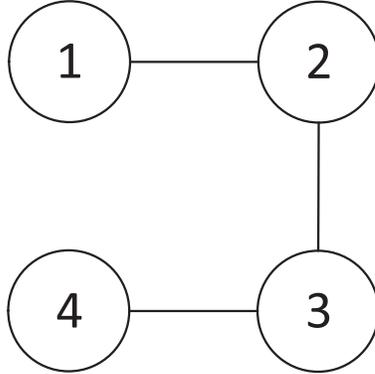


Figure 4.8: The communication topology in Example 4.1.

$$A = \begin{bmatrix} 0.9531 & 0 \\ 0 & -12.0397 \end{bmatrix}, \quad B = \begin{bmatrix} 4.7655 \\ 3.4399 \end{bmatrix}.$$

It is easy to verify that the matrix pair (A, B) is stabilizable. The controller we adopted is

$$K = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix}.$$

In the simulation, the initial conditions are set as $x_{10} = [2 \ 1]^T$, $x_{20} = [3 \ -2]^T$, $x_{30} = [-0.5 \ 1]^T$, and $x_{40} = [-1 \ -3]^T$. The bound of the time delay is set as $\tau_{max} = 0.12$, and $\tau(t)$ is generated randomly in the simulation. Figure 4.9 and Figure 4.10 are the singular value plots based on Theorem 4.1 and Theorem 4.2, respectively. Obviously, Theorem 4.1 is satisfied, but Theorem 4.2 is not satisfied. The system reaches consensus asymptotically, as observed from Figure 4.11 and Figure 4.12, respectively. This implies that Theorem 4.2 is more conservative than Theorem 4.1.

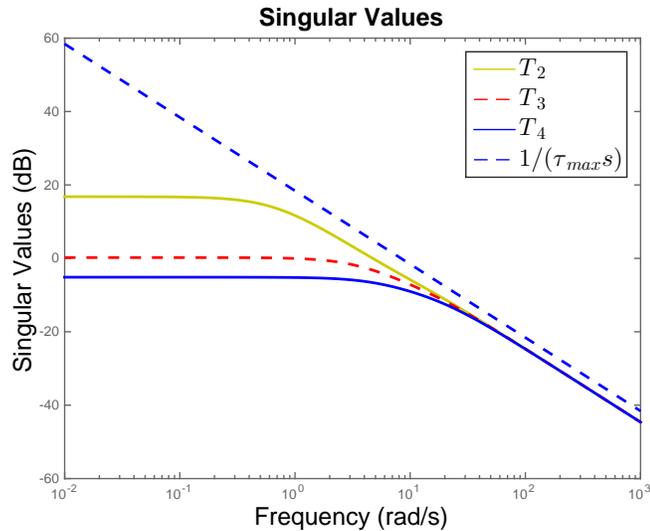


Figure 4.9: The singular value plot based on Theorem 4.1 in Example 4.1.

Example 4.2. Suppose that the system consists of five agents. Figure 4.13 shows the communication topology of the agents. The eigenvalues of the Laplacian matrix are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 3$, and $\lambda_4 = \lambda_5 = 5$. The agent's model is

$$x(k+1) = 1.01x(k) + 0.01u(k).$$

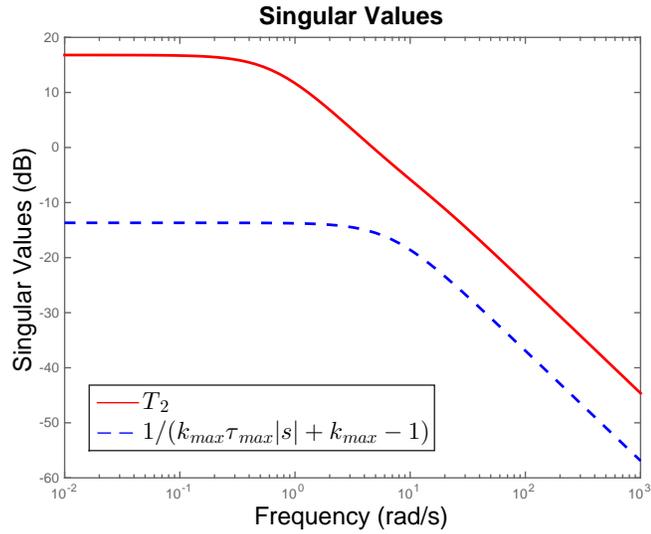


Figure 4.10: The singular value plot based on Theorem 4.2 in Example 4.1.

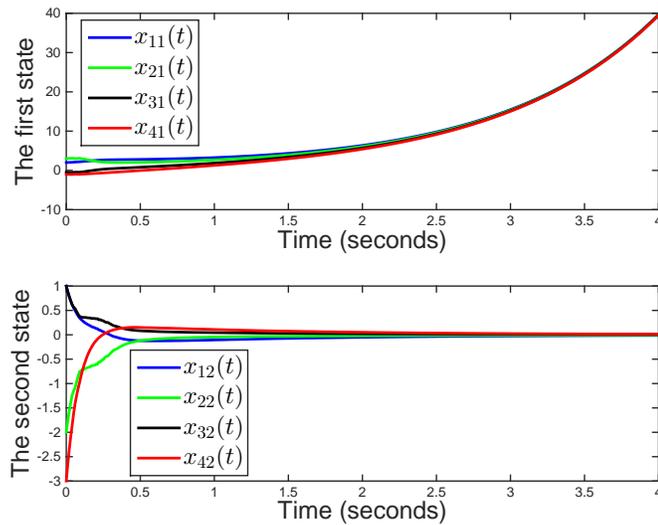


Figure 4.11: State trajectories of the system in Example 4.1.

Again, the agent can be stabilizable. The dynamical controller is

$$K(z) = \frac{0.1047}{z - 0.9048}.$$

The initial conditions are set as $x_{10} = 0.2$, $x_{20} = -1.3$, $x_{30} = 0.5$, $x_{40} = 0.8$, and $x_{50} = -1$. The bound of the time delay is set as $\tau_{max} = 10$, and $\tau(k)$ is generated

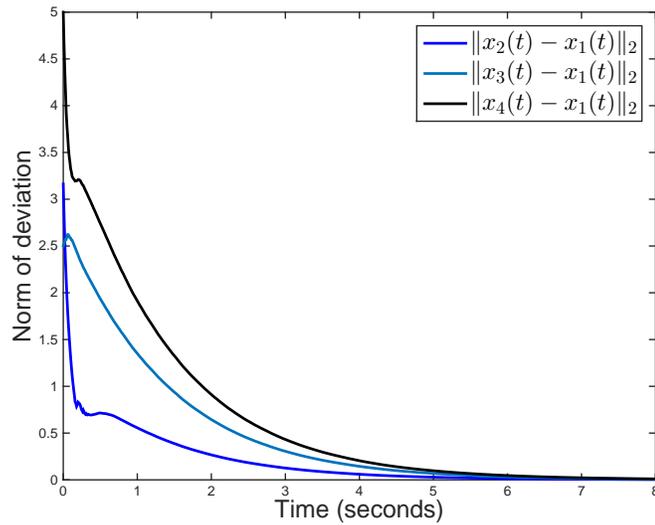


Figure 4.12: Deviation trajectories of the system in Example 4.1.

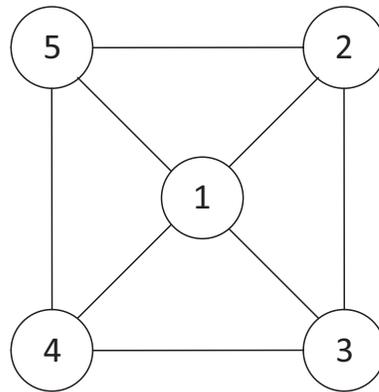


Figure 4.13: The communication topology in Example 4.2.

randomly. The sampling period is 0.01. Figure 4.14 and Figure 4.15 are the singular value plots based on Theorem 4.3 and Theorem 4.4, respectively. Both Theorem 4.3 and Theorem 4.4 are satisfied. And obviously, the gain margin in Figure 4.14 is larger than that in Figure 4.15, which means Theorem 4.3 is less conservative. Figure 4.16 and Figure 4.17 show that the system reaches consensus asymptotically.

Remark 4.4. The results in [82] only considered the case that all the eigenvalues of the system matrices lay on the imaginary axis. From our simulation examples, the eigenvalues of the system matrices can lie off the imaginary axis. Hence, our results are less conservative in terms of system dynamics.

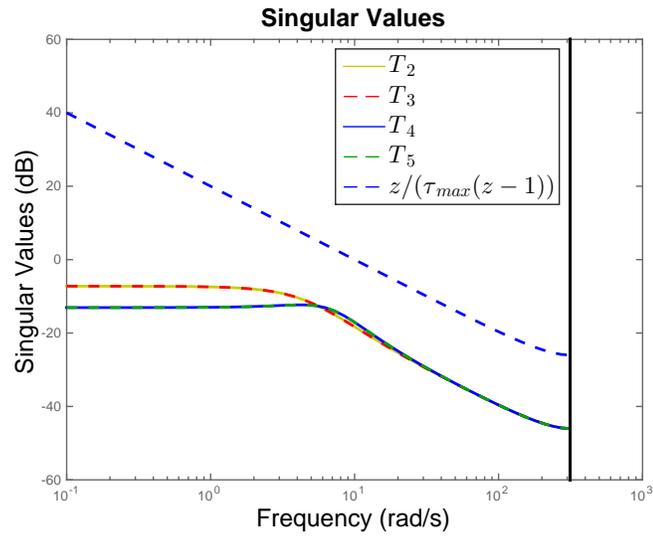


Figure 4.14: The singular value plot based on Theorem 4.3 in Example 4.2.

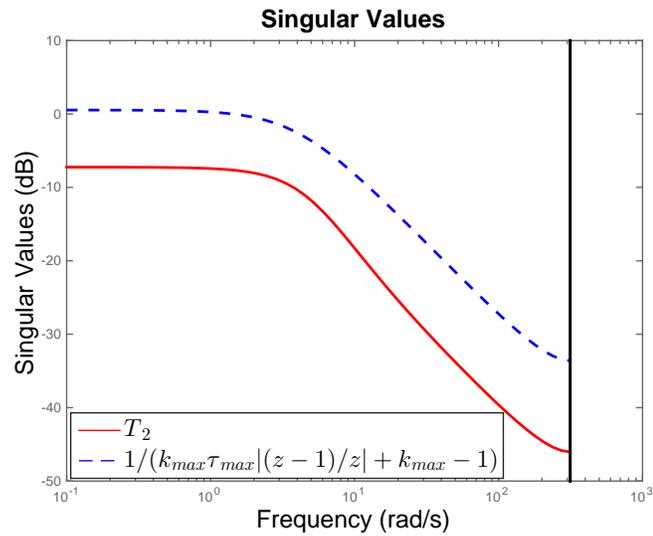


Figure 4.15: The singular value plot based on Theorem 4.4 in Example 4.2.

Remark 4.5. From the singular value plots in Figure 4.9, Figure 4.10, Figure 4.14 and Figure 4.15, we can find that the closed-loop systems in the examples have relative large gain margins especially for low frequencies. This also shows that the obtained results are not very conservative.

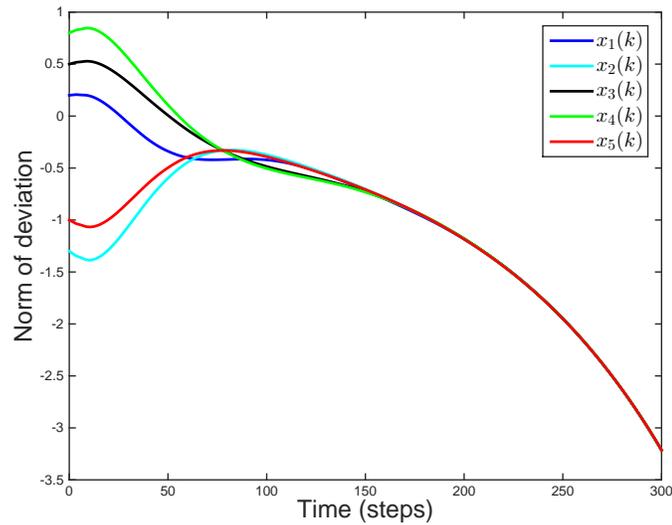


Figure 4.16: State trajectories of the system in Example 4.2.

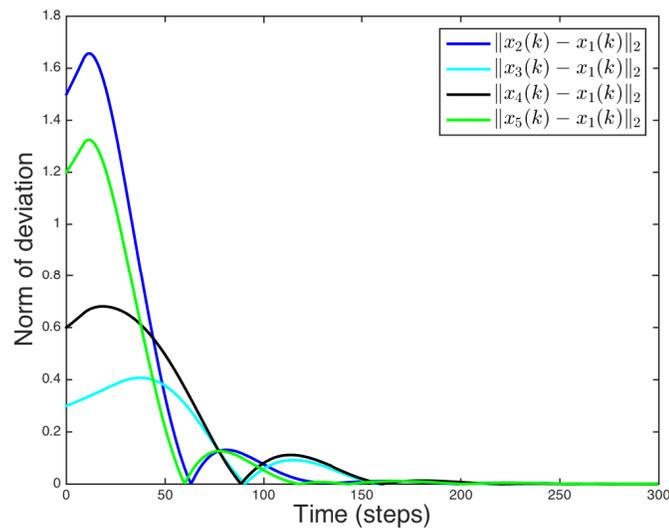


Figure 4.17: Deviation trajectories of the system in Example 4.2.

4.6 Conclusion

In this chapter, the consensus problem for general linear MASs with time-varying delays is considered from the frequency domain perspective. The average state feedback protocol with dynamic controller is proposed for a fixed and undirected communication topology. By adopting an appropriate state transformation, the consensus prob-

lem is transformed into a robust stability problem. Based on small-gain theorem, sufficient consensus criteria are established for both continuous-time and discrete-time systems. It is shown that the consensus conditions are only dependent on the upper bounds of time delays, no matter how fast the delays are varying. Moreover, the dynamic controller can be designed by employing well-developed frequency domain methods. The effectiveness of the proposed theoretical results are verified by numerical examples. Note that in this work, only the perturbations caused by the time-varying delays are studied. Hence a potential future topic would be: To consider more practical issues such as the multiplicative uncertainties.

Chapter 5

Distributed Consensus of Linear Multi-Agent Systems: A Laplacian Spectra Based Method

5.1 Introduction

The last decade has witnessed the prosperous development in distributed cooperative control of MASs. Related researches have widely spread among a variety of industrial application areas, such as smart grid [22, 140] and vehicle platoons [23]. The implementation of the distributed coordination requires the agents to cooperate based on the information exchanged via a communication network to achieve consensus, namely, all of the agents converging to a common state.

Consensus has become an attractive research area in recent years. Most of the existing literatures focus on the consensus problems of low-order dynamics, integral dynamics, and some types of nonlinear dynamics agents, like [44, 46, 69, 52, 106, 127, 141, 128, 142, 143, 144]. Lately, some studies draw attention to the general linear dynamics agents, like [145, 48, 74, 146, 147]. In [145] and [146], average consensus protocols with static control gains are adopted. The stability margins of the closed-loop feedback system considering the individual agent and the consensus gain are analyzed. Region based consensus conditions are further established for general linear dynamics under the undirected communication topologies. In [48] and [58], consensus protocols with dynamic controllers are introduced to improve the performance. In order to design dynamic controllers, the problems are studied in frequency

domain. Some gain margins based frequency domain consensus results are developed only considering undirected communication topologies. In [74], the authors further explore the case of systems with directed topologies based on the result in [48] by considering the gain and phase margins optimization problem. However, the problem of determining the stability margins and regions, especially for directed communication topologies, still requires further investigation. In addition, different from most of the consensus results, the consensusability problem, fundamental but challenging, in terms of the existence and synthesis of distributed feedback control protocols for achieving the consensus, has also been studied in [27, 48, 74].

Motivated by the above discussions, in this chapter, we develop some more general consensus results. Specifically, we aim to tackle the following two problems.

1. To establish Laplacian spectra based consensus conditions for general linear dynamics with directed communication topology.
2. To establish Laplacian spectra based conditions and conduct consensusability analysis and controller synthesis.

In this chapter, the dynamic feedback protocol is adopted, which can greatly improve the consensusability conditions [48]. To take advantage of the consensus protocols with dynamic controllers, and to study the consensusability, the consensus problems are studied in frequency domain. We first cast the consensus problems into the equivalent stability problems in terms of the error dynamics. In light of the Laplacian spectra properties, the stability regions can be determined. For the systems of general linear dynamics, the consensus can be ensured by the stability of several complex weighted closed-loop systems. Moreover, by bridging the Laplacian spectra based stability region to gain and phase margins, we also establish stability margin based result for agents of single-input dynamics. The convergence can be proved by the conformal mapping based method. The result is further extended to a condition directly depending on the unstable poles of the agents' dynamics, in which the conservativeness is reduced by introducing an appropriate tuning parameter.

The main contribution in this work is three-fold:

- A Laplacian spectra based consensus criterion of general linear agents is established by checking the stability of several complex weighted closed-loop systems.
- A consensusability criterion of single-input linear agents is developed based on the gain and phase margins determined by the Laplacian spectra.

- The consensusability criterion directly depending on the unstable poles of the agents' dynamics is proposed, and the corresponding design procedure is provided.

The rest of this chapter is structured as follows. In Section 5.2, some preliminaries are provided, and the problem formulation is introduced. The main results for general linear dynamics agents and single-input linear dynamics agents are discussed in Section 5.3 and Section 5.4, respectively. A simulation study is demonstrated in Section 5.5 to validate the theoretical results. Finally Section 5.6 concludes the chapter.

Notation in this chapter: We let \mathbb{R} represent the real number space. $\text{Re}(\cdot)$ denotes the real part of a complex number, and $\text{Im}(\cdot)$ denotes the imaginary part of a complex number. $\|\cdot\|_2$ denotes the Euclidean norm of a vector. $\|\cdot\|_\infty$ stands for the \mathcal{H}_∞ norm of a transfer function. \otimes represents Kronecker product. T represents the transpose of a matrix. $\mathbf{0}$ stands for the zero matrix with proper dimensions, and I stands for the identity matrix with proper dimensions. $\mathbf{1}$ is a vector of compatible dimension with all entries to be one. \cap is the intersection operator of two sets. $\min\{\cdot\}$ and $\max\{\cdot\}$ denote the minimum and maximum value in the set, respectively. j is the imaginary unit, where $j^2 = -1$.

5.2 Problem Formulation

We study an MAS consisting of N agents. The state-space model of each agent is

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \quad x_i(0) = x_{i0}, \quad (5.1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$ are the system matrices. $i \in \mathcal{N} = \{1, 2, \dots, N\}$. $x_i(k) \in \mathbb{R}^n$ is the state vector of the i th agent, and $u_i(k) \in \mathbb{R}^m$ is the input of the i th agent. Hence, the dynamics of each agent can also be represented by the transfer matrix in frequency domain as

$$P(z) = (zI - A)^{-1}B,$$

which is the linear mapping of z -transform for the input $u_i(k)$ to the state $x_i(k)$.

A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is adopted to describe the information exchange between different agents. $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of index standing for the N agents, $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix of the

graph. If agent i can receive information from agent j , $a_{ij} > 0$, otherwise $a_{ij} = 0$. We assume that there is no edge from an agent to itself, which indicates $a_{ii} = 0$. A directed graph contains a directed spanning tree if there exists a node such that there exists a directed path from this node to every other node. The set of agent i 's neighbours is denoted by $\mathcal{N}_i = \{j \mid i, j \in \mathcal{E}\}$. The in-degree of the i -th agent is represented by $\deg_i = \sum_{j=1}^N a_{ij}$. Denote $\mathcal{D} := \text{diag}\{\deg_1, \deg_2, \dots, \deg_N\}$, and the Laplacian matrix of the graph is $\mathcal{L}_{\mathcal{G}} := \mathcal{D} - \mathcal{A}$. The Laplacian matrix of the graph has the following spectra property [148].

Lemma 5.1. *Denote $\bar{a} := \max\{a_{ij}\}$. The eigenvalues of the Laplacian matrix $\mathcal{L}_{\mathcal{G}}$ are distributed within the region Υ , where Υ is bounded by*

- two closed disks, one centred at \bar{a} , the other centred at $\bar{a}(N - 1)$, each having radius $\bar{a}(N - 1)$;
- two closed smaller angles, one bounded with the two half-lines drawn from $\bar{a}N$ through $\bar{a}Ne^{-\frac{2\pi j}{N}}$ and $\bar{a}Ne^{\frac{2\pi j}{N}}$, the other bounded with the half-lines drawn from 0 through $e^{-(\frac{\pi}{2} - \frac{\pi}{N})j}$ and $e^{(\frac{\pi}{2} - \frac{\pi}{N})j}$;
- the band $|\text{Im}(\lambda_i)| \leq \frac{\bar{a}}{2} \cot \frac{\pi}{2N}$.

