APPROXIMATION BY NORMAL ELEMENTS WITH FINITE SPECTRA IN SIMPLE AF–ALGEBRAS

by

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DMS–595–IR December 1991

Approximation by Normal Elements with Finite Spectra in Simple AF-Algebras

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Abstract

We show that normal elements in simple AF-algebras with countably many extremal traces (such as UHF-algebras and matroid algebras) can be approximated by normal elements with finite spectra. Other AF-algebras are shown to have the same property.

Key words: Simple AF-algebras,approximation, normal elements, finite spectrum. 1990 AMS Mathematical Subject Classificaion 46L05

1. Introduction

A C^*-algebra A is called an AF-algebra if for any $\epsilon > 0$ and finitely many elements $a_1, a_2, ..., a_n \in A$, there are a finite dimensional C^*-subalgebra B of A and elements $b_1, b_2, ..., b_n \in B$ such that

$$\|a_i - b_i\| < \epsilon, \;\; i = 1, 2, ..., n.$$

AF-algebras have been intensively studied (See [Br],[Eff],[Ell] etc. It is almost impossible to give a complete list.) and appear to be most understandable C^*-algebras. The most interesting AF-algebras are those simple separable ones, such as matroid algebras (see [Dix]) and UHF-algebras. One question conserning AF-algebras is the following question:

$\mathbf{Q_1}$ *Can every normal element x in an AF-algebra A be approximated (in the norm topology) by normal elements in A with finite spectra?*

The question was mentioned in [P, 3.11] but we fail to locate its origional source. In [Bl1,2.6], a C^*-algebra A is said to have (FN) if every normal element in A is a norm limit of elements with the form $\sum_{k=1}^{n} \lambda_k p_k$, where λ_k are complex numbers in the spectrum of x and p_k are mutually orthogonal projections in A. So if the answer to the $\mathbf{Q_1}$ is affirmative, then every AF -algebra has (FN). It is known that every Von Neumann algebra and every AW^*-algebra have (FN). It is recently proved that the corona algebras of finite matroid algebras have (FN) ([Lin2]). In [P], Chris Phillips constructs two separable simple C^*-algebras have (FN) (They are not AF). To this author's knowledge, there are no other non-commutative C^*-algebras are known to have (FN).

If x is a self adjoint element in an AF-algebra, then for any $\epsilon > 0$, there are a finite dimensional C^*-subalgebra and an element $y \in B$ such that

$$\|x - y\| < \epsilon.$$

Set $y_1 = 1/2(y + y^*)$, then $\|x - y_1\| < \epsilon$. Since $y_1 \in B$, y_1 has finite spectrum. So the question $\mathbf{Q_1}$ has an affirmative answer for selfadjoint elements. If x

is a unitary, then the unitary part of the polar decomposition of y (in B) is close to x, provided that ϵ is small enough. So the question $\mathbf{Q_1}$ has an affirmative answer for unitaries too. In general, if x is a normal element, one hopes that y is close to a normal element (in B). In fact,

$$\|yy^* - y^*y\| < \epsilon.$$

Therefore if the following finite dimensional problem:

$\mathbf{Q_2}$ *Given $\epsilon > 0$, is there a $\delta > 0$ so that whenever y is a norm 1 element in a finite dimensional C^*-algebra B such that $\|yy^* - y^*y\| < \delta$, there is a normal element $z \in B$ satisfying $\|y - z\| < \epsilon$?*

has a positive solution, then one can also give an affirmative answer to the question $\mathbf{Q_1}$. The problem $\mathbf{Q_2}$ is equivalent to the question whether two almost commuting Hermitian matrices are close to a commuting pair of Hermitian matrices. But this is an old problem in linear algebra (see [V1],[V2] and [D]) and, unfortunately, remains open today.

Without making any effort to solve the problem $\mathbf{Q_2}$, we would like to shed some light on the question $\mathbf{Q_1}$. We will show that the answer to $\mathbf{Q_1}$ is affirmative, if A is a matroid algebra (in particular, A is a UHF-algebra). In fact, we show that for a more general class of simple AF-algebras, the answer to the question $\mathbf{Q_1}$ is affirmative.

