Dominating Broadcasts in Graphs

by

Sarada Rachelle Anne Herke
Bachelor of Science, University of Victoria, 2007

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

© Sarada Rachelle Anne Herke, 2009
University of Victoria
All rights reserved. This thesis may not be reproduced in whole or in part, by photocopying or other means, without the permission of the author.
Dominating Broadcasts in Graphs

by

Sarada Rachelle Anne Herke
Bachelor of Science, University of Victoria, 2007

Supervisory Committee

Dr. Kieka Mynhardt, Supervisor
(Department of Mathematics and Statistics)

Dr. Gary MacGillivray, Co-Supervisor or Departmental Member
(Department of Mathematics and Statistics)

Dr. Ernie Cockayne, Departmental Member
(Department of Mathematics and Statistics)
Abstract

A broadcast is a function $f : V \rightarrow \{0, ..., \text{diam } G\}$ that assigns an integer value to each vertex such that, for each $v \in V$, $f(v) \leq e(v)$, the eccentricity of $v$. The broadcast number of a graph is the minimum value of $\sum_{v \in V} f(v)$ among all broadcasts $f$ for which each vertex of the graph is within distance $f(v)$ from some vertex $v$ having $f(v) \geq 1$. This number is bounded above by the radius of the graph, as well as by its domination number. Graphs for which the broadcast number is equal to the radius are called radial. We prove a new upper bound on the broadcast number of a graph and motivate the study of radial trees by proving a relationship between the broadcast number of a graph and those of its spanning subtrees. We describe some classes of radial trees and then provide a characterization of radial trees, as well as a geometric interpretation of our characterization.
Contents

Supervisory Committee ii

Abstract iii

Contents iv

List of Figures vi

1 Introduction 1

2 Background Results 6
  2.1 Basic Facts ................................................. 6
  2.2 Background on Radial Graphs ................................. 10
  2.3 Algorithms and Complexity .................................... 12

3 Broadcast Number of Graphs vs. Trees 15

4 A New Upper Bound 22
  4.1 The Upper Bound ............................................. 23
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2</td>
<td>Equality in the Upper Bound</td>
<td>24</td>
</tr>
<tr>
<td>4.3</td>
<td>A Characterization of Radial Caterpillars</td>
<td>29</td>
</tr>
<tr>
<td>5</td>
<td>Long Paths Added at Vertices of $P_n$</td>
<td>34</td>
</tr>
<tr>
<td>5.1</td>
<td>The Central Case</td>
<td>35</td>
</tr>
<tr>
<td>5.2</td>
<td>The Bicentral Case</td>
<td>42</td>
</tr>
<tr>
<td>5.3</td>
<td>Corollaries of Theorems 5.1 and 5.2</td>
<td>46</td>
</tr>
<tr>
<td>6</td>
<td>Characterization</td>
<td>50</td>
</tr>
<tr>
<td>6.1</td>
<td>Motivation for the Characterization</td>
<td>51</td>
</tr>
<tr>
<td>6.2</td>
<td>Very Efficient Broadcasts</td>
<td>53</td>
</tr>
<tr>
<td>6.3</td>
<td>Proof of Characterization</td>
<td>70</td>
</tr>
<tr>
<td>6.4</td>
<td>A Geometrical Interpretation of the Characterization</td>
<td>75</td>
</tr>
<tr>
<td>6.5</td>
<td>Applications of Theorem 6.5</td>
<td>80</td>
</tr>
<tr>
<td>6.5.1</td>
<td>Generalized Coronas</td>
<td>80</td>
</tr>
<tr>
<td>6.5.2</td>
<td>Graphs with Radial Subtrees</td>
<td>82</td>
</tr>
<tr>
<td>6.5.3</td>
<td>Determining $\gamma_b(T)$</td>
<td>83</td>
</tr>
<tr>
<td>6.5.4</td>
<td>An Interpolation Result</td>
<td>84</td>
</tr>
<tr>
<td>7</td>
<td>Future Research</td>
<td>86</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>88</td>
</tr>
</tbody>
</table>
List of Figures

2.1 A $\gamma_b$-broadcast for $P_9$ ........................................ 9
2.2 A $\gamma_b$-broadcast for $P_6$ ........................................ 9
2.3 Radial trees not satisfying Proposition 2.10 ......................... 11
2.4 Star $K_{1,3}$ and the 2-subdivided graph $S_{2,3}$ of $K_{1,3}$ .......... 11
2.5 The ball graph of a broadcast $f$ on a tree $T$ ...................... 13

3.1 A radial spanning subtree $T$ with rad($T$) = rad($G$) = 4 ........ 16
3.2 Graph $G$ of Example 3.2 .............................................. 19
3.3 A spanning subgraph $H$ obtained from $G$ in Example 3.2 ... 19
3.4 Counterexample to Question 3.4 .................................... 21
3.5 Counterexample to Question 3.6 .................................... 21

4.1 The caterpillars $F_8$ and $F_9$ ........................................ 25

5.1 Labeling of vertices for $P_{2k+1}$ and $P_{2k}$ ....................... 35
5.2 Broadcast $f$ for Theorem 5.1 (i) .................................... 36
5.3 Broadcast $f$ for Theorem 5.1 (ii) a .................................. 38
LIST OF FIGURES

5.4 Broadcast \( f \) for Theorem 5.1 (ii) \( b \) ........................................ 41
5.5 Broadcast \( f \) for Theorem 5.1 (iii) ........................................ 41
5.6 Broadcast \( f \) for Theorem 5.2 Part (i) ...................................... 43
5.7 Broadcast \( f \) for Theorem 5.2 Part (ii) ...................................... 45
5.8 \( T \) and \( T + uv \) ................................................................. 49

6.1 Nonradial central trees ......................................................... 51
6.2 Nonradial bicentral trees ...................................................... 52
6.3 A tree with split-sets \( M = \{u,v\} \) and \( M' = \{x,y\} \) .............. 52
6.4 Subcase 1.1 of Theorem 6.1 ................................................... 56
6.5 Subcase 1.2 of Theorem 6.1 ................................................... 58
6.6 Case 2 of Theorem 6.1 ......................................................... 65
6.7 Case 2 of Theorem 6.1 redrawn .............................................. 66
6.8 The cycle on 9 vertices ......................................................... 69
6.9 A tree with two very efficient \( \gamma_b \)-broadcasts ....................... 70
6.10 The complete binary tree on three levels ................................. 74
6.11 A shadow tree of the tree in Figure 6.9 ................................. 75
6.12 Tree with many diametrical paths ......................................... 76
6.13 Vertices of nonradial trees covered by isosceles right triangles ... 79
6.14 Vertices of radial trees covered by isosceles right triangles .......... 80
6.15 Counterexample to the converse of Corollary 6.12 ...................... 82
Chapter 1

Introduction, Definitions and Notation

Suppose that a radio station wishes to broadcast at several locations so that its station may be heard by a certain region of the country. This situation can be modeled by a graph $G$ whose vertices denote the sections of the region in which to broadcast, where an edge between two vertices indicates that these two areas are close to each other. If a broadcast tower is built at any of these locations, then the nearby neighbourhoods can hear the broadcast (vertices at distance 1). The goal for the company is to broadcast to the entire region using the fewest number of broadcasting towers. This goal is achieved by finding the minimum cardinality of a dominating set $S$, which is a set such that every vertex of the graph is either in $S$, or adjacent to a vertex in $S$. Finding such a set is a typical domination problem, a subject that has
been studied extensively in recent years. For an overview of domination, see [10]. Variations on domination include distance \( k \)-domination, in which vertices within distance \( k \) of a vertex in \( S \) are dominated by \( S \). Distance domination is discussed in [12, 13, 14], for example. Broadcasting in graphs is another variation of domination where vertices in \( S \) dominate vertices within varying distances. Now we allow the radio station the option of building more powerful broadcast stations, but at an additional cost.

Any undefined terms and notations can be found in [3]. Let \( G \) be a graph. We assume throughout that \( G \) is nontrivial and connected. Let \( \Delta(G) \) and \( \delta(G) \) denote the maximum and minimum degree of the vertices of \( G \), respectively. The eccentricity of a vertex \( v \), denoted \( e(v) \), is the greatest distance between \( v \) and another vertex of \( G \). We use \( N(v) \) and \( N[v] \) to denote the open neighbourhood and the closed neighbourhood of a vertex \( v \), respectively. For \( a, b \in \mathbb{Z}^+ \) we use \([a, b]\) to denote the integer interval \( \{a, a+1, \ldots, b\} \) if \( a \leq b \), or the empty set if \( a > b \).

A broadcast on a connected graph \( G \) is a function \( f : V(G) \to [0, \text{diam}(G)] \) such that for every vertex \( v \in V(G) \), \( f(v) \leq e(v) \). Given a broadcast \( f \), an \( f \)-dominating vertex or broadcast vertex is a vertex \( v \) for which \( f(v) > 0 \). The set of all \( f \)-dominating vertices is called the \( f \)-dominating set and is denoted \( V_f^+(G) \), or \( V_f^+ \) when the graph under consideration is clear. An \( f \)-dominating vertex \( v \) \( f \)-dominates (or broadcasts to) every vertex \( u \) such that \( d(u, v) \leq f(v) \). For a given \( v \in V_f^+ \), we define the open \( f \)-neighborhood of \( v \) as \( N_f(v) = \{u \in V(G) - \{v\} : u \text{ is } f \text{-dominated by } v\} \). The closed
CHAPTER 1. INTRODUCTION

The \( f \)-neighborhood of \( v \) is \( N_f[v] = N_f(v) \cup \{v\} \). A vertex \( u \) is overdominated if \( f(v) - d(u, v) > 0 \) for some \( v \in V_f^+ \).

A broadcast \( f \) is a dominating broadcast if every vertex in \( V(G) - V_f^+ \) is \( f \)-dominated by some vertex in \( V_f^+ \). The cost of a broadcast \( f \) is defined as \( \sum_{v \in V_f^+} f(v) \) and is denoted \( \sigma(f) \). The broadcast number of a given graph \( G \) is thus defined as

\[
\gamma_b(G) = \min \{\sigma(f) : f \text{ is a dominating broadcast of } G\}.
\]

A broadcast \( f \) on \( G \) for which \( \sigma(f) = \gamma_b(G) \) is called a minimum dominating broadcast, or a \( \gamma_b \)-broadcast.

The topic of broadcasting in graphs was first considered in a thesis by D.J. Erwin [8] in 2001, using the term cost domination. In his thesis, Erwin established some sharp upper and lower bounds on the broadcast number of a graph and characterized those graphs with broadcast number at most 3. He also discussed several other types of broadcasts, such as minimal broadcasts and independent broadcasts. Erwin’s results can also be found in [9]. The following is a basic upper bound first noted by Erwin [8].

**Proposition 1.1** [8] For every nontrivial connected graph \( G \),

\[
\left\lceil \frac{\text{diam}(G) + 1}{3} \right\rceil \leq \gamma_b(G) \leq \min \{\text{rad}(G), \gamma(G)\}.
\]

We call graphs for which \( \gamma_b(G) = \text{rad}(G) \) Type 1 graphs or radial graphs. Graphs for which \( \gamma_b(G) = \gamma(G) \) are called Type 2, and graphs for which
\(\gamma_b(G) < \min\{\text{rad}(G), \gamma(G)\}\) are called Type 3. It was proved in [8] that there are infinitely many graphs of Type 3:

**Proposition 1.2** [8] *For every* \(t \in \mathbb{Z}^+\), *there exists a connected graph* \(G\) *for which*

\[
\min\{\text{rad}(G), \gamma(G)\} - \gamma_b(G) \geq t.
\]

In 2003 Dunbar, Hedetniemi and Hedetniemi [7] considered the problem of characterizing Type 1 and Type 2 trees and they achieved some partial results to this end. In 2005 Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [6] provided bounds on the minimum and maximum costs of broadcasts in graphs as well as for other types of broadcasts, and listed the characterization of Type 1 and Type 2 graphs as unsolved. In 2008 Seager [16] characterized caterpillars of Types 1, 2, and 3 respectively.

The focus of this thesis is to provide a characterization of radial trees and it is outlined as follows. In Chapter 2 we discuss relevant background material as well as the algorithmic complexity of the problem. We motivate the study of radial trees by providing a relationship between the broadcast number of a graph and those of its spanning subtrees in Chapter 3. Then in Chapter 4 we prove a new upper bound on the broadcast number of a graph, which leads to a characterization of radial caterpillars. We next provide some results about classes of radial trees with several long paths in Chapter 5. In Chapter 6 we motivate and prove a characterization of radial trees and discuss a geometrical interpretation of our characterization. We conclude Chapter
6 with an application of our characterization to general corona graphs. In Chapter 7 we list some open problems for further research.
Chapter 2

Background Results

In this chapter we begin with some basic background facts about broadcasts in Section 2.1. In Section 2.2 we discuss the work by Dunbar et al. [7] that begins to classify types of radial graphs. Then in Section 2.3 we provide a history of the study of the complexity of the broadcast problem.

2.1 Basic Facts

We begin with an important definition. An efficient broadcast $f$ is a broadcast such that each vertex is $f$-dominated by exactly one vertex of $V_j^+$. 

**Proposition 2.1** [6] Every graph $G$ has a $\gamma_b$-broadcast that is efficient.

The above proposition is interesting because the same result is not true for domination; not every graph has an efficient dominating set. For example, the tree obtained by joining a new leaf to a central vertex of $P_4$ has no
efficient dominating set. The next result is frequently used to show that a
given broadcast is not efficient.

**Proposition 2.2** Suppose $f$ is a broadcast on a connected graph $G$. If, for
some $v \in V_f^+(G)$, $G - N_f[v]$ contains an isolated vertex, then $f$ is not efficient.

**Proof.** Let $f$ be a broadcast on $G$ and let $v \in V_f^+$ such that $G - N_f[z]$ has an isolated vertex, $w$. In order to $f$-dominate $w$, either $f(w) \geq 1$ or there is a vertex $q$ at distance $\ell$ from $w$ such that $f(q) \geq \ell$. Let $w'$ be any neighbour of $w$ in $T$. Then in either case, $w'$ is $f$-dominated by more than one vertex of $V_f^+$; therefore $f$ is not an efficient broadcast.

\[\square\]

The next two results concern the broadcast number of a tree and (certain

types of) its subtrees, and mirror the corresponding results for the domination

number of trees.

**Proposition 2.3** [7] If $T$ is a tree with subtree $T'$, then $\gamma_b(T') \leq \gamma_b(T)$.

The following two definitions are required for the next result. A *leaf* is a
vertex of a tree with degree 1 and a *support vertex* is a non-leaf vertex that
is adjacent to a leaf.