The region in Lemma 5.1 becomes a line when $N = 2$, and a quadrilateral when $N = 3$. The region is a hexagon when $4 \leq N \leq 18$, as shown in Figure 5.1, and its boundary contains arcs when $N > 18$. Denote the polygon region $\hat{\Upsilon}$, which is bound by

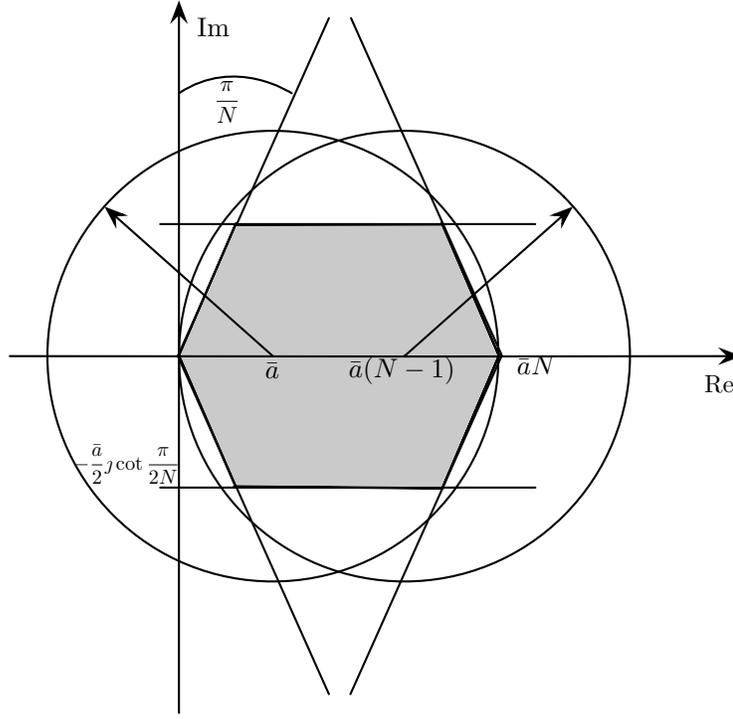
- two closed smaller angles, one bounded with the two half-lines drawn from $\bar{a}N$ through $\bar{a}Ne^{-\frac{2\pi j}{N}}$ and $\bar{a}Ne^{\frac{2\pi j}{N}}$, the other bounded with the half-lines drawn from 0 through $e^{-(\frac{\pi}{2} - \frac{\pi}{N})j}$ and $e^{(\frac{\pi}{2} - \frac{\pi}{N})j}$;
- the band $|\text{Im}(\lambda_i)| \leq \frac{\bar{a}}{2} \cot \frac{\pi}{2N}$.

It can be verified that $\Upsilon = \hat{\Upsilon}$ when $N \leq 18$, and $\Upsilon \subset \hat{\Upsilon}$ when $N > 18$.

And we also adopt the following preliminary result from [149].

Lemma 5.2. *The digraph \mathcal{G} contains a spanning tree if and only if 0 is a simple eigenvalue of the Laplacian matrix $\mathcal{L}_{\mathcal{G}}$ with corresponding right eigenvector, and all the other eigenvalues have positive real parts.*

Suppose that the following assumption stands throughout this chapter.

Figure 5.1: The region of Υ .

Assumption 5.1. *The graph \mathcal{G} is fixed and contains a spanning tree.*

Hence, we can arrange the the eigenvalues of $\mathcal{L}_{\mathcal{G}}$ as $0 = \text{Re}(\lambda_1) \leq \text{Re}(\lambda_2) \leq \dots \leq \text{Re}(\lambda_N)$.

Rather than the average based control protocol with a static controller, we adopt a local consensus protocol containing a dynamic controller,

$$u_i(k) = \sum_{l=0}^{\infty} K(k-l) \left(\sum_{j=1}^N a_{ij}(x_j(l)) - x_i(l) \right), \quad (5.2)$$

where $K(l)$ represents the impulse response of the dynamic feedback controller in time domain. In frequency domain, we denote the corresponding transfer matrix as $K(z)$ via z -transform. In order to facilitate the implementation, we assume that $K(z)$ is proper and finite-order. Let $x_{iK}(k) \in \mathbb{R}^{n_K}$ represent the state of the controller for the i th agent, then

$$K(z) = D_K + C_K(zI - A_K)^{-1}B_K.$$

The input for the controller is $v_i(k) = \sum_{j=1}^N a_{ij}(x_j(k) - x_i(k))$.

Under the communication network depicted by \mathcal{G} , our aim is to develop some criteria such that the overall system achieves consensus, which can be equivalently represented by

$$\lim_{k \rightarrow \infty} \|x_j(k) - x_i(k)\|_2 = 0, i, j = 1, 2, \dots, N. \quad (5.3)$$

5.3 Laplacian Spectra Based Consensus of General Linear Agents

Denote the augmented state $X_i(k) := [x_i^T(k), x_{iK}^T(k)]^T$. The system in (5.1) with the control protocol in (5.2) can be derived as an augmented system

$$\begin{aligned} X_i(k+1) &= \hat{A}X_i(k) + \hat{B}v_i(k), \\ v_i(k) &= \sum_{j=1}^N a_{ij}(X_j(k) - X_i(k)), \end{aligned}$$

where

$$\hat{A} = \begin{bmatrix} A & BC_K \\ \mathbf{0} & A_K \end{bmatrix}, \quad \text{and} \quad \hat{B} = \begin{bmatrix} BD_K & \mathbf{0} \\ B_K & \mathbf{0} \end{bmatrix}.$$

With the above preparation, we have the first result as summarized in the following theorem.

Theorem 5.1. *Consider the MAS in (5.1) with the control protocol in (5.2). Suppose Assumption 5.1 is satisfied. The consensus is reachable if there exists a controller $K(z)$ such that for all $\alpha_i, i = 1, 2, \dots, n_p$, the closed-loop systems consisting of $\alpha_i P(z)$ and $K(z)$ are stable. Here, α_i is the vertices of the polygon region $\tilde{\Upsilon} = \hat{\Upsilon} \cap \{\lambda | \text{Re}(\lambda) \geq \epsilon\}$, where $\epsilon \in \mathbb{R}, 0 \leq \epsilon \leq \text{Re}(\lambda_2)$, and n_p is the vertices number of the polygon region $\hat{\Upsilon}$.*

Proof. Denote

$$X(k) := [X_1^T(k), X_2^T(k), \dots, X_N^T(k)]^T.$$

With the control protocol in (5.2), the closed-loop system can be represented in terms

of the following state-space model:

$$X(k+1) = (I \otimes \hat{A} - \mathcal{L}_{\mathcal{G}} \otimes \hat{B})X(k). \quad (5.4)$$

Denote the average augmented state as

$$\bar{X}(k) := \frac{1}{N} \sum_{i=1}^N X_i(k) = \frac{1}{N} (\mathbf{1}^T \otimes I)X(k).$$

Denote the deviation between the augmented state of agent i and the average augmented state as

$$\delta_i(k) := X_i(k) - \bar{X}(k),$$

and let $\delta(k) := [\delta_1^T(k), \delta_2^T(k), \dots, \delta_N^T(k)]^T$. Hence we have

$$\delta(k+1) = (I \otimes \hat{A} - \mathcal{L}_{\mathcal{G}} \otimes \hat{B})\delta(k). \quad (5.5)$$

Let $\mathcal{U} = [\mathbf{1}/\sqrt{N}, \phi_2, \dots, \phi_N]$ be a unitary matrix such that

$$\mathcal{U}^T \mathcal{L}_{\mathcal{G}} \mathcal{U} = \begin{bmatrix} \lambda_1 & \bar{l}_{12} & \cdots & \bar{l}_{1N} \\ & \lambda_2 & \cdots & \bar{l}_{2N} \\ & & \ddots & \vdots \\ & & & \lambda_N \end{bmatrix}$$

is an upper triangular matrix. By left multiplying $I \otimes \mathcal{U}^T$ on both sides of (5.5), we have

$$(I \otimes \mathcal{U}^T)\delta(k+1) = (I \otimes \mathcal{U}^T)(I \otimes \hat{A} - \mathcal{L}_{\mathcal{G}} \otimes \hat{B})\delta(k). \quad (5.6)$$

Denote $\hat{\delta}(k) = (I \otimes \mathcal{U}^T)\delta(k)$, (5.6) can be transformed as

$$\hat{\delta}(k+1) = \left(I \otimes \hat{A} - \begin{bmatrix} \lambda_1 & \bar{l}_{12} & \cdots & \bar{l}_{1N} \\ & \lambda_2 & \cdots & \bar{l}_{2N} \\ & & \ddots & \vdots \\ & & & \lambda_N \end{bmatrix} \otimes \hat{B} \right) \hat{\delta}(k),$$

and

$$\hat{\delta}_1(k) = \frac{1}{\sqrt{N}} \mathbf{1}^T \delta(k) = \sum_{i=1}^N \delta_i(k) = 0. \quad (5.7)$$

Since (5.7) stands, consensus can be achieved if and only if

$$\hat{A} - \lambda_i \hat{B}, i = 2, 3, \dots, N, \quad (5.8)$$

are Hurwitz.

In light of Lemma 5.1, it can be found that $\lambda_i \in \{\lambda | \lambda \in \Upsilon, \text{Re}(\lambda) \geq \text{Re}(\lambda_2)\}$, $i = 2, 3, \dots, N$. Therefore (5.8) holds if

$$\hat{A} - \lambda \hat{B}, \forall \lambda \in \{\lambda | \lambda \in \Upsilon, \text{Re}(\lambda) \geq \text{Re}(\lambda_2)\},$$

are Hurwitz, which indicates

$$\det(zI - \hat{A} + \lambda \hat{B}) \neq 0, \forall z \in \{z | |z| \geq 1\}, \forall \lambda \in \{\lambda | \lambda \in \Upsilon, \text{Re}(\lambda) \geq \text{Re}(\lambda_2)\}. \quad (5.9)$$

By substituting \hat{A} and \hat{B} , (5.9) can be simplified as

$$I + \lambda K(z)P(z) \neq 0, \forall z \in \{z | |z| \geq 1\}, \forall \lambda \in \{\lambda | \lambda \in \Upsilon, \text{Re}(\lambda) \geq \text{Re}(\lambda_2)\}. \quad (5.10)$$

Denote $\bar{\Upsilon} := \hat{\Upsilon} \cap \{\lambda | \text{Re}(\lambda) \geq \epsilon\}$, where $\epsilon \in \mathbb{R}$, $0 \leq \epsilon \leq \text{Re}(\lambda_2)$. $\bar{\Upsilon}$ is a convex polygon region with vertices α_i , $i = 1, 2, \dots, n_p$, where n_p is the vertices number of the polygon region $\bar{\Upsilon}$. It can be verified that $\{\lambda | \lambda \in \Upsilon, \text{Re}(\lambda) \geq \text{Re}(\lambda_2)\} \subset \bar{\Upsilon}$. Therefore $\forall \lambda \in \{\lambda | \lambda \in \Upsilon, \text{Re}(\lambda) \geq \text{Re}(\lambda_2)\}$ can be written as a linear combination of α_i :

$$\lambda = \beta_1 \alpha_1 + \beta_2 \alpha_2 + \dots + \beta_{n_p} \alpha_{n_p}, \quad (5.11)$$

where

$$\sum_{i=1}^{n_p} \beta_i = 1, \text{ and } 0 \leq \beta_i \leq 1.$$

If there exists a controller $K(z)$ such that for all α_i , $i = 1, 2, \dots, n_p$, with each

closed-loop system of $\alpha_i P(z)$ and $K(z)$ stable, we have

$$I + \alpha_i K(z)P(z) \neq 0, \forall z \in \{z \mid |z| \geq 1\}, i = 1, 2, \dots, n_p. \quad (5.12)$$

Hence the condition in (5.10) holds in light of (5.11) and (5.12), which completes the proof. \square

Theorem 5.1 provides a verification method as well as a consensus condition of general linear MASs for dynamic consensus protocols. For any given $K(z)$, we can examine the stability of closed-loop systems consisting of $\alpha_i P(z)$ and $K(z)$ to verify the reachability of consensus.

5.4 Consensusability of Single-Input Agents

This section is devoted to the spectra based consensusability conditions for single-input dynamics agents. We first extend the analysis in the previous section to develop a gain and phase margins based consensusability criterion for single-input dynamic systems. We suppose that the following condition holds in this section.

Assumption 5.2. *Each agent of the system is a single-input system.*

Consensus can be reached for such single-input MAS if and only if

$$1 + \lambda_i K(z)P(z) \neq 0, \forall z \in \{z \mid |z| \geq 1\}, i = 2, 3, \dots, N. \quad (5.13)$$

There exists a nonnegative real number $\epsilon < \text{Re}(\lambda_2)$, then $\lambda_i \in \tilde{\Upsilon}$, $i = 2, 3, \dots, N$. Hence the condition in (5.13) stands if the closed-loop system consisting of $\bar{a}\sqrt{\epsilon N}P(z)$ and $K(z)$ has a gain margin $[\sqrt{\frac{\epsilon}{N}}, \sqrt{\frac{N}{\epsilon}}]$, and a phase margin $[\frac{\pi}{2} - \frac{\pi}{N}, -\frac{\pi}{2} + \frac{\pi}{N}]$. Denote $a := \sqrt{\frac{N}{\epsilon}}$, $\theta := \frac{\pi}{2} - \frac{\pi}{N}$, and

$$\tilde{T}(z) := \frac{\bar{a}\sqrt{\epsilon N}K(z)P(z)}{1 + \bar{a}\sqrt{\epsilon N}K(z)P(z)}$$

is the complementary sensitivity function of $\bar{a}\sqrt{\epsilon N}P(z)$ and $K(z)$. The following lemma on gain and phase margins problem is adopted in [150], which provides a necessary and sufficient condition.

Lemma 5.3. *Suppose $P(z)$ is a transfer function of a discrete-time system. There exists a state feedback controller $K(z)$ such that $\mathcal{P}(z) = kP(z)$ can be stabilized for all*

$$\forall k \in \{|k|e^{j\omega}, |k| \in [p, q], 0 < p < 1 < q, \omega \in [\omega_1, \omega_2], -\pi < \omega_1 < 0 < \omega_2 < \pi\},$$

if and only if $1 + P(z)K(z) \neq 0, \forall |z| > 1$, and

$$T(z) = \frac{K(z)P(z)}{1 + K(z)P(z)} \neq \frac{1}{1 - k}.$$

By applying Lemma 5.3, we present the consensusability result in Theorem 5.2.

Theorem 5.2. *Consider the MAS in (5.1) with the control protocol in (5.2). Assumption 5.1 and 5.2 are satisfied. The system is consensusable if*

$$\gamma^* \leq \bar{\gamma},$$

where

$$\begin{aligned} \gamma^* &= \inf_{K(z)} \|T(z)\|_\infty, \\ \bar{\gamma} &= \frac{1 + a^2}{\sqrt{1 + a^4 - 2a^2 \cos \theta + 2a^2 \sin \theta}}. \end{aligned}$$

Proof. If the closed-loop system consisted of $\bar{a}\sqrt{\epsilon N}P(z)$ and $K(z)$ has a gain margin of $[\frac{1}{a}, a]$, and a phase margin of $[-\theta, \theta]$, based on Lemma 5.3 and the result in [151], the complementary sensitivity function $\tilde{T}(z)$ maps the following region $\{z \mid |z| > 1\}$ to the region

$$\Omega = \left\{ z \mid \left| z - \frac{1}{2} - j \frac{\cos \theta}{2 \sin \theta} \right| < \frac{1}{2 \sin \theta}, \left| z - \frac{1}{2} + j \frac{\cos \theta}{2 \sin \theta} \right| < \frac{1}{2 \sin \theta}, \right. \\ \left. \left| z - \frac{1}{1 - a^2} \right| < \frac{a}{a^2 - 1}, \left| z - \frac{a^2}{a^2 - 1} \right| < \frac{a}{a^2 - 1} \right\},$$

and

$$\frac{1}{1 - k} \notin \Omega.$$

where

$$r = \frac{1}{2} \frac{\sqrt{a^4 + 1 - 2a^2 \cos^2 \theta + 2a^2 \sin^2 \theta}}{a^2 + 1 - 2a \cos \theta}.$$

If $\|\bar{T}(z)\|_\infty \leq 1$, which is equivalent to

$$\gamma \leq \frac{1 + a^2}{\sqrt{1 + a^4 - 2a^2 \cos \theta + 2a^2 \sin \theta}}, \quad (5.14)$$

$\bar{T}(z)$ is a mapping from the region outside the unit disk to the unit disk. Then it can be verified that $\tilde{T}(z)$ is a mapping from the region outside the unit disk to the set \mathbf{G} , namely, the maximum inscribed circle of Ω . The region of Ω and \mathbf{G} are shown in Figure 5.2. Hence

$$\tilde{T}(z) \neq \frac{1}{1-k}.$$

And it can be verified that

$$\tilde{T}(p_m) = 1, \quad \tilde{T}(z_n) = 0.$$

This means that under the condition in (5.14), $\tilde{T}(z)$ is a complementary sensitivity function for the closed-loop system consisting of $\bar{a}\sqrt{\epsilon\bar{N}}P(z)$ and $K(z)$. In light of Lemma 5.3, there always exists a controller $K(z)$ such that the closed-loop system consisting of $\bar{a}\sqrt{\epsilon\bar{N}}P(z)$ and $K(z)$ have the gain margin of $[\frac{1}{a}, a]$, and a phase margin of $[-\theta, \theta]$, which ensures the consensus. The proof is completed. \square

Theorem 5.2 provides a distributed consensusability conditions depending on the \mathcal{H}_∞ norm of the complementary sensitivity function, which is solvable by analytical method [74] or numerical method [152]. It is shown that the \mathcal{H}_∞ norm of the complementary sensitivity function closely depends the unstable poles. Note that the discrete-time system has the property presented in the following lemma.

Lemma 5.4. *Denote $T_0(z) := K_1(zI - A + BK_1)^{-1}B$ as the complementary sensitivity function under the static state feedback controller K_1 , and $K_1^T, B \in \mathbb{R}^n$. If (A, B) is stabilizable, that is to say, all unstable modes of the system are controllable, then*

$$\gamma_{\text{opt}} = \inf_{K_1} \|T_0(z)\|_\infty = \mu(A),$$

where

$$\mu(A) = \prod_{j=1}^n \max\{|\lambda_j(A)|, 1\}.$$

For each $\gamma > \gamma_{\text{opt}}$, there exists a static stabilizing controller

$$K_1 = (1 + (1 - \gamma^{-2})B^T X B)^{-1} B^T X A,$$

such that $\|T_0\|_\infty < \gamma$, where $X \geq 0$ is the stabilizing solution to

$$\begin{aligned} X &= A^T X (I + (1 - \gamma^{-2})B B^T X)^{-1} A, \\ \gamma^2 &> B^T X B. \end{aligned}$$

Following Lemma 5.4, we obtain the second result of this section.

Theorem 5.3. *Suppose that Assumption 5.1 and 5.2 stand. The MAS (1), with the control protocol (3), is consensusable if*

$$\gamma_{\text{opt}} < \min\{\bar{\gamma}, \bar{\gamma}'\},$$

where

$$\bar{\gamma}' = \frac{1 - 2\bar{\gamma}^\beta + \bar{\gamma}^{2\beta} + 2\bar{\gamma}^{1+\beta}}{2\bar{\gamma}} - \frac{(\bar{\gamma}^\beta - 1)\sqrt{1 - 2\bar{\gamma}^\beta + \bar{\gamma}^{2\beta} + 4\bar{\gamma}^{1+\beta}}}{2\bar{\gamma}},$$

and $\beta \in \mathbb{R}$ is a tuning parameter.