Let A be a separable simple AF-algebra. Fix a nonzero projection $e \in A$, let T be the set of those (lower semi-continuous and semi-finite) traces τ such that $\tau(e) = 1$. With weak*-topology, T is a compact convex set. It is easy to see that the compact convex space T does not depend on the choices of the projection $e \in A$. If $\tau \in T$ is an extreme point of T, then we say that τ is an extremal trace. The main result of this note is the following:

Theorem A *Let A be a separable simple C^*-algebra. Suppose that there is a countable subset $\{\tau_n\}$ of T such that every $\tau \in T$ has the following form:*

$$\tau = \sum_{n=1}^{\infty} \alpha_n \tau_n,$$

where $\alpha_n \geq 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = 1$. *Then for every normal element* $x \in A$ *and* $\epsilon > 0$, *there are complex numbers* $\lambda_1, \lambda_2, ..., \lambda_n$ *in the spectrum of* x *and mutually orthogonal projections* $p_1, p_2, ..., p_n \in A$ *such that*

$$\|x - \sum_{i=1}^{n} \lambda_i p_i\| < \epsilon.$$

It is known ([S, 3.1.8]) that T is a metrizable Choquet simplex. By [Al, I.49], every point in T is a barycenter of a measure concentrated on its extreme points. Therefore we have the following:

Corollary B *Let* A *be a separable simple* AF*-algebra with countably many extremal traces. Then for every normal elements* $x \in A$ *and* $\epsilon > 0$, *there are complex numbers* $\lambda_1, \lambda_2, ... \lambda_n$ *in the spectrum of* x *and mutually orthognal projections* $p_1, p_2, ..., p_n \in A$ *such that*

$$\|x - \sum_{i=1}^{n} \lambda_i p_i\| < \epsilon.$$

Since matroid algebra has unique trace (up to the scalar mutiples), we immediately have the following :

Corollary C *Every matroid algebra has* (FN).

Corollary D *Every* UHF*-algebra has* (FN).

We also have the following result for non-simple AF-algebras:

Theorem E *Let* A *be a separable unital* AF*-algebra. Suppose that every (lower semi-continuous and semi-finite) trace* τ, $\tau(1) < \infty$. *Set* T *be the set of (lower semi-continuous and semi-finite) traces* τ *such that* $\tau(1) = 1$. *If there is a countable subset* $\{\tau_n\} \subset T$ *such that for every trace* $\tau \in T$, *there is a sequence of nonnegative numbers* $\{\alpha_n\}$ *such that* $\sum_{n=1}^{n} \alpha_n = 1$ *and*

$$\tau = \sum_{n=1}^{\infty} \alpha_n \tau_n.$$

Then A *has* (FN).

Before we end the introduction, we would like to state a few terminologies we will use later.

Let A be a separable AF-algebra and $p \in A$ be a nonzero projection. We use the notation $[p]$ for the equivalence class of projections containing p. Those are the projections q such that there is a partial isometry $u \in A$ with the property that $u^*u = p$, $uu^* = q$. We write $[p] > [q]$, if there is a partial isometry $u \in A$ such that $u^*u = q$, $uu^* \leq p$ and $uu^* \neq p$.

Let $p \in A^{**}$ be an open projection, where A^{**} is the enveloping von-Neumann algebra of A. We use the notation $Her(p)$ for the hereditary C^*-subalgebra $pA^{**}p \cap A$.