**Proposition 2.4** Let $T$ be any tree. If a tree $T'$ is obtained from $T$ by joining

a new leaf to a support vertex of $T$, then $\gamma_b(T') = \gamma_b(T)$.
CHAPTER 2. BACKGROUND RESULTS

Proof. By Proposition 2.3, \( \gamma_b(T) \leq \gamma_b(T') \). Amongst all \( \gamma_b \)-broadcasts on \( T \), let \( f \) be one such that \( V^+_f \) has the minimum number of leaves. Suppose that \( v \) is a leaf of \( T \) and that \( v \in V^+_f \). Let \( u \) be the support vertex of \( v \), and define the following broadcast on \( T \):

\[
g(x) = \begin{cases} 
0 & \text{if } x = v \\
f(u) + f(v) & \text{if } x = u \\
f(x) & \text{otherwise.}
\end{cases}
\]

Then \( g \) is a \( \gamma_b \)-broadcast with fewer leaves as broadcast vertices than \( f \), which is a contradiction. Thus \( V^+_f \) contains no leaf of \( T \), so \( V^+_f \) contains all support vertices. Therefore \( f \) broadcasts to all vertices of \( T' \) as well, and \( \gamma_b(T') \leq \gamma_b(T) \).

\[\square\]

It is not surprising that the broadcast number of a path is equal to its domination number.

**Proposition 2.5** [9] For every integer \( n \geq 2 \),

\[ \gamma_b(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil. \]  \hfill (2.1)

It is easy to see that all trees with radius at most 2 are radial. From the results stated thus far, we obtain the following lemma.
Lemma 2.6 If $T$ is a central tree with radius 3, then $T$ is radial.

**Proof.** Since $T$ is central with radius 3, $P_7$ is a subtree of $T$. It is clear that $\gamma_b(P_7) = 3$, so by Lemma 2.3, $3 \leq \gamma_b(T)$. However, by Proposition 1.1, $\gamma_b(T) \leq \min\{\text{rad}(T), \gamma(T)\}$. We are given that rad$(T) = 3$, and it is clear that $\gamma(T) \geq 3$ since at least three vertices are needed to dominate the subtree $P_7$. Thus, $\min\{\text{rad}(T), \gamma(T)\} = 3$, and so $\gamma_b(T) \leq 3$. Therefore $\gamma_b(T) = 3 = \text{rad}(T)$ and $T$ is radial.

\[\square\]

We note that Lemma 2.6 does not hold for trees of radius 4. For example, $P_9$ has radius 4 but $\gamma_b(P_9) = \gamma(P_9) = 3$ (see Figure 2.1). Also, the path $P_6$ shows that bicentral trees of radius 3 are not necessarily radial (see Figure 2.2).

![Figure 2.1: A $\gamma_b$-broadcast for $P_9$](image1)

![Figure 2.2: A $\gamma_b$-broadcast for $P_6$](image2)
2.2 Background on Radial Graphs

It is mentioned in [7] that of all nontrivial trees of order at most 9, only 11 are nonradial. This fact leads us to believe that there are more radial trees than nonradial ones of fixed order. In this section we state some previously discovered results about radial graphs.

The corona of two graphs $G_1$ and $G_2$, denoted $G_1 \circ G_2$, is the graph obtained from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where the $i$th vertex of $G_1$ is adjacent to every vertex in the $i$th copy of $G_2$. The generalized corona of a connected graph $G$ with $V(G) = \{v_1, ..., v_n\}$ and $n$ arbitrary graphs $G_1, ..., G_n$ is the graph $H = G \circ (G_1, ..., G_n)$; that is, $v_i$ is adjacent to each vertex of $G_i$.

**Proposition 2.7** [7] For any connected graph $G_1$ and any graph $G_2$, the graph $G = G_1 \circ G_2$ is radial.

The proof of Proposition 2.7 also gives the following result.

**Corollary 2.8** [7] For any connected graph $G$ of order $n$ and any $n$ graphs $G_1, ..., G_n$, the generalized corona $G \circ (G_1, ..., G_n)$ is radial.

**Proposition 2.9** [7] If $T_1$ and $T_2$ are two radial trees, then the tree formed by adding an edge between a central vertex of $T_1$ and a central vertex of $T_2$, is radial.

**Proposition 2.10** [7] If $T$ is a tree containing three vertices $u, v, w$ satisfying $d(u, v) = \text{diam}(T)$ and $d(u, w), d(v, w) \geq \text{diam}(T) - 1$, then $T$ is radial.
However, Proposition 2.10 does not account for all radial trees. For example, Figure 2.3 shows three radial trees with radius 5 that do not satisfy the conditions of Proposition 2.10.

For a graph $G$ and a positive integer $k$, we define the $k$-subdivided graph of $G$, denoted $S_k(G)$, as the graph obtained from $G$ by inserting $k$ vertices into every edge of $G$. For positive integers $k$ and $t$, we let $S_{k,t} = S_k(K_{1,t})$, where $K_{1,t}$ is the star consisting of a central vertex adjacent to $t$ leaves. For example, see Figure 2.4.

For a graph $G$ and a positive integer $k$, we define the $k$-subdivided graph of $G$, denoted $S_k(G)$, as the graph obtained from $G$ by inserting $k$ vertices into every edge of $G$. For positive integers $k$ and $t$, we let $S_{k,t} = S_k(K_{1,t})$, where $K_{1,t}$ is the star consisting of a central vertex adjacent to $t$ leaves. For example, see Figure 2.4.
In 2001, Erwin [8] proved that $S_{k,t}$ is radial for $k \geq 0$ and $t \geq 5$, and he conjectured this property for $k \geq 0$ and $t \in \{3, 4\}$. In 2009, Bouchemakh and Sahbi [2] proved the following proposition, thus proving Erwin’s conjecture.

**Proposition 2.11** For every integer $k \geq 0$ and $t \geq 3$,

$$\gamma_b(S_{k,t}) = \text{rad}(S_{k,t}) = k + 1.$$  

However, we note that Proposition 2.11 follows immediately from Proposition 2.10 by taking $u, v$ and $w$ to be any three of the $t$ leaves of $S_{k,t}$.

### 2.3 Algorithms and Complexity

There are many varieties of domination and many of these problems are NP-hard. Thus, when the topic of broadcast domination on graphs was introduced, it was generally believed that the computational complexity of finding $\gamma_b(G)$ for a general graph $G$ would also be in the class NP. However, this is not the case. The complexity of computing $\gamma_b$ was studied by Horton, Meneses, Mukhegjee and Ulucakli [15] and by Blair, Heggernes, Horton and Maine [1]. These two groups of authors found some polynomial time algorithms for specific types of graphs. Then, in 2006, Heggernes and Lokshtanov [11] showed that minimum broadcast domination is solvable in polynomial time for any graph. Their algorithm runs in $O(n^6)$ time for a graph with $n$ vertices. The algorithm depends largely on two properties of minimum dominating broadcasts. The first of these is Proposition 2.1 by Dunbar et
al. [6] that every graph has an efficient minimum dominating broadcast. In order to state the second of these results, a definition is required.

In [11], the closed \( f \)-neighbourhood \( N_f[v] \) is also called the ball with centre \( v \), where \( v \in V_f^+ \). For an efficient dominating broadcast \( f \) on a graph \( G \), the ball graph \( B(f) \) of \( G \) is the graph obtained by contracting the vertices in every ball \( N_f[v] \) for \( f(v) > 0 \) down to a single vertex.

**Lemma 2.12** [11] For any graph \( G \), there is an efficient minimum dominating broadcast \( f \) on \( G \) such that the ball graph \( B(f) \) has maximum degree 2.

**Corollary 2.13** [4] Every tree \( T \) has an efficient minimum dominating broadcast \( f \) on \( T \) such that the ball graph \( B(f) \) is a path.

Note that Corollary 2.13 does not imply that the broadcast vertices all lie on the same path. However, we will prove this stronger result in Chapter 6 as a crucial result for our eventual characterization of radial trees.

For example, Figure 2.5 shows an efficient broadcast \( f \) on a tree \( T \), and the corresponding ball graph \( B(f) \).

![Figure 2.5: The ball graph of a broadcast \( f \) on a tree \( T \)](image)
In 2007, J.R. Dabney [4] showed that for trees, $\gamma_b$ can be found by an algorithm that runs in $O(n)$ time. To do this, Dabney required non-standard methods to make decisions based on non-local information. He also made use of the structure described by Corollary 2.13. This result is presented by Dabney, Dean and Hedetnimi in [5]. However, even with this algorithm, he was unable to determine a characterization of radial graphs.
Chapter 3

Broadcast Number of Graphs vs. Trees

In this chapter we motivate the study of broadcasts in trees by exploring the relationship between the broadcast number of a graph and those of its spanning subtrees. Let $G$ be a connected graph and $T$ a spanning tree of $G$ with $\text{rad}(T) = \text{rad}(G)$. It is clear that if $G$ is radial then $T$ is radial. However, if $T$ is radial then $G$ is not necessarily radial. For example, $T$ in Figure 3.1 has $\text{rad}(T) = 4$ and is a spanning subtree of $G$ with $\text{rad}(T) = \text{rad}(G)$. But $G$ is not radial, as illustrated by the dominating broadcast given in Figure 3.1 with cost 3.

We note that $G$ does have a spanning subtree with the same broadcast number. Consider the spanning subtree $T'$ of the graph $G$ from Figure 3.1 obtained by deleting edge $e$ of $G$. This spanning tree has the same radius as
Figure 3.1: A radial spanning subtree $T$ with $\text{rad}(T) = \text{rad}(G) = 4$

$G$ but, like $G$, is not radial because the same broadcast used on $G$ with cost 3 dominates $T'$.

If $G$ is any connected graph and $T$ a spanning tree of $G$ with $\text{diam}(T) = \text{diam}(G)$, and if $T$ is radial, one might wonder if it follows that $G$ is radial. This is not the case, as illustrated by the same graphs in Figure 3.1. Here $\text{diam}(G) = \text{diam}(T) = 8$ and while $T$ is radial, $G$ is not.

The following theorem describes the relationship between the broadcast number of a graph and the broadcast numbers of its spanning subtrees. It is because of this relationship that the study of radial trees is vital to the characterization of radial graphs. For a connected graph $G$, we use $S(G)$ to denote the set of spanning subtrees of $G$. 


CHAPTER 3. BROADCAST NUMBER OF GRAPHS VS. TREES

Theorem 3.1 Suppose $G$ is a connected graph. Then

$$\gamma_b(G) = \min_{T \in S(G)} \{\gamma_b(T)\}.$$ 

Proof. Let $\gamma_b(G) = k$ and $\min_{T \in S(G)} \{\gamma_b(T)\} = t$. A minimum broadcast on a spanning subtree $T$ also dominates the graph $G$, since $G$ is obtained from $T$ by adding more edges. Thus it is clear that $k \leq t$. We wish to show that $k \geq t$. For the purpose of deriving a contradiction, suppose $k < t$. By Proposition 2.1, $G$ has an efficient $\gamma_b$-broadcast; call it $f$. Now consider the vertices of $G$ partitioned into $V_f^+(G)$ and $V(G) - V_f^+(G)$. Since $f$ is efficient, there are no edges between vertices of $V_f^+$. For a given vertex $u \in V_f^+$, with $f(u) = r$, we define

$$L_i(u) = \{v \in N_f(u) : d(u, v) = i\},$$

for $i = 1, ..., r$. We obtain a spanning subgraph $H$ of $G$ in the following way:

- For every $u \in V_f^+(G)$ with $f(u) = r$:
  - If there is an edge $vv' \in E(G)$ where $v, v' \in L_s(u)$ for some $s \leq r$, then delete this edge.
  - If for some $v \in L_s(u)$ there are $\ell$ edges $vv_1, vv_2, ..., vv_\ell \in E(G)$ where $v_i \in L_{s'}(u)$ for some $s' \leq s$, then delete any $\ell - 1$ of these edges, so that only one edge remains.

- For every $u, u' \in V_f^+(G)$ with $f(u) = r, f(u') = r'$:
- If there are $\ell$ edges of the form $vv' \in E(G)$ where $v \in L_r(u), v' \in L_{r'}(u')$, then delete any $\ell - 1$ of these edges, so that only one edge remains.

Note that there are no edges between vertices $v \in L_s(u)$ and $v' \in L_{s'}(u')$ where $s < r, s' < r'$ since the existence of such an edge would contradict the fact that $f$ is efficient.

It is clear that the graph $H$ is a tree. Hence $H \in S(G)$, so $\gamma_b(H) \geq t > k$.

But by the way in which $H$ was constructed, $f$ dominates $H$, so $\gamma_b(H) \leq k$.

This is a contradiction. So we have shown that $k \geq t$. Therefore $k = t$.

$\square$

Example 3.2 To illustrate the technique used in the proof of Theorem 3.1, consider the example of a graph $G$ in Figure 3.2 and one possible resulting spanning subtree $H$ in Figure 3.3.

Let $R(G)$ be the set of all spanning trees of a connected graph $G$ such that $\text{rad}(T) = \text{rad}(G)$. It is of interest to notice that there is always such a spanning subtree.

Proposition 3.3 Every connected graph $G$ has a spanning tree $T$ with $\text{rad}(T) = \text{rad}(G)$.

Proof. Let $x$ be any central vertex of $G$. For each $v \in V$, let $P_v$ be a shortest $x - v$ path and define $G' = \langle \bigcup_{v \in V} E(P_v) \rangle$. Then $G'$ is a connected spanning subgraph of $G$ with $\text{rad}(G') = \text{rad}(G)$. If $G'$ is acyclic, we are done,
so assume $C$ is a cycle of $G'$. Then $C$ is a subgraph of $P_u \cup P_v$ for some $u, v \in V$. Let $w_1, w_2$ be the two vertices of $C$ common to both $P_u$ and $P_v$; say $d(x, w_1) < d(x, w_2)$. Since $P_u$ and $P_v$ are shortest $x-u$ and $x-v$ paths, the two $w_1 - w_2$ paths on $C$ have the same length. Let $e$ be any of the two
edges of $C$ incident with $w_2$ and define $G'' = G' - e$. Then $G''$ is connected and $e_{G''}(x) = e_{G'}(x)$, so that $\text{rad}(G'') = \text{rad}(G')$. Repeating this process until no cycles remain yields the desired tree $T$.

\[\square\]

We might then wonder if Theorem 3.1 can be strengthened by answering the following question in the affirmative.

**Question 3.4** Is it true that for any connected graph $G$

\[
\gamma_b(G) = \min_{T \in \mathcal{R}(G)} \{\gamma_b(T)\}?
\]

The answer to Question 3.4 is no. Consider the graph $G$ given in Figure 3.4 with $\gamma_b(G) = 9$ and $\text{rad}(G) = 10$. By using the method of the proof of Theorem 3.1, we see that the only possible tree $T \in \mathcal{S}(G)$ for which the same (or any other) minimum broadcast of $G$ will work is $T = G - e_1 - e_2$. However, $\text{rad}(T) = 11 \neq \text{rad}(G)$.

The situation for radial graphs is different, though.

**Lemma 3.5** If $G$ is radial, then there exists $T \in \mathcal{R}(G)$ with $\gamma_b(T) = \gamma_b(G)$.

**Proof.** Let $T \in \mathcal{R}(G)$. Then $\gamma_b(T) \leq \text{rad}(G) = \gamma_b(G)$. But $\gamma_b(G) = \min_{T \in \mathcal{S}(G)} \{\gamma_b(T)\}$, so $\gamma_b(T) \geq \text{rad}(G)$. Therefore $\gamma_b(T) = \gamma_b(G)$.

\[\square\]

Another natural question concerns the monotonicity of the broadcast number of trees with the same order and different radii.
CHAPTER 3. BROADCAST NUMBER OF GRAPHS VS. TREES

Figure 3.4: Counterexample to Question 3.4

**Question 3.6** Is it true that if $T_1$ and $T_2$ are trees of order $n$ such that \( \text{rad}(T_1) \leq \text{rad}(T_2) \), then $\gamma_b(T_1) \leq \gamma_b(T_2)$?