Proof. The controller can be represented as $K(z) = K_2(z)K_1$, where K_1 is a static gain and $K_2(z)$ is a dynamic controller. In light of Lemma 5.4, there exists a stabilizing K_1 such that $\|T_0(z)\| < \bar{\gamma}$ and $\bar{\gamma} > \gamma_{\text{opt}}$. A specific static gain has the following form

$$K_1 = (1 + (1 - \bar{\gamma}^{-2})B^T X B)^{-1} B^T X A,$$

where $X \geq 0$ is solution to

$$\begin{aligned} X &= A^T X (I + (1 - \bar{\gamma}^{-2})B B^T X)^{-1} A, \\ \bar{\gamma}^2 &> B^T X B. \end{aligned}$$

Let

$$K_2(z) = \frac{(1 - \bar{\gamma}^{-\beta})^2}{1 - \bar{\gamma}^{-2\beta}T_0(z)}. \quad (5.15)$$

It can be verified by small-gain theorem that $K_2(z)$ and $K_2^{-1}(z)$ are stable. The complementary sensitivity function can be written as

$$T(z) = \frac{K_1P(z)K_2(z)}{1 + K_1P(z)K_2(z)}. \quad (5.16)$$

Substituting (5.15) into (5.16), $T(z)$ can be simplified as

$$T(z) = \frac{(1 - \bar{\gamma}^{-\beta})^2T_0(z)}{(1 - \bar{\gamma}^{-\beta}T_0(z))^2}. \quad (5.17)$$

It can be verified that (5.17) is stable and proper. Based on Theorem 5.2, the system can reach consensus if

$$\|T(z)\|_\infty \leq \bar{\gamma}. \quad (5.18)$$

Since

$$\|T(z)\|_\infty \leq \frac{(1 - \bar{\gamma}^{-\beta})^2\|T_0(z)\|_\infty}{(1 - \bar{\gamma}^{-\beta}\|T_0(z)\|_\infty)^2},$$

the system can reach consensus if

$$\frac{(1 - \bar{\gamma}^{-\beta})^2\|T_0(z)\|_\infty}{(1 - \bar{\gamma}^{-\beta}\|T_0(z)\|_\infty)^2} \leq \bar{\gamma}. \quad (5.19)$$

By solving the inequality (5.19), we have

$$\inf_{K_1} \|T_0(z)\|_\infty \leq \bar{\gamma}',$$

where

$$\bar{\gamma}' = \frac{1 - 2\bar{\gamma}^\beta + \bar{\gamma}^{2\beta} + 2\bar{\gamma}^{1+\beta}}{2\bar{\gamma}} - \frac{(\bar{\gamma}^\beta - 1)\sqrt{1 - 2\bar{\gamma}^\beta + \bar{\gamma}^{2\beta} + 4\bar{\gamma}^{1+\beta}}}{2\bar{\gamma}}.$$

The proof is completed. \square

Remark 5.1. $\bar{\gamma}'$ is a function of β . Although the function may not be convex, $\max_{\beta} \bar{\gamma}'(\beta)$ can be solved locally using numerical methods, which is applicable. By maximizing $\bar{\gamma}'$, it can potentially increase the bound $\min\{\bar{\gamma}, \bar{\gamma}'\}$, and accordingly, can improve the consensusability conditions.

Theorem 5.3 reveals that the agent's unstable poles directly yield constraints on the consensusability. The proof part also provides an explicit controller design procedure. Furthermore, the introduction of tuning parameter β , compared to that in [48], poses a distinct advantage on reducing the conservativeness.

Remark 5.2. The result in this section is established based on the gain and phase margins analysis, which are more challenging for MIMO systems. Corresponding results for MIMO systems will be further explored using multivariable control system theory and techniques.

5.5 Simulation Example

In this section, a simulation example is presented to validate the derived consensusability results. We consider an MAS containing 5 agents for example. The dynamics for each agent are defined as

$$P(s) = \frac{0.05}{z - 1.002}.$$

Assuming that the MAS is connected via a graph \mathcal{G} , and the Laplacian matrix of \mathcal{G} is

$$\mathcal{L}_{\mathcal{G}} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We choose $\epsilon = 0.6$. Through computation, we have $\bar{\gamma} = 1.1548$, $\gamma_{\text{opt}} = 1.002$. It can be found that $\bar{\gamma}'$ increases as β increases when $\beta \in [0, 70]$, and $\sup_{\beta} \bar{\gamma}' = 1.15478$. Hence the system is consensusable.

Remark 5.3. When $\beta = 1$, the corresponding $\bar{\gamma}'_{\beta=1} = 1.01003 < \sup_{\beta} \bar{\gamma}'$. Hence by introducing the tuning parameter β , the consensusability condition is improved as $\gamma_{\text{opt}} < 1.15478$ rather than $\gamma_{\text{opt}} < 1.01003$.

Base on the procedures in proof of Theorem 5.3, we can design $K_1 = 10$, $K_2 = 0.008281 + \frac{0.0328}{z-0.9187}$ with $\beta = 1$.

In this simulation, we set the sampling period as 0.01s, and each initial state variable is generated randomly from the set $[-5, 5]$. The state trajectories are shown in Figure 5.3. Figure 5.4 shows that state deviations of the agents converge to 0, which indicates that the consensus is achieved.

Remark 5.4. The simulation result demonstrates the effectiveness of our result for an MAS with directed communication topology, compared to undirected communication topology in [48]. Furthermore, different from the result in [74] only considering the consensusability condition, the simulation result also validates the controller synthesis method as proposed in this chapter.

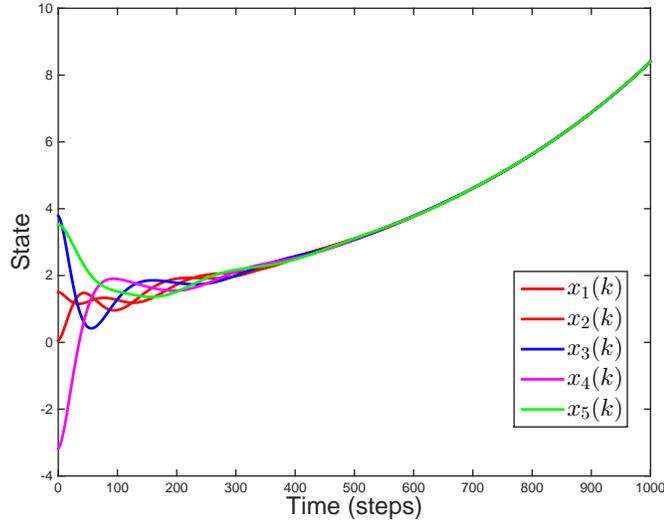


Figure 5.3: The state trajectories.

5.6 Conclusion

The consensus problem of general linear MASs have been investigated in this chapter. Taking the advantages of the Laplacian spectra properties, we have proposed that the

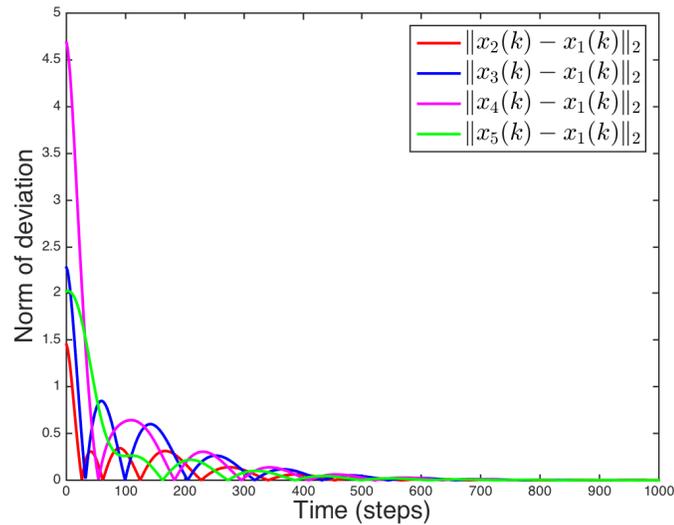


Figure 5.4: The state deviation trajectories.

reachability of consensus can be verified by checking the stability of several complex weighted closed-loop systems. Moreover, we have extended the aforementioned analysis to single-input multiple-output systems, and have provided some gain and phase margins based analysis in frequency domain to establish the consensusability conditions. The development has also led to an explicit controller synthesis procedure. The established consensusability results have been further validated by a simulation example.

Chapter 6

Simultaneous Stabilization of Discrete-Time Delay Systems: Bounds on Delay Margin

When trying to explore the consensusability on MASs governed by the time delays, it is found that the performance limitation problem on time delays is still an open problem in networked control area [19], which is very challenging even for the conventional single-plant systems. In this chapter, we investigate the fundamental limitations impacted by the time delays considering the conventional networked control setting, which builds up the base for our further studies on the consensusability of MASs subject to the time-delay constraints.

6.1 Introduction

Time delays are found in many engineering systems, especially in modern interconnected networks, which may result from communication delays, measurement delays, and computational delays. The presence of time delays can degrade the performance and robustness of control systems, and in the extreme lead to instability. There has been a large body of literature documenting the advances on time-delay control systems in the recent years; see, e.g., [153, 154, 155, 156, 157].

The *delay margin* furnishes a fundamental robustness measure in stabilizing a system against unknown, uncertain, and possibly time-varying delays, which concerns the question [19]: For a fixed finite-dimensional LTI plant, is there an upper bound on the

uncertain delay that can be tolerated by an LTI stabilizing controller? The problem dwells on one of the fundamental limitations of LTI feedback controllers, and it bears close similarity to the classical measures of gain and phase margin. Unlike the gain and phase margin problems [158], however, the delay margin problem proves to be fundamentally more challenging due to the difficulty of controlling infinite-dimensional systems; delay systems constitute a subclass of infinite-dimensional systems and are inherently more difficult to control.

Most of the existing work on the delay margin problem has been focused on continuous-time systems. In [15], explicit upper bounds are derived for single-input single-output (SISO) systems determined by the unstable poles and nonminimum phase zeros of the plant. The bounds become tight for plants containing one unstable pole and one nonminimum phase zero. Subsequent improvements are made in [17], for plants with two or more real unstable poles. On the other hand, [18] obtained lower bounds for systems with an arbitrary number of unstable poles and nonminimum phase zeros, and developed an operator-interpolation approach applicable to SISO and multi-input multi-output systems. Other works show that by employing more sophisticated control laws, such as linear periodic controllers [159], nonlinear periodic controllers [160], and nonlinear adaptive controllers [161, 162], the delay margin can be made infinite.

In this chapter we examine the delay margin problem for LTI discrete-time plants with LTI controllers, which, alternatively, can be posed as a *simultaneous stabilization* problem. Surprisingly, unlike the parallelism one typically expects between continuous-time and discrete-time results, the discrete-time delay margin problem turns out to be fundamentally more difficult. Indeed, for continuous-time systems, if an LTI plant can be stabilized by an LTI controller free of delay, then the system can always tolerate a sufficiently small delay; in other words, the delay margin is always greater than zero. This is no longer true for discrete-time systems. In stark contrast, it is shown in [16] that a discrete-time plant has a zero delay margin whenever it contains a negative real unstable pole. Fundamentally, unlike its continuous-time counterpart, the discrete-time delay margin problem amounts to stabilizing simultaneously multiple plants, each differing from another by its length of delay. Problems in this class are generically difficult; in general, simultaneous stabilization of multiple systems poses a *NP-hard* decision problem, and is considered intractable due to its prohibitive computational complexity [19]. The difficulty in solving the discrete-time delay margin problem, despite being a special class of simultaneous stabilization

problems, can be attributed to this complexity.

Our purpose in this chapter is to establish lower bounds on the delay margin, which consequently provide a guaranteed range of delays ensuring the robust stabilization of the systems within that range, or equivalently, conditions that guarantee the simultaneous stabilization of multiple delay systems. Based on the small-gain stability condition and rational approximation techniques, we cast the delay margin problem as one of parameter-dependent \mathcal{H}_∞ optimization problems. The problem is then tackled and solved by employing analytic interpolation theory ([163]). Our main contributions can be summarized as follows:

- 1) A crucial step in our solution approach is to construct low-order rational function approximations of certain delay transfer functions, which are constructed for both constant and time-varying delays.
- 2) With the rational approximations so constructed, we then develop a computational formula and explicit bounds on the delay margin. This is accomplished by solving a delay-dependent \mathcal{H}_∞ optimal control problem using Nevanlinna-Pick interpolation techniques, whose solution in turn can be obtained explicitly by solving an eigenvalue problem.
- 3) In yet another contribution, we derive bounds on the delay margin of low-order unstable systems achievable by PID controllers. The results consequently shed lights into the limitation of PID controllers in controlling delay systems and in simultaneous stabilization.

The notation used in this paper is standard. Let \mathbb{N} denote the set of nonnegative integers. Let $\mathbb{D} := \{z : |z| < 1\}$, $\overline{\mathbb{D}} := \{z : |z| \leq 1\}$, and $\mathbb{D}^C := \{z : |z| \geq 1\}$. For any complex number z , we denote its conjugate by \bar{z} . j is the imaginary unit, where $j^2 = -1$. For a matrix A , we denote by A^H its conjugate transpose. If A is a Hermitian matrix, its largest eigenvalue will be denoted as $\bar{\lambda}(A)$. At times, where no confusion may arise, we also denote by $\bar{\lambda}(A)$ the largest real eigenvalue provided a real eigenvalue exists, even though A may not be symmetric. The matrix inequality $A \geq 0$ ($A \leq 0$) indicates that A is nonnegative (nonpositive) definite, and $A > 0$ ($A < 0$) indicates that A is positive (negative) definite. $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor function and ceiling functions, i.e., the largest integer less than x and the smallest integer greater than x , respectively. The notation $m \pmod n$ represents the remainder m/n of two integers m and n . Let \mathcal{L}_2 (see, e.g., [164]) be the space of

square summable sequences $u = \{u(0), u(1), \dots\}$ with the norm

$$\|u\|_2 = \sqrt{\sum_{k=0}^{\infty} \|u(k)\|^2},$$

where $\|u(k)\|$ is the Hölder ℓ_2 norm of the vector $u(k)$. For any stable linear system G , we may define the \mathcal{L}_2 induced system norm as

$$\|G\|_{2,2} = \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2}.$$

If G is a stable LTI system, then it can be represented by its transfer function $G(z)$, and $\|G\|_{2,2}$ reduces to the \mathcal{H}_∞ norm of $G(z)$:

$$\|G(z)\|_\infty = \sup_{z \in \mathbb{D}} |G(z)|.$$

The set of all stable transfer functions with \mathcal{H}_∞ norm bounded by a certain $\gamma > 0$ is denoted as

$$B\mathcal{H}_\infty(\gamma) := \{G(z) : \|G(z)\|_\infty \leq \gamma\}.$$

Finally, the identity operator is denoted by \mathcal{I} .

6.2 Problem formulation

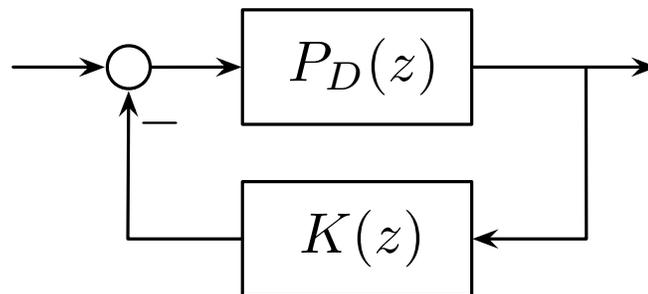


Figure 6.1: Standard feedback control structure.

We consider the discrete-time delay feedback system shown in Fig. 6.1, where

$P_D(z)$ represents a class of plants depending on an uncertain delay $D \in \mathbb{N}$,

$$P_D(z) = z^{-D}P_0(z), \quad D \in \mathbb{N},$$

and $P_0(z)$ is the delay-free plant. Suppose that $P_0(z)$ can be stabilized by a certain finite-dimensional LTI controller $K(z)$. Then the delay margin is defined as

$$D^* = \max \{N \in \mathbb{N} : \text{there exists some } K(z) \text{ that} \\ \text{stabilizes } P_D(z), \forall D \in \{0, 1, \dots, N\}\}.$$

Stated in words, this amounts to determining the maximum number of delay plants $P_D(z)$ that can be stabilized by a certain finite-dimensional LTI controller $K(z)$; that is, the controller $K(z)$ can stabilize simultaneously D^*+1 plants $\{P_0(z), P_1(z), \dots, P_{D^*}(z)\}$.

Evidently, for $K(z)$ to stabilize $P_D(z)$ for some given $D \in \mathbb{N}$, it is both necessary and sufficient that

$$1 + P_D(z)K(z) \neq 0, \quad z \in \mathbb{D}^C. \quad (6.1)$$

If $K(z)$ stabilizes $P_0(z)$, then this condition is equivalent to

$$1 + T_0(z)(z^{-D} - 1) \neq 0, \quad z \in \mathbb{D}^C, \quad (6.2)$$

where

$$T_0(z) = \frac{P_0(z)K(z)}{1 + P_0(z)K(z)} \quad (6.3)$$

is the system's complementary sensitivity function. Thus, $P_D(z)$ can be stabilized by some $K(z)$ for all $D \in \{0, 1, \dots, \underline{D}\}$ if

$$\max_{D \in \{0, 1, \dots, \underline{D}\}} \inf_{K(z) \text{ stabilizes } P_0(z)} \|T_0(z)(z^{-D} - 1)\|_\infty < 1. \quad (6.4)$$

Denote $\phi_D(\omega) = |e^{-j\omega D} - 1|$, and define

$$\phi(\omega) = \max_{D \in \{0, 1, \dots, \underline{D}\}} \phi_D(\omega).$$

Obviously, the condition in (6.4) holds whenever

$$\inf_{K(z) \text{ stabilizes } P_0(z)} |T_0(e^{j\omega})\phi(\omega)| < 1, \quad \forall \omega \geq 0. \quad (6.5)$$

Unfortunately, since $\phi(\omega)$ is a transcendental function, the problem in (6.5) poses a difficult problem. To mitigate this difficulty, in a crucial step of our development we construct a delay-dependent rational function

$$w_D(z) = \frac{b_D(z)}{a_D(z)} = \frac{b_q(z)D^q + \dots + b_1(z)D + b_0(z)}{a_q(z)D^q + \dots + a_1(z)D + a_0(z)}, \quad (6.6)$$

such that

$$\phi(\omega) \leq |w_D(e^{j\omega})|, \quad \forall \omega \geq 0,$$

where $a_i(z)$ and $b_i(z)$, $i = 0, 1, \dots, q$ are both polynomials in z . For this purpose, $w_D(z)$ is required to be stable and have no nonminimum phase zero, except a possible zero at $z = 1$. That $w_D(z)$ may have a nonminimum phase zero at $z = 1$, i.e., $w_D(1) = 0$, albeit unnecessary, ensures a close approximation of $\phi(\omega)$ by $w_D(e^{j\omega})$ at low frequencies. Note also that we may write $w_D(z)$ in the alternative form

$$w_D(z) = M \frac{\prod_{i=1}^r (z - \beta_i(D))}{\prod_{i=1}^q (z - \alpha_i(D))}, \quad (6.7)$$

where without loss of generality, we assume that $M > 0$. Since $w_D(z)$ is required to be stable, it is necessary that $\alpha_i(D) \in \mathbb{D}$, $i = 1, \dots, q$. Similarly, we assume that $\beta_i(D) \in \overline{\mathbb{D}}$, $i = 1, \dots, r$.

By making use of the rational approximation in (6.6), we may then attempt to compute

$$\underline{D} = \max \left\{ D \in \mathbb{N} : \inf_{K(z) \text{ stabilizes } P_0(z)} \|T_0(z)w_D(z)\|_\infty < 1 \right\}. \quad (6.8)$$

The condition in (6.5) holds whenever

$$\inf_{K(z) \text{ stabilizes } P_0(z)} \|T_0(z)w_{\underline{D}}(z)\|_\infty < 1.$$

Since $\phi_D(\omega)$ is monotonically increasing with D , \underline{D} is a lower bound on the delay margin D^* guaranteeing the stabilizability of $P_D(z)$; in other words, there exists a certain controller $K(z)$ that can stabilize $P_D(z)$ for all $D \in \{0, 1, \dots, \underline{D}\}$.