2. Proof of the results

Lemma 1 *Let A be a unital C^*-algebra and x be a normal element in A. Then for any $\epsilon > 0$, there is $\delta > 0$ for any finitely many points $\lambda_1, \lambda_2, ..., \lambda_n \in sp(x)$, if S_k are open subsets of $sp(x)$ such that*

$$|z - \lambda_k| < \delta$$

for all $z \in S_k$, $S_k \cap S_{k'} = \emptyset$ if $k \neq k'$, $k = 1, 2, ..., n$ and projections $p_k \in Her(q_k), k = 1, 2, ..., n$, where q_k is the spectral projection in A^{**} corresponding the open subset S_k, then

$$\|x - (y + \sum_{k=1}^{n} \lambda_k p_k)\| < \epsilon,$$

where $y = (1 - \sum_{i=1}^{n} p_i)x(1 - \sum_{i=1}^{n} p_i)$ and

$$\|(1 - \sum_{i=1}^{n} p_i)x - x(1 - \sum_{i=1}^{n} p_i)\| < \epsilon.$$

Proof. We have

$$\|x(\sum_{k=1}^{n} p_k) - \sum_{k=1}^{n} \lambda_k p_k\| = \|x(\sum_{k=1}^{n} q_k)(\sum_{k=1}^{n} p_k) - (\sum_{k=1}^{n} \lambda_k p_k)\|$$

$$= \| \sum_{k=1}^{n} q_k(x - \lambda_k q_k)p_k\| < \delta.$$

Similarly,

$$\|(\sum_{k=1}^{n} p_k)x - \sum_{k=1}^{n} \lambda_k p_k\| < \delta.$$

Moreover,

$$\|(1 - \sum_{k=1}^{n} p_k)x(1 - \sum_{k=1}^{n} p_k) - x(1 - \sum_{k=1}^{n} p_k)\|$$

$$= \| \sum_{k=1}^{n} p_k x(1 - \sum_{k=1}^{n} p_k)\|$$

$$= \|[(\sum_{k=1}^{n} p_k)x - \sum_{k=1}^{n} \lambda_k p_k](1 - \sum_{k=1}^{n} p_k)\| < \delta.$$

Similarly,

$$\|(1 - \sum_{i=1}^{n} p_i)x - (1 - \sum_{i=1}^{n} p_i)x(1 - \sum_{i=1}^{n} \| < \delta.$$

Set

$$y = (1 - \sum_{k=1}^{n} p_k)x(1 - \sum_{k=1}^{n} p_k).$$

Then

$$\|x - (y + \sum_{k=1} \lambda_k p_k)\| < 2\delta$$

and

$$\|(1 - \sum_{i=1}^{n} p_i)x - x(1 - \sum_{i=1}^{n} p_i)\| < 2\delta.$$

So take $\delta = \epsilon/2$.
Q.E.D.

Lemma 2 *Let A be a separable simple unital AF-algebra satisfying the trace condition described in Theorem A and x be a normal element in A. For any $\epsilon > 0$ and $K > 0$ there are open subsets $O_1, O_2, ..., O_n$ such that*

$$O_i \cap O_j = \emptyset, \ [\cup_{i=1}^{n} O_i]^- = sp(x),$$

$\lambda_i \in O_i$, projections $p_i \in Her(q_i)$, where q_i are spectral projections of x in A^{**} corresponding to the open subsets O_i such that

$$\|x - (y + \sum_{i=1} \lambda_i p_i)\| < \epsilon,$$

where $y = (1 - \sum_{i=1}^{n} p_i)x(1 - \sum_{i=1}^{n} p_i)$,

$$\|(1 - \sum_{i=1}^{n} p_i)x - x(1 - \sum_{i=1}^{n} p_i)\| < \epsilon$$

and

$$[p_k] > K[1 - \sum_{i=1}^{n} p_i].$$

Proof: Without loss of generality, we may assume that $\|x\| \leq 1$. Let D denote the unit disk. For any open subset $O \subset D$, let q_o be the spectral projection of x in A^{**} corresponding to the open subset O. The projection q_o is an open projection in A^{**}. Let B_o be the hereditary C^*-subalgebra of A corresponding to the open projection q_o and let $\{e_n^o\}$ be an approximate identity for B_o consisting of projections. Let τ be a trace on A with $\tau(1) = 1$. Define

$$\mu_\tau(O) = sup\{\tau(e_n^o)\}.$$

By defining

$$\mu_\tau(B) = sup\{\mu_\tau(O) : B \subset O, O \text{ open}\},$$

we know that, from measure theory, this μ_τ defines a normalized Borel measure on D. Let T_0 denote the countable subset $\{\tau_n\}$. For the simplicity, we use the notation μ_i for the mesure μ_{τ_i}.