The answer to Question 3.6 is also no. Consider the example given in Figure 3.5 in which two trees $T_1$ and $T_2$ are each of order 15, and $5 < 6 = \text{rad}(T_2)$, but $\gamma_b(T_1) = 5 > 4 = \gamma_b(T_2)$.

Figure 3.5: Counterexample to Question 3.6
Chapter 4

A New Upper Bound on the Broadcast Number

In Section 4.1 we show that the broadcast number of a tree of order $n$ is bounded above by $\lceil \frac{n}{3} \rceil$, the exact value of $\gamma_b(P_n)$ (see Proposition 2.5). The same bound for general graphs then follows from Theorem 3.1. In Section 4.2 we consider other classes of trees for which the bound is exact. Our results here lead us to a characterization of radial caterpillars, which we prove in Section 4.3.
4.1 The Upper Bound

**Theorem 4.1** For any tree $T$ of order $n$, $\gamma_b(T) \leq \left\lceil \frac{n}{3} \right\rceil$.

**Proof.** The result is obviously true for $n \leq 3$. Suppose the result is not true in general and let $T$ be a counterexample of minimum order $n \geq 4$. We first show that $T$ has no adjacent vertices of degree two.

Suppose $u_1$ and $u_2$ are adjacent vertices of degree two. Let $v_i$ be the other neighbour of $u_i$, $i = 1, 2$, so that $v_1, u_1, u_2, v_2$ is a path in $T$. Let $T_1$ be the component of $T - u_1u_2$ containing $u_i$. If $|V(T_1)| \equiv 0 \pmod{3}$, let $T' = T_1$ and $T'' = T_2$. If $|V(T_1)| \equiv 1 \pmod{3}$, let $T' = T_1 - u_1$ and $T'' = T - T'$. If $|V(T_1)| \equiv 2 \pmod{3}$, let $T'' = T_2 - u_2$ and $T' = T - T''$. In each case $T'$ and $T''$ are trees where $|V(T')| \equiv 0 \pmod{3}$; say $|V(T')| = 3t$. By the minimality of $T$, $\gamma_b(T') \leq t$ and $\gamma_b(T'') \leq \left\lceil \frac{n-3t}{3} \right\rceil$, so that

$$\gamma_b(T) \leq \left\lceil \frac{n-3t}{3} \right\rceil + t \leq \left\lceil \frac{n}{3} \right\rceil,$$

a contradiction.

Now assume $T$ has radius $k$ and let $P$ be a diametrical path. Then $\gamma_b(T) \leq k$ and, since $P$ is a subtree of $T$ with the same radius, $\gamma_b(P) \leq k$.

If $T$ is central, let $P = v_1, \ldots, v_{2k+1}$. Since $T$ does not have adjacent vertices of degree two, an application of the pigeonhole principle shows that at least $\left\lceil \frac{2k-1}{2} \right\rceil = k - 1$ of the vertices $v_i$, $i = 2, \ldots, 2k$, are adjacent to vertices not on $P$. Hence $n \geq 2k + 1 + k - 1 = 3k$. Therefore $\gamma_b(T) \leq k \leq \left\lceil \frac{n}{3} \right\rceil$, a
contradiction.

If \( T \) is bicentral, let \( P = v_1, \ldots, v_{2k} \). As above, at least \( \frac{2k-2}{2} = k - 1 \) of the vertices \( v_i, i = 2, \ldots, 2k - 1 \), are adjacent to vertices not on \( P \). So \( n \geq 2k + k - 1 = 3k - 1 \), i.e. \( \gamma_b(T) \leq k \leq \lceil \frac{n}{3} \rceil \). This final contradiction proves the theorem.

By Theorem 3.1, we obtain the following corollary.

**Corollary 4.2** For any connected graph \( G \) of order \( n \), \( \gamma_b(G) \leq \lceil \frac{n}{3} \rceil \).

**Corollary 4.3** If \( T \) is a radial tree of radius \( k \), then \( T \) has at least \( 3k - 2 \) vertices.

**Proof.** If \( T \) has at most \( 3k - 3 \) vertices, then by Theorem 4.1, \( \gamma_b(T) \leq k - 1 \) and so \( T \) is not radial.

\[ \square \]

### 4.2 Equality in the Upper Bound

We consider a class of radial trees that satisfy equality in the bound given in Theorem 4.1. We define a *caterpillar* to be a tree \( T \) consisting of a path with singleton vertices adjacent to any subset of the non-leaf vertices of the path. The path associated with a given caterpillar \( T \) is called its *spine*. Note that our definition differs from the conventional definition of a caterpillar in that
we require the spine to have the same diameter as the caterpillar. We define two specific classes of caterpillars. Seager [16] characterized caterpillars of Types 1, 2, and 3 respectively. Here we use a different approach which eventually leads to a characterization of radial trees.

Consider $P_{2k+1}$ with a labeling of the vertices $v_1, \ldots, v_{2k+1}$. Add a single leaf vertex to each of $v_3, v_5, v_7, \ldots, v_{2k-1}$ of this path. Thus $\lceil \frac{2k+1-4}{2} \rceil$ vertices are added to $P_{2k+1}$, resulting in a tree with $2k+1 + \lceil \frac{2k-3}{2} \rceil = 2k+1+k-1 = 3k$ vertices. We call the resulting caterpillar $F_{3k}$, with spine $P_{2k+1}$. We define $F_{3k-1}$ similarly, with spine $P_{2k}$ and singletons adjacent to $v_3, v_5, v_7, \ldots, v_{2k-1}$, so that the total number of vertices is $2k + \lceil \frac{2k-2}{2} \rceil = 3k - 1$. For example, $F_8$ and $F_9$ are given in Figure 4.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{caterpillar.png}
\caption{The caterpillars $F_8$ and $F_9$}
\end{figure}

\textbf{Theorem 4.4} For any $k \in \mathbb{Z}^+$, $\gamma_b(F_{3k}) = \gamma_b(F_{3k-1}) = k$.

\textbf{Proof.} Consider $T = F_{3k}$ and let $f$ be an efficient $\gamma_b$-broadcast of $T$. Let $v_1, \ldots, v_{2k+1}$ be a labeling of the spine of $T$ and let $u_i$ be the leaf adjacent to $v_i$, ...
i = 3, 5, ..., 2k − 1. Note that Cen(T) = {v_{k+1}}. We prove that V^+_f = {v_{k+1}}; the result will follow immediately.

We first prove that

\[
\text{if } v_i \in V^+_f \text{ and } f(v_i) = m, \text{ where } i - m > 1 \text{ or } i + m < 2k + 1, \\text{then } i \equiv m \pmod{2}. \tag{4.1}
\]

Suppose \( i \not\equiv m \pmod{2} \). Then \( i + m \) and \( i - m \) are odd. Assume \( i + m < 2k + 1 \); the proof in the case \( 1 < i - m \) is the same. Then \( v_i \) does not broadcast to \( v_{2k+1} \). Since \( i + m \) is odd, \( v_{i+m} \) is adjacent to \( u_{i+m} \). Moreover, \( v_i \) broadcasts to \( v_{i+m} \) but not to \( u_{i+m} \). But now \( u_{i+m} \) is an isolated vertex of \( T - N_f[v_i] \), so that Proposition 2.2 provides a contradiction of the efficiency of \( f \). Thus (4.1) holds.

We prove next that

\[
\text{if } u_i \in V^+_f, \text{ then } f(u_i) = 1. \tag{4.2}
\]

If \( f(u_i) \geq 2 \), define the broadcast \( g \) on \( T \) by

\[
g(x) = \begin{cases} 
0 & \text{if } x = u_i \\
f(u_i) - 1 & \text{if } x = v_i \\
f(x) & \text{otherwise.}
\end{cases}
\]

Clearly, \( g \) is a dominating broadcast of \( T \), which is impossible because \( \sigma(g) < \sigma(f) = \gamma_b(T) \). Therefore (4.2) holds.
Let \( v \) be the vertex that broadcasts to \( v_1 \). Then (4.2) implies that \( v = v_i \) for some \( i \); say \( f(v_i) = m \). If \( i = k + 1 \), then \( m = k \) and we are done, so assume \( i \leq k \). Now \( m \leq \sigma(f) \leq \text{rad} T = k \), so \( i + m < 2k + 1 \); hence \( v_{i+m+1} \in V(T) \) and \( v_i \) does not broadcast to \( v_{i+m+1} \). Moreover, Proposition 2.2 and the efficiency of \( f \) imply that \( i + m + 1 \neq 2k + 1 \). By (4.1), \( i + m \) is even, so \( i + m + 1 \) is odd and hence \( v_{i+m+1} \) is adjacent to \( u_{i+m+1} \). Let \( u \) be the vertex that broadcasts to \( v_{i+m+1} \); say \( f(u) = l \). If \( u = v_j \) for some \( j \), then, by the efficiency of \( f \), \( v_j \) does not broadcast to \( v_{i+m} \) and \( j - l = i + m + 1 \). But by (4.1), \( j - l \) is even, a contradiction. Hence \( u = u_{i+m+1} \) and by (4.2) \( l = 1 \).

Also note that since \( i + m \) is even, \( d(v_1, v_{i+m}) = i + m - 1 \) is odd, so \( v_i \) overdominates \( v_1 \).

Let \( w \) be the vertex that broadcasts to \( v_{i+m+2} \). Again (4.2) implies that \( w = v_r \) for some \( r \); say \( f(v_r) = p \). If \( v_r \) also broadcasts to \( v_{2k+1} \), then as in the case of \( v_1 \), \( v_{2k+1} \) is overdominated. If \( v_r \) does not broadcast to \( v_{2k+1} \), then \( r + p \) is even and less than \( 2k \) (by Proposition 2.2 and the efficiency of \( f \)), and as before, \( v_{r+p+1} \) is \( f \)-dominated by \( u_{r+p+1} \) where \( f(u_{r+p+1}) = 1 \).

By repeating the above arguments we eventually obtain a sequence

\[ v_i = v_{i_1}, u_{j_1}, v_{i_2}, u_{j_2}, \ldots, u_{j_{s-1}}, v_{i_s} \in V_f^+, \quad s \geq 2, \]

where

- \( j_\ell = i_\ell + f(v_{i_\ell}) + 1 \) and \( v_{i_{\ell+1}} \) broadcasts to \( v_{j_{\ell+1}} \).
• for $\ell \notin \{1, s\}$, $v_{i\ell}$ broadcasts to $2f(v_{i\ell}) + 1$ vertices (including itself) on the spine of $T$, and

• $v_{i_1}$ overdominates $v_1$; $v_{i_s}$ overdominates $v_{2k+1}$.

Now

$$\sigma(f) = \sum_{\ell=1}^{s} f(v_{i\ell}) + \sum_{\ell=1}^{s-1} f(u_{j\ell}) = \sum_{\ell=1}^{s} f(v_{i\ell}) + s - 1.$$ 

Also

$$2k + 1 \leq 2(f(v_{i_1}) + f(v_{i_s})) + \sum_{\ell=2}^{s-1} (2f(v_{i\ell}) + 1) + s - 1 = 2 \sum_{\ell=1}^{s} f(v_{i\ell}) + 2s - 3.$$ 

Therefore

$$\text{rad}(T) = k \leq \left\lfloor \frac{2 \sum_{\ell=1}^{s} f(v_{i\ell}) + 2(s - 1) - 1}{2} \right\rfloor = \sum_{\ell=1}^{s} f(v_{i\ell}) + s - 2 < \sigma(f),$$

which is a contradiction. Thus $f(v_i) = 0$ for $i \leq k$. The same arguments show that $f(v_i) = 0$ for $i \geq k + 2$. Suppose $f(v_{k+1}) < k$. Then, even if $\{u_3, u_5, \ldots, u_{2k-1}\} \subset V^+_f$, $f$ is not a dominating broadcast by (4.2), which is a contradiction. So $V^+_f = \{v_{k+1}\}$ where $f(v_{k+1}) = k$ and hence $T$ is radial.
A similar argument holds for \( T = F_{3k-1} \).

Proposition 2.3 now provides the following corollary.

**Corollary 4.5** If \( T \) is a tree of radius \( k \) that contains \( F_{3k} \) or \( F_{3k-1} \) as subgraph, then \( T \) is radial.

**Corollary 4.6** If a new vertex is joined to every non-leaf vertex of \( P_n \), then the resulting tree \( T \) is radial.

**Proof.** The result follows from Corollary 4.5.


**4.3 A Characterization of Radial Caterpillars**

The classes of radial caterpillars described by \( F_{3k} \) and \( F_{3k-1} \) are constructed such that no two vertices of degree two are adjacent. This property is significant for characterizing radial caterpillars. By considering adjacent vertices of degree two on the spine of a caterpillar, we are now able to determine exactly when a given caterpillar is radial.
Theorem 4.7 Let $T$ be any caterpillar with spine $P$ of order $n$.

1. If $n$ is even and there exists a vertex $v_i \in V(P)$ where $i \geq 3$ is odd such that $\text{deg}(v_i) = \text{deg}(v_{i+1}) = 2$; or

2. if $n$ is odd and there exist vertices $v_i, v_j \in V(P)$ where $i \geq 3$ is odd and $j \geq i + 3$ is even such that $\text{deg}(v_i) = \text{deg}(v_{i+1}) = \text{deg}(v_j) = \text{deg}(v_{j+1}) = 2$,

then $T$ is nonradial. Otherwise $T$ is radial.

Proof. Suppose (1) holds. Let $v_1, \ldots, v_{2k}$ be a labeling of the vertices of the spine $P$. Suppose that $i = 2\ell + 1$, so that $\text{deg}(v_{2\ell+1}) = \text{deg}(v_{2\ell+2}) = 2$. Let $e = v_{2\ell+1}v_{2\ell+2}$ and let $T_1$ and $T_2$ be the components of $T - e$ containing $v_{2\ell+1}$ and $v_{2\ell+2}$, respectively. Then $T_1$ has a spine of order $2\ell + 1$ and central vertex $v_{\ell+1}$; $T_2$ has a spine of order $2k - 2\ell - 1$ and central vertex $v_{\ell+k+1}$. Define the broadcast $f$ on $T$ by $f(v_{\ell+1}) = \ell, f(v_{\ell+k+1}) = k - \ell - 1$, and $f(v) = 0$ otherwise. Then $f$ is a dominating broadcast on $T$ and $\sigma(f) = k - 1$.

Therefore $T$ is nonradial.

