A NOTE ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH REAL PART GREATER THAN α

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ABSTRACT

The object of the present paper is to derive several further properties of the subclass $\mathscr{P}_b(\alpha)$ of analytic functions with real part greater than α, which was introduced and studied by C.P. McCarty ([1], [2]). The main results of this paper provide interesting generalizations of the corresponding observations made by D.Z. Pashkouleva [3].

1. INTRODUCTION

Let $\mathscr{P}(\alpha)$ be the class of functions of the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{1.1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z: |z| < 1\}$ and satisfy the inequality:

$$\text{Re}\{p(z)\} > \alpha \qquad (z \in \mathscr{U}) \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$. A function $p(z)$ belonging to the class $\mathscr{P}(\alpha)$ is said to be in the class $\mathscr{P}_b(\alpha)$ if it also satisfies the condition:

$$p'(0) \equiv p_1 = 2b(1-\alpha) \qquad (0 \leqq b \leqq 1). \tag{1.3}$$

The class $\mathscr{P}_b(\alpha)$ was introduced and studied systematically by McCarty ([1], [2]) who presented several interesting inequalities as well as distortion, covering, and radius of convexity properties associated with this class. In particular, the class $\mathscr{P}_b(0)$ was studied recently by Pashkouleva [3]. In the present paper we derive several further properties of the general class $\mathscr{P}_b(\alpha)$. Our main results (Theorems 1, 2, and 3 below) provide interesting generalizations of the corresponding results given earlier by Pashkouleva [3].

2. GENERAL INEQUALITIES FOR THE CLASS $\mathscr{P}_b(\alpha)$

In order to prove one of our main results, giving certain general inequalities associated with the class $\mathscr{P}_b(\alpha)$, we shall need the following lemmas.

LEMMA 1 (McCarty [1, p. 213, Corollary 2]). If the function $p(z)$ defined by (1.1) is in the class $\mathscr{P}_b(\alpha)$, then

$$|p'(z)| \leqq \frac{2(\mathrm{Re}\{p(z)\}-\alpha)(b+2r+br^2)}{(1-r^2)(1+2br+r^2)} \tag{2.1}$$

for $r = |z| < 1$.

LEMMA 2 (McCarty [2, p. 154, Lemma 1]). If the function $f(z)$ defined by (1.1) is in the class $\mathscr{P}_b(\alpha)$, then

$$|p(z)-A_b| \leqq D_b, \tag{2.2}$$

where

$$A_b = \frac{(1+br)^2 - (2\alpha-1)(b+r)^2\ r^2}{(1-r^2)(1+2br+r^2)}, \tag{2.3}$$

(2.4) $$D_b = \frac{2r(1-\alpha)(b+r)(1+br)}{(1-r^2)(1+2br+r^2)},$$

and $r = |z| < 1$.

Now we state our first main result contained in

THEOREM 1. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{P}_b(\alpha)$, *and if* $a \geqq 0$, *then*

(2.5) $$\mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)+a}\right\} \geqq (\mathrm{Re}\{p(z)\}-\alpha)(1-E_b) + \alpha$$

and

(2.6) $$\mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)+a}\right\} \leqq (\mathrm{Re}\{p(z)\}-\alpha)(1+E_b) + \alpha,$$

where

(2.7) $$E_b = \frac{2r(b+2r+br^2)}{1 + a + 2(\alpha+a)br - 2(1-\alpha)r^2 - 2(\alpha+a)br^3 - (2\alpha-1+a)r^4}$$

and $r = |z| < 1$.

PROOF. With the help of Lemma 2, we observe that

(2.8) $$|p(z)+a| \geqq |A_b+a| - D_b$$

$$= \frac{1 + a + 2(\alpha+a)br - 2(1-\alpha)r^2 - 2(\alpha+a)br^3 - (2\alpha+a-1)r^4}{(1-r^2)(1+2br+r^2)}.$$

Therefore, an application of Lemma 1 yields

(2.9) $$\mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)+a}\right\} \geqq \mathrm{Re}\{p(z)\} - \left|\frac{zp'(z)}{p(z)+a}\right|$$

$$\geqq \mathrm{Re}\{p(z)\} - (\mathrm{Re}\{p(z)\}-\alpha)E_b$$

$$= (\mathrm{Re}\{p(z)\}-\alpha)(1-E_b) + \alpha,$$

which proves the assertion (2.5).

Furthermore, we have

(2.10) $$\mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)+a}\right\} \leqq \mathrm{Re}\{p(z)\} + \left|\frac{zp'(z)}{p(z)+a}\right|$$

$$\leqq \mathrm{Re}\{p(z)\} + (\mathrm{Re}\{p(z)\}-\alpha)E_b$$

$$= (\mathrm{Re}\{p(z)\}-\alpha)(1+E_b) + \alpha,$$

which gives (2.6). Thus we complete the proof of Theorem 1.

