APPROXIMATION BY NORMAL ELEMENTS
WITH FINITE SPECTRA IN SIMPLE
AF–ALGEBRAS

by

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Approximation by Normal Elements with Finite Spectra in Simple AF-Algebras

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Abstract

We show that normal elements in simple $AF$-algebras with countably many extremal traces (such as $UHF$-algebras and matroid algebras) can be approximated by normal elements with finite spectra. Other $AF$-algebras are shown to have the same property.

Key words: Simple $AF$-algebras, approximation, normal elements, finite spectrum. 1990 AMS Mathematical Subject Classification 46L05
1. Introduction

A $C^*$-algebra $A$ is called an $AF$-algebra if for any $\epsilon > 0$ and finitely many elements $a_1, a_2, ..., a_n \in A$, there are a finite dimensional $C^*$-subalgebra $B$ of $A$ and elements $b_1, b_2, ..., b_n \in B$ such that

$$\|a_i - b_i\| < \epsilon, \ i = 1, 2, ..., n.$$ 

$AF$-algebras have been intensively studied (See [Br],[Eff],[Ell] etc. It is almost impossible to give a complete list.) and appear to be most understandable $C^*$-algebras. The most interesting $AF$-algebras are those simple separable ones, such as matroid algebras (see [Dix]) and UHF-algebras. One question concerning $AF$-algebras is the following question:

$Q_1$ Can every normal element $x$ in an $AF$-algebra $A$ be approximated (in the norm topology) by normal elements in $A$ with finite spectra?

The question was mentioned in [P, 3.11] but we fail to locate its origional source. In [Bl1,2.6], a $C^*$-algebra $A$ is said to have $(FN)$ if every normal element in $A$ is a norm limit of elements with the form $\sum_{k=1}^{n} \lambda_k p_k$, where $\lambda_k$ are complex numbers in the spectrum of $x$ and $p_k$ are mutually orthogonal projections in $A$. So if the answer to the $Q_1$ is affirmative, then every $AF$-algebra has $(FN)$. It is known that every Von Neumann algebra and every $AW^*$-algebra have $(FN)$. It is recently proved that the corona algebras of finite matroid algebras have $(FN)$ ([Lin2]). In [P], Chris Phillips constructs two separable simple $C^*$-algebras have $(FN)$ (They are not $AF$). To this author's knowledge, there are no other non-commutative $C^*$-algebras are known to have $(FN)$.

If $x$ is a self adjoint element in an $AF$-algebra, then for any $\epsilon > 0$, there are a finite dimensional $C^*$-subalgebra and an element $y \in B$ such that

$$\|x - y\| < \epsilon.$$ 

Set $y_1 = 1/2(y + y^*)$, then $\|x - y_1\| < \epsilon$. Since $y_1 \in B$, $y_1$ has finite spectrum. So the question $Q_1$ has an affirmative answer for selfadjoint elements. If $x$
is a unitary, then the unitary part of the polar decomposition of $y$ (in $B$) is close to $x$, provided that $\epsilon$ is small enough. So the question $Q_1$ has an affirmative answer for unitaries too. In general, if $x$ is a normal element, one hopes that $y$ is close to a normal element (in $B$). In fact,

$$\|yy^* - y^*y\| < \epsilon.$$ 

Therefore if the following finite dimensional problem:

$Q_2$ Given $\epsilon > 0$, is there a $\delta > 0$ so that whenever $y$ is a norm 1 element in a finite dimensional $C^*$-algebra $B$ such that $\|yy^* - y^*y\| < \delta$, there is a normal element $z \in B$ satisfying $\|y - z\| < \epsilon$?

has a positive solution, then one can also give an affirmative answer to the question $Q_1$. The problem $Q_2$ is equivalent to the question whether two almost commuting Hermitian matrices are close to a commuting pair of Hermitian matrices. But this is an old problem in linear algebra (see [V1],[V2] and [D]) and, unfortunately, remains open today.

Without making any effort to solve the problem $Q_2$, we would like to shed some light on the question $Q_1$. We will show that the answer to $Q_1$ is affirmative, if $A$ is a matroid algebra (in particular, $A$ is a $UHF$-algebra). In fact, we show that for a more general class of simple $AF$-algebras, the answer to the question $Q_1$ is affirmative.