Let $\{L_i\}$ be a set of finitely many straight line segments in D such that $D \setminus (\cup L_i \cup S^1)$ is a disjoint union of finitely many open subsets which have diameter $< \epsilon/4$. Take a finite subsets $\{\zeta_1, \zeta_2, ..., \zeta_m\}$ of $\cup L_i \cup S^1$ such that for any $\zeta \in D$, there is an integer i such that

$$|\zeta_i - \zeta| < \epsilon/32$$

and for any i, there is $j \neq i$ such that

$$|\zeta_i - \zeta_j| < \epsilon/16.$$

For each i set

$$D_i = \{\zeta : \epsilon/32 \leq |\zeta_i - \zeta| \leq \epsilon/16\}.$$

Fix i, for each $\epsilon/32 \leq r \leq \epsilon/16$, set

$$S_r = \{\zeta : |\zeta - \zeta_i| = r\}.$$

Since $\mu_k(D_i) \leq 1$ and $S_r \cap S_{r'} = \emptyset$, if $r \neq r'$, there are only countably many r in $(\epsilon/32, \epsilon/16)$ such that

$$\mu_k(S_r) > 0.$$

Since the union of countably many countable sets is still countable, we conclude that for each i, there is $r_i \in (\epsilon/32, \epsilon/16)$ such that

$$\mu_k(S_{r_i}) = 0$$

for all k.

Now $D \setminus \cup S_{r_i}$ is a disjoint union of finitely many open sets $O_1, O_2, ..., O_N$ such that the diameter of each O_i is $< \epsilon/4$ and

$$\mu_k(\cup S_{r_i}) = 0$$

for all k.

Let $\{e_n^{(i)}\}$ be an approximate identity for B_{O_i}. Then

$$\tau_j(e_n^{(i)}) \nearrow \mu_j(O_i)$$

$j = 1, 2, ...$ and $i = 1, 2, ..., N$. Since $\mu_j(D \setminus \cup_{i=1}^N O_i) = 0$,

$$\tau_j(\sum_{i=1}^N e_n^{(i)}) \nearrow 1,$$

as $n \to \infty$, $j = 1, 2, ...,$. Since every $\tau \in T$ has the form

$$\tau = \sum_{j=1}^{\infty} \alpha_j \tau_j,$$

where $\alpha_j \geq 0$ and $\sum_{j=1}^{\infty} \alpha_j = 1$, we conclude that

$$\tau(\sum_{i=1}^{N} e_n^{(i)}) \nearrow 1$$

for all $\tau \in T$. Since T is compact, by Dini's theorem, the continuous functions $\sum_{i=1}^{N} e_n^{(i)}(\tau)$ defined on T converges to the constant funiction 1 uniformly on T. Hence we have projections $p_i \in B_{O_i}$ such that

$$\tau(p_i) > K\tau(1 - \sum_{i=1}^{N} p_i)$$

for all i and $\tau \in T$. It follows from [Bl2, Prop. 4.1] that

$$[p_i] > K[1 - \sum_{i=1}^{k} p_i].$$

The rest of proof follows from Lemma 1.
Q.E.D.

Lemma 3 *Let A be a unital C^*-algebra and let x be a normal element in A. Suppose that there is a projection $p \in A$ such that*

$$\|px - xp\| < \epsilon/2$$

and there is an element $y \in pAp$ sucht that

$$\|y - pxp\| < \epsilon/2.$$

Then (in pAp)
(1) $sp(y) \subset \{\lambda : dis(\lambda, sp(x)) < \epsilon\}$;
(2)$\|(\lambda p - y)^{-1}\| < [dis(\lambda, sp(x)) - \epsilon]^{-1}$
for those λ such that $dis(\lambda, sp(x)) \geq \epsilon$.