A simple first-order approximation meeting the aforementioned requirement can be found in [136]:

$$w_{1D}(z) = D(1 - z^{-1}), \quad (6.9)$$

which, evidently, represents a discrete-time differentiator. This approximation, however, is rather crude. In this chapter we propose a first-order and a second-order approximation, given as

$$w_{2D}(z) = \frac{D(1 - z^{-1})}{1 + \frac{D-1}{6}(1 - z^{-1})}, \quad (6.10)$$

$$w_{3D}(z) = \frac{D(1 - z^{-1})(2 \times 0.125^2(D - 1)(1 - z^{-1}) + 1)}{(0.125(D - 1)(1 - z^{-1}) + 1)^2}. \quad (6.11)$$

Both approximations improve considerably $w_{1D}(z)$, especially when D is large. Fig. 6.2 shows the frequency responses of these functions within one period. It is useful

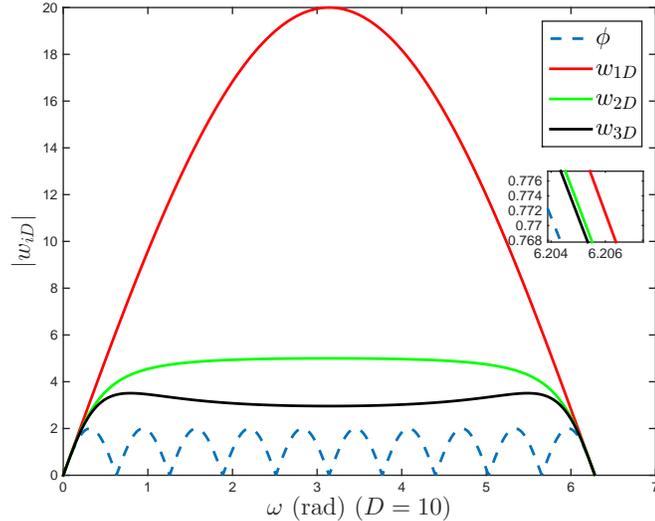


Figure 6.2: Rational approximation for $\phi(\omega)$.

to point out that $w_{2D}(z)$ and $w_{3D}(z)$ are not made available by discretizing their continuous-time counterparts such as those developed in [18]. It is also worth noting that these low-order approximations are particularly desirable, for they render

a lower complexity in solving the \mathcal{H}_∞ control problem in (6.8), and in synthesizing and implementing a lower-order controller $K(z)$, thus achieving a judicious balance between the tightness of the delay bounds and the computational and implementation complexities; indeed, it is well-known [164] that when solving an \mathcal{H}_∞ optimal control problem such as that in (6.8), the order of the optimal controller is proportional to that of the plant and the weighting function $w_D(z)$.

Lemma 6.1. *Let $a(k)$, $b(k)$, f_k and g_k , for $k \in \mathbb{N}$ be given matrix-valued sequences. Suppose that $z_k \in \mathbb{D}$, $z_k \neq 0$ for $k \in \{0, 1, \dots, L\}$, and that z_k are distinct. Then there exists an $H(z) \in B\mathcal{H}_\infty(1)$ such that*

$$a(k) = \sum_{i=0}^k H(k-i)b(i), \quad k = 0, 1, \dots, N,$$

and

$$H(z_k)g_k = f_k, \quad k = 0, 1, \dots, L,$$

if and only if

$$\begin{bmatrix} \mathcal{T}_a^H \\ \mathcal{D}_f^H \end{bmatrix} \begin{bmatrix} I & Z^H \\ Z & \Lambda \end{bmatrix} \begin{bmatrix} \mathcal{T}_a \\ \mathcal{D}_f \end{bmatrix} < \begin{bmatrix} \mathcal{T}_b^H \\ \mathcal{D}_g^H \end{bmatrix} \begin{bmatrix} I & Z^H \\ Z & \Lambda \end{bmatrix} \begin{bmatrix} \mathcal{T}_b \\ \mathcal{D}_g \end{bmatrix},$$

where

$$\mathcal{T}_a = \begin{bmatrix} a(0) & 0 & \cdots & 0 \\ a(1) & a(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a(N) & a(N-1) & \cdots & a(0) \end{bmatrix}, \quad \mathcal{T}_b = \begin{bmatrix} b(0) & 0 & \cdots & 0 \\ b(1) & b(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b(N) & b(N-1) & \cdots & b(0) \end{bmatrix},$$

$$\mathcal{D}_f = \text{diag}(f_0, f_1, \dots, f_L), \quad \mathcal{D}_g = \text{diag}(g_0, g_1, \dots, g_L),$$

$$\Lambda = \begin{bmatrix} \frac{1}{1-z_0 z_0} & \cdots & \frac{1}{1-z_0 z_L} \\ \vdots & \ddots & \vdots \\ \frac{1}{1-z_L z_0} & \cdots & \frac{1}{1-z_L z_L} \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & z_0 & \cdots & z_0^N \\ 1 & z_1 & \cdots & z_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_L & \cdots & z_L^N \end{bmatrix}.$$

Based on Lemma 6.1, in what follows we solve the problem in (6.8) as a mixed Nevanlinna-Pick and Carathéodory-Fejér interpolation problem, which leads to the

main result of this section.

Theorem 6.1. *Let $p_i \in \mathbb{D}^C$, $i = 1, 2, \dots, n$, and $z_j \in \mathbb{D}^C$, $j = 1, 2, \dots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(z)$, respectively. Let also d be the relative degree of $P_0(z)$. Assume that $p_i \neq z_j$, for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Denote*

$$D_w = \text{diag}(w_D(p_1), w_D(p_2), \dots, w_D(p_n))$$

and write $D_w = A^{-1}B$, where

$$A = \sum_{k=0}^q A_k D^k, \quad B = \sum_{k=0}^q B_k D^k. \quad (6.12)$$

Then for any stable rational function $w_D(z)$ such that $\|T_0(z)w_0(z)\|_\infty < 1$ for some $K(z)$ that stabilizes $P_0(z)$,

$$\underline{D} = \lfloor \bar{\lambda}^{-1}(U^{-1}V) \rfloor,$$

where

$$V = \text{diag}(I, \dots, I, \Phi_{2q}), \quad U = \begin{bmatrix} 0 & I & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & I \\ -\Phi_0 & -\Phi_1 & \dots & -\Phi_{2q-1} \end{bmatrix}.$$

The matrices Φ_k are constructed such that

$$\sum_{k=0}^{2q} D^k \Phi_k = \begin{bmatrix} Q_2 & B^H \\ B & A Q_1^{-1} A^H \end{bmatrix}, \quad (6.13)$$

with

$$Q_1 = D_p^d Q_p (D_p^d)^H, \quad D_p = \text{diag}(p_1, \dots, p_n), \quad Q_2 = Q_p - Q_{pz} Q_z^{-1} Q_{pz}^H, \\ Q_p = \begin{bmatrix} p_i \bar{p}_j \\ p_i \bar{p}_j - 1 \end{bmatrix}, \quad Q_{pz} = \begin{bmatrix} p_i \bar{z}_j \\ p_i \bar{z}_j - 1 \end{bmatrix}, \quad Q_z = \begin{bmatrix} z_i \bar{z}_j \\ z_i \bar{z}_j - 1 \end{bmatrix}.$$

Proof. It is well-known that the complementary sensitivity function $T_0(z)$ satisfies

the interpolation conditions

$$\begin{aligned} T_0(p_i) &= 1, \quad i = 1, 2, \dots, n, \\ T_0(z_i) &= 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

Define $H(z) = \sum_{k=0}^{\infty} h(k)z^k$ by

$$H(z) = \frac{1}{\gamma} T_0 \left(\frac{1}{z} \right) w_D \left(\frac{1}{z} \right).$$

Then the interpolation conditions become

$$H \left(\frac{1}{p_i} \right) = \frac{1}{\gamma} w_D(p_i), \quad (6.14)$$

$$H \left(\frac{1}{z_i} \right) = 0, \quad (6.15)$$

$$h(k) = 0, \quad k = 0, 1, \dots, d-1. \quad (6.16)$$

Clearly, there exists some $K(z)$ such that it stabilizes $P_0(z)$ and that $\|T_0(z)w_D(z)\|_{\infty} \leq \gamma$ if and only if $H(z)$ is analytic in \mathbb{D} , $\|H(z)\|_{\infty} \leq 1$, and $H(z)$ satisfies the interpolation conditions given in (6.14)-(6.16). In light of Lemma 6.1, the latter statement is equivalent to the condition

$$\gamma^2 \begin{bmatrix} D_p^{-d} & \\ & D_z^{-d} \end{bmatrix} \begin{bmatrix} Q_p & Q_{pz} \\ Q_{pz}^H & Q_z \end{bmatrix} \begin{bmatrix} D_p^{-d} & \\ & D_z^{-d} \end{bmatrix}^H \geq \begin{bmatrix} D_w^H Q_p D_w & 0 \\ 0 & 0 \end{bmatrix},$$

where $D_z = \text{diag}(z_1, \dots, z_m)$. By pre-multiplying $\text{diag}(D_p^d, D_z^d)$ and post-multiplying $\text{diag}(D_p^d, D_z^d)^H$, we have

$$\gamma^2 \begin{bmatrix} Q_p & Q_{pz} \\ Q_{pz}^H & Q_z \end{bmatrix} \geq \begin{bmatrix} D_w^H D_p^d Q_p (D_p^d)^H D_w & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $Q_z > 0$, with the help of Schur complement [152], we obtain the equivalent condition

$$\gamma^2 Q_2 \geq D_w^H Q_1 D_w.$$

Note also that $Q_2 > 0$ [74]. Hence by pre- and post-multiplying $Q_2^{-1/2}$, we may rewrite this condition as

$$\gamma^2 I \geq Q_2^{-1/2} D_w^H Q_1 D_w Q_2^{-1/2}.$$

We have thus found that

$$\begin{aligned} \gamma_{\inf} &= \inf_{K(z) \text{ stabilizes } P_0(z)} \|T_0(z)w_D(z)\|_{\infty} \\ &= \inf \{ \gamma : K(z) \text{ stabilizes } P_0(z) \text{ and } \|T_0(z)w_D(z)\|_{\infty} \leq \gamma \} \\ &= \inf \{ \gamma : H(z) \text{ analytic in } \mathbb{D}, \text{ satisfies (6.14)-(6.16), and } \|H(z)\|_{\infty} \leq 1 \} \\ &= \bar{\lambda}^{\frac{1}{2}} (Q_2^{-1/2} D_w^H Q_1 D_w Q_2^{-1/2}). \end{aligned}$$

Consider now $w_D(z)$ given by (5), with D relaxed to a nonnegative real number. We are led to

$$\begin{aligned} \underline{D} &= \max \left\{ [D] \geq 0 : \bar{\lambda}^{\frac{1}{2}} (Q_2^{-1/2} D_w^H Q_1 D_w Q_2^{-1/2}) \leq 1 \right\} \\ &= \max \left\{ [D] \geq 0 : Q_2 - D_w^H Q_1 D_w \geq 0 \right\} \\ &= \max \left\{ [D] \geq 0 : \begin{bmatrix} Q_2 & D_w^H \\ D_w & Q_1^{-1} \end{bmatrix} \geq 0 \right\}. \end{aligned} \tag{6.17}$$

Note that

$$\begin{bmatrix} Q_2 & D_w^H \\ D_w & Q_1^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} Q_2 & B^H \\ B & A Q_1^{-1} A^H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A^{-H} \end{bmatrix}.$$

This gives rise to

$$\underline{D} = \sup \left\{ [D] \geq 0 : \sum_{k=0}^{2q} D^k \Phi_k \geq 0 \right\}.$$

Since by assumption $\|T_0(z)w_0(z)\|_{\infty} < 1$, it follows that

$$\Phi_0 = \begin{bmatrix} Q_2 & B_0^H \\ B_0 & A_0 Q_1^{-1} A_0^H \end{bmatrix} > 0.$$

Hence by continuity, \underline{D} can be calculated as

$$\underline{D} = \inf \left\{ \lfloor D \rfloor : \det \left(\sum_{k=0}^{2q} D^k \Phi_k \right) = 0 \text{ for } D > 0 \right\}.$$

Using the polynomial linearization formula in [153], we have

$$\det \left(\sum_{k=0}^{2q} D^k \Phi_k \right) = \det(DV - U).$$

That $\Phi_0 > 0$ implies that the matrix U is invertible. We then arrive at

$$\begin{aligned} \underline{D} &= \inf \{ \lfloor D \rfloor : \det(DV - U) = 0 \text{ for } D > 0 \} \\ &= \inf \{ \lfloor D \rfloor : \det(DU^{-1}V - I) = 0 \text{ for } D > 0 \} \\ &= \inf \{ \lfloor D \rfloor : \det(D^{-1}I - U^{-1}V) = 0 \text{ for } D > 0 \}. \end{aligned}$$

This completes the proof. \square

It is useful to note that for $w_D(z) = w_{iD}(z)$, $i = 1, 2, 3$, $w_0(z) = 0$ and hence $\|T_0(z)w_0(z)\|_\infty = 0$. As a result, for these approximations, Theorem 6.1 always yields a nontrivial lower bound \underline{D} , which guarantees the existence of a controller $K(z)$ that can simultaneously stabilize $P_D(z)$ for all $D \in \{0, 1, \dots, \underline{D}\}$. This lower bound can be computed by solving an eigenvalue problem of a constant matrix. In turn, the controller $K(z)$ can be synthesized by solving an \mathcal{H}_∞ optimization problem. Specifically, the eigenvalue problem gives rise to a positive eigenvalue $\bar{\lambda}^{-1}(U^{-1}V)$, which need not be an integer. One may then solve the \mathcal{H}_∞ optimal synthesis problem

$$\inf_{K(z) \text{ stabilizes } P_0(z)} \|T_0(z)w_D(z)\|_\infty,$$

with $D = \bar{\lambda}^{-1}(U^{-1}V)$. According to Theorem 6.1, this controller will guarantee the simultaneous stabilization of $P_D(z)$ for all $D \in \{0, 1, \dots, \lfloor \bar{\lambda}^{-1}(U^{-1}V) \rfloor\}$.

In what follows we derive more explicit expressions of the delay margin under more special circumstances. The following corollary concerns the case where the plant has only one positive unstable pole. Note that the delay margin can be nonzero only when $P_0(z)$ has no real negative unstable poles [16].

Corollary 6.1. *Suppose that $P_0(z)$ has only one real unstable pole $p \in \mathbb{D}^C$ and has*

no nonminimum phase zeros. Then for any stable rational function $w_D(z)$ such that $\|T_0(z)w_0(z)\|_\infty < 1$ for some $K(z)$ that stabilizes $P_0(z)$,

$$\underline{D} = \lfloor \underline{\lambda} \rfloor \quad (6.18)$$

where $\underline{\lambda}$ is the smallest positive root of the polynomial equation

$$\sum_{k=0}^q (b_k(p) - p^{-d}a_k(p)) D^k = 0. \quad (6.19)$$

If $w_D(z) = w_{iD}(z)$, $i = 1, 2$, then $\underline{D} = \underline{D}_i$, with

$$\underline{D}_1 = \left\lfloor \frac{p^{-d+1}}{p-1} \right\rfloor, \quad \underline{D}_2 = \left\lfloor \frac{p^{-d}(1+5p)}{(p-1)(6-p^{-d})} \right\rfloor.$$

Furthermore, if $d = 0$, then for $w_D(z) = w_{iD}(z)$, $i = 1, 2, 3$, we have

$$\underline{D}_1 = \left\lfloor \frac{p}{p-1} \right\rfloor, \quad \underline{D}_2 = \left\lfloor \frac{5p+1}{5(p-1)} \right\rfloor, \quad \underline{D}_3 = \left\lfloor \frac{5\sqrt{0.04 + 0.56p + 25p^2} - 24p}{(p-1)} \right\rfloor.$$

Proof. In this case,

$$Q_1 = \frac{p^2}{p^2-1}, \quad Q_2 = \frac{p^{2(d+1)}}{p^2-1}, \quad D_w = w_D(p).$$

Hence, it follows from (6.17) that

$$\underline{D} = \max \{ \lfloor D \rfloor \geq 0 : w_D^2(p) \leq p^{-2d} \}.$$

Since by assumption the above inequality holds at $D = 0$, it follows by continuity that

$$\underline{D} = \min \{ \lfloor D \rfloor : w_D^2(p) = p^{-2d} \text{ for } D > 0 \}.$$

Note from (6.18) that $w_D(p) > 0$ for all $p > 1$. Thus,

$$\underline{D} = \inf \{ \lfloor D \rfloor : w_D(p) = p^{-d} \text{ for } D > 0 \}.$$

Rewriting the equation $w_D(p) = p^{-d}$ in the form of the polynomial equation (6.19)

gives rise to (6.18). The rest of the proof is completed by solving the polynomial equations corresponding to $w_D(z) = w_{iD}(z)$, $i = 1, 2, 3$.

□

Remark 6.1. For $p > 1$, it can be verified that

$$\frac{p}{p-1} < \frac{5p+1}{5(p-1)} < \frac{5\sqrt{0.04 + 0.56p + 25p^2} - 24p}{(p-1)}.$$

This indicates that a more accurate approximation $w_D(z)$ results in a potentially larger bound. On the other hand, for $p < -1$, we found that

$$\left\lfloor \frac{p}{p-1} \right\rfloor = \left\lfloor \frac{5p+1}{5(p-1)} \right\rfloor = \left\lfloor \frac{5\sqrt{0.04 + 0.56p + 25p^2} - 24p}{(p-1)} \right\rfloor = 0.$$

This is consistent with the result in [16].

Next, we consider systems containing nonminimum phase zeros.

Corollary 6.2. Suppose that $P_0(z)$ has only one real unstable pole $p \in \mathbb{D}^C$ and distinct nonminimum phase zeros $z_i \in \mathbb{D}^C$, $i = 1, \dots, m$, and $p \neq z_i$ for all $i = 1, \dots, m$. Then for any stable rational function $w_D(z)$ such that $\|T_0(z)w_0(z)\|_\infty < 1$ for some $K(z)$ that stabilizes $P_0(z)$,

$$\underline{D} = \lfloor \underline{\lambda} \rfloor \tag{6.20}$$

where $\underline{\lambda}$ is the smallest positive root of the polynomial equation

$$\sum_{k=0}^q (b_k(p) - \mu p^{-d} a_k(p)) D^k = 0,$$

where

$$\mu = \prod_{i=1}^m \left| \frac{p - z_i}{1 - p\bar{z}_i} \right|.$$

If $w_D(z) = w_{iD}(z)$, $i = 1, 2$, then $\underline{D} = \underline{D}_i$, with

$$\underline{D}_1 = \left\lfloor \frac{p^{-d+1}\mu}{p-1} \right\rfloor, \quad \underline{D}_2 = \left\lfloor \frac{p^{-d}(1+5p)\mu}{(p-1)(6-p^{-d}\mu)} \right\rfloor.$$

Proof. We first identify the matrices

$$Q_p = \frac{p^2}{p^2 - 1}, \quad Q_{pz} = \left[\frac{p\bar{z}_1}{p\bar{z}_1 - 1} \quad \cdots \quad \frac{p\bar{z}_m}{p\bar{z}_m - 1} \right], \quad Q_z = \left[\frac{z_i \bar{z}_j}{z_i \bar{z}_j - 1} \right].$$

Next, we construct the function

$$f(z) = \frac{1}{z - p} \prod_{i=1}^m \frac{z - (1/z_i)}{1 - (1/\bar{z}_i)z}. \quad (6.21)$$

Performing a partial fraction expansion, we may write

$$f(z) = \frac{f_0}{z - p} + \sum_{i=1}^m \frac{f_i}{1 - (1/\bar{z}_i)z}, \quad (6.22)$$

where

$$f_0 = \prod_{i=1}^m \frac{p - (1/z_i)}{1 - (1/\bar{z}_i)p} = \prod_{i=1}^m \left(\frac{\bar{z}_i}{z_i} \right) \frac{pz_i - 1}{\bar{z}_i - p},$$

while f_i , $i = 1, \dots, m$ will prove immaterial. Evaluating $f(1/z_i)$, $i = 1, \dots, m$, we obtain

$$Q_z \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} - \frac{f_0}{p} Q_{pz}^H = 0.$$

This leads to

$$Q_{pz} Q_z^{-1} Q_{pz}^H = \frac{p}{f_0} Q_{pz} \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \frac{p}{f_0} \sum_{i=1}^m \frac{p\bar{z}_i}{p\bar{z}_i - 1} f_i.$$

However, in view of (6.20), we have

$$\sum_{i=1}^m \frac{p\bar{z}_i}{p\bar{z}_i - 1} f_i = f(1/p) - \frac{f_0}{(1/p) - p} = f(1/p) + \frac{p}{p^2 - 1} f_0.$$

Using (6.18), we obtain

$$f(1/p) = -\frac{p}{p^2 - 1} \prod_{i=1}^m \left(\frac{\bar{z}_i}{z_i} \right) \frac{z_i - p}{p\bar{z}_i - 1} = -\frac{1}{f_0} \left(\frac{p}{p^2 - 1} \right).$$

Consequently,

$$Q_{pz}Q_z^{-1}Q_{pz}^H = \frac{p}{f_0} \left(\frac{p}{p^2-1} \right) \left(-\frac{1}{f_0} + f_0 \right) = \frac{p^2}{p^2-1} \left(1 - \frac{1}{|f_0|^2} \right).$$

It thus follows that

$$\begin{aligned} Q_2 &= Q_p - Q_{pz}Q_z^{-1}Q_{pz}^H \\ &= \frac{1}{|f_0|^2} \left(\frac{p^2}{p^2-1} \right) \\ &= \prod_{i=1}^m \left| \frac{p-z_i}{1-p\bar{z}_i} \right|^2 \left(\frac{p^2}{p^2-1} \right) \\ &= \mu^2 \left(\frac{p^2}{p^2-1} \right). \end{aligned}$$

According to (6.17), we have

$$\begin{aligned} \underline{D} &= \max \{ \lfloor D \rfloor : w_D(p) \leq \mu p^{-d} \text{ for } D > 0 \} \\ &= \min \{ \lfloor D \rfloor : w_D(p) = \mu p^{-d} \text{ for } D > 0 \}. \end{aligned} \tag{6.23}$$

The rest of the proof may then be completed as in that of Corollary 6.1. □

Remark 6.2. *Since $\mu < 1$, it follows trivially from (6.23) that*

$$\{D > 0 : w_D(p) \leq \mu p^{-d}\} \subset \{D > 0 : w_D(p) \leq p^{-d}\}.$$

Hence in general, the presence of any nonminimum phase zero z_i will reduce the lower bound \underline{D} , demonstrating potentially the restriction imposed by nonminimum phase zeros on the delay margin. This restriction can be especially severe when there is a pair of closely located nonminimum phase zero and unstable pole, for then μ may become rather small; in the limit when $\mu \rightarrow 0$, $\underline{D} \rightarrow 0$.