Let $A$ be a separable simple $AF$-algebra. Fix a nonzero projection $e \in A$, let $T$ be the set of those (lower semi-continuous and semi-finite) traces $\tau$ such that $\tau(e) = 1$. With weak*-topology, $T$ is a compact convex set. It is easy to see that the compact convex space $T$ does not depend on the choices of the projection $e \in A$. If $\tau \in T$ is an extreme point of $T$, then we say that $\tau$ is an extremal trace. The main result of this note is the following:

**Theorem A** Let $A$ be a separable simple $C^*$-algebra. Suppose that there is a countable subset $\{\tau_n\}$ of $T$ such that every $\tau \in T$ has the following form:

$$\tau = \sum_{n=1}^{\infty} \alpha_n \tau_n,$$
where $\alpha_n \geq 0$ and $\sum_{n=1}^{\infty} \alpha_n = 1$. Then for every normal element $x \in A$ and $\epsilon > 0$, there are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the spectrum of $x$ and mutually orthogonal projections $p_1, p_2, \ldots, p_n \in A$ such that

$$\|x - \sum_{i=1}^{n} \lambda_i p_i\| < \epsilon.$$ 

It is known ([S, 3.1.8]) that $T$ is a metrizable Choquet simplex. By [Al, I.49], every point in $T$ is a barycenter of a measure concentrated on its extreme points. Therefore we have the following:

**Corollary B** Let $A$ be a separable simple AF-algebra with countably many extremal traces. Then for every normal elements $x \in A$ and $\epsilon > 0$, there are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ in the spectrum of $x$ and mutually orthogonal projections $p_1, p_2, \ldots, p_n \in A$ such that

$$\|x - \sum_{i=1}^{n} \lambda_i p_i\| < \epsilon.$$ 

Since matroid algebra has unique trace (up to the scalar multiples), we immediately have the following:

**Corollary C** Every matroid algebra has $(FN)$.

**Corollary D** Every UHF-algebra has $(FN)$.

We also have the following result for non-simple AF-algebras:

**Theorem E** Let $A$ be a separable unital AF-algebra. Suppose that every (lower semi-continuous and semi-finite) trace $\tau$, $\tau(1) < \infty$. Set $T$ be the set of (lower semi-continuous and semi-finite) traces $\tau$ such that $\tau(1) = 1$. If there is a countable subset $\{\tau_n\} \subset T$ such that for every trace $\tau \in T$, there is a sequence of nonnegative numbers $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$ and

$$\tau = \sum_{n=1}^{\infty} \alpha_n \tau_n.$$ 

Then $A$ has $(FN)$. 

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Before we end the introduction, we would like to state a few terminologies we will use later.

Let $A$ be a separable $AF$-algebra and $p \in A$ be a nonzero projection. We use the notation $[p]$ for the equivalence class of projections containing $p$. Those are the projections $q$ such that there is a partial isometry $u \in A$ with the property that $u^*u = p$, $uu^* = q$. We write $[p] > [q]$, if there is a partial isometry $u \in A$ such that $u^*u = q$, $uu^* \leq p$ and $uu^* \neq p$.

Let $p \in A^{**}$ be an open projection, where $A^{**}$ is the enveloping von-Neumann algebra of $A$. We use the notation $Her(p)$ for the hereditary $C^*$-subalgebra $pA^{**}p \cap A$.

2. Proof of the results

**Lemma 1** Let $A$ be a unital $C^*$-algebra and $x$ be a normal element in $A$. Then for any $\epsilon > 0$, there is $\delta > 0$ for any finitely many points $\lambda_1, \lambda_2, \ldots, \lambda_n \in sp(x)$, if $S_k$ are open subsets of $sp(x)$ such that

$$|z - \lambda_k| < \delta$$

for all $z \in S_k$, $S_k \cap S_{k'} = \emptyset$ if $k \neq k'$, $k = 1, 2, \ldots, n$ and projections $p_k \in Her(q_k), k = 1, 2, \ldots, n$, where $q_k$ is the spectral projection in $A^{**}$ corresponding the open subset $S_k$, then

$$\|x - (y + \sum_{k=1}^{n} \lambda_k p_k)\| < \epsilon,$$

where $y = (1 - \sum_{i=1}^{n} p_i)x(1 - \sum_{i=1}^{n} p_i)$ and

$$\|(1 - \sum_{i=1}^{n} p_i)x - x(1 - \sum_{i=1}^{n} p_i)\| < \epsilon.$$

**Proof:** We have

$$\|x(\sum_{k=1}^{n} p_k) - \sum_{k=1}^{n} \lambda_k p_k\| = \|x(\sum_{k=1}^{n} q_k)(\sum_{k=1}^{n} p_k) - (\sum_{k=1}^{n} \lambda_k p_k)\|.$$
\[ = \left\| \sum_{k=1}^{n} q_k (x - \lambda_k q_k) p_k \right\| \leq \delta. \]

