

SPECTRUM PRESERVING MAPS OF FACTORS

By

C. ROBERT MIERS

DM-431-IR

JANUARY 1987

Spectrum Preserving Maps of Factors¹C. Robert Miers²

In Memory of Henry Dye

ABSTRACT. Let M and N be von Neumann algebras with M a factor, $S(M)$ and $S(N)$ the real linear spaces of self-adjoint operators in M and N respectively, and let $\varphi: S(M) \rightarrow S(N)$ be a spectrum preserving surjective linear map. It is shown that $\varphi(A) = \tilde{\varphi}(A)$ for all $A \in S(M)$ where $\tilde{\varphi}$ is a $*$ -isomorphism or a $*$ -anti-isomorphism of M onto N .

1. INTRODUCTION. In [4] Jafarian and Sourour characterized spectrum preserving linear maps between the real linear spaces $S(H_1)$ and $S(H_2)$ of self-adjoint operators on the (real or complex) Hilbert spaces H_1 and H_2 . They showed that such a map φ is either of the form $\varphi(A) = UAU^*$ or $\varphi(A) = UA^tU^*$ where U is a unitary operator and A^t is the transpose with respect to some fixed orthonormal basis. In the proof of their theorem for complex Hilbert spaces they used the fact that every self-adjoint operator is a real linear combination of a finite number of projections and also a result of Fillmore and Longstaff [3, Theorem 1] characterizing isomorphisms between the lattices of closed subspaces of infinite dimensional complex normed linear spaces. Our proof for von Neumann factors parallels this development with the following orthoisomorphism theorem of Dye [1, Theorem 1, Corollary] taking the place of

¹1980 Mathematics Subject Classification (1985 Revision) Primary 46L10.
Key words and phrases. von Neumann algebra, spectrum preserving map, orthomorphism.

²This research was partially supported by a grant from the NSERC of Canada.

the Fillmore-Longstaff result:

THEOREM. Let M be a W^* -algebra with no direct summands of type I_2 . Then any projection orthoisomorphism of M on a W^* -algebra N is implemented by the direct sum of a $*$ -isomorphism and a $*$ -anti-isomorphism.

2. NOTATION AND TERMINOLOGY.

A von Neumann algebra M is a weakly closed, self-adjoint algebra of bounded linear operators on a complex Hilbert space H containing the identity operator I . The set $Z_M = \{S \in M \mid ST = TS \text{ for all } T \in M\}$ is called the center of M . If $Z_M = \{\lambda I \mid \lambda \in \mathbb{C}\}$ then M is called a factor. By a projection we mean a self-adjoint idempotent. A projection orthoisomorphism between von Neumann algebras M and N is a one-one mapping θ of the set of projections in M on that in N which preserves orthogonality in the sense that $PQ = 0$ iff $\theta(P)\theta(Q) = 0$. If T is a bounded linear operator on H we let $\sigma(T)$ be its spectrum. If M and N are von Neumann algebras and $S(M)$, $S(N)$ the real linear spaces of all self-adjoint operators in M and N respectively, we say that a surjective map $\varphi: S(M) \rightarrow S(N)$ is spectrum preserving if $\sigma(\varphi(A)) = \sigma(A)$ for each $A \in S(M)$.

3. THE THEOREM.

The first lemma is part of [4, Lemma 1]. The proof is included for completeness.

Lemma 1. Let $\varphi: S(M) \rightarrow S(N)$ be a surjective spectrum preserving map where M and N are von Neumann algebras. Then

- (i) φ is one-one
- (ii) $\varphi(I) = I$
- (iii) $\varphi(P)$ is a projection iff P is a projection
- (iv) If P and Q are projections in M then $PQ = 0$ iff $\varphi(P)\varphi(Q) = 0$.

Proof: (i) If $\varphi(A) = 0$ then $\{0\} = \sigma(\varphi(A)) = \sigma(A)$. Since A is self-adjoint, $A = 0$.

(ii) $\varphi(I)$ is a self-adjoint operator whose spectrum is $\sigma(I) = \{1\}$.

(iii) Since φ is one-one it is immediate that φ^{-1} is also spectrum preserving. A self-adjoint operator is a projection iff its spectrum is a subset of $\{0,1\}$, hence the result.

(iv) If both P and Q are projections, then $P + Q$ is a projection iff $PQ = 0$. Since $\varphi(P+Q) = \varphi(P) + \varphi(Q)$ the result follows from (iii).

Lemma 2. Let φ be as in Lemma 1 and assume that M has no direct summands of type I_2 . There exists a map $\tilde{\varphi}$ of M onto N which coincides with φ on the projections of M and which is the direct sum of a $*$ -isomorphism and a $*$ -anti-isomorphism.

Proof: This is Dye's theorem applied to the projection orthoisomorphism φ .

THEOREM. Let $\varphi: S(M) \rightarrow S(N)$ be a surjective spectrum preserving map where M is a factor. For each $A \in S(M)$, $\varphi(A) = \tilde{\varphi}(A)$ where $\tilde{\varphi}$ is a $*$ -isomorphism or a $*$ -anti-isomorphism of M onto N . In particular, N is a factor which is $*$ -isomorphic or $*$ -anti-isomorphic to M .

Proof: We divide the proof into two cases.

Case 1. M is a factor of type I_2 . (i.e. $M \cong M_2(\mathbb{C})$, the 2×2 matrices over \mathbb{C}). $S(M)$ is thus a four-dimensional real space and, therefore, so is $S(N)$. By the structure theory for von Neumann algebras, N is either a factor of type I_2 , or N is the direct sum of four one-dimensional algebras. In the latter case N would be abelian so that if P' and Q' were projections in N we would have $\sigma(P'+Q') \subseteq \{0,1,2\}$. But if $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ in M , then $\sigma(P+Q) = \sigma(\varphi(P+Q)) = \sigma(\varphi(P)+\varphi(Q)) = \sigma(P'+Q') \notin \{0,1,2\}$. Hence N is of type I_2 . The result now follows by the Jafarian-Sourour theorem.

Case 2. M is not a factor of type I_2 . Since M is a factor there are no central projections, save 0 and I , so that Lemma 2 implies the existence of a map $\tilde{\varphi}$ which is either a $*$ -isomorphism or a $*$ -anti-isomorphism of M onto N and such that $\varphi(P) = \tilde{\varphi}(P)$ on the projections of M . If M is a factor of type I_n , the finite-dimensional spectral theorem implies that each self-adjoint $A \in M$ is the real linear combination of projections. This is also true for any properly infinite von Neumann algebra by [5]. It was proved in [2] that any operator in a factor of type II_1 is a finite linear combination of projections so that any self-adjoint operator in such a factor is a finite real linear combination of projections. Thus, since φ is real linear, $\varphi(A) = \tilde{\varphi}(A)$ for all $A \in S(M)$.

Bibliography

1. H.A. Dye, On the geometry of projections in certain operator algebras, *Ann. of Math.* (2) 61(1955), 73-89.
2. T. Fack, and P. de la Harpe, Sommes de commutateurs dans le algebres de von Neumann finies continues, *Ann. Inst. Fourier* 30(1980), 49-73.
3. P.A. Fillmore, and W.E. Longstaff, On isomorphisms of lattices of closed subspaces, *Can. J. Math.* 36(1984), 820-829.
4. A.A. Jafarian, and A.R. Sourour, Spectrum preserving linear maps on the space of self-adjoint operators, *J. Functional Analysis* 66(1986), 255-261.
5. C. Pearcy, and D. Topping, Sums of small numbers of idempotents, *Mich. Math. J.* 14(1967), 453-465.