Suppose (2) holds. Let $v_1, \ldots, v_{2k+1}$ be a labeling of the vertices of the spine $P$. Suppose that $i = 2\ell + 1$ and $j = 2m$, so that $\text{deg}(v_{2\ell+1}) = \text{deg}(v_{2\ell+2}) = \text{deg}(v_{2m}) = \text{deg}(v_{2m+1}) = 2$. Let $e_1 = v_{2\ell+1}v_{2\ell+2}, e_2 = v_{2m}v_{2m+1}$ and let $T_1$, $T_2$ and $T_3$ be the components of $T - e_1 - e_2$ containing $v_{2\ell+1}, v_{2\ell+2},$ and $v_{2m+2}$ respectively. Then $T_1$ has a spine of order $2\ell + 1$ and central vertex $v_{\ell+1}$; $T_2$ has a spine of order $2m - 2\ell - 1$ and central vertex $v_{\ell+m+1}$; $T_3$ has a spine of order $2k - 2m + 1$ and central vertex $v_{m+k+1}$. Define the broadcast
CHAPTER 4. A NEW UPPER BOUND

\( f \) on \( T \) by \( f(v_{i+1}) = \ell, f(v_{i+m+1}) = m - \ell - 1, f(v_{m+k+1}) = k - m \) and \( f(v) = 0 \) otherwise. Then \( f \) is a dominating broadcast on \( T \) and \( \sigma(f) = k - 1 \). Therefore \( T \) is nonradial.

Now suppose that caterpillar \( T \) with spine \( P \) of order \( n \) is nonradial. Let \( f \) be an efficient \( \gamma_b \)-broadcast with \( |V_f^+| \) minimized. Since \( T \) is nonradial, \( |V_f^+| \geq 2 \).

We wish to show that every \( v \in V_f^+ \) is on the spine \( P \). Suppose not. Then for some \( i \geq 2, v_i \in V(P) \) is adjacent to \( w_i \in V(T) - V(P) \) and \( w_i \in V_f^+ \). If \( f(w_i) > 1 \), then the broadcast defined by \( g(v_i) = f(w_i) - 1, g(w_i) = 0, \) and \( g(z) = f(z) \) otherwise is a dominating broadcast with lower cost than \( f \), which is a contradiction. Hence \( f(w_i) = 1 \). Also, since \( f \) is efficient, there are vertices \( u,v \in V_f \) such that \( f(u) = \ell = d(u, v_{i-1}) \) and \( f(v) = m = d(v, v_{i+2}) \). Then \( u,v,w_i \) broadcast to a subtree of \( T \) with spine \( P_1 \) of order at most \( 2(\ell + m) + 3 \). Assume firstly that \( P_1 \) has order equal to \( 2(\ell + m) + 3 \). Let \( x \) be the central vertex of \( P_1 \) and define a broadcast \( g \) by

\[
 g(z) = \begin{cases} 
 \ell + m + 1 & \text{if } z = x \\
 0 & \text{if } z \in \{w_i, u, v\} \\
 f(z) & \text{otherwise.}
\end{cases}
\]

Then \( g \) is an efficient \( \gamma_b \)-broadcast with fewer broadcast vertices than \( f \), which is a contradiction. Now assume \( P_1 \) has order less than \( 2(\ell + m) + 3 \). Then one (or both) leaves of \( P \) are overdominated by \( u \) or \( v \). If one leaf of \( P \) is overdominated then we can choose a vertex \( x \) on \( P \) such that the
broadcast $g$ as defined above is efficient, which is again a contradiction. If both leaves are overdominated, then the broadcast $g'$ defined by $g'(x) = \ell + m$ and $g(z) = 0$ otherwise, where $x$ is the central vertex of $P_1$, is a dominating broadcast with cost less than $f$, which is a contradiction.

So we may assume that $f$ is an efficient $\gamma_b$-broadcast with $|V_f^+|$ minimized such that every $v \in V_f^+$ is a vertex of the spine $P$.

Suppose that a leaf of $P$, say $v_1$, is overdominated by $f$. Let $v_i$ be the vertex that $f$-dominates $v_1$, and suppose $f(v_i) = \ell$. Then $v_i$ broadcasts to a subtree $T_1$ of $T$ with spine $P_1$ of order at most $2\ell$. Consider the nearest broadcasting vertex $v_j$ to $v_i$ and suppose $f(v_j) = m$. Then $v_i$ and $v_j$ together broadcast to a subtree $T_2$ of $T$ with spine $P_2$ of order at most $2\ell + 2m + 1$. Let $y \in P_2$ be the vertex at maximum distance from $v_1$ and let $x \in P_2$ be the unique vertex with $d(x, y) = m + \ell$. Define the broadcast $g$ by $g(x) = m + \ell$, $g(v_i) = g(v_j) = 0$ and $g(v) = f(v)$ otherwise. Then $g$ is an efficient broadcast with $\sigma(g) = \sigma(f)$ but $g$ has fewer broadcasting vertices, which is a contradiction.

So we may assume that $f$ is an efficient $\gamma_b$-broadcast with $|V_f^+|$ minimized such that every $v \in V_f^+$ is a vertex of the spine $P$ and no leaf of $P$ is overdominated.

Suppose $n$ is even. For every $v \in V_f^+$, $N_f[v]$ is a subtree of $T$ with a spine of odd order. Since $n$ is even, there is an even number of such balls, i.e. an even number of broadcasting vertices. Let $u, v \in V_f^+$ be such that $v_1 \in N_f[u]$ and $N_f[u]$ and $N_f[v]$ are adjacent in the ball graph of $T$. Since
$v_1$ is not overdominated, $v_{2f(u)+1}$ is the vertex at distance $f(u)$ from $u$, and thus $\deg(v_{2f(u)+1}) = \deg(v_{2f(u)+2}) = 2$. So (1) holds with $i = 2f(u) + 1$.

Suppose $n$ is odd. For every $v \in V_f^+$, $N_f[v]$ is a subtree of $T$ with a spine of odd order. Since $n$ is odd, there is an odd number of such balls, i.e. an odd number of broadcasting vertices. Since $T$ is nonradial, there are at least 3 broadcasting vertices. Let $u, v, w \in V_f^+$ be such that $v_1 \in N_f[u]$ and in the ball graph of $T$, $N_f[v]$ is adjacent to $N_f[u]$ and $N_f[w]$. Since $v_1$ is not overdominated, $v_{2f(u)+1}$ is the vertex at distance $f(u)$ from $u$. Let $i = 2f(u) + 1$ and $j = 2(f(u) + f(v)) + 2$. Thus $\deg(v_i) = \deg(v_{i+1}) = \deg(v_j) = \deg(v_{j+1}) = 2$, and so (2) holds. Therefore, $T$ is nonradial if and only if (1) or (2) is satisfied.

\[\square\]

**Corollary 4.8** If $T$ is a caterpillar with no adjacent vertices of degree 2, then $T$ is radial.

In the proof of Theorem 4.7 we also proved that if $f$ is an efficient $\gamma_b$-broadcast with the minimum number of broadcast vertices, then every broadcast vertex lies on the spine (thus on a diametrical path) and no leaf of the spine is overdominated. In Section 6.2 we generalize this crucial result to arbitrary trees. This generalization provides the key to characterizing radial trees (see Theorem 6.1).
Chapter 5

Long Paths Added at Vertices of $P_n$

In this chapter we prove stronger results than in [7]. We consider classes of trees obtained by joining a long path to vertices of $P_n$. In Sections 5.1 and 5.2 we deal with the case when $n$ is odd and even, respectively. In Section 5.3 we provide some corollaries of these results.

We use the following notation for the vertices of $P_n$ (see Figure 5.1). If $n = 2k + 1$, let $x$ be the central vertex of $P_n$ and let $u_j$ and $v_j$ be the vertices in $P_n$ at distance $j$ from $x$ on the left and right, respectively. If $n = 2k$, let $x$ and $y$ be the two central vertices of $P_n$, on the left and right, respectively. Then let $u_j$ be the vertex of $P_n$ at distance $j$ from $x$ and distance $j + 1$ from $y$. Similarly, let $v_j$ be the vertex of $P_n$ at distance $j$ from $y$ and distance $j + 1$ from $x$. We say that we add a path of length $k$ at vertex $u_j$ of $P_n$ if we...
CHAPTER 5. LONG PATHS ADDED AT VERTEXES OF $P_N$

identify a leaf of $P_{k+1}$ with $u_j$, and that a path added to vertex $u_j$ of $P_n$ is of allowable length if $P_n$ is a diametrical path of the resulting tree.

![Diagram showing labeling of vertices for $P_{2k+1}$ and $P_{2k}$](image)

Figure 5.1: Labeling of vertices for $P_{2k+1}$ and $P_{2k}$

5.1 The Central Case

Theorem 5.1 Let $n = 2k + 1$ and consider $P_n$, which has radius $k$.

(i) Adding a single path of length $\ell \leq k - 3$ at $x$ does not make the resulting tree radial; adding a single path of any longer allowable length does make the resulting tree radial.

(ii) Adding a single path of length $\ell \leq k - 4$ at $u_1$ or to $u_2$ does not make the resulting tree radial; adding a single path of any longer allowable length does make the resulting tree radial.

(iii) Adding a single path of any allowable length at $u_j$, where $j \geq 3$, does not make the resulting tree radial.
CHAPTER 5. LONG PATHS ADDED AT VERTEICES OF $P_N$  

Proof.

(i) Add a path of length $\ell \leq k - 3$ to vertex $x$ of $P_n$ and let $T$ be the resulting tree. Now define the following broadcast on $T$:

$$f(v) = \begin{cases} 
  k - 3 & \text{if } v = x \\
  1 & \text{if } v \in \{u_{k-1}, v_{k-1}\} \\
  0 & \text{otherwise.}
\end{cases}$$

![Figure 5.2: Broadcast $f$ for Theorem 5.1 (i)](image)

Then it is clear that $f$ is a dominating broadcast with cost $\sigma(f) = k - 1$, so $T$ is not radial.

Now add a path of length $\ell = k - 2$ to vertex $x$ of $P_n$ and let $T$ be the resulting tree. Let $w$ be the leaf of the added path. By Proposition 2.1, $T$ has an efficient $\gamma_b$-broadcast $f$. Let $T_u = [u_k, x]$, the path from the leaf $u_k$ to vertex $x$, of length $k$. Similarly, let $T_v = [v_k, x]$ and $T_w = [w, x]$, with lengths $k$ and $k - 2$, respectively. Now, let $z$ be the vertex that $f$-dominates $x$ and
Chapter 5. Long Paths Added at Vertices of $P_N$

Let $d(z, x) = t$. Suppose that $z \in T_u$ (the proof is the same if $z \in T_v$). Then $T - N_f[z]$ is a collection $\mathbb{P} = \{P_u, P_v, P_w\}$ of three disjoint paths. Since $f$ is efficient, it follows that

$$\sigma(f) = f(z) + \sum_{P \in \mathbb{P}} \gamma_b(P). \quad (5.1)$$

We note that $P_u$, $P_v$, and $P_w$ have orders

$$k - f(z) - t, \quad k - f(z) + t, \quad k - f(z) + t - 2.$$ 

By Proposition 2.5,

$$\sum_{P \in \mathbb{P}} \gamma_b(P) \geq \left\lfloor \frac{k - f(z) - t}{3} \right\rfloor + \left\lfloor \frac{k - f(z) + t}{3} \right\rfloor + \left\lfloor \frac{k - f(z) + t - 2}{3} \right\rfloor$$

$$\geq \frac{3k - 3f(z) + t - 2}{3}$$

$$\geq k - f(z) - 2/3.$$ 

If $z \in T_w$, then the only difference in the above proof is that paths $P_u$, $P_v$, and $P_w$ have orders

$$k - f(z) + t, \quad k - f(z) + t, \quad k - f(z) - t - 2.$$ 

In this case we also have $\sum_{P \in \mathbb{P}} \gamma_b(P) \geq k - f(z) - 2/3$. Now, by (5.1), $\sigma(f) \geq k - 2/3$. But $f$ is a $\gamma_b$-broadcast, so $\sigma(f) = \gamma_b(T)$. Therefore, $\gamma_b(T) \geq k - 2/3$, and so $\gamma_b(T) = k = \text{rad}(T)$. Thus $T$ is a radial tree. If a path of length $k - 1$ or $k$ is added to $x$, then the result follows from Propo-
(ii) a) Add a path of length $\ell \leq k - 4$ to vertex $u_1$ of $P_n$ and let $T$ be the resulting tree. Now define the following broadcast on $T$:

$$f(v) = \begin{cases} 
  k - 4 & \text{if } v = u_1 \\
  1 & \text{if } v \in \{u_{k-1}, v_{k-1}, v_{k-3}\} \\
  0 & \text{otherwise.} 
\end{cases}$$

Then it is clear that $f$ is a dominating broadcast with cost $\sigma(f) = k - 1$, so $T$ is not radial.

Now add a path of length $\ell = k - 3$ to vertex $u_1$ of $P_n$ and let $T$ be the resulting tree. Let $w$ be the leaf of the added path. By Proposition 2.1, $T$ has an efficient $\gamma_b$-broadcast $f$. Let $T_u = [u_k, u_1]$, the path from the leaf $u_k$ to vertex $x$, of length $k - 1$. Similarly, let $T_v = [v_k, u_1]$ and $T_w = [w, u_1]$. 

Figure 5.3: Broadcast $f$ for Theorem 5.1 (ii)a
with lengths \( k + 1 \) and \( k - 3 \), respectively. Now, let \( z \) be the vertex that \( f \)-dominates \( x \) and let \( d(z, x) = t \). Suppose firstly that \( z \in T_u \). Then \( T - N_f[z] \) is a collection \( \mathcal{P} = \{P_u, P_v, P_w\} \) of three disjoint paths. Since \( f \) is efficient, (5.1) holds. We note that \( P_u, P_v, \) and \( P_w \) have orders

\[
 k - f(z) - t - 1, \quad k - f(z) + t + 1, \quad k - f(z) + t - 3.
\]

By Proposition 2.5,

\[
\sum_{P \in \mathcal{P}} \gamma_b(P) = \left\lceil \frac{k - f(z) - t - 1}{3} \right\rceil + \left\lceil \frac{k - f(z) + t + 1}{3} \right\rceil + \left\lceil \frac{k - f(z) + t - 3}{3} \right\rceil \\
\geq \frac{3k - 3f(z) + t - 3}{3} \\
= k - f(z) + \frac{t - 3}{3}.
\]

If \( z \in T_v \), then the paths \( P_u, P_v, \) and \( P_w \) have orders

\[
 k - f(z) + t - 1, \quad k - f(z) - t + 1, \quad k - f(z) + t - 3.
\]

If \( z \in T_w \), then the paths \( P_u, P_v, \) and \( P_w \) have orders

\[
 k - f(z) + t - 1, \quad k - f(z) + t + 1, \quad k - f(z) - t - 3.
\]

Therefore, in any case, \( \sum_{P \in \mathcal{P}} \gamma_b(P) \geq k - f(z) + \frac{t - 3}{3} \). Thus \( \sigma(f) \geq k + \frac{t - 3}{3} \), where \( t \geq 0 \). If \( 1 \leq t \leq 3 \), then \( \sigma(f) \geq k \) so \( \sigma(f) = \gamma_b(T) = k = \text{rad}(T) \), and we’re done. If \( t \geq 4 \), then \( \sigma(f) > k \), which is impossible. So suppose \( t = 0 \) and let \( s = k - f(z) \).