Similarly,
\[ \left\| \left( \sum_{k=1}^{n} p_k \right) x - \sum_{k=1}^{n} \lambda_k p_k \right\| \leq \delta. \]

Moreover,
\[
\begin{align*}
\left\| \left( 1 - \sum_{k=1}^{n} p_k \right) x \left( 1 - \sum_{k=1}^{n} p_k \right) - x \left( 1 - \sum_{k=1}^{n} p_k \right) \right\| \\
= \left\| \sum_{k=1}^{n} p_k x \left( 1 - \sum_{k=1}^{n} p_k \right) \right\| \\
= \left\| \sum_{k=1}^{n} p_k x - \sum_{k=1}^{n} \lambda_k p_k \left( 1 - \sum_{k=1}^{n} p_k \right) \right\| < \delta.
\end{align*}
\]

Similarly,
\[ \left\| \left( 1 - \sum_{i=1}^{n} p_i \right) x - \left( 1 - \sum_{i=1}^{n} p_i \right) x \left( 1 - \sum_{i=1}^{n} p_i \right) \right\| < \delta. \]

Set
\[ y = \left( 1 - \sum_{k=1}^{n} p_k \right) x \left( 1 - \sum_{k=1}^{n} p_k \right). \]

Then
\[ \left\| x - \left( y + \sum_{k=1}^{n} \lambda_k p_k \right) \right\| < 2\delta \]

and
\[ \left\| \left( 1 - \sum_{i=1}^{n} p_i \right) x - x \left( 1 - \sum_{i=1}^{n} p_i \right) \right\| < 2\delta. \]

So take \( \delta = \epsilon/2 \).

Q.E.D.

**Lemma 2** Let \( A \) be a separable simple unital AF-algebra satisfying the trace condition described in Theorem A and \( x \) be a normal element in \( A \). For any \( \epsilon > 0 \) and \( K > 0 \) there are open subsets \( O_1, O_2, ..., O_n \) such that
\[ O_i \cap O_j = \emptyset, \ [\bigcup_{i=1}^{n} O_i]^- = sp(x), \]

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\[ \lambda_i \in O_i, \text{ projections } p_i \in \text{Her}(q_i), \text{ where } q_i \text{ are spectral projections of } x \text{ in } A^{**} \text{ corresponding to the open subsets } O_i \text{ such that} \]
\[ \| x - (y + \sum_{i=1}^{n} \lambda_i p_i) \| < \epsilon, \]
where \[ y = (1 - \sum_{i=1}^{n} p_i) x (1 - \sum_{i=1}^{n} p_i), \]
\[ \|(1 - \sum_{i=1}^{n} p_i) x - x (1 - \sum_{i=1}^{n} p_i)\| < \epsilon \]
and
\[ [p_k] > K [1 - \sum_{i=1}^{n} p_i]. \]

Proof: Without loss of generality, we may assume that \( \|x\| \leq 1 \). Let \( D \) denote the unit disk. For any open subset \( O \subset D \), let \( q_o \) be the spectral projection of \( x \) in \( A^{**} \) corresponding to the open subset \( O \). The projection \( q_o \) is an open projection in \( A^{**} \). Let \( B_o \) be the hereditary \( C^* \)-subalgebra of \( A \) corresponding to the open projection \( q_o \) and let \( \{ e_n^o \} \) be an approximate identity for \( B_o \) consisting of projections. Let \( \tau \) be a trace on \( A \) with \( \tau(1) = 1 \). Define
\[ \mu_\tau(O) = \sup \{ \tau(e_n^o) \}. \]

By defining
\[ \mu_\tau(B) = \sup \{ \mu_\tau(O) : B \subset O, O \text{ open} \}, \]
we know that, from measure theory, this \( \mu_\tau \) defines a normalized Borel measure on \( D \). Let \( T_0 \) denote the countable subset \( \{ \tau_n \} \). For the simplicity, we use the notation \( \mu_i \) for the measure \( \mu_\tau \).