We conclude this section by discussing a number of monotonicity properties of \underline{D} pursuant to Corollary 6.1 and Corollary 6.2. First, let $w_D(z)$ be so constructed that $w_D(p)$ is monotonically increasing with $p > 1$. One such case corresponds to $w_D(z)$ described by (6), where $\alpha_i(D)$ and $\beta_i(D)$, $i = 0, 1, \dots, q$ are all real numbers

and satisfy the inequality $\alpha_i(D) \leq \beta_i(D)$. Under this circumstance, we have

$$\frac{w'_D(p)}{w_D(p)} = \sum_{i=0}^q \frac{1}{p - \beta_i(D)} - \sum_{i=0}^q \frac{1}{p - \alpha_i(D)} \geq 0.$$

Since $w_D(p) > 0$, it follows that $w'_D(p) \geq 0$. The approximations $w_{1D}(z)$ and $w_{2D}(z)$ both belong to this category. Denote \underline{D} by $\underline{D}(p)$, as a function of $p > 1$. Evidently, with the monotonicity of $w_D(p)$,

$$\{D > 0 : w_D(p_2) \leq p_2^{-d}\} \subset \{D > 0 : w_D(p_1) \leq p_1^{-d}\}.$$

Hence, $\underline{D}(p_2) \leq \underline{D}(p_1)$ for $p_2 \geq p_1 > 1$; that is, a more unstable $P_0(z)$ will tend to result in a smaller bound on the delay margin.

It is also of interest to see how \underline{D} may vary with choices of $w_D(z)$. In this vein, let $w_{iD}(z)$ and $w_{jD}(z)$ be two approximations such that

$$|w_{iD}(e^{j\omega})| \leq |w_{jD}(e^{j\omega})|, \quad \forall \omega \geq 0,$$

or equivalently,

$$\left| \frac{w_{iD}(e^{j\omega})}{w_{jD}(e^{j\omega})} \right| \leq 1, \quad \forall \omega \geq 0.$$

Correspondingly, denote the lower bounds by \underline{D}_i , and \underline{D}_j , respectively. It follows from the *maximum modulus principle* ([165]) that

$$\frac{w_{iD}(p)}{w_{jD}(p)} \leq 1, \quad \forall p > 1.$$

As such,

$$\{D > 0 : w_{jD}(p) \leq \mu p^{-d}\} \subset \{D > 0 : w_{iD}(p) \leq \mu p^{-d}\}.$$

In other words, it holds that $\underline{D}_j \leq \underline{D}_i$; that is, a better approximation $w_{iD}(z)$ tends to yield a larger lower bound.

6.3 Systems with time-varying delays

With a distinctive feature, the preceding interpolation-based method can also be extended to analyze linear systems with time-varying delays. Consider the system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k - D(k)), \\ y(k) = Cx(k) + Eu(k - D(k)), \end{cases} \quad (6.24)$$

where $D(k)$ indicates a time-varying delay. We assume that $D(k)$ satisfies the bound

$$0 \leq D(k) \leq \bar{D}, \quad (6.25)$$

and its variation rate is bounded by the m -step total variation [166]

$$\delta_m(k) = \sum_{i=0}^{m-1} |D(k+i+1) - D(k+i)| \leq \delta, \quad (6.26)$$

where $\delta \leq m$, and $\delta \leq \bar{D}$. Denote

$$\Delta u(k) = u(k - D(k)). \quad (6.27)$$

Then the time-varying delay can be represented by the linear operator Δ . Denote also by \mathcal{I} the identity operator. The following properties of Δ are known from [136, 166].

Lemma 6.2. *Let Δ be defined in (6.27). Then,*

1. $\|\Delta\|_{2,2} \leq \sqrt{1 + \delta}$, for all $0 \leq D(k) \leq \bar{D}$ and $\delta_m(k) \leq \delta$.
2. $\|(\Delta - \mathcal{I})w_{1\bar{D}}^{-1}(z)\|_{2,2} \leq 1$, for all $0 \leq D(k) \leq \bar{D}$ and $\delta_m(k) \leq \delta$.

Let $P_0(z) = C(zI - A)^{-1}B + E$ denote the transfer function of the delay-free system. In a similar goal, we seek to find fundamental ranges of (\bar{D}, δ) so that the system in (6.24) may be robustly stabilized by an LTI output feedback controller $K(z)$. Toward this end, we first note that the system can be represented by the block diagram in Fig. 6.3. By rearranging the feedback loop, it can be represented as in Fig. 6.4, an equivalent configuration amenable to small-gain analysis. Indeed, using the small-gain theorem [164], it follows at once that the system in (6.24) can be stabilized by a certain controller $K(z)$ provided that the \mathcal{L}_2 -induced norm

$$\|(\Delta - \mathcal{I})T_0(z)\|_{2,2} < 1,$$

where $T_0(z)$ is the complementary sensitivity function given in (6.3). Since by Lemma 6.2, the composite linear operator $(\Delta - \mathcal{I})w_{1\bar{D}}^{-1}(z)$ is bounded, it follows that

$$\begin{aligned} \|(\Delta - \mathcal{I})T_0(z)\|_{2,2} &\leq \|(\Delta - \mathcal{I})w_{1\bar{D}}^{-1}(z)\|_{2,2}\|w_{1\bar{D}}(z)T_0(z)\|_{2,2} \\ &\leq \bar{D}\|(1 - z^{-1})T_0(z)\|_{\infty}. \end{aligned}$$

As such, we conclude that the system in (6.24) can be stabilized by $K(z)$ for all $0 \leq D(k) \leq \bar{D}$ and $\delta_m(k) \leq \delta$ whenever $K(z)$ stabilizes $P_0(z)$ and satisfies the small-gain condition

$$\|\bar{D}(1 - z^{-1})T_0(z)\|_{\infty} < 1. \tag{6.28}$$

A lower bound on \bar{D} can then be determined by solving the \mathcal{H}_{∞} optimization problem

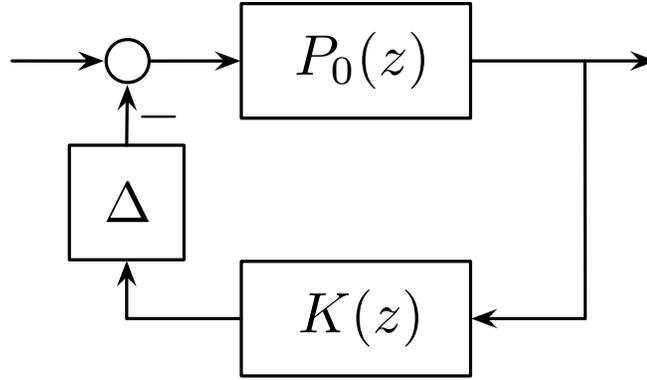


Figure 6.3: Feedback control systems with time-varying delay.

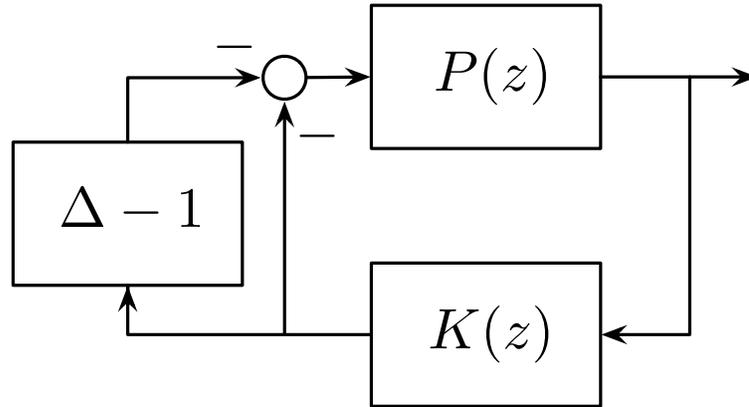


Figure 6.4: Small-gain setup of systems with time-varying delay.

in (6.28) regardless of δ , thus casting the stabilization of the system (6.24) in the

same spirit as the problem in (6.8), with a rational approximant given as $w_{1D}(z) = D(1 - z^{-1})$. As a result, we are led to the following result.

Theorem 6.2. *Let $p_i \in \mathbb{D}^C$, $i = 1, 2, \dots, n$, and $z_j \in \mathbb{D}^C$, $j = 1, 2, \dots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(z)$, respectively. Let also d be the relative degree of $P_0(z)$. Assume that $p_i \neq z_j$, for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then the system in (6.24) can be stabilized by some $K(z)$ for all $D(k) \in \{0, 1, \dots, \bar{D}\}$ if*

$$\bar{D} = \left\lfloor \bar{\lambda}^{-\frac{1}{2}} (Q_2^{-1/2} \hat{Q}_p^H Q_1 \hat{Q}_p Q_2^{-1/2}) \right\rfloor,$$

where Q_1, Q_2 are defined in Theorem 6.1, and $\hat{Q}_p = \text{diag}(1 - 1/p_1, \dots, 1 - 1/p_n)$.

It is also possible to improve Theorem 6.2 by seeking alternative, more elaborate approximations of the operator $\Delta - \mathcal{I}$. In particular, it will prove advantageous to incorporate the delay variation rate in the approximation. For this purpose, it suffices to construct rational approximants $w_{\bar{D},\delta}(z)$ such that it satisfies the property similar to Lemma 6.2 (2), that is, to find a proper stable, minimum phase rational function except with a zero at $z = 1$, such that

$$\|(\Delta - \mathcal{I})w_{\bar{D},\delta}^{-1}(z)\|_{2,2} \leq 1 \quad (6.29)$$

for all $0 \leq D(k) \leq \bar{D}$ and $\delta(k) \leq \delta$. With such a $w_{\bar{D},\delta}(z)$, it follows analogously that the system (6.24) can be stabilized by $K(z)$ for all $0 \leq D(k) \leq \bar{D}$ and $\delta_m(k) \leq \delta$ provided that $K(z)$ stabilizes $P_0(z)$ and satisfies the condition

$$\|w_{\bar{D},\delta} T_0(z)\|_{\infty} < 1. \quad (6.30)$$

As a result, a set of \bar{D} and δ can be obtained in a manner analogous to Theorem 6.1, by solving a similar \mathcal{H}_{∞} problem.

In light of the conservatism of Theorem 6.2, which largely results from the use of the simple approximant $w_{1D}(z)$, we next construct $w_{D,\delta}(z)$ with the property

$$\begin{cases} |w_{D,\delta}(e^{j\omega})| \geq 1 + \sqrt{1 + \delta} & \text{if } D|1 - e^{-j\omega}| > 1 + \sqrt{1 + \delta} \\ |w_{D,\delta}(e^{j\omega})| \geq D|1 - e^{-j\omega}| & \text{if } D|1 - e^{-j\omega}| \leq 1 + \sqrt{1 + \delta} \end{cases}. \quad (6.31)$$

Thus, while sacrificing moderately the approximation accuracy of $w_{1D}(z)$ at low frequencies, $w_{D,\delta}(z)$ may gain a significantly better accuracy in the high frequency

range. In this paper, we have constructed one such approximant:

$$w_{D,\delta}(z) = \frac{\alpha(D^2(1 - z^{-1})^2 + 4.5D(1 - z^{-1}))}{D^2(1 - z^{-1})^2 + \beta(1 - z^{-1}) + 4.5\alpha}, \quad (6.32)$$

where $\alpha = \sqrt{\delta + 1} + 1$, and $\beta = 4.5 - (2.25\alpha)/D$. Fig. 6.5 shows the frequency response of the function within one period.

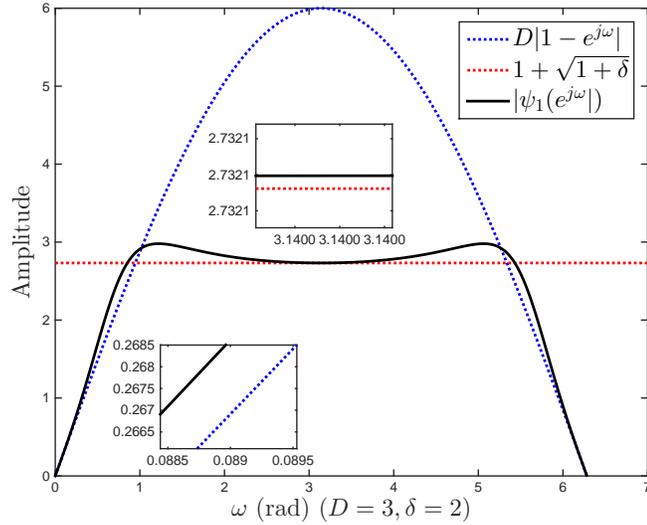


Figure 6.5: The frequency response of $\phi_\delta(\omega)$.

The following lemma states that any proper stable rational function with one nonminimum phase zero at $z = 1$, which possesses the property (6.31), does satisfy the inequality (6.29).

Lemma 6.3. *Let $w_{D,\delta}(z)$ be a proper stable rational function which has one nonminimum phase zero at $z = 1$ and satisfies (6.31). Then for any Δ defined in (6.27), the condition (6.29) holds.*

Proof. The proof follows the idea of [167], which was carried out for continuous-time systems. First, since $w_{D,\delta}(z)$ is proper, stable and has one nonminimum phase zero at $z = 1$, it follows from Lemma 6.2 (2) that $(\Delta - \mathcal{I})w_{D,\delta}^{-1}(z)$ is a bounded linear operator. Define

$$\begin{aligned} \Omega_1 &= \left\{ \omega : D|1 - e^{-j\omega}| > 1 + \sqrt{1 + \delta} \right\}, \\ \Omega_2 &= \left\{ \omega : D|1 - e^{-j\omega}| \leq 1 + \sqrt{1 + \delta} \right\}. \end{aligned}$$

For any $u \in \mathcal{L}_2$, decompose $u = u_1 + u_2$, so that the Fourier transform of u_1 is equal to zero in Ω_2 and that of u_2 is equal to zero in Ω_1 . In view of (6.31), we have

$$\begin{aligned} \|(\Delta - \mathcal{I})w_{D,\delta}^{-1}(z)u_1\|_2 &\leq \|(1 + \sqrt{1 + \delta})^{-1}(\Delta - \mathcal{I})\|_{2,2} \cdot \|(1 + \sqrt{1 + \delta})w_{D,\delta}^{-1}(z)u_1\|_2 \\ &\leq \|(1 + \sqrt{1 + \delta})w_{D,\delta}^{-1}(z)u_1\|_2 \\ &\leq \|u_1\|_2. \end{aligned}$$

Here the second inequality follows from Lemma 6.2 (1), and the third inequality is a direct consequence of (6.31). Similarly, we can show that $\|(\Delta - \mathcal{I})w_{D,\delta}^{-1}(z)u_2\|_2 \leq \|u\|_2$. It thus follows that

$$\begin{aligned} \|(\Delta - \mathcal{I})w_{D,\delta}^{-1}(z)u\|_2 &\leq \|(\Delta - \mathcal{I})w_{D,\delta}^{-1}(z)u_1\|_2 + \|(\Delta - \mathcal{I})w_{D,\delta}^{-1}(z)u_2\|_2 \\ &\leq \|u_1\|_2 + \|u_2\|_2 = \|u\|_2. \end{aligned}$$

This completes the proof. \square

With a $w_{D,\delta}(z)$ given as in Lemma 6.3, the range of (\bar{D}, δ) over which the system is robustly and simultaneously stabilized can be determined by solving the \mathcal{H}_∞ optimal control problem

$$\inf_{K(z) \text{ stabilizes } P_0(z)} \|w_{\bar{D},\delta}(z)T_0(z)\|_\infty,$$

leading to a condition in much the same spirit as Theorem 6.1. To proceed, define analogously

$$D_w = \text{diag}(w_{\bar{D},\delta}(p_1), w_{\bar{D},\delta}(p_2), \dots, w_{\bar{D},\delta}(p_n)), \quad (6.33)$$

and $D_w = A^{-1}B$, where A and B are polynomial matrices satisfying (6.12).

We summarize the result in the following theorem. The proof is analogous to that of Theorem 6.1 and hence is omitted.

Theorem 6.3. *Let $p_i \in \mathbb{D}^C$, $i = 1, 2, \dots, n$, and $z_j \in \mathbb{D}^C$, $j = 1, 2, \dots, m$ be the distinct unstable poles and nonminimum phase zeros of $P_0(z)$, respectively. Let also d be the relative degree of $P_0(z)$. Assume that $p_i \neq z_j$, for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then the system in (6.24) can be stabilized by some $K(z)$ for all $D(k) \in \{0, 1, \dots, \bar{D}\}$ with $\delta_m(k) \leq \delta$ if $\bar{D} < \underline{D}$, where*

$$\underline{D} = \lfloor \bar{\lambda}^{-1}(U^{-1}V) \rfloor,$$

with

$$V = \text{diag}\{I, \dots, I, \Phi_{2q}\}, \quad U = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -\Phi_0 & -\Phi_1 & \cdots & -\Phi_{2q-1} \end{bmatrix},$$

where Φ_k are constructed as

$$\sum_{k=0}^{2q} D^k \Phi_k = \begin{bmatrix} Q_2 & B^H \\ B & A Q_1^{-1} A^H \end{bmatrix}, \quad (6.34)$$

and Q_1, Q_2, A and B are defined in Theorem 6.1.

Compared to Theorem 6.2, Theorem 6.3 exploits also the additional information on the delay variation rate, and hence can be potentially less conservative. Note that similarly, explicit bounds analogous to those in Corollary 6.1 and Corollary 6.2 can also be derived from Theorem 6.2 and Theorem 6.3. We leave these points to the reader.

6.4 Delay margin with PID controllers

Also of interest in this chapter is the delay margin achievable by PID controllers. PID control has been widely used in industrial and process control industries, which typically postulate dynamic models of low order [168, 169]. For continuous-time systems, the delay margin was found in [157] (pp. 154) for first-order systems achievable by proportional static feedback. By using more general PID controllers, the delay margin of first-order systems was found to double that attained by proportional control [170].

We also consider the first-order delay plant, given by

$$P_D(z) = \frac{z^{-D}}{z - p}, \quad D \in \mathbb{N}, \quad (6.35)$$

where $p > 1$. Denote $P_0(z) = 1/(z - p)$. It follows that $P_D(z) = z^{-D} P_0(z)$. We consider proportional feedback and PD controllers

$$K_P(z) = k_p$$

and

$$K_{PD}(z) = k_p + k_d(1 - z^{-1}),$$

and define the corresponding delay margin

$$D_P^* = \max \{N \in \mathbb{N} : \text{there exists some } K_P(z) \text{ that} \\ \text{stabilizes } P_D(z), \forall D \in \{0, 1, \dots, N\}\},$$

and

$$D_{PD}^* = \max \{N \in \mathbb{N} : \text{there exists some } K_{PD}(z) \text{ that} \\ \text{stabilizes } P_D(z), \forall D \in \{0, 1, \dots, N\}\},$$

respectively. It is known that the addition of the integral control will perform no better, if not worse, than PD controllers for feedback stabilization ([171]); as such, it suffices to consider PD control only.

Our main results in this section consist of lower bounds on the delay margin achievable by proportional and PD controllers.