Proof: Suppose that $dis(\lambda, sp(x)) \geq \epsilon$. Then

$$\|p - p(\lambda - x)^{-1}(\lambda - p)\|$$

$$\leq \|p(\lambda - x)^{-1}[(\lambda - y) - (\lambda - x)p\|$$

$$\leq \|(\lambda - x)^{-1}\|(\|y - pxp\|| + \|pxp - xp\| < \epsilon/dis(\lambda, sp(x)) \leq 1.$$

Similarly,

$$\|p - (\lambda p - y)(\lambda - x)^{-1}p\| < 1.$$

Therefore $\lambda p - y$ is invertible in pAp. This proves (1).

For (2), we have the following inequalities:

$$\|(\lambda p - y)^{-1}\| \leq \|(\lambda p - y)^{-1} - p(\lambda - x)^{-1}p\| + \|p(\lambda - x)^{-1}p\|$$

$$\leq \|(\lambda p - y)^{-1}\|\|p - (\lambda p - y)(\lambda - x)^{-1}p\| + \|(\lambda - x)^{-1}\|$$

$$< \|(\lambda p - y)^{-1}\| \cdot \epsilon/dis(\lambda, sp(x)) + 1/dis(\lambda, sp(x)).$$

So, we have

$$\|(\lambda p - y)^{-1}\| < 1/[dis(\lambda, sp(x)) - \epsilon].$$

Q.E.D.

Lemma 4 *Let X be a closed subset of the square S, where*

$$S = \{\alpha + i\beta : -1 \leq \alpha \leq 1, -1 \leq \beta \leq 1\}.$$

Let

$$-1 = t_0 < t_1 < ... < t_k = 1$$

be a partition of the interval $[-1, 1]$ and denote

$$D_i = S \cap \{\lambda : t_{i-1} \leq Re\lambda \leq t_i\},$$

$i = 1, 2, ..., k$. For any $0 < \delta < 1/2(min\{(t_i - t_{i-1})\})$ and $\eta > 0$ there is $\epsilon > 0$, for any finite-dimensional C^*-algebras B, if $x \in B$ satisfies

(1) $sp(x) = X$;

(2) $\|x^*x - xx^*\|^{1/2} < \epsilon$;
(3) $\|(\lambda - x)^{-1}\| < [dis(\lambda, X_\delta) - \epsilon]^{-1}$,
where $X_\delta = \{\lambda : dis(\lambda, X) < \delta/2\}$;
then there are normal elements $y_1, y_2, ..., y_{k-1} \in B$ *with* $sp(y_i) \subset R_i$, *where*

$$R_i = \{\alpha + i\beta : t_i - \delta/2 \le \alpha \le t_{i+1} + \delta/2, |\beta| \le 1\}$$

$i = 1, 2, ..., k-1$, *and* $x_1, x_2, ..., x_k \in B$ *with*

$$sp(x_i) \subset \{\lambda : dis(\lambda, X_i) < \delta/2\},$$

where $X_i = [X \cap D_i] \cup R_i$ *and there is a unitary* u *such that*

$$\|x \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \cdots \oplus x_k)u\| < \eta.$$

Proof : If ϵ is small enough, by applying Lemma 5.2 of [BD] repeatedly, we obtain the following: there are normal elements $y_1, y_2, ..., y_{k-1}$ in B, $x_1, x_2, ..., x_k \in B$ and a unitary u such that $sp(y_i) \subset R_i, i = 1, 2, ..., k-1$,

$$sp(x_i) \subset \{\lambda : dis(\lambda, D_i) < \delta/2\}$$

and

$$\|x \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u\| < \eta.$$

Let $x' = x \oplus y_1 \oplus \cdots \oplus y_{k-1}$ and $x'' = u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u$. If $dis(\lambda, X \cup \cup_{i=1}^{k-1} R_i) \ge \delta/2$ and η is small enough (so ϵ is small), then, by (3),

$$\|1 - (\lambda - x')^{-1}(\lambda - x'')\|$$

$$\le \|(\lambda - x')^{-1}[(\lambda - x'') - (\lambda - x')\|$$

$$< \eta/(dis(\lambda, X_\delta) - \epsilon) < \eta/(\delta/2 - \epsilon) < 1.$$

Similarly,

$$\|1 - (\lambda - x'')(\lambda - x')^{-1}\| < 1.$$

So, $\lambda \notin sp(x'')$. In other words,

$$sp(x'') \subset \{\lambda : dis(\lambda, X \cup \cup_{i=1}^{k-1} R_i) < \delta/2\}.$$