Let \( \{ L_i \} \) be a set of finitely many straight line segments in \( D \) such that \( D \setminus (\cup L_i \cup S^1) \) is a disjoint union of finitely many open subsets which have diameter \( < \epsilon/4 \). Take a finite subsets \( \{ \zeta_1, \zeta_2, ..., \zeta_m \} \) of \( \cup L_i \cup S^1 \) such that for any \( \zeta \in D \), there is an integer \( i \) such that
\[ |\zeta_i - \zeta| < \epsilon/32 \]
and for any \( i \), there is \( j \neq i \) such that
\[
|\zeta_i - \zeta_j| < \epsilon/16.
\]
For each \( i \) set
\[
D_i = \{ \zeta : \epsilon/32 \leq |\zeta_i - \zeta| \leq \epsilon/16 \}. 
\]
Fix \( i \), for each \( \epsilon/32 \leq r \leq \epsilon/16 \), set
\[
S_r = \{ \zeta : |\zeta - \zeta_i| = r \}. 
\]
Since \( \mu_k(D_i) \leq 1 \) and \( S_r \cap S_{r'} = \emptyset \), if \( r \neq r' \), there are only countably many \( r \) in \( (\epsilon/32, \epsilon/16) \) such that
\[
\mu_k(S_r) > 0. 
\]
Since the union of countably many countable sets is still countable, we conclude that for each \( i \), there is \( r_i \in (\epsilon/32, \epsilon/16) \) such that
\[
\mu_k(S_{r_i}) = 0 
\]
for all \( k \).

Now \( D \setminus \cup S_{r_j} \) is a disjoint union of finitely many open sets \( O_1, O_2, \ldots, O_N \) such that the diameter of each \( O_i \) is \( < \epsilon/4 \) and
\[
\mu_k(\cup S_{r_i}) = 0 
\]
for all \( k \)

Let \( \{ e^{(i)}_n \} \) be an approximate identity for \( B_{O_i} \). Then
\[
\tau_j(e^{(i)}_n) \nearrow \mu_j(O_i)
\]
\( j = 1, 2, \ldots \) and \( i = 1, 2, \ldots, N \). Since \( \mu_j(D \setminus \cup_{i=1}^N O_i) = 0 \),
\[
\tau_j(\sum_{i=1}^N e^{(i)}_n) \nearrow 1,
\]
as \( n \to \infty, \ j = 1, 2, \ldots \). Since every \( \tau \in T \) has the form

\[
\tau = \sum_{j=1}^{\infty} \alpha_j \tau_j,
\]

where \( \alpha_j \geq 0 \) and \( \sum_{j=1}^{\infty} \alpha_j = 1 \), we conclude that

\[
\tau\left(\sum_{i=1}^{N} e^{(i)}_n\right) \not\to 1
\]

for all \( \tau \in T \). Since \( T \) is compact, by Dini's theorem, the continuous functions \( \sum_{i=1}^{N} e^{(i)}_n(\tau) \) defined on \( T \) converges to the constant function 1 uniformly on \( T \). Hence we have projections \( p_i \in B_0 \), such that

\[
\tau(p_i) > K\tau\left(1 - \sum_{i=1}^{N} p_i\right)
\]

for all \( i \) and \( \tau \in T \). It follows from [Bl2, Prop. 4.1] that

\[
[p_i] > K\left[1 - \sum_{i=1}^{k} p_i\right].
\]

The rest of proof follows from Lemma 1.

Q.E.D.

**Lemma 3** Let \( A \) be a unital \( C^* \)-algebra and let \( x \) be a normal element in \( A \). Suppose that there is a projection \( p \in A \) such that

\[
\|px - xp\| < \epsilon/2
\]

and there is an element \( y \in pAp \) such that

\[
\|y - pxp\| < \epsilon/2.
\]

Then (in \( pAp \))