Taking $\alpha = 0$ in Theorem 1, we have

COROLLARY 1 (cf. Pashkouleva [3, p. 507, Theorem 1]). If the function p(z) defined by (1.1) is in the class $\mathscr{P}_b(0)$, then

(2.11) $$\operatorname{Re}\{p(z)\}(1-F_b) \leqq \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)+a}\right\} \leqq \operatorname{Re}\{p(z)\}(1+F_b),$$

where

(2.12) $$F_b = \frac{2r(b+2r+br^2)}{1 + a + 2abr - 2r^2 - 2abr^3 - (a-1)r^4}$$

and $r = |z| < 1$.

Further, letting $a = 0$, Theorem 1 becomes

COROLLARY 2. If the function p(z) defined by (1.1) is in the class $\mathscr{P}_b(\alpha)$, then

(2.13) $$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} \geqq (\operatorname{Re}\{p(z)\}-\alpha)(1-G_b) + \alpha$$

and

(2.14) $$\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} \leqq (\operatorname{Re}\{p(z)\}-\alpha)(1+G_b) + \alpha,$$

where

(2.15) $$G_b = \frac{2r(b+2r+br^2)}{1 + 2\alpha br - 2(1-\alpha)r^2 - 2\alpha br^3 - (2\alpha-1)r^4}$$

and $r = |z| < 1$.

3. FURTHER CLASSES RELATED TO THE CLASS $\mathcal{P}_b(\alpha)$

Suppose that the function

(3.1) $$F(z) = z + 2b(1-\alpha)z^2 + q_3 z^3 + \cdots$$

is analytic in the open unit disk $\mathcal{U}$. Then $F(z)$ defined by (3.1) is said to be in the class $\mathcal{S}_b^*(\alpha)$ if and only if

$$\frac{zF'(z)}{F(z)} \in \mathcal{P}_b(\alpha) \quad \text{for } z \in \mathcal{U}.$$

Further, let $\mathcal{C}_b(\alpha)$ be the class of functions $F(z)$ defined by (3.1) such that

$$1 + \frac{zF''(z)}{F'(z)} \in \mathcal{P}_b(\alpha) \quad \text{for } z \in \mathcal{U}.$$

We begin by proving

THEOREM 2. *If the function* $F(z)$ *defined by* (3.1) *is in the class* $\mathcal{S}_b^*(\alpha)$, *then the function* $f(z)$ *given by*

(3.2) $$f(z) = \frac{1}{a+1}\,[aF(z)+zF'(z)] \qquad (a \geq 0)$$

<u>is starlike of order</u> α <u>for</u> $|z| < r_b$, <u>where</u> r_b <u>is the smallest root in the closed interval</u> $[0, 1]$ <u>of the equation</u>

$$(3.3) \qquad 1 + a + 2(\alpha+a-b)r + 2(\alpha-3)r^2 - 2b(\alpha+a+1)r^3 - (2\alpha+a-1)r^4 = 0.$$

PROOF. For $F(z) \in \mathscr{S}_b^*(\alpha)$, we may write

$$(3.4) \qquad \frac{zF'(z)}{F(z)} = p(z) \qquad [p(z) \in \mathscr{P}_b(\alpha)].$$

Therefore, using Theorem 1, we have

$$(3.5) \qquad \begin{aligned} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &= \mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z) + a}\right\} \\ &\geqq (\mathrm{Re}\{p(z)\}-\alpha)(1-E_b) + \alpha \\ &> \alpha \end{aligned}$$

for $|z| < r_b$, where r_b denotes the smallest root in $[0, 1]$ of Equation (3.3).

This proves the assertion that $f(z)$ defined by (3.2) is starlike of order α for $|z| < r_b$.

REMARK 1. As already observed by Pashkouleva [3, p. 508], Theorem 2 when $\alpha = 0$ is sharp in case the function $F(z)$ is given by

$$(3.6) \qquad F(z) = \frac{z}{1 - 2bz + z^2}.$$

Finally, we prove

THEOREM 3. *If the function* F(z) *defined by* (3.1) *is in the class* $\mathscr{C}_b(\alpha)$, *then the function* f(z) *given by* (3.2) *is convex of order* α *for* $|z| < r_b$, *where* r_b *is the smallest root in the closed interval* [0, 1] *of Equation* (3.3).

PROOF. Letting

$$1 + \frac{zF''(z)}{F'(z)} = p(z) \qquad [p(z) \in \mathscr{P}_b(\alpha)], \tag{3.7}$$

and using Theorem 1, we see that

$$\begin{aligned} \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &= \mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z) + a}\right\} \\ &\geqq (\mathrm{Re}\{p(z)\} - \alpha)(1 - E_b) + \alpha \\ &> \alpha \end{aligned} \tag{3.8}$$

for $|z| < r_b$, where r_b denotes the smallest root in [0, 1] of Equation (3.3).

This evidently completes the proof of Theorem 3.

REMARK 2. Theorem 3 when $\alpha = 0$ is sharp for the function F(z) given by (3.6) (*cf.* [3]).

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