INTERPOLATION OF D–STABILITY AND SIGN STABILITY

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An n–by–n real matrix $A = [a_{ij}]$ is called (positive) *stable* if each eigenvalue of A has positive real part. The matrix A is further called *D–stable* if DA is stable whenever D is a diagonal matrix with positive diagonal entries. We denote the set of n–by–n stable matrices by S_n and it closure, the set of semistable matrices, by $\bar{S}_n$.

We say that $B = [b_{ij}] \in M_n(\mathbb{R})$ has the same sign pattern as A if b_{ij} is positive <respectively, zero, negative> iff a_{ij} is positive <respectively, zero, negative> for all $i, j \in \{1,\cdots,n\}$. Matrix A is called *sign stable* if each matrix with the same sign pattern as A is stable. The sign stable matrices have been characterized in finitely computable terms [2], as have the sign semistable matrices [4,5], but the D–stable matrices have not been so characterized; see [3] for results on D–stability.

It is clear that a sign stable matrix is D–stable, but the converse is not in general true as the example $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ shows. We now define a parameterized sequence of classes of matrices that is nested; it turns out that the D–stable matrices lie at one end of this sequence while the sign stable matrices lie at the other. This raises natural questions as to the relationships between the intervening classes. These are especially interesting as the sign stable matrices are a combinatorially based class, while the D–stable matrices are more analytic (magnitudes matter). To make these ideas explicit we introduce some more notation; see [1] for any notation not defined here.

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Recall that the *Hadamard product* of A and B is denoted and defined by $A \circ B = [a_{ij}b_{ij}]$. For $k \leq n$, we let

$$P_{n,k} = \{B \in M_n(\mathbb{R}) \colon B > 0; \ \text{rank } B \leq k\}$$

in which $B > 0$ means entry–wise positivity. We may then define

$$D_{n,k} = \{A \in M_n(\mathbb{R}) \colon \ B \in P_{n,k} \Rightarrow A \circ B \in S_n\}$$

and

$$\hat{D}_{n,k} = \{A \in M_n(\mathbb{R}) \colon \ B \in P_{n.k} \Rightarrow A \circ B \in \bar{S}_n\}.$$

The set $D_{n,k}$ is the one of main interest to us, and we now give some observations.

Observation 1. $D_{n,1} = \{A \in M_n(\mathbb{R}) \colon \ A$ is D–stable$\}$.
This is true because $B \in P_{n,1}$ can be written as $B = xy^T$ in which x and y are n–vectors with positive components. Then

$$A \circ B = \text{diag}(x) \ A \ \text{diag}(y)$$

which is similar to $\text{diag}(y)\text{diag}(x)A$. As xy^T runs through $P_{n,1}$, $\text{diag}(y)\text{diag}(x)$ runs through all positive diagonal matrices.

Observation 2. $D_{n,n} = \{A \in M_n \ (\mathbb{R}) \colon \ A$ is sign stable$\}$.
This is true because as B runs through $P_{n,n}$, $A \circ B$ runs through all matrices with the same sign pattern as A.

Observation 3. $P_0^+ \supseteq D_{n,1} \supseteq D_{n,2} \supseteq \cdots \supseteq D_{n,n}$, in which

$$P_0^+ = \{A \in M_n(\mathbb{R}): \text{ every principal minor of } A \text{ is } \geq 0 \text{ and at least one of each order is } > 0\}$$

This observation shows that the classes $D_{n,k}$ "interpolate" the D–stable and sign stable matrices.

Observation 4. $A \in D_{n,k}$ iff

(a) $A^T \in D_{n,k}$,

(b) $DAE \in D_{n,k}$ for D, E any positive diagonal matrices, or

(c) $P^TAP \in D_{n,k}$ for P any permutation matrix.

(a) holds as $A^T \circ B^T = (A \circ B)^T$ and $B \in P_{n,k}$ iff $B^T \in P_{n,k}$.
(b) holds because $DAE \circ B = A \circ DBE$ and $DBE \in P_{n,k}$ iff $B \in P_{n,k}$.
(c) holds because $A \circ B$ is similar to $P^T(A \circ B)P = P^TAP \circ P^TBP$.

Observation 5. The inclusion between $D_{n,1}$ and $D_{n,2}$ is strict as can be seen by considering the following example with $n = 3$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 0.5 \end{bmatrix}.$$

From [3], $A \in D_{3,1}$, as $A + A^T$ is positive definite. But $A \notin D_{3,2}$, because if

$$B = \begin{bmatrix} 1 & 8 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(which has rank 2), then $A \circ B$ has negative determinant, and so is not stable.

That $D_{n,1} \not\supseteq D_{n,2}$ for all $n \geq 4$ can be seen from the above example by taking the above A and forming the direct sum $A \oplus I_{n-3}$. Then $(A \oplus I_{n-3}) \circ \begin{bmatrix} B & K \\ K^T & J \end{bmatrix}$, where K, J are matrices with every entry 1, has negative determinant.

Consider the class of matrices

$$A(\alpha,\beta,\gamma) = \begin{bmatrix} \alpha & 1 & 1 \\ -1 & \beta & 1 \\ -1 & -1 & \gamma \end{bmatrix}$$

where α, β, $\gamma > 0$; note that the example above is in this class. $A(\alpha,\beta,\gamma) \in D_{3,1}$, but $A(\alpha,\beta,\gamma) \notin D_{3,3}$ as its digraph $D(A)$ has a cycle of length 3, see [2]. For this particular class, is it possible to find α, β, γ so that $A(\alpha,\beta,\gamma) \in D_{3,2}$?