Then \( \sigma(f) = f(z) + y \), where \( y = \left\lceil \frac{s - 1}{3} \right\rceil + \left\lceil \frac{s + 1}{3} \right\rceil + \left\lceil \frac{s - 3}{3} \right\rceil \). We consider the congruence classes of \( s \) modulo 3.
CHAPTER 5. LONG PATHS ADDED AT VERTICES OF $P_N$

- If $s \equiv 0 \pmod{3}$ then $s = 3r$ for some $r \in \mathbb{Z}^+$. So $y = \left\lfloor \frac{3r-1}{3} \right\rfloor + \left\lfloor \frac{3r-2}{3} \right\rfloor + \left\lfloor \frac{3r-3}{3} \right\rfloor = r + r + 1 + r - 1 = 3r = s$.

- If $s \equiv 1 \pmod{3}$ then $s = 3r + 1$ for some $r \in \mathbb{Z}^+$. So $y = \left\lfloor \frac{3r}{3} \right\rfloor + \left\lfloor \frac{3r+2}{3} \right\rfloor + \left\lfloor \frac{3r-2}{3} \right\rfloor = r + r + 1 + r = 3r + 1 = s$.

- If $s \equiv 2 \pmod{3}$ then $s = 3r + 2$ for some $r \in \mathbb{Z}^+$. So $y = \left\lfloor \frac{3r+1}{3} \right\rfloor + \left\lfloor \frac{3r+3}{3} \right\rfloor + \left\lfloor \frac{3r-1}{3} \right\rfloor = r + 1 + r + 1 + r = 3r + 2 = s$.

Thus, $\sigma(f) = f(z) + s = f(z) + k - f(z) = k$. So $\gamma_b(T) = k = \text{rad}(T)$ and therefore $T$ is a radial tree. Again, if an allowable path of length greater than $k - 3$ is added at $u_1$, the result follows from Proposition 2.3.

(ii) b) Add a path of length $\ell \leq k - 4$ to vertex $u_2$ of $P_n$ and let $T$ be the resulting tree. Now define the following broadcast on $T$:

$$f(v) = \begin{cases} 
  k - 4 & \text{if } v = u_2 \\
  1 & \text{if } v \in \{v_{k-1}, v_{k-1}, v_{k-4}\} \\
  0 & \text{otherwise}.
\end{cases}$$

Then it is clear that $f$ is a dominating broadcast with cost $\sigma(f) = k - 1$, so $T$ is not radial.

As in the proof of part (ii) a), the tree resulting from adding a path of allowable length at least $k - 3$ at vertex $u_2$ of $P_n$ is radial.

(iii) Add a path of any allowable length at any vertex $u_j$, where $j \geq 3$, of $P_n$.
and let $T$ be the resulting tree. Define the following broadcast on $T$:

$$f(v) = \begin{cases} 
    k - 3 & \text{if } v = u_3 \\
    1 & \text{if } v \in \{v_{k-1}, v_{k-4}\} \\
    0 & \text{otherwise}
\end{cases}$$

Then it is clear that $f$ is a dominating broadcast with cost $\sigma(f) = k - 1$, 

Figure 5.4: Broadcast $f$ for Theorem 5.1 (ii)

Figure 5.5: Broadcast $f$ for Theorem 5.1 (iii)
5.2 The Bicentral Case

Theorem 5.2 Let \( n = 2k \) and consider \( P_n \), which has radius \( k \).

(i) Adding a single path of length \( \ell \leq k - 3 \) at \( x \) does not make the tree radial; adding a single path of any longer allowable length does make the tree radial.

(ii) Adding a single path of any allowable length at \( u_j \), where \( j \geq 1 \), does not make the tree radial.

Proof.

(i) Add a path of length \( \ell \leq k - 3 \) at vertex \( x \) of \( P_n \) and let \( T \) be the resulting tree. Define the following broadcast on \( T \):

\[
  f(v) = \begin{cases} 
    k - 3 & \text{if } v = x \\
    1 & \text{if } v \in \{u_{k-2}, v_{k-2}\} \\
    0 & \text{otherwise.}
  \end{cases}
\]

Then it is clear that \( f \) is a dominating broadcast with cost \( \sigma(f) = k - 1 \), so \( T \) is not radial.

Now add a path of length \( \ell = k - 2 \) to vertex \( x \) of \( P_n \) and let \( T \) be the resulting tree. Let \( w \) be the leaf of the added path. By Proposition 2.1, \( T \)
has an efficient $\gamma_b$-broadcast $f$. Let $T_u = [u_{k-1}, x]$, the path from the leaf $u_{k-1}$ to vertex $x$, of length $k$. Similarly, let $T_v = [v_{k-1}, x]$ and $T_w = [w, x]$, with lengths $k + 1$ and $k - 1$, respectively. Now, let $z$ be the vertex that $f$-dominates $x$ and let $d(z, x) = t$. Suppose that $z \in T_u$ (the proof is the same in the other cases). Then $T - N_f[z]$ is a collection $\mathcal{P} = \{P_u, P_v, P_w\}$ of three disjoint paths. Since $f$ is efficient, $(5.1)$ from the proof of Theorem 5.1 holds. We note that $P_u, P_v$, and $P_w$ have orders

$$k - f(z) - t - 1, \quad k - f(z) + t, \quad k - f(z) + t - 2.$$
CHAPTER 5. LONG PATHS ADDED AT VERTICES OF $P_N$

By Proposition 2.5,

$$\sum_{P \in \mathcal{P}} \gamma_b(P) = \left\lceil \frac{k - f(z) - t - 1}{3} \right\rceil + \left\lceil \frac{k - f(z) + t}{3} \right\rceil + \left\lceil \frac{k - f(z) + t - 2}{3} \right\rceil$$

$$\geq \frac{3k - 3f(z) + t - 3}{3}$$

$$\geq k - f(z) - 1. \quad (5.2)$$

(Note that Inequality (5.2) is strict because at least one of the ceiling functions is rounded up.)

So, by (5.1), $\sigma(f) > k - 1$. But $f$ is a $\gamma_b$-broadcast, so $\sigma(f) = \gamma_b(T)$. Therefore, $k - 1 < \gamma_b(T) \leq k$, and so $\gamma_b(T) = k = \text{rad}(T)$. Thus $T$ is a radial tree. If a path of length $k - 2$ or $k - 1$ is added to $x$, then the result follows from Proposition 2.3.

(ii) Add a path of any allowable length to any vertex $u_j$, where $j \geq 1$, of $P_n$ and let $T$ be the resulting tree. Define the following broadcast on $T$:

$$f(v) = \begin{cases} 
    k - 2 & \text{if } v = u_1 \\
    1 & \text{if } v = u_{k-2} \\
    0 & \text{otherwise.}
\end{cases}$$
Then it is clear that $f$ is a dominating broadcast with cost $\sigma(f) = k - 1$, so $T$ is not radial.

\[\square\]

We saw in Section 4.3 that adjacent vertices of degree two on the spine of a caterpillar play an important role in determining whether the caterpillar is radial or not. When a single path is joined to a vertex of $P_n$, there are many adjacent vertices of degree two, and their role is not immediately clear. Note, however, that in the case where the resulting tree is bicentral and nonradial, we can delete the edge $v_{k-4}v_{k-3}$ (or $u_{k-4}u_{k-3}$), where $\deg(v_{k-4}) = \deg(v_{k-3}) = 2$, so that $u_{k-1},...,u_1,x,y,v_1,...,v_{k-4}$ is a diametrical path of the new tree. In the central nonradial case we can similarly delete either three vertices from each end of $P_n$, or two consecutive paths $P_3$ from the same end. In all instances the endvertices of the edges where the “cut” is performed have degree two. This significant observation is explored in Section 6.1.
CHAPTER 5. LONG PATHS ADDED AT VERTECIES OF $P_N$

5.3 Corollaries of Theorems 5.1 and 5.2

Let $P$ be a path in a tree $T$ and $w \in V(T) - V(P)$. Define the distance $d(P, w)$ of $w$ from $P$ by $d(P, w) = \min\{d(v, w) : v \in V(P)\}$. We now state a corollary that is stronger than Proposition 2.10, which was proved in [7].

**Corollary 5.3** Let $T$ be a tree and let $P$ be a diametrical path of $T$. If $d(P, w) \geq k - 2$ for some $w \in V(T) - V(P)$, then $T$ is radial.

**Corollary 5.4** Let $T$ be a central tree with diametrical path $P$ and let $v$ be a non-central vertex of $P$. If $v$ is the initial vertex of a path of length at least $k - 3$ that is internally disjoint from $P$, then $T$ is radial.

The next result follows from the broadcast given in Figure 5.7.

**Corollary 5.5** Let $n = 2k$ and consider $P_n$ with radius $k$. If $T$ is formed by adding any number of paths of any allowable lengths to any subset of the vertices $u_1, ..., u_k$, then $T$ is nonradial.

Similarly, the broadcasts given in Figures 5.2 to 5.7 imply the following results.
Corollary 5.6 Let \( T \) be any central tree with \( \text{rad}(T) = k \geq 4 \), \( \text{Cen}(T) = \{ x \} \), and peripheral vertices \( u_k \) and \( v_k \) such that \( d(u_k, v_k) = \text{diam}(T) = 2 \text{rad}(T) \). Let \( P : u_k, u_{k-1}, \ldots, x, \ldots, v_k \) be the \( u_k - v_k \) path in \( T \). If one of the following situations occurs, then \( T \) is nonradial:

- each vertex \( w \in V(T) - V(P) \) is either adjacent to \( u_{k-1} \) or \( v_{k-1} \), or satisfies \( d(x, w) \leq k - 3 \);
- each vertex \( w \in V(T) - V(P) \) is either adjacent to \( u_{k-1}, v_{k-3} \) or \( v_{k-1} \), or satisfies \( d(u_1, w) \leq k - 4 \);
- each vertex \( w \in V(T) - V(P) \) is either adjacent to \( u_{k-1}, v_{k-4} \) or \( v_{k-1} \), or satisfies \( d(u_2, w) \leq k - 4 \); or
- each vertex \( w \in V(T) - V(P) \) is either adjacent to \( v_{k-4} \) or \( v_{k-1} \), or satisfies \( d(u_3, w) \leq k - 3 \).

Corollary 5.7 Let \( T \) be any bicentral tree with \( \text{rad}(T) = k \geq 4 \), \( \text{Cen}(T) = \{ x, y \} \), and peripheral vertices \( u_{k-1} \) and \( v_{k-1} \) such that \( d(u_{k-1}, v_{k-1}) = \text{diam}(T) = 2 \text{rad}(T) - 1 \). Let \( P : u_{k-1}, u_{k-2}, \ldots, x, \ldots, v_{k-2}, v_{k-1} \) be the \( u_{k-1} - v_{k-1} \) path in \( T \). If one of the following situations occurs, then \( T \) is nonradial:

- each vertex \( w \in V(T) - V(P) \) is either adjacent to \( u_{k-2} \) or \( v_{k-2} \), or satisfies \( d(x, w) \leq k - 3 \);
- each vertex \( w \in V(T) - V(P) \) is either adjacent to \( v_{k-2} \), or satisfies \( d(u_1, w) \leq k - 2 \).
We use Theorems 5.1 and 5.2 to prove a sufficient condition for a tree to be radial.

**Theorem 5.8** Let $T$ be a tree. If $\text{rad}(T) = \text{rad}(T + uv)$ for every pair of distinct vertices $u, v \in V(T)$ such that $uv \notin E(T)$, then $T$ is radial.

**Proof.** Suppose $T$ is nonradial. We show that there is a pair of vertices $u, v \in V(T)$ such that $uv \notin E(T)$ and $\text{rad}(T) > \text{rad}(T + uv)$. Let $P$ be a diametrical path in $T$. Let $\text{rad}(T) = k$.

**Case 1:** $T$ is central. Say $P = u_k, ..., u_1, x, v_1, ..., v_k$. Then $u_2v_2 \notin E(T)$, otherwise $u_2, u_1, x, v_1, v_2$ is a cycle. Root $T$ at $x$ and consider $T' = T + u_2v_2$. We calculate $e(u_2)$ in $T'$. If $w$ is a descendant of $u_2$, then $d(u_2, w) \leq k - 2$. If $w$ is a descendant of $u_1$ but not $u_2$, then $d(u_1, w) \leq k - 4$, by Theorem 5.1, so $d(u_2, w) \leq k - 3$. If $w$ is a descendant of $x$ but not of $u_1$ or $v_1$, then $d(x, w) \leq k - 3$, by Theorem 5.1, so $d(u_2, w) \leq k - 1$. If $w$ is a descendant of $v_1$ but not $v_2$, then $d(v_1, w) \leq k - 4$, by Theorem 5.1, so $d(u_2, w) \leq k - 3$. Thus $e(u_2) \leq k - 1$ and so $\text{rad}(T') \leq k - 1$.

**Case 2:** $T$ is bicentral. Say $P = u_{k-1}, ..., u_1, x, y, v_1, ..., v_{k-1}$. Then $u_1v_1 \notin E(T)$, otherwise $u_1, x, y, v_1$ is a cycle. Root $T$ at $x$ and consider $T' = T + u_1v_1$. We calculate $e(u_1)$ in $T'$. If $w$ is a descendant of $u_1$, then $d(u_1, w) \leq k - 2$. If $w$ is a descendant of $x$ but not $u_1$ or $y$, then $d(x, w) \leq k - 3$, by Theorem 5.2, so $d(u_1, w) \leq k - 2$. If $w$ is a descendant of $y$ but not $v_1$, then $d(y, w) \leq k - 3$, by Theorem 5.2, so $d(u_1, w) \leq k - 1$. If $w$ is a descendant of $v_1$,
then \( d(v_1, w) \leq k - 2 \), so \( d(u_1, w) \leq k - 1 \). Thus \( e(u_1) \leq k - 1 \) and so \( \text{rad}(T') \leq k - 1 \).

\[ \Box \]

Note that the converse of Theorem 5.8 is not true. For example, consider the tree in Figure 5.8, where \( T \) is radial since \( \gamma_b(T) = \text{rad}(T) = 2 \), but \( \text{rad}(T + uv) = 1 \neq \text{rad}(T) \).

![Figure 5.8: T and T + uv](image-url)
Chapter 6

A Characterization of Radial Trees

In this chapter we prove the main result of this thesis. In Section 6.1 we look at some examples that compare results from the previous chapters and motivate the characterization. We require a crucial result given in Section 6.2. Then in Section 6.3 we prove the characterization and obtain two related formulas for the broadcast number of a tree as corollaries. In Section 6.4 we provide a geometric interpretation of the characterization. In Section 6.5 we apply this characterization of radial trees to show that general coronas of graphs are radial, a result first proved in [7]. We also describe a method for calculating the broadcast number of a tree, and close with a proof that every tree $T$ has an efficient broadcast with cost $k$, for any $k$ between $\gamma_b(T)$ and $\text{rad}(T)$. 
6.1 Motivation for the Characterization

We begin this section by comparing our results about caterpillars and long paths added to $P_n$. The central trees with radius 5 in Figure 6.1 are nonradial by Corollary 5.6 (for the trees on the left) and by Theorem 4.7 (for the caterpillars on the right). We notice that for each of these nonradial trees, the vertices can be covered by three isosceles right triangles whose hypotenuses have even integer lengths, as shown. Similarly, by Corollary 5.7 and Theorem 4.7, the bicentral trees with radius 5 in Figure 6.2 are nonradial, and we notice that the vertices of these trees can be covered by two isosceles right triangles whose hypotenuses have even integer lengths. In either the central or the bicentral case, when more vertices are added to the tree such that the tree becomes radial (by Theorem 5.1 or Theorem 4.7), it then becomes impossible to cover the vertices with triangles as described above. This rough description motivates our next definition, and is refined in Section 6.4.