1. \( sp(y) \subset \{ \lambda : dis(\lambda, sp(x)) < \epsilon \} \);
2. \( \|(\lambda p - y)^{-1}\| < \|dis(\lambda, sp(x)) - \epsilon\|^{-1} \)

for those \( \lambda \) such that \( dis(\lambda, sp(x)) \geq \epsilon \).
Proof: Suppose that \( \text{dis}(\lambda, sp(x)) \geq \epsilon \). Then

\[
\|p - p(\lambda - x)^{-1}(\lambda - p)\| \\
\leq \|p(\lambda - x)^{-1}[(\lambda - y) - (\lambda - x)p]\| \\
\leq \|((\lambda - x)^{-1}\|\|y - pxp\|\| + \|pxp - xp\| < \epsilon/\text{dis}(\lambda, sp(x)) \leq 1.
\]

Similarly,

\[
\|p - (\lambda p - y)(\lambda - x)^{-1}p\| < 1.
\]

Therefore \( \lambda p - y \) is invertible in \( pAp \). This proves (1).

For (2), we have the following inequalities:

\[
\|((\lambda p - y)^{-1}\| \leq \|((\lambda p - y)^{-1} - p(\lambda - x)^{-1}p\| + \|p(\lambda - x)^{-1}p\|
\]

\[
\leq \|((\lambda p - y)^{-1}\|\|p - (\lambda p - y)(\lambda - x)^{-1}p\| + \|((\lambda - x)^{-1}\|
\]

\[
< \|((\lambda p - y)^{-1}\| \cdot \epsilon/\text{dis}(\lambda, sp(x)) + 1/\text{dis}(\lambda, sp(x)).
\]

So, we have

\[
\|((\lambda p - y)^{-1}\| < 1/[\text{dis}(\lambda, sp(x)) - \epsilon].
\]

Q.E.D.

**Lemma 4** Let \( X \) be a closed subset of the square \( S \), where

\[
S = \{\alpha + i\beta : -1 \leq \alpha \leq 1, -1 \leq \beta \leq 1\}.
\]

Let

\[
-1 = t_0 < t_1 < ... < t_k = 1
\]

be a partition of the interval \([-1, 1]\) and denote

\[
D_i = S \cap \{\lambda : t_{i-1} \leq \text{Re}\lambda \leq t_i\},
\]

\( i = 1, 2, ..., k. \) For any \( 0 < \delta < 1/2(\min\{t_i - t_{i-1}\}) \) and \( \eta > 0 \) there is \( \epsilon > 0 \), for any finite-dimensional \( C^* \)-algebras \( B \), if \( x \in B \) satisfies

(1) \( sp(x) = X \);
(2) \( \|x^*x - xx^*\|^{1/2} < \epsilon; \)

(3) \( \| (\lambda - x)^{-1} \| < \left[ \text{dis}(\lambda, X_\delta) - \epsilon \right]^{-1}, \)

where \( X_\delta = \{ \lambda : \text{dis}(\lambda, X) < \delta/2 \}; \)

then there are normal elements \( y_1, y_2, ..., y_{k-1} \in B \) with \( sp(y_i) \subset R_i, \) where

\[
R_i = \{ \alpha + i\beta : t_i - \delta/2 \leq \alpha \leq t_{i+1} + \delta/2, |\beta| \leq 1 \}
\]

\( i = 1, 2, ..., k - 1, \) and \( x_1, x_2, ..., x_k \in B \) with

\[
sp(x_i) \subset \{ \lambda : \text{dis}(\lambda, X_i) < \delta/2 \},
\]

where \( X_i = [X \cap D_i] \cup R_i \) and there is a unitary \( u \) such that

\[
\|x \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u\| < \eta.
\]

**Proof:** If \( \epsilon \) is small enough, by applying Lemma 5.2 of [BD] repeatedly, we obtain the following: there are normal elements \( y_1, y_2, ..., y_{k-1} \) in \( B, \)

\( x_1, x_2, ..., x_k \in B \) and a unitary \( u \) such that \( sp(y_i) \subset R_i, i = 1, 2, ..., k - 1, \)

\[
sp(x_i) \subset \{ \lambda : \text{dis}(\lambda, D_i) < \delta/2 \}
\]

and

\[
\|x \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u\| < \eta.
\]

Let \( x' = x \oplus y_1 \oplus \cdots \oplus y_{k-1} \) and \( x'' = u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u. \) If \( \text{dis}(\lambda, X \cup \cup_{i=1}^{k-1} R_i) \geq \delta/2 \) and \( \eta \) is small enough (so \( \epsilon \) is small), then, by (3),

\[
\|1 - (\lambda - x')^{-1}(\lambda - x'')\|
\]

\[
\leq \| (\lambda - x')^{-1}[(\lambda - x'') - (\lambda - x')] \|
\]

\[
< \eta/(\text{dis}(\lambda, X_\delta) - \epsilon) < \eta/(\delta/2 - \epsilon) < 1.
\]