Theorem 6.4. *Let $P_D(z)$ be given by (6.35), and $p > 1$.*

1. *If $1/(p-1)$ is not an integer, then a lower bound on the delay margin achievable by the proportional controller $K_P(z)$, for $p-1 < k_p < p+1$, is*

$$\underline{D}_P = \left\lfloor \frac{1}{p-1} \right\rfloor.$$

If $1/(p-1)$ is an integer, then the exact delay margin $D_P^ = \frac{1}{p-1} - 1$.*

2. *If $2/(p-1)$ is not an integer, then a lower bound on the delay margin achievable by the PD controller $K_{PD}(z)$, for $k_p > p-1$ and $|k_d| < 1$, is*

$$\underline{D}_{PD} = \left\lfloor \frac{2}{p-1} \right\rfloor.$$

If $2/(p-1)$ is an integer, then the exact delay margin $D_{PD}^ = \frac{2}{p-1} - 1$.*

Proof. Consider first the proportional controller $K_P(z) = k_p$. It is trivial to find that the closed-loop system has a pole at $z = p - k_p$. Hence, in the absence of delay, the

system is stable if and only if

$$p - 1 < k_p < p + 1. \quad (6.36)$$

Let us then inspect the open-loop frequency response

$$L_0(e^{j\omega}) = P_0(e^{j\omega})K_P(e^{j\omega}) = \frac{k_p}{e^{j\omega} - p}.$$

The open-loop gain is

$$|L_0(e^{j\omega})|^2 = \frac{k_p^2}{p^2 - 2p \cos \omega + 1}.$$

Setting $|L_0(e^{j\omega_0})| = 1$ yields

$$\cos \omega = \frac{p^2 + 1 - k_p^2}{2p}.$$

This indicates that there exists a unique $0 < \omega_0 < \pi$, such that $|L_0(e^{j\omega_0})| = 1$. At $\omega = \omega_0$, we have

$$\angle L_0(e^{j\omega_0}) = \pi + \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0}.$$

Set

$$\omega_0 D_0 = \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0}.$$

It can be readily seen that

$$1 + P_{D_0}(e^{j\omega_0})K_P(e^{j\omega_0}) = 1 + P_0(e^{j\omega_0})K_P(e^{j\omega_0})e^{-j\omega_0 D_0} = 0,$$

and for any $D < D_0$, $1 + P_D(e^{j\omega_0})K_P(e^{j\omega_0}) \neq 0$, $\forall \omega \geq 0$. Equivalently, the system is stable $\forall D < D_0$. We now set out to calculate D_0 . In view of (6.36), we construct the controller

$$K_P(z) = \sqrt{(p-1)^2 + \epsilon}, \quad 0 < \epsilon < 4p.$$

A straightforward calculation gives rise to

$$\cos \omega_0(\epsilon) = \frac{2p - \epsilon}{2p}, \quad (6.37)$$

$$\sin \omega_0(\epsilon) = \sqrt{\frac{4p\epsilon - \epsilon^2}{4p^2}}. \quad (6.38)$$

As $\epsilon \rightarrow 0$, $\omega_0(\epsilon) \rightarrow 0$. Hence,

$$\begin{aligned} \sup_{\omega_0 > 0} D_0 &= \lim_{\omega_0 \rightarrow 0} \frac{1}{\omega_0} \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\sin \omega_0(\epsilon)}{\omega_0(\epsilon) (p - \cos \omega_0(\epsilon))} \\ &= \frac{1}{p-1}. \end{aligned}$$

Evidently, $\underline{D}_P = \left\lfloor \frac{1}{p-1} \right\rfloor$ is a lower bound on the delay margin if $1/(p-1)$ is not an integer. Otherwise, $D_P^* = \frac{1}{p-1} - 1$ is the exact delay margin.

We next consider the PD controller $K_{PD}(z)$. For the delay-free plant $P_0(z)$, the closed-loop characteristic equation is given by

$$z^2 + (k_p + k_d - p)z - k_d = 0.$$

By Jury's test ([172]), the closed-loop system is stable if and only if $|k_d| < 1$, $k_p > p-1$, and $k_p + 2k_d < p + 1$. Consider analogously the open-loop gain

$$|L_0(e^{j\omega})|^2 = \frac{(k_p + k_d)^2 - 2k_d(k_p + k_d) \cos \omega + k_d^2}{1 - 2p \cos \omega + p^2}.$$

The solution of $|L_0(e^{j\omega_0})| = 1$ is given by

$$\cos \omega_0 = \frac{(k_p + k_d)^2 + k_d^2 - 1 - p^2}{2k_d(k_p + k_d) - 2p}.$$

Correspondingly, the open-loop phase at ω_0 is found as

$$\angle L_0(e^{j\omega_0}) = \pi + \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} + \tan^{-1} \frac{k_d \sin \omega_0}{k_p + k_d - k_d \cos \omega_0}$$

Let us construct $K_{PD}(z)$ with the coefficients given as

$$k_p = p - 1 + \epsilon^2, \quad k_d = 1 - \epsilon$$

for some sufficiently small $\epsilon > 0$. A direct calculation then yields

$$\cos(\omega_0(\epsilon)) = 1 + o(\epsilon),$$

that is, $\omega_0(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Let

$$\omega_0 D_0 = \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} + \tan^{-1} \frac{\sin \omega_0}{k_p + k_d - k_d \cos \omega_0}.$$

It follows that

$$\begin{aligned} \sup_{\omega_0 > 0} D_0 &= \lim_{\omega_0 \rightarrow 0} \frac{1}{\omega_0} \left(\tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} + \tan^{-1} \frac{k_d \sin \omega_0}{k_p + k_d - k_d \cos \omega_0} \right) \\ &= \frac{2}{p-1}. \end{aligned}$$

This indicates that $\underline{D}_{PD} = \left\lfloor \frac{2}{p-1} \right\rfloor$ is a lower bound on the delay margin if $2/(p-1)$ is not an integer, and is the exact delay margin $D_{PD}^* = \frac{2}{p-1} - 1$ if otherwise. The proof is now completed. □

Remark 6.3. *Much like its continuous-time counterpart, Theorem 6.4 shows that with a PD controller, the lower bound on the delay margin can at least double that achievable by a proportional controller. Indeed, it is easy to recognize that*

$$\left\lfloor \frac{2}{p-1} \right\rfloor = \begin{cases} 2 \left\lfloor \frac{1}{p-1} \right\rfloor & \text{if } 1 \pmod{(p-1)} < 1/2 \\ 2 \left\lfloor \frac{1}{p-1} \right\rfloor + 1 & \text{if } 1 \pmod{(p-1)} \geq 1/2 \end{cases}$$

On the other hand, the result also exhibits its limitation; it follows instantly that $\underline{D}_P = 0$ whenever $p \geq 2$, and $\underline{D}_{PD} = 0$ whenever $p \geq 3$. Nevertheless, a further analysis shows that this restriction is intrinsic of the hard limitation in controlling discrete-time delay systems, or in simultaneously stabilizing several discrete-time plants by PID controllers, instead of the bounds alone. The following result summarizes this finding.

Theorem 6.5. Let $P_D(z)$ be given by (6.35). Then for $p \geq 2$,

$$D_P^* = 0, \quad (6.39)$$

and

$$D_{PD}^* = \begin{cases} 1 & \text{if } 2 \leq p < 3 \\ 0 & \text{if } p \geq 3 \end{cases} \quad (6.40)$$

Proof. Assume that $p \geq 2$ and $D_P^* \geq 1$. Then $K_P(z)$ stabilizes $P_0(z)$; that is, k_p satisfies the condition in (29): $p - 1 < k_p < p + 1$. Furthermore, it stabilizes $P_1(z)$. For this to be true, it follows similarly from Jury's test that k_p must be such that $|k_p| < 1$, $k_p > p - 1$. Hence, for $K_P(z)$ to stabilize simultaneously $P_0(z)$ and $P_1(z)$, it is necessary that $p - 1 < 1$, i.e., $p < 2$, hence resulting in a contradiction. This establishes that $D_P^* = 0$. The proof of (6.40) is analogous, albeit more tedious, and hence is omitted. □

We now end our discussion by studying how nonminimum phase zeros may limit the delay margin achievable by PID controllers. Toward this end, we analyze the first-order delay plant

$$P_D(z) = \frac{z - s}{z - p} z^{-D}, \quad D \in \mathbb{N}, \quad (6.41)$$

where $p > 1$, $s > 1$, and $p \neq s$ are the plant unstable pole and nonminimum phase zero, respectively. We derive a lower bound achievable by proportional controllers. The result shows as well that the presence of a nonminimum phase zero is likely to reduce the delay margin.

Theorem 6.6. Consider the delay plant given in (6.41). Let $s > 1$, $p > 1$.

1. For $s > p$, if $\frac{s - p}{(s - 1)(p - 1)}$ is not an integer, then a lower bound on the delay margin achievable by the proportional controller $K_P(z)$, for $-\frac{p + 1}{s + 1} < k_p < -\frac{p - 1}{s - 1}$, is

$$\underline{D}_P = \left\lfloor \frac{s - p}{(s - 1)(p - 1)} \right\rfloor.$$

If $\frac{s - p}{(s - 1)(p - 1)}$ is an integer, then the exact delay margin is $D_P^* = \frac{s - p}{(s - 1)(p - 1)} - 1$.

2. For $1 < s \leq p$, $D_p^* = \underline{D}_p = 0$.

Proof. First, for the proportional controller $K_P(z) = k_p$ to stabilize $P_0(z) = \frac{z-s}{z-p}$, it is both necessary and sufficient that

$$-1 < \frac{p + k_p s}{1 + k_p} < 1. \quad (6.42)$$

Let $s > 1$. It can be verified that for (6.42) to hold, k_p must assume a negative value. In particular,

$$-\frac{p+1}{s+1} < k_p < -\frac{p-1}{s-1}.$$

A trivial calculation yields

$$|L_0(e^{j\omega})|^2 = k_p^2 \frac{s^2 + 1 - 2s \cos \omega}{p^2 + 1 - 2p \cos \omega}.$$

For $s > p$, $|L_0(e^{j\omega})|^2$ is monotonically decreasing for $\omega \in [0, \pi]$. Setting $|L_0(e^{j\omega_0})| = 1$, we then obtain the unique ω_0 , given as

$$\cos(\omega_0) = \frac{k_p^2(s^2 + 1) - p^2 - 1}{2k_p^2 s - 2p}. \quad (6.43)$$

Correspondingly,

$$\angle L_0(e^{j\omega_0}) = \pi + \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} - \tan^{-1} \frac{\sin \omega_0}{s - \cos \omega_0}.$$

Let D_0 be some real number such that

$$\omega_0 D_0 = \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} - \tan^{-1} \frac{\sin \omega_0}{s - \cos \omega_0},$$

and construct

$$K_P(z) = -\frac{p-1}{s-1} - \epsilon \quad (6.44)$$

for some sufficiently small $\epsilon > 0$. It follows that $\cos \omega(\epsilon) = 1 + o(\epsilon)$, and hence

$\omega(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. We may then evaluate

$$\begin{aligned} \sup_{\omega_0 > 0} D_0 &= \lim_{\omega_0 \rightarrow 0} \frac{1}{\omega_0} \left(\tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} - \tan^{-1} \frac{\sin \omega_0}{s - \cos \omega_0} \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\omega_0(\epsilon)} \left(\tan^{-1} \frac{\omega_0(\epsilon)}{p - \cos \omega_0(\epsilon)} - \tan^{-1} \frac{\omega_0(\epsilon)}{s - \cos \omega_0(\epsilon)} \right) \\ &= \frac{s - p}{(s - 1)(p - 1)}. \end{aligned}$$

This indicates that

$$\underline{D}_P = \left\lfloor \frac{s - p}{(s - 1)(p - 1)} \right\rfloor$$

is a lower bound on the delay margin whenever $\frac{s - p}{(s - 1)(p - 1)}$ is not an integer, and otherwise

$$\frac{s - p}{(s - 1)(p - 1)} - 1$$

is the exact delay margin.

On the other hand, when $1 < s \leq p$, $K_P(z)$ can stabilize $P_0(z)$ if and only if

$$-\frac{p - 1}{s - 1} < k_p < -\frac{p + 1}{s + 1}.$$

In this case,

$$\tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} - \tan^{-1} \frac{\sin \omega_0}{s - \cos \omega_0} < 0.$$

As such, there exists no $D_0 > 0$ such that

$$\omega_0 D_0 = \tan^{-1} \frac{\sin \omega_0}{p - \cos \omega_0} - \tan^{-1} \frac{\sin \omega_0}{s - \cos \omega_0},$$

implying that the achievable delay margin is 0. The proof is now completed. \square

6.5 Illustrative examples

We now present a number of examples to illustrate our results.

Example 6.1. *We first consider the plant*

$$P_0(z) = \frac{(z - s)(z^2 - 0.4z + 0.0425)}{(z - p)(z - 0.2)(z + 0.3)}, \quad (6.45)$$

For different values of (p, s) , the lower bounds \underline{D}_2 in Corollary 6.1 and \underline{D}_2 in Corollary 6.2 are calculated and shown in Figure 6.1. It is rather evident that increase in the value of the unstable pole p tends to decrease the lower bounds. Moreover, the presence of the nonminimum phase zero s generally leads to further reduction of the bound. Consider next, fixing $s = -1.4$ and $p = 1.6$. The corresponding lower bound

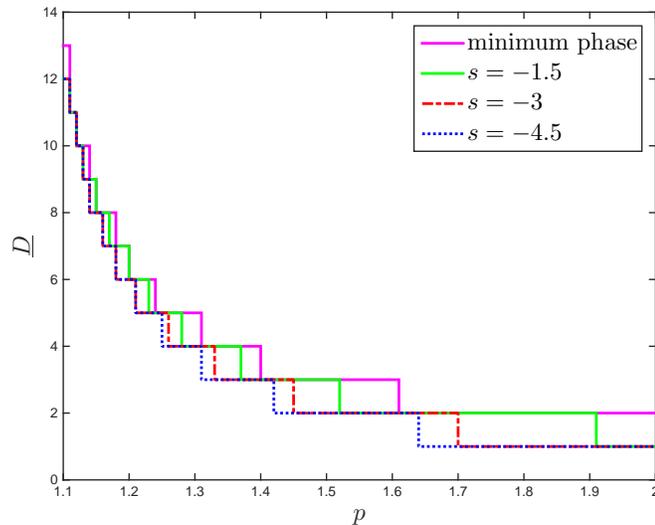


Figure 6.6: Lower bounds on the delay margin of system (6.45).

is found as $\underline{D} = 2$, and the optimal, simultaneously stabilizing controller achieving the bound is synthesized as

$$K(z) = \frac{0.5070z^2 + 0.5070z + 3.10 \times 10^{-13}}{z^2 + 1.621z - 0.6232} \times \frac{z^2 + 0.1z - 0.06}{z^2 - 0.4z + 0.0425}. \quad (6.46)$$

Fig. 6.7 shows that the three plants $P_0(z)$, $P_1(z)$, and $P_2(z)$ are indeed simultaneously stabilized.

Example 6.2. We next consider the following plant with a time-varying delay:

$$\begin{cases} x(k+1) = \begin{bmatrix} 2.2 & -0.21 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k - D(k)), \\ y(k) = \begin{bmatrix} 2.5 & -0.22 \end{bmatrix} x(k) + u(k - D(k)), \end{cases} \quad (6.47)$$

where the time-varying delay is a periodic function shown in Fig 6.8 and satisfies the specification $\delta \leq 1$ and $m = 2$. The transfer function for the delay-free plant is

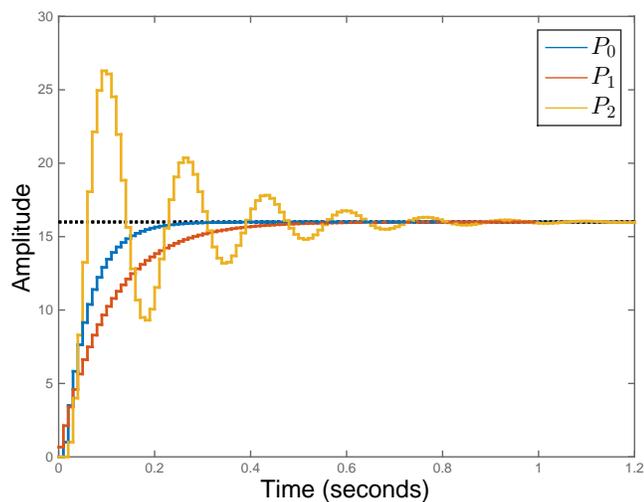


Figure 6.7: Step response of system (6.45) with controller (6.46).

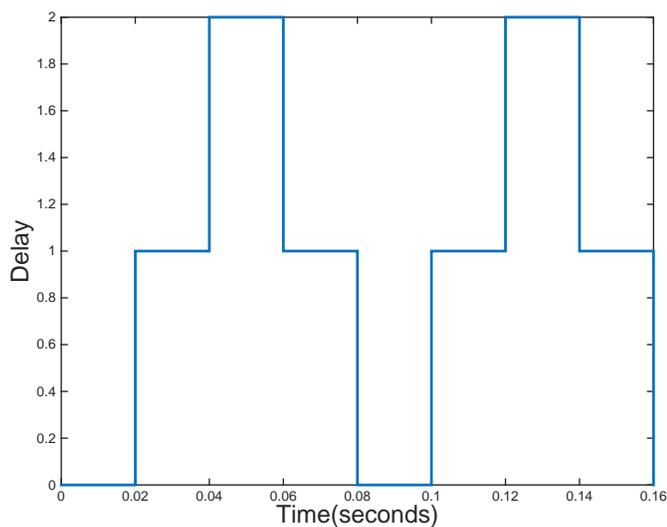


Figure 6.8: The time-varying delay.

$$P(z) = \frac{(z + 0.5)(z - 0.2)}{(z - 2.1)(z - 0.1)}, \quad (6.48)$$

which is minimum phase but contains an unstable pole $p = 2.1$. Using Theorem 6.2, we find $\bar{D}_1 = 1$. While by Theorem 6.3 with the approximant $\psi_{1\delta}(z)$, we could

calculate $\bar{D}_2 = 2$ and obtain the stabilizing controller

$$K(z) = \frac{2.775z^3 + 2.497z^2 - 0.2775z - 2.6 \times 10^{-14}}{z^3 + 1.294z^2 + 0.2882z - 0.009941}. \quad (6.49)$$

This controller generates the unit step response of the stable closed-loop system given in Fig. 6.9.

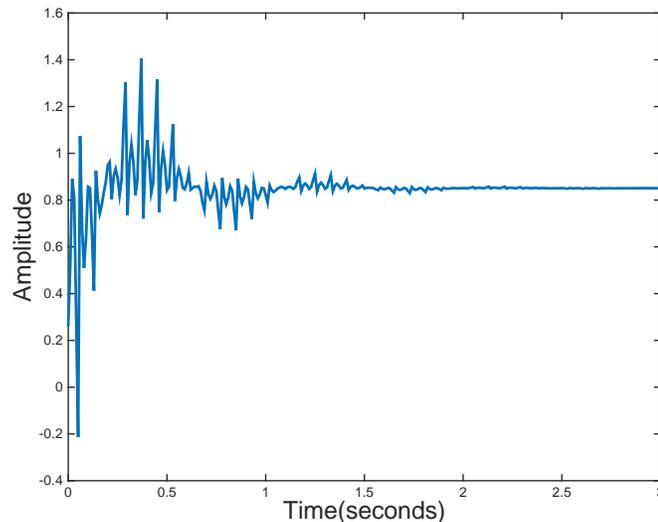


Figure 6.9: Step response of system (6.47) with controller (6.49).

Example 6.3. Finally, we consider the plant

$$P_0(z) = \frac{1}{z - 1.501}. \quad (6.50)$$

According to Theorem 6.4, a lower bound achievable by a proportional controller is $\underline{D}_P = 1$. Let the controller be

$$K_P(z) = 0.501 + \epsilon, \quad (6.51)$$

where $\epsilon = 10^{-2}$. Fig. 6.10 gives the step responses corresponding to $P_0(z)$ and $P_1(z)$, showing that $P_0(z)$ and $P_1(z)$ are simultaneously stabilized, with the final value converging to 100. Fig. 6.11 shows however, that $P_2(z)$ is not stabilized by $K_P(z)$, with the response diverging and oscillating around 100.

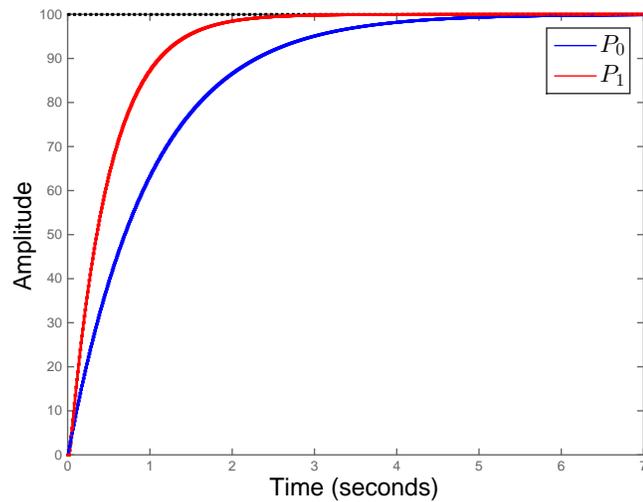


Figure 6.10: Step response of system (6.50) with controller (6.51).