Figure 6.1: Nonradial central trees
Figure 6.2: Nonradial bicentral trees

Figure 6.3: A tree with split-sets $M = \{uv\}$ and $M' = \{xy\}$

Let $P$ be a diametrical path of the tree $T$. A set $M$ of edges of $P$ is a split-$P$ set if the endvertices of each edge in $M$ have degree two in $T$, and each component $T'$ of $T - M$ has even positive diameter, the path $P' = P \cap T'$ being a diametrical path of $T'$. For example, the sets $M = \{uv\}$ and $M' = \{xy\}$ are split-$P$ sets of the tree in Figure 6.3, where $P$ is the path of black vertices. A split-set of $T$ is a split-$P$ set for some diametrical path $P$ of $T$, and a maximum split-set of $T$ is a split-set of maximum cardinality.
§6.2 Very Efficient Broadcasts

We now prove that a stronger result than Theorem 2.12 holds for trees (Theorem 6.1). Our characterization of radial trees, our formulas for the broadcast number of a tree, and our recursive method for calculating $\gamma_b$ depend on this result. For these reasons, Theorem 6.1 is the most important result of this thesis.

**Theorem 6.1** Let $P$ be a diametrical path of a tree $T$ and amongst all $\gamma_b$-broadcasts of $T$, let $f$ be one with the minimum number of broadcast vertices. Then

(i) $f$ is efficient,

(ii) every broadcast vertex lies on $P$, and

(iii) unless $P$ is a bicentral radial tree, neither endvertex of $P$ is overdominated.

Conversely, every $\gamma_b$-broadcast that satisfies (i), (ii) and (iii) is a $\gamma_b$-broadcast with the minimum number of broadcast vertices.

Note that because of the conditions imposed on $f$, (i) does not follow from Proposition 2.1, and Lemma 2.12 does not imply that $\Delta(B(f)) \leq 2$. 
Nevertheless, similar proofs establish these properties of \( f \). We include them here for completeness.

**Proof.** Let \( f \) and \( P = v_1, v_2, ..., v_n \) satisfy the hypothesis of the theorem.

\((i)\) Suppose, to the contrary, that \( N_f[u] \cap N_f[w] \neq \emptyset \). Then this intersection contains a vertex \( v \) of the \( u - w \) path (possibly an endvertex of the path) chosen as follows:

(a) if \( f(w) \geq d(u, w) \), let \( v = w \),

(b) otherwise choose \( v \) so that \( d(u, v) = f(w) \).

In each case \( d(u, v) \leq f(w) \). With choice (a), \( d(v, w) = 0 < f(u) \). With choice (b), if \( d(v, w) > f(u) \), then \( d(u, w) = d(u, v) + d(v, w) > f(w) + f(u) \), which implies that \( N_f[u] \cap N_f[w] = \emptyset \), a contradiction. Hence

\[
d(u, v) \leq f(w) \text{ and } d(v, w) \leq f(u).
\]

Define a broadcast \( g \) on \( T \) by

\[
g(x) = \begin{cases} 
0 & \text{if } x \in \{u, w\} - \{v\} \\
f(u) + f(w) & \text{if } x = v \\
f(x) & \text{otherwise.}
\end{cases}
\]
We show that $v$ broadcasts to all of $N_f[u]$; the proof that $v$ broadcasts to all of $N_f[w]$ is identical. If $x \in N_f[u]$, then $d(x, u) \leq f(u)$, hence

$$d(x, v) \leq d(x, u) + d(u, v) \leq f(u) + f(w) \quad \text{(by (6.1))}$$

$$= g(v).$$

Therefore $g$ is a $\gamma_b$-broadcast of $T$ with fewer broadcast vertices than $f$, a contradiction. This proves (i).

(ii) Suppose, to the contrary, that there is a broadcast vertex $u \in V(T) - V(P)$. Assume without loss of generality that there is no other broadcast vertex not on $P$ that lies between $u$ and $P$, and let $v$ be the vertex on $P$ at minimum distance from $u$.

**Case 1:** $N_f[u] \cap V(P) = \emptyset$. Let $w$ be the vertex of $T$ that broadcasts to $v$.

**Subcase 1.1:** $w = v$. We define a number of vertices as follows (see Figure 6.4):

- $w'$ is the vertex with $d(w', u) = f(u)$ and $d(w', w) = f(w) + 1$,
- $w'_1$ and $w'_2$ are the vertices on $P$ at distance $f(v) + 1$ from $v$ on the $v - v_1$ and $v - v_n$ subpaths of $P$, respectively; $w'_1$ and $w'_2$ exist because $P$ is a diametrical path,
- for $i = 1, 2$, $w_i$ is the vertex that broadcasts to $w'_i$.  

Note that $w_1$ and $w_2$ do not necessarily lie on $P$. Let $Q, Q', Q''$ be the paths in $T$ between $u$ and $w_1$, $u$ and $w_2$, $w_1$ and $w_2$, respectively. Denote the lengths
of the paths by $\ell(Q), \ell(Q'), \ell(Q'')$, respectively.

Without loss of generality, assume that $Q$ is the longest of these paths (the proof works the same in the other two cases). Then

$$f(w_2) \leq f(u) \quad (6.2)$$

and

$$\ell(Q) = f(u) + 2f(v) + f(w_1) + 2. \quad (6.3)$$

Let $v' \in V(Q)$ be such that $d(v', w_1) = f(u) + f(v) + 1$ and define a broadcast $g$ on $T$ as follows:

$$g(x) = \begin{cases} 
0 & \text{if } x \in \{u, v, w_1, w_2\} - \{v'\} \\
 f(v) + f(u) + f(w_1) + f(w_2) & \text{if } x = v' \\
 f(x) & \text{otherwise}. 
\end{cases}$$

Figure 6.4: Subcase 1.1 of Theorem 6.1
For each $x \in N_f[w_1]$,

$$d(v', x) \leq d(v', w_1) + f(w_1) = f(u) + f(v) + 1 + f(w_1) \leq g(v') \quad \text{(since } f(w_2) \geq 1).$$

Thus $v'$ broadcasts to all of $N_f[w_1]$. Also, for each $x \in N_f[u]$,

$$d(v', x) \leq \ell(Q) + f(u) - d(w_1, v')$$

$$= f(u) + 2f(v) + f(w_1) + 2 + f(u) - [f(u) + f(v) + 1] \quad \text{(by (6.3))}$$

$$= f(u) + f(v) + f(w_1) + 1 \leq g(v'),$$

so $v'$ broadcasts to all of $N_f[u]$. Since $N_f[v]$ lies between $N_f[w_1]$ and $N_f[u]$, it follows that $v'$ broadcasts to all of $N_f[v]$. We show that $v'$ broadcasts to all of $N_f[w_2]$. If $f(w_1) \geq f(u)$, then $v'$ lies on the $w_1 - v$ path in $T$ and $d(v', v) = f(w_1) - f(u)$. Hence for each $x \in N_f[w_2]$,

$$d(v', x) = d(v', v) + d(v, x)$$

$$\leq f(w_1) - f(u) + f(v) + 2f(w_2) + 1$$

$$\leq f(w_1) + f(w_2) + f(v) + 1 \quad \text{(by (6.2))}$$

$$\leq g(v') \quad \text{(} f(u) \geq 1\text{).}$$

Similarly, when $f(w_1) < f(u)$, $d(v', x) \leq g(v')$ for each $x \in N_f[w_2]$. Therefore $v'$ broadcasts to all of $N_f[w_2]$. Hence $g$ is a $\gamma_b$-broadcast on $T$ with fewer broadcast vertices than $f$, which is a contradiction.
Subcase 1.2: $w \neq v$. Define $u'$ as in Subcase 1.1. We define a number of other vertices as follows (see Figure 6.5):

- $w' \in N_f[w]$ is the vertex adjacent to $u'$ (i.e. $d(w, w') = f(w)$),
- $w'' \in N_f[w]$ is the vertex on $P$ at maximum distance from $w$ such that the $w - w''$ path contains $v$,
- $w'_1 \in V(P)$ is the vertex adjacent to $w''$ such that $d(w, w'_1) = f(w) + 1$ ($w'_1$ exists because $P$ is a diametrical path),
- $w_1$ is the vertex that broadcasts to $w'_1$.

Then $d(w, w'') = f(w)$. Note that $w$ and $w_1$ do not necessarily lie on $P$ and that $w' = w'' = v$ is possible. Let $Q, Q', Q''$ be the paths in $T$ between $u$ and $w, u$ and $w_1, w$ and $w_1$, respectively.

- Suppose firstly that $Q$ is the longest of these paths. (The proof is similar if $Q''$ is the longest path.)
Then
\[ \ell(Q) = f(w) + f(u) + 1. \] (6.4)

Let \( v' \in V(Q) \) be such that \( d(w, v') = f(u) + 1 \) (and \( d(u, v') = f(w) \)) and define a broadcast \( g \) on \( T \) as follows:

\[
g(x) = \begin{cases} 
0 & \text{if } x \in \{w, u, w_1\} - \{v'\} \\
 f(w) + f(u) + f(w_1) & \text{if } x = v' \\
 f(x) & \text{otherwise.}
\end{cases}
\] (6.5)

For each \( x \in N_f[w] \), \( d(v', x) \leq f(w) + f(u) + 1 \leq g(v') \) since \( f(w_1) \geq 1 \), so \( v' \) broadcasts to all of \( N_f[w] \). It also follows from (6.4) that for each \( x \in N_f[u] \),

\[
d(v', x) = \ell(Q) - d(w, v') + d(u, x) \leq f(w) + f(u) < g(v'),
\]

so \( v' \) broadcasts to all of \( N_f[u] \). We show that \( v' \) broadcasts to all of \( N_f[w_1] \).

By the choice of \( Q \),

\[
\begin{array}{c}
\{ d(w, v) \\
\min_{v''} d(u, v)
\end{array} \geq d(v, w_1) = d(v, w'') + f(w_1) + 1. \] (6.6)
CHAPTER 6. CHARACTERIZATION

If \( v' \) lies on the \( v - u \) path in \( T \), then

\[
d(v', w'') = d(v', w) - d(w, v) + d(v, w'')
\]

\[
\leq f(u) + 1 - [d(v, w'') + f(w_1) + 1] + d(v, w'') \quad \text{(by (6.6))}
\]

\[
= f(u) - f(w_1),
\]

from which it follows that for each \( x \in N_f[w_1] \),

\[
d(v', x) = d(v', w'') + d(w'', x) \leq [f(u) - f(w_1)] + [2f(w_1) + 1]
\]

\[
= f(u) + f(w_1) + 1 \leq g(v').
\]

If \( v' \) lies on the \( w - v \) path in \( T \), then

\[
d(v', w'') = d(v', u) - d(u, v) + d(v, w'')
\]

\[
\leq f(w) - [d(v, w'') + f(w_1) + 1] + d(v, w'') \quad \text{(by (6.6))}
\]

\[
= f(w) - f(w_1) - 1,
\]

so for each \( x \in N_f[w_1] \),

\[
d(v', x) = d(v', w'') + d(w'', x) \leq [f(w) - f(w_1) - 1] + [2f(w_1) + 1]
\]

\[
= f(w) + f(w_1) < g(v').
\]
Hence $v'$ broadcasts to all of $N_f[w_1]$ and so $g$ is a $\gamma_b$-broadcast on $T$ with fewer broadcast vertices than $f$, which is a contradiction.

- Suppose $Q'$ is the longest path.

Let $d = d(v, w') = d(v, w'')$. Then $d = f(w) - d(w, v) < f(w)$, i.e.

$$d + 1 \leq f(w), \quad (6.7)$$

and

$$\ell(Q') = f(w_1) + 2d + f(u) + 2. \quad (6.8)$$

We assume that $f(u) \geq f(w_1)$; the proof is similar if $f(u) < f(w_1)$. We choose a vertex $v'$ as described below, and in each case define the broadcast $g$ on $T$ as in (6.5).

- If $d(v, w) \leq 2f(w_1)$, choose $v' \in V(Q')$ such that $d(w_1, v') = d + f(u) + 1$ (and $d(u, v') = d + f(w_1) + 1$).

Then $v'$ lies on the $v - u$ path in $T$. For each $x \in N_f[w_1]$,

$$d(v', x) \leq d(v', w_1) + d(w_1, x)$$

$$\leq d + f(u) + 1 + f(w_1)$$

$$\leq f(w) + f(u) + f(w_1) \quad \text{(by (6.7))}$$

$$= g(v').$$
So \( v' \) broadcasts to all of \( N_f[w_1] \). Moreover, for any \( x \in N_f[u] \),

\[
d(v', x) \leq d(v', u) + d(u, x) \\
\leq d + f(w_1) + 1 + f(u) \leq g(v') \quad \text{(by (6.7))}.
\]

Hence \( v' \) broadcasts to all of \( N_f[u] \). We show that \( v' \) broadcasts to all of \( N_f[w] \). Since

\[
d(v, v') = d(v', w_1) - d(v, w_1) = d + f(u) + 1 - [d + f(w_1) + 1] = f(u) - f(w_1),
\]

it follows that for any \( x \in N_f[w] \),

\[
d(v', x) \leq d(v', v) + d(v, w) + d(w, x) \\
\leq f(u) - f(w_1) + 2f(w_1) + f(w) \quad \text{(by the choice } \star \text{)} \\
= f(u) + f(w_1) + f(w) = g(v')
\]

and so \( N_f[w] \subseteq N_g[v'] \).