Observation 6. If $A \in D_{n,1}$ then $A^{-1} \in D_{n,1}$.
The corresponding result does not hold in general for $D_{n,k}$, $k > 1$, as can be seen from the example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ with } B = \begin{bmatrix} 1 & 8 & 22 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}, \text{ so } B \text{ has rank 2.}$$

The matrix $A^{-1} \circ B$ has eigenvalues $\{2.216, -.108 \pm .831i\}$ and so is unstable, thus $A^{-1} \notin D_{3,2}$. Note that A is sign stable, but its inverse is full and so it is obviously not in $D_{3,3}$.

Observation 7. If $A \in D_{n,k}$, then $A[\{i_1,\cdots,i_m\}] \in \hat{D}_{m,k}$ for $k \le m \le n$, where $A[\{i_1,\cdots,i_m\}]$ is the principal submatrix of A including rows (and columns) indicated.

Note that $\hat{D}_{m,k}$ cannot in general be replaced by $D_{m,k}$ in this observation, as shown by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. This observation is proved by partitioning B in the form $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ where $B_{11} \in M_m(\mathbb{R})$ has rank k, and (w.l.o.g. by Observation 4(c)) $\{i_1,\cdots,i_m\} = \{1,\cdots,m\}$. Then B_{12}, B_{21}, B_{22} can be chosen so that rank B is still k and $A \circ B$ is arbitrarily close to $\begin{bmatrix} A[\{1,\cdots,m\}] \circ B_{11} & 0 \\ 0 & 0 \end{bmatrix}$. The result then follows by continuity.

Observation 8. If $A \in D_{n,2}$, then in $D(A)$ there are:

(i) no negative 1–cycles,

(ii) no positive 2–cycles, and

(iii) no minimal simple cycles of length ≥ 3.

Here a *minimal simple cycle* is defined as a simple cycle such that no proper subset of ≥ 2 nodes of the cycle are themselves nodes of a cycle; note that 1 cycles are not counted. Assume that $A \in D_{n,2}$, then by Observation 7, every principal submatrix $\in \hat{D}_{2,2} =$ {2–by–2 sign semistable matrices}. So (i) and (ii) follow from [4, 5]. Part (iii) follows by obtaining a contradiction with a rank 2 matrix B which has $b_{ij} = 1$ for all i, j except for a large entry α on the minimal simple cycle of length $\ell \ge 3$ in $D(A)$. The eigenvalues of $A \circ B$ are then arbitrarily close to the roots of $\lambda^{\ell} = \pm\alpha$. For example, if $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, take $B = \begin{bmatrix} 1 & \alpha & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ as $D(A)$ has the 3–cycle from $a_{12}a_{23}a_{31}$. For $\alpha > 8$, $A \notin D_{3,2}$.

Observation 9. If $A \in D_{n,k}$ for $k \geq 3$, then $D(A)$ has no cycle of length ℓ where $3 \leq \ell \leq k$.

This is true because, by Observation 7, any k–by–k principal submatrix of $A \in \hat{D}_{k,k}$, and so (by [4,5]) can contain no cycle of length ≥ 3 in its digraph.

These observations lead to the following problems.

Problem 1. As a general question, what else can be said about $D_{n,k}$, $1 < k < n$, and effective characterizations of each class? How do transitions between one class and the next occur? In particular, is each inclusion in Observation 3 strict, or, perhaps is there a single transition (from $D_{n,2}$ to $D_{n,1}$) with all remaining inclusions equalities? Most specifically, is $D_{3,2} = D_{3,3}$ or does $D_{3,2} \supsetneq D_{3,3}$?

Problem 2. Where do other classes of matrices (such as those with diagonal Lyapunov solutions or the totally stable matrices, etc.) fit amongst the classes $D_{n,k}$?

Problem 3. Which classes $D_{n,k}$ are monotone with respect to the diagonal entries, i.e. $A \in D_{n,k}$, E a nonnegative diagonal matrix implies $A + E \in D_{n,k}$? For $n \geq 4$, $D_{n,1}$ is known not to be monotone, but, of course $D_{n,n}$ is always.

Problem 4. For which n,k is the topological closure of $D_{n,k}$ (call it $\bar{D}_{n,k}$) equal to $\hat{D}_{n,k}$?

Problem 5. Define $D^{+}_{n,k} = \{A \in M_n(\mathbb{R})\colon\ A \circ B \in S_n$ for all $B = B_1 + \cdots + B_k$ with $B_i > 0$ and rank $B_i = 1$, $i = 1,\cdots,k\}$. Then $D^{+}_{n,1} = D_{n,1}$, and $D^{+}_{n,k} \supseteq D_{n,k}$, but is the latter containment strict or an equality for $k \geq 2$?

Problem 6. Note that the same kind of interpolation may be considered between ordinary matrices and qualitative matrices defined by other properties of interest. For example, we suggest for study another sequence of classes of matrices by defining

$$L_{n,k} = \{A \in M_n(\mathbb{R}):\ B \in P_{n,k} \Rightarrow A \circ B \text{ is nonsingular}\}.$$

Note that $L_{n,1}$ is the set of nonsingular matrices, and $L_{n,n}$ is the set of sign nonsingular matrices (*cf.* Observations 1 and 2). In particular, how do transitions occur between these classes?

Several of these questions re–occurred to the first author after a pleasant conversation with Prof. Bruce Clarke, Dept. Chemistry, University of Alberta, about the stability of chemical reactions at a workshop, CRM, Montréal, Sept. 1986.

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