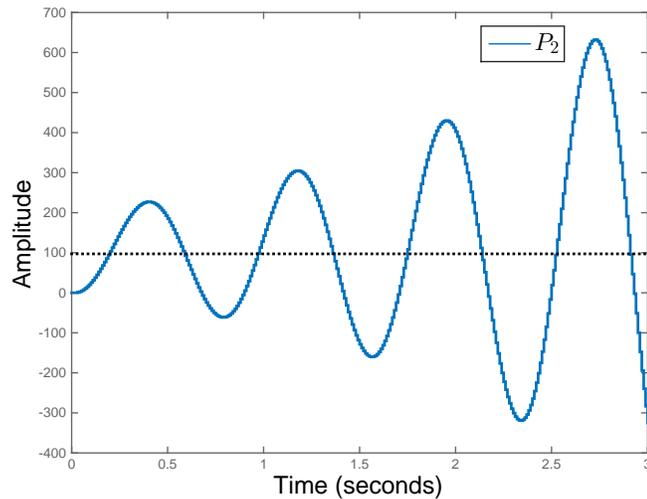


Figure 6.11: Step response of system (6.50) with controller (6.51).

6.6 Conclusion

In this chapter we have studied the delay margin problem for discrete-time systems. The problem is to find the fundamental margin of delay, or equivalently, the intrinsic limit under which an LTI controller may exist to stabilize simultaneously multiple plants with different delay lengths. By employing rational approximation and analyt-

ical interpolation techniques, we showed that for general LTI plants with an arbitrary number of unstable poles and nonminimum phase zeros, lower bounds on the delay margin can be obtained by solving an eigenvalue problem, and simultaneously stabilizing controllers can be duly synthesized to achieve the bounds. The results are then extended to systems with time-varying delays, and to those controlled by PID controllers. Generally and in a coherent manner, our results show how plant unstable poles and nonminimum phase zeros may fundamentally restrict the range of delay tolerable before losing stability, and in turn the limitations of LTI and in particular, PID controllers in simultaneously stabilizing unstable delay systems.

Chapter 7

Conclusions and Future Work

This thesis mainly investigates the consensus and consensusability problems for MASs under network-induced constraints; in addition, the delay margins of discrete-time LTI systems are studied. Furthermore, the theoretical analysis and synthesis have been verified by simulation examples. However, some more research topics are still remaining for further exploration in addition to the results in this thesis. This chapter includes a summary of the thesis, followed by summarizing several topics for future research.

7.1 Summary of the Thesis

Chapter 2 incorporates time-varying delays and switching topologies for the leader-following general linear MASs. We have introduced a unified switched systems scheme to handle the constraints of time delays and switching topologies simultaneously. The sufficient consensus conditions are derived in terms of state-transition matrices based on the results of switched control theory.

Chapter 3 takes the constraint of multiple communication channels into consideration for general linear MASs. By applying the regulation based consensus protocols, the overall closed-loop system can be cast as a cascade system. Hence the convergence can be analyzed by investigating the consensus of first-order systems and the stability of feedback systems. On this basis, necessary and sufficient consensus conditions are established.

Chapter 4 investigates the consensusability problem for linear MASs under time-varying delays. In order to analyze the performance limitation, we employ the fre-

quency domain methods. Loop transformation methods for feedback systems are utilized to isolate the delay dependent process. Based on small-gain theorem, sufficient consensusability condition are derived.

Chapter 5 studies the directed topology which is more general and practical. By adopting the properties of directed Laplacian spectra, we transform the consensusability analysis into the robust performance analysis of feedback control system in terms of gain and phase margins. A systematic controller synthesis method is also provided.

In Chapter 6, the delay margin problems for discrete-time LTI systems, a special class of simultaneous stabilization problems, is studied. We cope with the time delay using rational approximation. Then by applying the analytical interpolation methods, the upper bound on delay margin dependent on the unstable poles and nonminimum phase zeros is developed explicitly. We also study the delay margin for first-order unstable plant using PID control. Our results reveal how the unstable poles and nonminimum phase zeros constrain the delay margins.

7.2 Future Work

7.2.1 Consensus of Heterogenous Multi-Agent Systems

Up to now, most of the consensus results on MASs are proposed for homogeneous systems, in which all the agents share the exactly identical dynamics. However, in real applications, the dynamics of different agents are always different for many constraints. Hence, researches on consensus problems of heterogenous MASs have been paid more and more attentions. Most of the existing works are based on agents with simple dynamics, such as first-order, second-order dynamics and high-order integrator dynamics [35, 173, 49]. And some of these work have considered the networked induced constraints. Very recently, heterogenous consensus problems with general linear dynamics are studied. The Lyapunov based methods are commonly adopted to solve such kind of problems. For example, the authors in [174] investigate the consensus problem of linear heterogenous multi-agents systems with the same order. However, the Lyapunov based results are sufficient consensus conditions and usually very conservative.

Figure 7.1 shows the scenario for a heterogenous MAS. Similarly to homogeneous MASs, each agent will communicate only with its neighbours. However, the dynamics

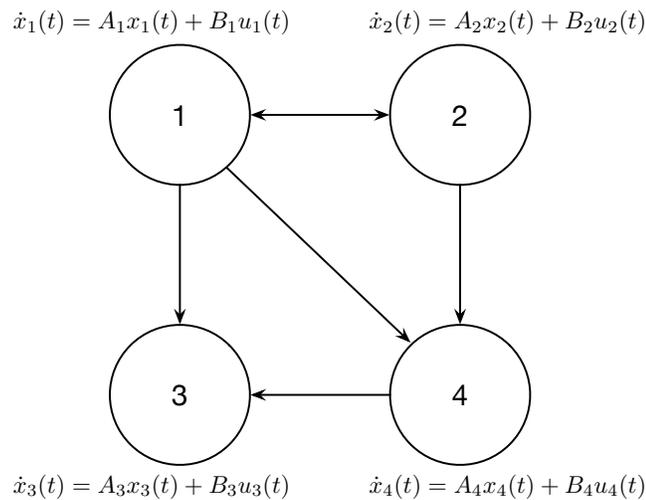


Figure 7.1: The heterogenous MASs.

of each agent may have different parameters, or even have different orders.

Consider a set of heterogenous agents with dynamics

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), x_i(0) = x_{i0}, \quad (7.1)$$

where $x_i(t) \in \mathbb{R}^{n_i}$, and $u_i(t) \in \mathbb{R}^{m_i}$. Then a fundamental problem is to design some appropriate control scheme and criteria for the system in (7.1) to reach heterogenous consensus, which is described by Definition 7.1

Definition 7.1. *The system represented by (7.1) achieves heterogenous consensus if for each agent i , there exists a local feedback u_i such that*

$$\lim_{k \rightarrow \infty} \|x_{i\hat{n}}(t) - \bar{x}_{\hat{n}}(t)\| = 0, \hat{n} = 1, 2, \dots, \max\{n_i\}, \quad (7.2)$$

for any finite initial condition $x_i(0)$, where $x_{i\hat{n}}(t)$ denotes the \hat{n} th element of vector $x_i(t)$, and $\bar{x}_{\hat{n}}(t)$ denotes the average state of $x_{i\hat{n}}(t)$ in the heterogenous multi-agent system.

Potential research could focus on the following aspects:

- To develop the consensus protocols and criteria to ensure the heterogenous consensus.
- To study the relationship between the network-induced constraints and the consensusability.

- To improve the consensus performance considering the network-induced constraints.

The decomposition technique based on the property of Laplacian matrix has been widely applied to deal with the consensus problem of homogenous MASs. The basic idea is to transform the MAS to a set of single agent control problems. To deal with the heterogenous MASs, the decomposition based method may be also applied.

Some frequency domain analyses methods may also help to get some necessary and sufficient like results, which tend to be less conservative compared to those derived by Lyapunov stability based methods. Furthermore, networked induced constraints, like time delays and packet dropouts, could be further considered. With the help of networked control methods and robust control methods, the consensus protocol would be appropriated designed to balanced the trade-off between the consensus and the system performance.

7.2.2 Other Directions of Future Work

Below are some other directions for further exploration.

- **Consensus of MASs considering network security:** Like all large-scale distributed systems, MASs also have many entry points for malicious attacks or intrusions. Only one agent that is affected by security issues can cause cascading performance faults and disrupt the normal operation of the whole MAS. Hence, it is worthwhile to study the problem considering network security.
- **Consensus of MASs considering scalability, agents join-in and drop-out:** In applications, the MASs could includes a large number of agents, like some wireless sensor networks. It is important to ensure the system performance considering the scalability and changes in the network topology. Furthermore, in literatures, almost all the existing researches are concentrating on the consensus problems with a fixed number of agents. However, in practical applications, some agents may join in a system, or drop out from a system. Thus it is important to design some control scheme which is robust when may experience agents join-in and drop-out.
- **Delay margins with advanced control methods:** Since LTI controllers are friendly for design and implement, most of the research effort focuses on

the delay margin problems considering LTI controllers. However, the simple structure of LTI controllers also poses some constraints on performance. The delay margins could be further improved using some more advanced control methods, like predictor feedback control and linear periodic control. These topics also deserve further exploration.

Appendix A

Publications

The following is a list of publications during the PhD studies.

- Journal papers

- J1. **Y. Chen**, and Y. Shi, “Consensus for linear multi-agent systems with time-varying delays: A frequency domain perspective,” *IEEE Transactions on Cybernetics*, vol. 47, no. 8, pp. 2143-2150, Aug. 2017.
- J2. **Y. Chen**, and Y. Shi, “Distributed consensus of linear multi-agent systems: A Laplacian spectra based method,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, accepted.
- J3. **Y. Chen**, and Y. Shi, “A fully distributed approach for consensus of multi-agent systems under multiple state-variables-dependent interaction topologies,” *Automatica*, under third-round review.
- J4. **Y. Chen**, A. Zulfiqar, D. Ma, Y. Shi, J. Chen, and F. Allgöwer, “Simultaneous stabilization of discrete-time delay systems: Bounds on delay margin,” *Automatica*, submitted.

- Conference papers

- C1. **Y. Chen**, and Y. Shi, “Leader-following consensus for multi-agent systems with switching topologies and time-varying delays: A switched system perspective,” in *2015 54th Conference on Decision and Control*, Osaka, 2015, pp. 374-379.

- C2. **Y. Chen**, A. Zulfiqar, D. Ma, Y. Shi, and J. Chen “On delay margin bounds of discrete-time systems,” in *2017 56th Conference on Decision and Control*, Melbourne, 2017, accepted.

Bibliography

- [1] M. Chen, S. Gonzalez, and V. C. M. Leung, “Applications and design issues for mobile agents in wireless sensor networks,” *IEEE Wireless Communications*, vol. 14, no. 6, pp. 20–26, 2007.
- [2] R. Olfati-Saber, “Distributed Kalman filtering for sensor networks,” in *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 5492–5498, 2007.
- [3] S. Kar and J. Moura, “Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise,” *IEEE Transactions on Signal Processing*, vol. 57, no. 1, pp. 355–369, 2009.
- [4] R. Olfati-Saber and P. Jalalkamali, “Coupled distributed estimation and control for mobile sensor networks,” *IEEE Transactions on Automatic Control*, vol. 57, no. 10, pp. 2609–2614, 2012.
- [5] D. M. de la Pena and P. D. Christofides, “Model-based control of nonlinear systems subject to sensor data losses: A chemical process case study,” in *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 3333–3338, 2007.
- [6] P. Arcara and C. Melchiorri, “Control schemes for teleoperation with time delay: A comparative study,” *Robotics and Autonomous Systems*, vol. 38, no. 1, pp. 49–64, 2002.
- [7] S. Wang, X. Meng, and T. Chen, “Wide-area control of power systems through delayed network communication,” *IEEE Transactions on Control Systems Technology*, vol. 20, no. 2, pp. 495–503, 2012.

- [8] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control*, vol. 46, no. 9, pp. 1384–1400, 2001.
- [9] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1391–1403, 2007.
- [10] N. C. Martins and M. A. Dahleh, "Feedback control in the presence of noisy channels: "Bode-like" fundamental limitations of performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1604–1615, 2008.
- [11] S. Fang, H. Ishii, and J. Chen, "Tradeoffs in networked feedback systems: From information-theoretic measures to Bode-type integrals," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1046–1061, 2017.
- [12] L. Zhang, Y. Shi, T. Chen, and B. Huang, "A new method for stabilization of networked control systems with random delays," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1177–1181, 2005.
- [13] L. Xiao, A. Hassibi, and J. P. How, "Control with random communication delays via a discrete-time jump system approach," in *Proceedings of the 2000 American Control Conference*, pp. 2199–2204, 2000.
- [14] P. L. Tang and C. W. de Silva, "Compensation for transmission delays in an ethernet-based control network using variable-horizon predictive control," *IEEE Transactions on Control Systems Technology*, vol. 14, no. 4, pp. 707–718, 2006.
- [15] R. H. Middleton and D. E. Miller, "On the achievable delay margin using LTI control for unstable plants," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1194–1207, 2007.
- [16] D. L. Gaudette and D. E. Miller, "When is the achievable discrete-time delay margin nonzero?," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 886–890, 2011.
- [17] P. Ju and H. Zhang, "Further results on the achievable delay margin using LTI control," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3134–3139, 2016.

- [18] T. Qi, J. Zhu, and J. Chen, “Fundamental limits on uncertain delays: When is a delay system stabilizable by LTI controllers?,” *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1314–1328, 2017.
- [19] V. Blondel and A. Megretski, *Unsolved Problems in Mathematical Systems and Control Theory*. Princeton University Press, 2009.
- [20] A. Aquino-Lugo, R. Klump, and T. Overbye, “A control framework for the smart grid for voltage support using agent-based technologies,” *IEEE Transactions on Smart Grid*, vol. 2, no. 1, pp. 173–180, 2011.
- [21] G. Zhabelova and V. Vyatkin, “Multiagent smart grid automation architecture based on IEC 61850/61499 intelligent logical nodes,” *IEEE Transactions on Industrial Electronics*, vol. 59, no. 5, pp. 2351–2362, 2012.
- [22] V. P. Singh, N. Kishor, and P. Samuel, “Distributed multi-agent system-based load frequency control for multi-area power system in smart grid,” *IEEE Transactions on Industrial Electronics*, vol. 64, pp. 5151–5160, June 2017.
- [23] Y. Zheng, S. E. Li, K. Li, F. Borrelli, and J. K. Hedrick, “Distributed model predictive control for heterogeneous vehicle platoons under unidirectional topologies,” *IEEE Transactions on Control Systems Technology*, vol. 25, pp. 899–910, May 2017.
- [24] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
- [25] L. Xiao and S. Boyd, “Fast linear iterations for distributed averaging,” *Systems & Control Letters*, vol. 53, no. 1, pp. 65–78, 2004.
- [26] W. Ren, “On consensus algorithms for double-integrator dynamics,” *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1503–1509, 2008.
- [27] K. You and L. Xie, “Network topology and communication data rate for consensusability of discrete-time multi-agent systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2262–2275, 2011.

- [28] Z. Li, W. Ren, X. Liu, and L. Xie, “Distributed consensus of linear multi-agent systems with adaptive dynamic protocols,” *Automatica*, vol. 49, no. 7, pp. 1986–1995, 2013.
- [29] D. V. Dimarogonas and K. J. Kyriakopoulos, “On the rendezvous problem for multiple nonholonomic agents,” *IEEE Transactions on Automatic Control*, vol. 52, pp. 916–922, May 2007.
- [30] H. Li, W. Yan, and Y. Shi, “A receding horizon stabilization approach to constrained nonholonomic systems in power form,” *Systems & Control Letters*, vol. 99, no. Supplement C, pp. 47–56, 2017.
- [31] S. J. Chung and J. J. E. Slotine, “Cooperative robot control and concurrent synchronization of Lagrangian systems,” *IEEE Transactions on Robotics*, vol. 25, pp. 686–700, June 2009.
- [32] J. Yao, Z.-H. Guan, and D. J. Hill, “Passivity-based control and synchronization of general complex dynamical networks,” *Automatica*, vol. 45, no. 9, pp. 2107–2113, 2009.
- [33] N. Chopra and M. W. Spong, “On exponential synchronization of Kuramoto oscillators,” *IEEE Transactions on Automatic Control*, vol. 54, pp. 353–357, Feb 2009.
- [34] C.-L. Liu and F. Liu, “Stationary consensus of heterogeneous multi-agent systems with bounded communication delays,” *Automatica*, vol. 47, no. 9, pp. 2130–2133, 2011.
- [35] Y. Zheng, Y. Zhu, and L. Wang, “Consensus of heterogeneous multi-agent systems,” *IET Control Theory & Applications*, vol. 5, no. 16, pp. 1881–1888, 2011.
- [36] Y. Zheng and L. Wang, “Consensus of heterogeneous multi-agent systems without velocity measurements,” *International Journal of Control*, vol. 85, no. 7, pp. 906–914, 2012.
- [37] W. Ren, “Collective motion from consensus with Cartesian coordinate coupling,” *IEEE Transactions on Automatic Control*, vol. 54, pp. 1330–1335, June 2009.

- [38] R. Olfati-Saber, “Flocking for multi-agent dynamic systems: Algorithms and theory,” *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 401–420, 2006.
- [39] W. Ren, “Multi-vehicle consensus with a time-varying reference state,” *Systems & Control Letters*, vol. 56, no. 7-8, pp. 474–483, 2007.
- [40] W. Yu, G. Chen, and M. Cao, “Distributed leader-follower flocking control for multi-agent dynamical systems with time-varying velocities,” *Systems & Control Letters*, vol. 59, no. 9, pp. 543–552, 2010.
- [41] X. Yin, Z. Li, L. Zhang, and M. Han, “Distributed state estimation of sensor-network systems subject to Markovian channel switching with application to a chemical process,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2016. Advance online publication. DOI: 10.1109/TSMC.2016.2632155.
- [42] A. Jadbabaie, J. Lin, and A. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.
- [43] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, “Novel type of phase transition in a system of self-driven particles,” *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.
- [44] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.
- [45] L. Moreau, “Stability of multiagent systems with time-dependent communication links,” *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.
- [46] W. Ren and R. W. Beard, “Consensus seeking in multiagent systems under dynamically changing interaction topologies,” *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.
- [47] J. H. Seo, H. Shim, and J. Back, “Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach,” *Automatica*, vol. 45, no. 11, pp. 2659–2664, 2009.

- [48] G. Gu, L. Marinovici, and F. L. Lewis, “Consensusability of discrete-time dynamic multiagent systems,” *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2085–2089, 2012.
- [49] Y. P. Tian and Y. Zhang, “High-order consensus of heterogeneous multi-agent systems with unknown communication delays,” *Automatica*, vol. 48, no. 6, pp. 1205–1212, 2012.
- [50] J. Sun and Z. Geng, “Adaptive output feedback consensus tracking for linear multi-agent systems with unknown dynamics,” *International Journal of Control*, vol. 88, no. 9, pp. 1735–1745, 2015.
- [51] M. Porfiri and D. Stilwell, “Consensus seeking over random weighted directed graphs,” *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1767–1773, 2007.
- [52] W. Yu, G. Chen, and M. Cao, “Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems,” *Automatica*, vol. 46, no. 6, pp. 1089–1095, 2010.
- [53] W. Yu, W. Ren, W. X. Zheng, G. Chen, and J. Lü, “Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics,” *Automatica*, vol. 49, no. 7, pp. 2107–2115, 2013.
- [54] W. Ren, K. L. Moore, and Y. Q. Chen, “High-order and model reference consensus algorithms in cooperative control of multivehicle systems,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 5, pp. 678–688, 2007.
- [55] P. Wieland, J.-S. Kim, and F. Allgöwer, “On topology and dynamics of consensus among linear high-order agents,” *International Journal of Systems Science*, vol. 42, no. 10, pp. 1831–1842, 2011.
- [56] B. Zhou and Z. Lin, “Consensus of high-order multi-agent systems with large input and communication delays,” *Automatica*, vol. 50, no. 2, pp. 452–464, 2014.
- [57] L. Cheng, Z.-G. Hou, Y. Lin, M. Tan, and W. Zhang, “Solving a modified consensus problem of linear multi-agent systems,” *Automatica*, vol. 47, no. 10, pp. 2218–2223, 2011.