Similarly,

\[
\|1 - (\lambda - x'')(\lambda - x')^{-1}\| < 1.
\]
So, $\lambda \not\in sp(x^n)$. In other words,

$$sp(x^n) \subset \{ \lambda : dis(\lambda, X \cup \bigcup_{i=1}^{k-1} R_i) < \delta/2 \}.$$ 

This implies that

$$sp(x_i) \subset \{ \lambda : dis(\lambda, [X \cap D_i] \cup R_i) < \delta/2 \}.$$ 

Q.E.D.

**Proof of the Theorem A**

We first assume that $A$ is unital.

For any $1 > \epsilon > \delta > 0$, since $sp(x)$ is compact, there are finitely many open balls $B_1, B_2, \ldots, B_n$ with diameters less then $\delta/4$ such that

$$sp(x) \subset \bigcup_{i=1}^{n} B_i.$$ 

So we may write $x = \sum_{i=1}^{n} \oplus x_i$, where each $x_i$ is normal and $sp(x_i)$ is a subset of a nice region $X_i$ which is comformally equivalent to a rectangular region with finitely many rectangular holes. Furthermore, for any $\lambda \in X_i$ there is a $\zeta \in sp(x_i)$ such that

$$dis(\lambda, \zeta) < \delta/4.$$ 

Without loss of generality, we may assume that $sp(x)$ is a subset of one of those $X_i$ and denote it by $X$. Moreover, we may assume that $X$ itself is a rectangular region with $k$ rectangular holes ($k$ could be zero).

Notice now $X$ is fixed. For any $\epsilon_1 > 0$, by applying lemma 2, there are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, mutually orthogonal projections $p_1, p_2, \ldots, p_n$ such that

$$\|x - (y + \sum_{i=1}^{n} \lambda_i p_i)\| < \epsilon_1,$$

where $y = (1 - \sum_{i=1}^{n} p_i)x(1 - \sum_{i=1}^{n} p_i)$,

$$\|(1 - \sum_{i=1}^{n} p_i)x - x(1 - \sum_{i=1}^{n} p_i)\| < \epsilon_1,$$

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\[ [p_i] > 2(k + 1)[1 - \sum_{i=1}^{n} p_i] \]

for \( i = 1, 2, \ldots, n \) and for any \( \zeta \in \text{sp}(x) \), there is \( \lambda_i \) such that \( |\zeta - \lambda_i| < \epsilon_1 \).

For the simplicity, without loss of generality, since \( A \) is an \( AF \)-algebra, we may assume that \( y \in B \), where \( B \) is a finite-dimensinal \( C^* \)-subalgebra of \( A \).

Now we are ready to cut the spectrum. If \( \epsilon_1 \) is small enough, combining Lemma 3 and Lemma 4, by cutting the spectrum properly, we obtain normal elements \( y_1, y_2, \ldots, y_{k-1} \in B \) with \( \text{sp}(y_i) \subset X \) and elements \( x_1, x_2, \ldots, x_k \in B \) with \( \text{sp}(x_i) \subset Y_i \) and a unitary \( u \) such that

\[ \|y \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u\| < \epsilon/4, \]

where \( Y_i \) is a closed subset \( X \) and \( Y_i \) is conformally equivalent to an annulus or a disk. (Notice that these \( Y_i \) depend on \( X \) and \( \epsilon_1 \) only.) If \( Y_i \) is conformally equivalent to the unit disk, then we will apply [D, Corollary 4.5]. If \( Y_i \) is conformally equivalent to an annulus, let \( f_i \) be the conformal mapping from \( Y_i \) onto the annulus \( \{ \lambda : a \geq |\lambda| \leq 1 \} \), where \( a > \delta \). Notice that each \( x_i \) satisfies the hypothesis of Lemma 3. By applying Lemma 3 and using some inequalities in the proof Lemma 4, if \( \epsilon_1 \) is small enough, we obtain

\[ \|f_i(x_i)^{-1}\| \leq 1/(a - \delta/2) \|f_i(x_i)\| \leq 1 + \delta/2. \]

Then, by applying Lemma 4.1 of [BD] ( or Theorem 1.5 of [BD]), if \( \epsilon_1 \) is small enough, we have normal elements \( y' \in M_{2k-1}(B) \) and \( y'' \in M_{2k}(B) \) such that

\[ \|y \oplus y' - y''\| < \epsilon/4 \]

and \( \text{sp}(y') \subset X_\delta = \{ \lambda : \text{dis}(\lambda, X) < \delta \} \).