\( \star \star \) If \( d(v, w) > 2f(w_1) \), choose \( v' \) on the \( w - u \) path such that \( d(w, v') = f(u) + f(w_1) \).
Obviously, \( N_f[w] \subseteq N_g[v'] \). We show that \( N_f[u] \cup N_f[w_1] \subseteq N_g[v'] \). Firstly, 

\[
d(v, v') = |d(w, v') - d(w, v)|
\]

\[
= \begin{cases} 
  f(u) + f(w_1) - f(w) + d & \text{if } d(w, v') \geq d(w, v) \\
  f(w) - f(u) - f(w_1) - d & \text{if } d(w, v') < d(w, v).
\end{cases}
\]

Hence, if \( d(w, v') \geq d(w, v) \), then \( v' \) lies on the \( v - u \) path, and

\[
d(v', w_1) = d(v', v) + d(v, w_1) = [f(u) + f(w_1) - f(w) + d] + [d + 1 + f(w_1)]
\]

\[
= f(u) + 2f(w_1) + 1 + d - [f(w) - d]
\]

\[
\leq f(u) + d(v, w) + d - [f(w) - d] \quad \text{(by the choice } \star \star \text{)}
\]

\[
= f(u) + f(w) - [f(w) - d] \quad \text{(by (6.7))}
\]

so that for any \( x \in N_f[w_1] \),

\[
d(v', x) = d(v', w_1) + d(w_1, x) < f(u) + f(w) + f(w_1) = g(v').
\]

Similarly,

\[
d(v', u) = d(v, u) - d(v', v) = [d + 1 + f(u)] - [f(u) + f(w_1) - f(w) + d]
\]

\[
= f(w) - f(w_1) + 1,
\]

(6.9)
so that for any \( x \in N_f[u] \),

\[
d(v', x) = d(v', u) + d(u, x) \leq f(w) - f(w_1) + 1 + f(u) < g(v').
\]

On the other hand, if \( d(w, v') < d(w, v) \), then \( v' \) lies on the \( w - v \) path, and

\[
d(v', w_1) = d(v', v) + d(v, w_1) = [f(w) - f(u) - f(w_1) - d] + [d + 1 + f(w_1)]
\]

\[
= f(w) - f(u) + 1,
\]

so that for any \( x \in N_f[w_1] \),

\[
d(v', x) = d(v', w_1) + d(w_1, x) \leq f(w) - f(u) + 1 + f(w_1) < g(v').
\]

Similarly,

\[
d(v', u) = d(v', v) + d(v, u) = [f(w) - f(u) - f(w_1) - d] + [d + 1 + f(u)]
\]

\[
= f(w) - f(w_1) + 1,
\]

which is the same as (6.9) and so \( d(v', x) < g(v') \) for all \( x \in N_f[u] \). Therefore in either case \( N_f[u] \cup N_f[w_1] \subseteq N_g[v'] \).

But then, for both choices of \( v' \), \( g \) is a \( \gamma_b \)-broadcast on \( T \) with fewer broadcast vertices than \( f \), a contradiction. Thus the proof of Case 1 is complete.
Case 2: \( N_f[u] \cap V(P) \neq \emptyset \). Now \( v \in N_f[u] \). Once again we define a number of vertices (see Figure 6.6):

- \( u', u'' \in N_f[u] \) are the vertices on \( P \) at maximum distance from \( v \) on the \( v-v_1 \) and \( v-v_n \) subpaths of \( P \), respectively; possibly \( u' = u'' = v \),

- \( w'_1, w'_2 \) are the vertices on \( P \) adjacent to \( u', u'' \), respectively; they exist because \( P \) is a diametrical path,

- for \( i = 1, 2 \), \( w_i \) is the vertex that broadcasts to \( w'_i \).

Similar to the other cases, \( w_1 \) and \( w_2 \) do not necessarily lie on \( P \). But Figure 6.6 can be redrawn as in Figure 6.7, which is the same as in Subcase 1.2 with some labels interchanged. Thus we obtain a contradiction as in Subcase 1.2. Therefore \((ii)\) holds.
(iii) Clearly, when \( T \) is a bicentral radial tree, a leaf of \( P \) is overdominated. Assume that \( T \) is not a bicentral radial tree and suppose without loss of generality that \( v_1 \) is overdominated. Then \( T \) is also not a central radial tree, otherwise all peripheral vertices of \( T \) are overdominated and there exists a dominating broadcast \( g \) of \( T \) with \( \sigma(g) < \sigma(f) \), a contradiction.

Assume therefore that \( T \) is not radial and let

- \( u \) be the vertex on \( P \) that broadcasts to \( v_1 \),
- \( u' \) be the vertex at maximum distance from \( v_1 \) such that \( u' \in N_f[u] \cap V(P) \),
- \( w' \) be the vertex on \( P \) adjacent to \( u' \) such that \( d(u, w') = f(u) + 1 \); \( w' \) exists because \( T \) is nonradial,
• $w \in V(P)$ be the vertex that broadcasts to $w'$,

• $Q$ be the $u - w$ subpath of $P$, and

• $v$ be the vertex on $Q$ at distance $f(w) + 1$ from $u$.

Since $f$ is efficient, $v$ is an internal vertex of $Q$. Define the broadcast $g$ on $T$ by

$$g(x) = \begin{cases} 
0 & \text{if } x \in \{u, w\} \\
f(u) + f(w) & \text{if } x = v \\
f(x) & \text{otherwise.}
\end{cases}$$

For each $x \in N_f[u]$ such that $x$ is joined to $u$ by a path internally disjoint from $Q$,

$$d(x, u) \leq d(v_1, u) \quad (v_1 \text{ is a peripheral vertex})$$

$$\leq f(u) - 1 \quad (v_1 \text{ is overdominated}).$$

Hence for each such $x$, $d(x, v) = d(x, u) + d(u, v) \leq f(u) + f(w) \leq g(v)$. If $x \in N_f[u]$ is joined to $u$ by a path that contains the internal vertex $y$ of $Q$, then

$$d(x, v) < d(x, y) + d(y, u) + d(u, v) \leq f(u) + f(w) + 1$$
and so \(d(x, v) \leq f(u) + f(w) = g(v)\) in this case as well. Thus \(v\) broadcasts to every vertex in \(N_f[u]\). Moreover, for each \(x \in N_f[w]\),

\[
d(v, x) \leq \ell(Q) - d(u, v) + d(w, x) \\
\leq f(u) + f(w) + 1 - [f(w) + 1] + f(w) \\
= f(u) + f(w) = g(v),
\]

so \(N_f[w] \subseteq N_g[v]\). Therefore \(g\) is a \(\gamma_b\)-broadcast of \(T\) with fewer broadcast vertices than \(f\). This contradiction concludes the proof of the necessity of conditions (i) – (iii).

For the converse, let \(P\) be a diametrical path of \(T\) and let \(f\) be a \(\gamma_b\)-broadcast that satisfies (i) – (iii); say \(V_f^+ = \{u_1, ..., u_r\}\). Let \(g\) be a \(\gamma_b\)-broadcast with the minimum number \(t\) of broadcast vertices \(w_1, ..., w_t\). As shown above, \(g\) also satisfies (i) – (iii). Then

\[
\gamma_b(T) = \sum_{i=1}^{r} f(u_i) = \sum_{i=1}^{t} g(w_i)
\]

and

\[
diam T = \left( \sum_{i=1}^{r} 2f(u_i) \right) + r - 1 = \left( \sum_{i=1}^{t} 2g(w_i) \right) + t - 1,
\]

from which it follows that \(r = t\). The proof of the theorem is now complete.

\(\square\)
Remark 6.2 We note that Theorem 6.1 does not hold for graphs in general. Consider $C_n$ with $n \geq 9$ if $n$ is odd or $n \geq 12$ if $n$ is even. For example, consider $C_9$. Each vertex $v$ has $e(v) = 4$, so $\text{rad}(C_9) = \text{diam}(C_9) = 4$ and it is clear that $\gamma_b(C_9) = 3$. In Figure 6.8, a diametrical path $P : v_1, \ldots, v_5$ is shown and it is straightforward to see that there is no $\gamma_b$-broadcast on $C_9$ such that every broadcast vertex lies on $P$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.8.png}
\caption{The cycle on 9 vertices}
\end{figure}

Remark 6.3 In the proof of Theorem 6.1, the fact that $f$ is a $\gamma_b$-broadcast is not used. In fact, what is proved is the following:

Suppose the tree $T$ has a dominating broadcast of cost $k$, where $\gamma_b(T) \leq k \leq \text{rad}(T)$. Amongst all such broadcasts, let $f$ be one with the minimum number $r$ of broadcast vertices. Let $P$ be any diametrical path of $T$. Then

(i) $f$ is efficient,

(ii) every broadcast vertex lies on $P$, and

(iii) unless $T$ is a bicentral tree and $r = 1$, neither endvertex of $P$ is over-dominated.
Conversely, every dominating broadcast with cost $k$ that satisfies (i), (ii) and (iii) is a dominating broadcast of cost $k$ with the minimum number of broadcast vertices.

In view of Theorem 6.1 we henceforth call a broadcast of a tree $T$ that satisfies (i), (ii) and (iii) a very efficient broadcast. Hence a $\gamma_b$-broadcast of $T$ with the minimum number of broadcast vertices is a very efficient $\gamma_b$-broadcast. As can be seen by considering the tree in Figure 6.9, a tree may have more than one very efficient $\gamma_b$-broadcast. However, the following corollary of Theorem 6.1 is obvious.

**Corollary 6.4** If $f$ is any very efficient $\gamma_b$-broadcast of a tree $T$, then every diametrical path of $T$ contains all broadcast vertices of $f$.

### 6.3 Proof of Characterization

We now prove the main result of this thesis.
Theorem 6.5  A tree \( T \) is radial if and only if it has no nonempty split-set.

Proof.  Suppose \( M \) is a nonempty split-P set for a diametrical path \( P \) of \( T \) and that \( T - M \) has \( r \) components \( T_1, \ldots, T_r \). Then \( r \geq 2 \) because \( M \neq \emptyset \), and each \( T_i \) has positive even diameter \( 2\ell_i \) with \( P_i = P \cap T_i \) being a diametrical path. Let \( v_i \) be the central vertex of \( P_i \) and define a broadcast \( f_M \) on \( T \) by

\[
f_M(v_i) = \ell_i \quad \text{and} \quad f_M(v) = 0 \quad \text{otherwise}. \tag{6.10}
\]

So

\[
\text{rad}(T) = \left\lfloor \frac{2\sum_{i=1}^{r} \ell_i + r}{2} \right\rfloor = \sum_{i=1}^{r} \ell_i + \left\lfloor \frac{r}{2} \right\rfloor = \sigma(f_M) + \left\lfloor \frac{r}{2} \right\rfloor. \tag{6.11}
\]

Hence \( \sigma(f_M) < \text{rad}(T) \) since \( r \geq 2 \). Therefore \( T \) is nonradial.

Conversely, suppose that \( T \) is nonradial. Let \( P \) be a diametrical path of \( T \) and let \( f \) be a very efficient \( \gamma_b \)-broadcast of \( T \). Since \( T \) is nonradial, Theorem 6.1 implies that \( |V_f^+| \geq 2 \), every broadcast vertex of \( f \) lies on \( P \), and no leaf of \( P \) is overdominated. Let \( V_f^+ = \{w_1, \ldots, w_r\} \), \( r \geq 2 \), where these vertices have been labelled in order of their appearance on \( P \). Since \( f \) is efficient, there exist consecutive vertices \( u_i, u_{i+1} \in V(P) \) such that \( u_i \in N_f[w_i] - N_f[w_{i+1}] \) and \( u_{i+1} \in N_f[w_{i+1}] - N_f[w_i], \ i = 1, \ldots, r - 1 \). Clearly, \( \deg_T u_i = \deg_T u_{i+1} = 2 \). Moreover, since no leaf of \( P \) is overdominated, \( \langle N_f[w_i] \rangle \) has a diametrical
path $P_i = \langle V(P) \cap N_f[w_i] \rangle$ of length $2f(w_i)$ for each $i$. Thus, if $e_i = u_iu_{i+1}$, $i = 1, \ldots, r - 1$, then $M_f = \{e_1, \ldots, e_{r-1}\}$ is a non-empty split-$P$ set.

We now obtain two formulas for $\gamma_b(T)$ as corollaries to Theorem 6.1. We need some further definitions and a lemma.

If $M$ is a split-set of a tree $T$ and $f_M$ is the broadcast as defined in the proof of Theorem 6.5, we call $f_M$ the broadcast associated with $M$. Similarly, if $f$ is a very efficient broadcast and $M_f$ is the split-set as defined in the proof of Theorem 6.5, we call $M_f$ the split-set associated with $f$.

By Theorem 6.1, a $\gamma_b$-broadcast is a very efficient $\gamma_b$-broadcast if and only if it is a $\gamma_b$-broadcast with the minimum number of broadcast vertices. We now show that a very efficient broadcast is a very efficient $\gamma_b$-broadcast if and only if it is a very efficient broadcast with the maximum number of broadcast vertices, that is, if and only if its associated split-set is a maximum split-set.

**Lemma 6.6** A split-set of a tree $T$ is a maximum split-set if and only if its associated broadcast is a very efficient $\gamma_b$-broadcast.

**Proof.** If $M$ is a split-set, then obviously $f_M$ is a very efficient dominating broadcast. Suppose $M$ is a maximum split-set, and let $g$ be a very efficient $\gamma_b$-broadcast of $T$ with associated split-set $M_g$. Let $r = |V_{f_M}^+|$ and $s = |V_g^+|$. Then

$$r = |M| + 1 \geq |M_g| + 1 = s$$
and, similar to (6.10),

\[ \text{diam}(T) = 2\sigma(f_M) + r - 1 = 2\sigma(g) + s - 1. \]

Therefore

\[ 2\sigma(f_M) = \text{diam}(T) - |M| \leq \text{diam}(T) - |M_g| = 2\sigma(g). \] (6.12)

Since \( g \) is a \( \gamma_b \)-broadcast and \( f \) is a dominating broadcast, it follows that \( \sigma(f) = \sigma(g) \) and so \( f_M \) is a very efficient \( \gamma_b \)-broadcast.

Conversely, assume that \( f \) is a very efficient \( \gamma_b \)-broadcast with associated split-set \( M_f \). Let \( M \) be a maximum split-set of \( T \) with associated broadcast \( g_M \). As proved above, \( g_M \) is a very efficient \( \gamma_b \)-broadcast, hence \( \sigma(f) = \sigma(g_M) \). As in (6.12),

\[ 2\sigma(f) = \text{diam}(T) - |M_f| = \text{diam}(T) - |M| = 2\sigma(g_M) \]

and the result follows.

\[ \square \]

The following corollaries of Theorem 6.1 and Lemma 6.6 are essentially restatements of (6.11) for very efficient \( \gamma_b \)-broadcasts.
Corollary 6.7  

(i) Let $f$ be a very efficient $\gamma_b$-broadcast of the tree $T$ with $r$ broadcast vertices. Then

$$\gamma_b(T) = \sigma(f) = \text{rad}(T) - \left\lfloor \frac{r}{2} \right\rfloor.$$ 

(ii) For any tree $T$, let $M$ be a split-set of maximum cardinality $m$. Then

$$\gamma_b(T) = \text{rad}(T) - \left\lceil \frac{m}{2} \right\rceil.$$ 

Let $T_{m,n}$ denote the complete $n$-ary tree on $m$ levels, defined as follows:

- Let $T_{1,n} = K_{1,n}$.

- For each $m \geq 2$, $T_{m,n}$ is obtained from $n$ copies of $T_{m-1,n}$ whose roots are each joined to a new vertex $r$.

![Figure 6.10: The complete binary tree on three levels](image)

In [2], Bouchemakh and Sahbi proved that for $m \geq 1$, $T_{m,2}$ is radial. By Theorem 6.5, the following corollary is clear.