- [58] T. Qi, L. Qiu, and J. Chen, “Multi-agent consensus under delayed feedback: Fundamental constraint on graph and fundamental bound on delay,” in *Proceedings of the 2013 American Control Conference*, pp. 952–957, 2013.
- [59] Z. Lin, B. Francis, and M. Maggiore, “State agreement for continuous-time coupled nonlinear systems,” *SIAM Journal on Control and Optimization*, vol. 46, no. 1, pp. 288–307, 2007.
- [60] G. Shi and Y. Hong, “Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies,” *Automatica*, vol. 45, no. 5, pp. 1165–1175, 2009.
- [61] Z. Li, W. Ren, X. Liu, and M. Fu, “Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols,” *IEEE Transactions on Automatic Control*, vol. 58, no. 7, pp. 1786–1791, 2013.
- [62] M. Franceschelli, A. Giua, and C. Seatzu, “A gossip-based algorithm for discrete consensus over heterogeneous networks,” *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1244–1249, 2010.
- [63] L. Mo, Y. Niu, and T. Pan, “Consensus of heterogeneous multi-agent systems with switching jointly-connected interconnection,” *Physica A: Statistical Mechanics and its Applications*, vol. 427, pp. 132–140, 2015.
- [64] Y. Hong, G. Chen, and L. Bushnell, “Distributed observers design for leader-following control of multi-agent networks,” *Automatica*, vol. 44, no. 3, pp. 846–850, 2008.
- [65] F. Xiao and L. Wang, “State consensus for multi-agent systems with switching topologies and time-varying delays,” *International Journal of Control*, vol. 79, no. 10, pp. 1277–1284, 2006.
- [66] Y. Gao, L. Wang, G. Xie, and B. Wu, “Consensus of multi-agent systems based on sampled-data control,” *International Journal of Control*, vol. 82, no. 12, pp. 2193–2205, 2009.
- [67] Y. Zhang and Y.-P. Tian, “Consensus of data-sampled multi-agent systems with random communication delay and packet loss,” *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 939–943, 2010.

- [68] W. Yu, W. X. Zheng, G. Chen, W. Ren, and J. Cao, "Second-order consensus in multi-agent dynamical systems with sampled position data," *Automatica*, vol. 47, no. 7, pp. 1496–1503, 2011.
- [69] T. Li and J.-F. Zhang, "Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions," *Automatica*, vol. 45, no. 8, pp. 1929–1936, 2009.
- [70] C.-Q. Ma and J.-F. Zhang, "Necessary and sufficient conditions for consensus-ability of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1263–1268, 2010.
- [71] Z. Meng, W. Ren, Y. Cao, and Z. You, "Leaderless and leader-following consensus with communication and input delays under a directed network topology," *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, vol. 41, no. 1, pp. 75–88, 2011.
- [72] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Systems & Control Letters*, vol. 59, no. 3-4, pp. 209–217, 2010.
- [73] W. Ren, "Distributed leaderless consensus algorithms for networked Euler-Lagrange systems," *International Journal of Control*, vol. 82, no. 11, pp. 2137–2149, 2009.
- [74] T. Qi, L. Qiu, and J. Chen, "Consensus over directed graph: Output feedback and topological constraints," in *2013 9th Asian Control Conference*, pp. 1–6, 2013.
- [75] D. Goldin, S. Attia, and J. Raisch, "Consensus for double integrator dynamics in heterogeneous networks," in *Proceedings of the 49th IEEE Conference on Decision and Control*, pp. 4504–4510, 2010.
- [76] J. Qin and C. Yu, "Coordination of multiagents interacting under independent position and velocity topologies," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 24, no. 10, pp. 1588–1597, 2013.
- [77] D. Goldin and J. Raisch, "Consensus for agents with double integrator dynamics in heterogeneous networks," *Asian Journal of Control*, vol. 16, no. 1, pp. 30–39, 2014.

- [78] J. Qin, W. X. Zheng, H. Gao, and Q. Ma, “Containment for double-integrator multi-agent systems in heterogeneous networks,” in *Proceedings of the 53rd IEEE Conference on Decision and Control*, pp. 6179–6184, 2014.
- [79] Y.-P. Tian and C.-L. Liu, “Consensus of multi-agent systems with diverse input and communication delays,” *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2122–2128, 2008.
- [80] F. Xiao and L. Wang, “Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays,” *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1804–1816, 2008.
- [81] X. Liu, W. Lu, and T. Chen, “Consensus of multi-agent systems with unbounded time-varying delays,” *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2396–2401, 2010.
- [82] Z. Wang, J. Xu, and H. Zhang, “Consensusability of multi-agent systems with time-varying communication delay,” *Systems & Control Letters*, vol. 65, pp. 37–42, 2014.
- [83] L. Fang and P. Antsaklis, “Information consensus of asynchronous discrete-time multi-agent systems,” in *Proceedings of the 2005 American Control Conference*, pp. 1883–1888 vol. 3, 2005.
- [84] J. Wu, Y. Shi, and H. Li, “Consensus in multi-agent systems with non-uniform sampling,” in *Proceedings of the 2013 American Control Conference*, pp. 3260–3265, 2013.
- [85] D. Dimarogonas and K. Johansson, “Event-triggered control for multi-agent systems,” in *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference*, pp. 7131–7136, 2009.
- [86] D. Dimarogonas, E. Frazzoli, and K. Johansson, “Distributed event-triggered control for multi-agent systems,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2012.
- [87] W. Zhu, Z.-P. Jiang, and G. Feng, “Event-based consensus of multi-agent systems with general linear models,” *Automatica*, vol. 50, no. 2, pp. 552–558, 2014.

- [88] Z. Wang, D. Ding, H. Dong, and H. Shu, " H_∞ consensus control for multi-agent systems with missing measurements: The finite-horizon case," *Systems & Control Letters*, vol. 62, no. 10, pp. 827–836, 2013.
- [89] B. Shen, Z. Wang, and Y. Hung, "Distributed H_∞ -consensus filtering in sensor networks with multiple missing measurements: The finite-horizon case," *Automatica*, vol. 46, no. 10, pp. 1682–1688, 2010.
- [90] Z.-H. Guan, C. Meng, R.-Q. Liao, and D.-X. Zhang, "Consensus of second-order multi-agent dynamic systems with quantized data," *Physics Letters A*, vol. 376, no. 4, pp. 387–393, 2012.
- [91] W. Chen, X. Li, and L. Jiao, "Quantized consensus of second-order continuous-time multi-agent systems with a directed topology via sampled data," *Automatica*, vol. 49, no. 7, pp. 2236–2242, 2013.
- [92] H. Li and W. Yan, "Receding horizon control based consensus scheme in general linear multi-agent systems," *Automatica*, vol. 56, no. Supplement C, pp. 12–18, 2015.
- [93] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Physica A: Statistical Mechanics and its Applications*, vol. 374, no. 2, pp. 853–863, 2007.
- [94] W. Zhu and D. Cheng, "Leader-following consensus of second-order agents with multiple time-varying delays," *Automatica*, vol. 46, no. 12, pp. 1994–1999, 2010.
- [95] A. J. Paulraj, D. A. Gore, R. U. Nabar, and H. Bolcskei, "An overview of MIMO communications—a key to gigabit wireless," *Proceedings of the IEEE*, vol. 92, no. 2, pp. 198–218, 2004.
- [96] J. Yick, B. Mukherjee, and D. Ghosal, "Wireless sensor network survey," *Computer Networks*, vol. 52, no. 12, pp. 2292–2330, 2008.
- [97] X. Zhou, R. Zhang, and C. K. Ho, "Wireless information and power transfer: Architecture design and rate-energy tradeoff," *IEEE Transactions on Communications*, vol. 61, no. 11, pp. 4754–4767, 2013.
- [98] C. X. Wang, F. Haider, X. Gao, X. H. You, Y. Yang, D. Yuan, H. M. Aggoune, H. Haas, S. Fletcher, and E. Hepsaydir, "Cellular architecture and key

- technologies for 5G wireless communication networks,” *IEEE Communications Magazine*, vol. 52, no. 2, pp. 122–130, 2014.
- [99] J. Chen, “Fundamental limitation of feedback control,” in *Encyclopedia of Systems and Control*, pp. 487–495, Springer, 2015.
- [100] J. Fax and R. Murray, “Information flow and cooperative control of vehicle formations,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.
- [101] Z. Lin, B. Francis, and M. Maggiore, “Necessary and sufficient graphical conditions for formation control of unicycles,” *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 121–127, 2005.
- [102] Z. Lin, L. Wang, Z. Han, and M. Fu, “Distributed formation control of multi-agent systems using complex Laplacian,” *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1765–1777, 2014.
- [103] J. Lin, A. S. Morse, and B. D. O. Anderson, “The multi-agent rendezvous problem,” in *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 1508–1513, 2003.
- [104] R. Olfati-Saber and J. Shamma, “Consensus filters for sensor networks and distributed sensor fusion,” in *Proceedings of the 44th IEEE Conference on Decision and Control, and 2005 European Control Conference*, pp. 6698–6703, 2005.
- [105] M. Cao, A. Morse, and B. Anderson, “Agreeing asynchronously,” *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1826–1838, 2008.
- [106] T. Li and J.-F. Zhang, “Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises,” *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2043–2057, 2010.
- [107] X. Wang and Y. Hong, “Finite-time consensus for multi-agent networks with second-order agent dynamics,” in *Proceedings of the IFAC World Congress*, pp. 15185–15190, 2008.

- [108] J. Wu and Y. Shi, "Consensus in multi-agent systems with random delays governed by a Markov chain," *Systems & Control Letters*, vol. 60, no. 10, pp. 863–870, 2011.
- [109] B. Mu, H. Li, J. Ding, and Y. Shi, "Consensus in second-order multiple flying vehicles with random delays governed by a Markov chain," *Journal of the Franklin Institute*, vol. 352, no. 9, pp. 3628–3644, 2015.
- [110] H. Li and Y. Shi, "Distributed receding horizon control of large-scale nonlinear systems: Handling communication delays and disturbances," *Automatica*, vol. 50, no. 4, pp. 1264–1271, 2014.
- [111] K. Peng and Y. Yang, "Leader-following consensus problem with a varying-velocity leader and time-varying delays," *Physica A: Statistical Mechanics and its Applications*, vol. 388, no. 23, pp. 193–208, 2009.
- [112] L. Ding, Q.-L. Han, and G. Guo, "Network-based leader-following consensus for distributed multi-agent systems," *Automatica*, vol. 49, no. 7, pp. 2281–2286, 2013.
- [113] G. Wen, G. Hu, W. Yu, J. Cao, and G. Chen, "Consensus tracking for higher-order multi-agent systems with switching directed topologies and occasionally missing control inputs," *Systems & Control Letters*, vol. 62, no. 12, pp. 1151–1158, 2013.
- [114] H. Lin and P. J. Antsaklis, "Persistent disturbance attenuation properties for networked control systems," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, pp. 953–958, 2004.
- [115] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems*, vol. 27, no. 2, pp. 71–82, 2007.
- [116] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.
- [117] J. Qin, H. Gao, and C. Yu, "On discrete-time convergence for general linear multi-agent systems under dynamic topology," *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 1054–1059, 2014.

- [118] Z. Meng, T. Yang, D. V. Dimarogonas, and K. H. Johansson, “Coordinated output regulation of heterogeneous linear systems under switching topologies,” *Automatica*, vol. 53, pp. 362–368, 2015.
- [119] W. Chen, S. Wang, and L. Qiu, “MIMO control using MIMO communication: A majorization condition for networked stabilizability,” in *Proceedings of the 2015 American Control Conference*, pp. 2205–2210, 2015.
- [120] L. Qiu, G. Gu, and W. Chen, “Stabilization of networked multi-input systems with channel resource allocation,” *IEEE Transactions on Automatic Control*, vol. 58, no. 3, pp. 554–568, 2013.
- [121] P. Wieland, R. Sepulchre, and F. Allgöwer, “An internal model principle is necessary and sufficient for linear output synchronization,” *Automatica*, vol. 47, no. 5, pp. 1068–1074, 2011.
- [122] Y. Su and J. Huang, “Cooperative output regulation with application to multi-agent consensus under switching network,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 42, no. 3, pp. 864–875, 2012.
- [123] W. Ren and R. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. Springer Publishing Company, Incorporated, 1st ed., 2010.
- [124] B. Wang, P. Shi, H. R. Karimi, Y. Song, and J. Wang, “Robust H_∞ synchronization of a hyper-chaotic system with disturbance input,” *Nonlinear Analysis: Real World Applications*, vol. 14, no. 3, pp. 1487–1495, 2013.
- [125] X. Liang, Y. Xiao, J. Zhang, H. Deng, and A. V. Vasilakos, “Stochastic event capturing with a single mobile robot in rectangular perimeters,” *Telecommunication Systems*, vol. 52, no. 4, pp. 2519–2532, 2013.
- [126] X. Liang and Y. Xiao, “Studying bio-inspired coalition formation of robots for detecting intrusions using game theory,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 40, no. 3, pp. 683–693, 2010.
- [127] F. Xiao, T. Chen, and H. Gao, “Synchronous hybrid event- and time-driven consensus in multiagent networks with time delays,” *IEEE Transactions on Cybernetics*, vol. 46, no. 5, pp. 1165–1174, 2016.

- [128] Y. Tang, H. Gao, W. Zou, and J. Kurths, “Distributed synchronization in networks of agent systems with nonlinearities and random switchings,” *IEEE Transactions on Cybernetics*, vol. 43, no. 1, pp. 358–370, 2013.
- [129] C. Zhou, H. Chen, N. Xiong, X. Huang, and A. V. Vasilakos, “Model-driven development of reconfigurable protocol stack for networked control systems,” *IEEE Transactions on Systems, Man, and Cybernetics, Part C: Applications and Reviews*, vol. 42, no. 6, pp. 1439–1453, 2012.
- [130] H. Karimi and H. Gao, “New delay-dependent exponential H_∞ synchronization for uncertain neural networks with mixed time delays,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 40, no. 1, pp. 173–185, 2010.
- [131] H. Karimi, “Robust delay-dependent H_∞ control of uncertain time-delay systems with mixed neutral, discrete, and distributed time-delays and Markovian switching parameters,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 58, no. 8, pp. 1910–1923, 2011.
- [132] J. Qin and H. Gao, “A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays,” *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2417–2422, 2012.
- [133] Z. Meng, Z. Li, A. Vasilakos, and S. Chen, “Delay-induced synchronization of identical linear multiagent systems,” *IEEE Transactions on Cybernetics*, vol. 43, no. 2, pp. 476–489, 2013.
- [134] C. Ma, P. Shi, X. Zhao, and Q. Zeng, “Consensus of Euler-Lagrange systems networked by sampled-data information with probabilistic time delays,” *IEEE Transactions on Cybernetics*, vol. 45, no. 6, pp. 1126–1133, 2015.
- [135] L. Wang, Y. Shen, and G. Zhang, “Synchronization of a class of switched neural networks with time-varying delays via nonlinear feedback control,” *IEEE Transactions on Cybernetics*, vol. 46, no. 10, pp. 2300–2310, 2016.
- [136] C.-Y. Kao and B. Lincoln, “Simple stability criteria for systems with time-varying delays,” *Automatica*, vol. 40, no. 8, pp. 1429–1434, 2004.

- [137] A. Barrat, M. Barthlemy, and A. Vespignani, *Dynamical Processes on Complex Networks*. New York, NY, USA: Cambridge University Press, 1st ed., 2008.
- [138] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*. Upper Saddle River, NJ, USA: Prentice Hall PTR, 1st ed., 2000.
- [139] F. Chung, *Spectral Graph Theory*. No. 92 in CBMS Regional Conference Series, AMS Publications, 1997.
- [140] M. Pipattanasomporn, H. Feroze, and S. Rahman, “Multi-agent systems in a distributed smart grid: Design and implementation,” in *2009 IEEE/PES Power Systems Conference and Exposition*, pp. 1–8, March 2009.
- [141] D. Zhang, S. K. Nguang, and L. Yu, “Distributed control of large-scale networked control systems with communication constraints and topology switching,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 7, pp. 1746–1757, 2017.
- [142] G. Wen, W. Yu, Y. Xia, X. Yu, and J. Hu, “Distributed tracking of nonlinear multiagent systems under directed switching topology: An observer-based protocol,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 47, no. 5, pp. 869–881, 2017.
- [143] W. Wang, D. Wang, Z. Peng, and T. Li, “Prescribed performance consensus of uncertain nonlinear strict-feedback systems with unknown control directions,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 46, no. 9, pp. 1279–1286, 2016.
- [144] Q. Shen and P. Shi, “Output consensus control of multiagent systems with unknown nonlinear dead zone,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 46, no. 10, pp. 1329–1337, 2016.
- [145] Y. Zhao, Z. Duan, G. Wen, and G. Chen, “Distributed H_∞ consensus of multi-agent systems: A performance region-based approach,” *International Journal of Control*, vol. 85, no. 3, pp. 332–341, 2012.
- [146] Z. Li and Z. Duan, “Distributed consensus protocol design for general linear multi-agent systems: A consensus region approach,” *IET Control Theory & Applications*, vol. 8, no. 18, pp. 2145–2161, 2014.

- [147] G. Wen, Y. Zhao, Z. Duan, W. Yu, and G. Chen, “Containment of higher-order multi-leader multi-agent systems: A dynamic output approach,” *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 1135–1140, 2016.
- [148] R. Agaev and P. Chebotarev, “On the spectra of nonsymmetric Laplacian matrices,” *Linear Algebra and its Applications*, vol. 399, pp. 157–168, 2005.
- [149] G. Lafferriere, A. Williams, J. Caughman, and J. Veerman, “Decentralized control of vehicle formations,” *Systems & Control Letters*, vol. 54, no. 9, pp. 899–910, 2005.
- [150] J. Cockburn, Y. Sidar, and A. Tannenbaum, “Stability margin optimization via interpolation and conformal mappings,” *IEEE Transactions on Automatic Control*, vol. 40, no. 6, pp. 1066–1070, 1995.
- [151] J. Cockburn, Y. Sidar, and A. R. Tannenbaum, “A constructive solution to the gain-phase margin problem,” in *Proceedings of the 31st IEEE Conference on Decision and Control*, pp. 682–683, 1992.
- [152] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *Studies in Applied Mathematics*. Philadelphia, PA: SIAM, 1994.
- [153] K. Gu, J. Chen, and V. L. Kharitonov, *Stability of time-delay systems*. Springer Science & Business Media, 2003.
- [154] J.-P. Richard, “Time-delay systems: An overview of some recent advances and open problems,” *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.
- [155] J. J. Loiseau, W. Michiels, S.-I. Niculescu, and R. Sipahi, “Topics in time delay systems,” *Lecture Notes in Control and Information Sciences*, no. 388, 2009.
- [156] S.-I. Niculescu and K. Gu, *Advances in Time-Delay Systems*, vol. 38. Springer Science & Business Media, 2012.
- [157] W. Michiels and S.-I. Niculescu, *Stability, Control, and Computation for Time-delay Systems: An Eigenvalue-based Approach*, vol. 27. SIAM, 2014.
- [158] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. New York, NY: Macmillan, 1992.

- [159] D. E. Miller and D. E. Davison, “Stabilization in the presence of an uncertain arbitrarily large delay,” *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1074–1089, 2005.
- [160] D. L. Gaudette and D. E. Miller, “Stabilizing a SISO LTI plant with gain and delay margins as large as desired,” *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2324–2339, 2014.
- [161] D. Bresch-Pietri, J. Chauvin, and N. Petit, “Adaptive control scheme for uncertain time-delay systems,” *Automatica*, vol. 48, no. 8, pp. 1536–1552, 2012.
- [162] N. Bekiaris-Liberis and M. Krstic, *Nonlinear Control Under Nonconstant Delays*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2013.
- [163] J. A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*. Birkhäuser Basel, 1990.
- [164] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Upper Saddle River, NJ, USA: Prentice Hall, 1996.
- [165] N. Levinson and R. M. Redheffer, *Complex variables*. Holden-Day, 1970.
- [166] C. Y. Kao, “On stability of discrete-time LTI systems with varying time delays,” *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1243–1248, 2012.
- [167] C.-Y. Kao and A. Rantzer, “Stability analysis of systems with uncertain time-varying delays,” *Automatica*, vol. 43, no. 6, pp. 959–970, 2007.
- [168] K. J. Åström and T. Häggglund, *Advanced PID Control*. ISA-The Instrumentation, Systems and Automation Society, 2006.
- [169] D. Xue, Y. Chen, and D. P. Atherton, *Linear Feedback Control: Analysis and Design with MATLAB*. SIAM, 2007.
- [170] G. J. Silva, A. Datta, and S. P. Bhattacharyya, “New results on the synthesis of PID controllers,” *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 241–252, 2002.
- [171] D. Ma and J. Chen, “Delay margin of low-order systems achievable by PID controllers,” *IEEE Transactions on Automatic Control*, 2017.

- [172] K. Ogata, *Discrete-time control systems (2nd edition)*. Prentice-Hall, 1995.
- [173] Y. Zheng and L. Wang, “Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements,” *Systems & Control Letters*, vol. 61, no. 8, pp. 871–878, 2012.
- [174] Q. Wei, D. Liu, and F. L. Lewis, “Optimal distributed synchronization control for continuous-time heterogeneous multi-agent differential graphical games,” *Information Sciences*, vol. 317, pp. 96–113, 2015.