Now, if \( \delta \) is small enough, since \( B \) is finite dimensional and \( \text{sp}(y') \subset X_\delta \), we may assume that there are mutually orthogonal projections \( q_i \in M_{2k-1}(B) \), \( i = 1, 2, \ldots, s_1 \) and mutually orthogonal projections \( q'_j \in M_{2k}(B) \) such that

\[ \|y \oplus \sum_{i=1}^{s_1} \lambda_i q_i - \sum_{j=1}^{s_2} \alpha_j q'_j\| < \epsilon/2, \]

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where \(0 \leq s_1 \leq n\). Since \([p_k] > 2(k+1)[1 - \sum_{i=1}^{n} p_i]\), we may write

\[
p_k = p_k^{(1)} \oplus p_k^{(2)}, \quad k = 1, 2, ..., s_1
\]

such that there are unitaries \(v_k \in A\) such that \(v_k p_k^{(1)} v_k^* = q_k, \quad k = 1, 2, ..., s_1\). Therefore, there is a unitary \(v \in M_{4k}(A)\) such that

\[
\| (\sum_{i=1}^{s_1} \lambda_i p_i^{(1)} \oplus y) - (\sum_{j=1}^{s_2} \alpha_j v^* q'_j v) \| < \epsilon/2.
\]

Thus we conclude that

\[
\| x - (\sum_{i=s_1+1}^{n} \lambda_i p_i \oplus \sum_{i=1}^{s_1} \lambda_i p_i^{(2)} \oplus \sum_{j=1}^{s_2} \alpha_j v^* q'_j v) \| < \epsilon.
\]

This completes the proof for the case that \(A\) is unital.

Now we assume that \(A\) is not unital. Then \(0 \in sp(x)\). Let \(h\) be a continuous function define on the unit disk \(D\) such that \(\|h\| \leq 1\), \(h(\zeta) = \zeta\) if \(|\zeta| > \epsilon/2\) and \(h(\zeta) = 0\) if \(|\zeta| < \epsilon/4\). Then

\[
\| h(x) - x \| < \epsilon/2.
\]

Now let \(p\) be the spectral projection of \(x\) in \(A^{**}\) corresponding to the open subset \(\{\zeta \in D : |\zeta| > \epsilon/16\}\) and \(q\) be the spectral projection of \(x\) corresponding to the closed subset \(\{\zeta \in D : |\zeta| \geq \epsilon/8\}\). Then \(p\) is an open projection in \(A^{**}\) and \(q\) is a closed projection in \(A^{**}\). Moreover, \(q \leq p\). Suppose that \(g\) is a continuous function defined on \(D\) such that \(\|g\| \leq 1\), \(g(\zeta) = 1\) if \(|\zeta| \geq \epsilon/8\) and \(g(\zeta) = 0\) if \(|\zeta| > \epsilon/16\). Then \(g(x) \in A\) and \(g(x) \geq q\). So \(q\) is compact. It follows from [Bn] that there is a projection \(e \in A\) such that

\[
q \leq e \leq p.
\]

Clearly,

\[
h(x)q = qh(x) = h(x).
\]
So $h(x) \in eAe$. Since $eAe$ is a unital simple $AF$-algebra with the same compact convex space $T$, from what we have established, there is a normal element $z \in eAe$ with finite spectrum contained in $sp(x)$ such that

$$||h(x) - z|| < \epsilon/2.$$ 

Therefore

$$||x - z|| < \epsilon.$$ 

Q.E.D.

**Proof of Theorem E**

By [Bl2, Prop.4.1], we know that if $p$ and $q$ are two projections in $A$ and for every (lower semi-continuous and semifinite) trace $\tau$ on $A$ $\tau(p) > \tau(q)$ then $[p] \geq [q]$. So the proof is exactly the same as that of Theorem A. Q.E.D.

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