**Corollary 6.8**  For any $m \geq 1$ and $n \geq 2$, $T_{m,n}$ is radial.
6.4 A Geometrical Interpretation of the Characterization

Let $P = v_1, ..., v_n$ be a diametrical path of the tree $T$. We define the shadow tree $S_T$ of $T$ with respect to $P$ as follows. For each $v_i \in V(P)$, let $V_i$ be the set of all vertices of $T$ that are connected to $v_i$ by a (possibly trivial) path internally disjoint from $P$. Let $w_i$ be a vertex in $V_i$ at maximum distance from $v_i$ and let $Q_i$ be the $v_i - w_i$ path in $T$. Then $S_T$ is the subtree of $T$ induced by $\bigcup_{i=1}^{n} Q_i$. Note that $\Delta(S_T) = 3$, and $\deg_{S_T} u = 3$ if and only if $u \in V(P)$ and $\deg_T u \geq 3$. The tree in Figure 6.11 is a shadow tree of the tree in Figure 6.9. Note that the shadow tree is not necessarily unique, and depends on the choice of $P$. For example, the tree in Figure 6.12 has many diametrical paths, and thus many shadow trees; two possible shadow trees are shown.

Since a set $M$ of edges is a split-$P$ set of $T$ if and only if it is a split-$P$ set of $S_T$, the next result follows immediately from Corollary 6.7(ii).
Corollary 6.9  For any tree $T$ and any shadow tree $S_T$ of $T$, $\gamma_b(T) = \gamma_b(S_T)$.

Let $S_T$ be a shadow tree of $T$ with respect to the diametrical path $P = v_1, ..., v_n$ of $T$. Draw $S_T$ in the positive $X - Y$ plane with $P$ on the $X$-axis, $v_1$ at the origin, each edge having unit length, and each edge not on $P$ being parallel to the $Y$-axis. Consider the vertices of $S_T$ to have zero dimension – they are only points in the plane with integer coordinates. We call this
representation the standard representation of $S_T$. The shadow tree in Figure 6.11 is in standard representation. A region $R$ in the $X-Y$ plane covers the vertices of $S_T$ if each vertex of $S_T$ lies on the boundary or in the interior of $R$.

**Corollary 6.10**  
(i) A central (bicentral, respectively) tree $T$ with shadow tree $S_T$ is nonradial if and only if the vertices of the standard representation of $S_T$ can be covered by three (two, respectively) isosceles right triangles whose hypotenuses have even integer lengths and lie on the $X$-axis, one unit length apart, their vertices corresponding to vertices of $S_T$.

(ii) A tree $T$ is radial if and only if the vertices of the standard representation of $S_T$ cannot be covered by isosceles right triangles whose hypotenuses have even integer lengths that sum to less than $\text{diam}(T)$.

(iii) Let $\rho$ be the maximum number of isosceles right triangles that can be used to cover the vertices of $S_T$ as described in (i). Then $\gamma_b(T) = \text{rad}(T) - \left\lfloor \frac{\rho}{2} \right\rfloor$.

**Proof.** (ii) Suppose $S_T$ is nonradial. Then $S_T$ has a non-empty split-set $M$. Let $f_M$ be the broadcast associated with $M$, where $V_{f_M}^+ = \{v_1, \ldots, v_r\}$ are the central vertices of the components $T_i$ of $S_T - M$. Then a right triangle with base $2f(v_i)$ and height $f(v_i)$ will cover $N_f[v_i]$ for all $1 \leq i \leq r$. Also, since
the $i^{th}$ triangle has diameter equal to the $\text{diam}(T_i)$ and 

$$\text{diam}(S_T) = \sum_{i=1}^{r} \text{diam}(T_i) + r - 1,$$

it follows that the sum of the diameters of the triangles is less than the diameter of $S_T$.

Next, suppose that the vertices of the standard representation of $S_T$ can be covered by isosceles right triangles whose hypotenuses have even integer lengths that sum to less than $\text{diam}(T)$. Let $T_1, ..., T_r$ be the components of $S_T$ contained within the $r$ triangles. Then $r \geq 2$ and $\text{diam}(T_i)$ is even, so let $u_i$ be the central vertex of $T_i$, $i = 1, ..., r$. Define $f$ on $S_T$ by $f(u_i) = \frac{\text{diam}(T_i)}{2}$. Then $N_f[u_i] = T_i$, so $f$ is a dominating broadcast with cost $\sigma(f) = \sum_{i=1}^{r} \frac{\text{diam}(T_i)}{2}$. Thus 

$$\text{rad}(S_T) = \left[ \frac{\text{diam}(S_T)}{2} \right] = \left[ \frac{\sum_{i=1}^{r} \text{diam}(T_i) + r - 1}{2} \right] > \frac{\sum_{i=1}^{r} \text{diam}(T_i)}{2} = \sigma(f).$$

(6.13) 

Therefore $S_T$ is nonradial. By Corollary 6.9, the proof of (ii) is complete.

(i) The vertices of $S_T$ can be covered by isosceles right triangles whose hypotenuses have even integer lengths that sum to less than $\text{diam}(T)$ if and only if there are at least two triangles. But if $T$ (and thus $S_T$) is central,
then there are at least three triangles, since otherwise \( r = 2 \) and \( \text{diam}(S_T) \) is odd, which is a contradiction. Thus, the proof of (i) is complete.

(iii) If \( T \) is radial, then \( \rho = 1 \) and the result holds. So suppose that \( T \) is nonradial. Then \( \rho > 1 \) and \( T \) has a maximum split-set \( M \) of size \( \rho - 1 \). Then by Lemma 6.6 the broadcast \( f_M \) associated with \( M \) is a very efficient \( \gamma_b \)-broadcast and \( f_M \) has \( \rho \) broadcast vertices. Hence, by Corollary 6.7, \( \gamma_b(T) = \text{rad}(T) - \left\lfloor \frac{\rho}{2} \right\rfloor \).

Thus the trees in Figure 6.13 are nonradial, while the trees in Figure 6.14 are radial.

![Figure 6.13: Vertices of nonradial trees covered by isosceles right triangles](image-url)
6.5 Applications of Theorem 6.5

6.5.1 Generalized Coronas

Let $G$ be any graph with $V(G) = \{v_1, ..., v_n\}$, and let $H_1, ..., H_n$ be any graphs. Recall that the generalized corona $G \circ H_1, ..., H_n$ is the graph obtained by joining $v_i$ to all vertices of $H_i$, $i = 1, ..., n$. As shown in [7], $G \circ H_1, ..., H_n$ is radial for all graphs $G$ and $H_i$ (see Corollary 2.8). We obtain this result as a corollary to Theorem 6.5.

**Corollary 6.11** For any graph $G$ of order $n$ and any graphs $H_1, ..., H_n$, the graph $G \circ H_1, ..., H_n$ is radial.

**Proof.** Let $T$ ($S$, respectively) be the set of all spanning trees of $G$ ($G \circ H_1, ..., H_n$, respectively), let $S^*$ be the set of spanning subtrees of $G \circ H_1, ..., H_n$ that consist of a tree in $T$ together with $|V(H_i)|$ leaves joined to $v_i$ for each $i$, and let $R$ be the set of all trees that consist of a tree in $T$
together with a leaf joined to $v_i$ for each $i$. Then

$$\gamma_b(G \circ H_1, ..., H_n) = \min_{T \in S} \{\gamma_b(T)\} \quad \text{(Theorem 3.1)}$$

$$\geq \min_{T \in R} \{\gamma_b(T)\} \quad \text{(Proposition 2.3)}$$

$$= \min_{T \in S^*} \{\gamma_b(T)\} \quad \text{(Proposition 2.4)}$$

$$\geq \min_{T \in S} \{\gamma_b(T)\} \quad (S^* \subseteq S).$$

Therefore equality holds throughout and $\gamma_b(G \circ H_1, ..., H_n) = \min_{T \in S^*} \{\gamma_b(T)\}$.

By Proposition 3.3 every graph has a spanning tree of the same radius, and for $G \circ H_1, ..., H_n$ such a subtree is in $S^*$. However, for any $T \in S^*$ and any diametrical path $P = v_1, ..., v_k$ of $T$, the only vertices that can possibly have degree two are $v_2$ and $v_{k-2}$. Hence $T$ has no nonempty split-set and so is radial, by Theorem 6.5. Thus

$$\min_{T \in S^*} \{\gamma_b(T)\} = \min_{T \in S^*} \{\text{rad}(T)\} = \text{rad}(G \circ H_1, ..., H_n)$$

and the result follows. \(\square\)
6.5.2 Graphs with Radial Subtrees

**Corollary 6.12** Let $G$ be a connected graph. If every spanning tree $T \in S(G)$ is radial, then $G$ is radial.

**Proof.** Let $\text{rad}(G) = k$. For any $T \in S(G)$, $\gamma_b(T) = \text{rad}(T) \geq k$. By Proposition 3.3, there exists $T' \in S(G)$ with $\text{rad}(T') = k$, and $T'$ has no nonempty split-set, so, by Theorem 6.5, $\gamma_b(T') = k$. Now, by Theorem 3.1, $\gamma_b(G) = \min_{T \in S(G)} \{\gamma_b(T)\} = k$.

Note that the converse of Corollary 6.12 is false. In Figure 6.15, $G$ is radial with radius 2, but $T$ is a spanning subtree of $G$ with a nonempty split-set $M = \{e\}$.

![Figure 6.15: Counterexample to the converse of Corollary 6.12](image)
CHAPTER 6. CHARACTERIZATION

6.5.3 Determining $\gamma_b(T)$

For a diametrical path $P = v_1, \ldots, v_n$ of a tree $T$, let $L_i$ denote the component of $T - v_i v_{i+1}$ that contains $v_i$, and $R_{i+1} = T - L_i$. We now describe an easy procedure to recursively determine a maximum split-$P$ set $M = M_k$ of $T$.

By Corollary 6.7 this also determines $\gamma_b(T)$.

Initially, let $T_0 = T$, $P_0 = v_{0,1}, \ldots, v_{0,n_0}$ be a diametrical path of $T_0$, and $M_0 = A_0 = \emptyset$.

Once the tree $T_i$, a diametrical path $P_i = v_{i,1}, \ldots, v_{i,n_i}$ of $T_i$ and the sets $M_i$ and $A_i$ have been constructed, construct $T_{i+1}$, $P_{i+1}$, $M_{i+1}$ and $A_{i+1}$ as follows:

1. Find the smallest odd integer $t$ such that $\deg_{T_i} v_{i,t} = \deg_{T_i} v_{i,t+1} = 2$, and the paths $P_{i,t} = P_i \cap L_{i,t}$ and $P_{i,t+1} = P_i \cap R_{i,t+1}$ are diametrical paths of $L_{i,t}$ and $R_{i,t+1}$, respectively.

2. If

   (a) $|V(P_{i,t+1})|$ is odd, then let $M_{i+1} = M_i \cup \{v_{i,t} v_{i,t+1}\} \cup A_i$, $T_{i+1} = R_{i,t+1}$, $P_{i+1} = P_{i,t+1} = v_{i+1,1}, \ldots, v_{i+1,n_{i+1}}$, and $A_{i+1} = \emptyset$;

   (b) $|V(P_{i,t+1})|$ is even, then let $M_{i+1} = M_i$, $T_{i+1} = R_{i,t+1}$, $P_{i+1} = P_{i,t+1} = v_{i+1,1}, \ldots, v_{i+1,n_{i+1}}$, and $A_{i+1} = \{v_{i,t} v_{i,t+1}\}$.

Repeat 1 and 2 until no such integer $t$ exists. The set $M = M_k$ is the last set $M_{i+1}$ thus obtained.
6.5.4 An Interpolation Result

Consider $K_{1,n}$. It has a minimal dominating set of cardinality 1, and a minimal dominating set of cardinality $n$, and no minimal dominating set of any other size. The same is not true for broadcasts:

**Proposition 6.13** If $k$ is any integer such that $\gamma_b(T) \leq k \leq \text{rad } G$, then $T$ has a very efficient dominating broadcast of cost $k$.

**Proof.** Consider a very efficient $\gamma_b$-broadcast $f = f_0$, and a diametrical path $P$ of $T$. By Theorem 6.1, all broadcast vertices lie on $P$. If there are only two broadcast vertices, $u_1$ and $u_2$, let $v \in \text{Cen}(T)$ and define the broadcast $f_1$ by $f_1(v) = f_0(u_1) + f_0(u_2) + 1$ and $f_1(x) = 0$ otherwise. If $|V_f^+| \geq 3$, let $u_1, u_2$ and $u_3$ be three consecutive broadcast vertices on $P$. Let $P_1$ be the subpath of $P$ that is $f$-dominated by $\{u_1, u_2, u_3\}$ and note that $P_1$ has even length. Let $v$ be the central vertex of $P_1$ and define $f_1$ by $f_1(v) = f_0(u_1) + f_0(u_2) + f_0(u_3) + 1$, $f_1(u_1) = f_1(u_2) = f_1(u_3) = 0$, and $f_1(x) = f_0(x)$ otherwise. Note that $f_1$ is a very efficient dominating broadcast of $T$ with $\sigma(f_1) = \sigma(f_0) + 1$.

Repeat the above steps until just one broadcast vertex remains. After the $i^{th}$ step, the broadcast $f_i$ is a very efficient dominating broadcast of $T$ with $\sigma(f_i) = \sigma(f_{i-1}) + 1$. The process ends when the broadcast has only one broadcast vertex and cost $\text{rad}(T)$.

\[ \square \]

In Chapters 4 and 5 we considered special classes of trees; we compared and contrasted the properties of radial and nonradial trees. These properties
lead to further ideas and the thesis culminates in Chapter 6 with the characterization of radial trees. Many unsolved problems remain and we list some of them in Chapter 7.
Chapter 7

Future Research

After the characterization of radial trees there are some natural questions that arise. We conclude this thesis by describing a few related open problems for future research.

**Problem 7.1** Use the characterization of radial trees to find classes of radial graphs.

A uniquely radial tree is a radial tree for which the only $\gamma_b$-broadcasts possible are such that $V^+_f \subseteq \text{Cen}(T)$. For example, $P_7$ is not uniquely radial, but $P_5$ with a leaf added to the central vertex is uniquely radial.

**Problem 7.2** Characterize uniquely radial trees.

**Problem 7.3** More generally, determine which trees have a unique $\gamma_b$-broadcast, and which trees have more than one $\gamma_b$-broadcast. Does the number of maximum split-sets play a role?
By Corollary 4.3, a radial tree of radius $k$ has at least $3k - 2$ vertices.

**Problem 7.4** Characterize radial trees with radius $k$ and order $3k - 2$.

Note that there does not exist a function $f(k)$ such that if $T$ has radius $k$ and at least $f(k)$ vertices, then $T$ is radial, because for $k \geq 4$, $P_{2k+1}$ together with any number of leaves added to its support vertices is nonradial.

**Problem 7.5** Characterize the class of trees $T$ with $\gamma_b(T) = \gamma(T)$.

**Problem 7.6** Characterize the class of trees $T$ of order $n$ with $\gamma_b(T) = \lceil \frac{n}{3} \rceil$.

**Problem 7.7** Characterize the class of graphs $G$ of order $n$ with $\gamma_b(G) = \lceil \frac{n}{3} \rceil$, or find classes of such graphs.

**Problem 7.8** Determine whether Proposition 6.13 holds for graphs in general. Does it hold for special classes of graphs?
Bibliography