This implies that

$$sp(x_i) \subset \{\lambda : dis(\lambda, [X \cap D_i] \cup R_i) < \delta/2\}.$$

Q.E.D.

Proof of the Theorem A

We first assume that A is unital.

For any $1 > \epsilon > \delta > 0$, since $sp(x)$ is compact, there are finitely many open balls $B_1, B_2, ..., B_n$ with diameters less then $\delta/4$ such that

$$sp(x) \subset \cup_{i=1}^{n} B_i.$$

So we may write $x = \sum_{i=1}^{m} \oplus x_i$, where each x_i is normal and $sp(x_i)$ is a subset of a nice region X_i which is comformally equivalent to a rectangular region with finitely many rectangular holes. Furthermore, for any $\lambda \in X_i$ there is a $\zeta \in sp(x_i)$ such that

$$dis(\lambda, \zeta) < \delta/4.$$

Without loss of generality, we may assume that $sp(x)$ is a subset of one of those X_i and denote it by X. Moreover, we may assume that X itself is a rectangular region with k rectangular holes (k could be zero).

Notice now X is fixed. For any $\epsilon_1 > 0$, by applying lemma 2, there are complex numbers $\lambda_1, \lambda_2, ..., \lambda_n$, mutually orthogonal projections $p_1, p_2, ..., p_n$ such that

$$\|x - (y + \sum_{i=1} \lambda_i p_i)\| < \epsilon_1,$$

where $y = (1 - \sum_{i=1}^{n} p_i) x (1 - \sum_{i=1}^{n} p_i)$,

$$\|(1 - \sum_{i=1}^{n} p_i) x - x(1 - \sum_{i=1}^{n} p_i)\| < \epsilon_1,$$

$$[p_i] > 2(k+1)[1 - \sum_{i=1}^{n} p_i]$$

for $i = 1, 2, ..., n$ and for any $\zeta \in sp(x)$, there is λ_i such that $|\zeta - \lambda_i| < \epsilon_1$. For the simplicity, without loss of generality, since A is an AF-algebra, we may assume that $y \in B$, where B is a finite-dimensinal C^*-subalgebra of A.

Now we are ready to cut the spectrum. If ϵ_1 is small enough, combining Lemma 3 and Lemma 4, by cutting the spectrum properly, we obtain normal elements $y_1, y_2, ..., y_{k-1} \in B$ with $sp(y_i) \subset X$ and elements $x_1, x_2, ..., x_k \in B$ with $sp(x_i) \subset Y_i$ and a unitary u such that

$$\|y \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u\| < \epsilon/4,$$

where Y_i is a closed subset X and Y_i is comformally equivalent to an annulus or a disk. (Notice that these Y_i depend on X and ϵ_1 only.) If Y_i is comformally equivalent to the unit disk, then we will apply [D, Corollary 4.5]. If Y_i is comformally equivalent to an annulus, let f_i be the comformal mapping from Y_i onto the annulus $\{\lambda : a \geq |\lambda| \leq 1\}$, where $a > \delta$. Notice that each x_i satisfies the hypothesis of Lemma 3. By applying Lemma 3 and using some inequalities in the proof Lemma 4, if ϵ_1 is small enough, we obtain

$$\|f_i(x_i)^{-1}\| \leq 1/(a - \delta/2) \quad \|f_i(x_i)\| \leq 1 + \delta/2.$$

Then, by applying Lemma 4.1 of [BD] (or Theorem 1.5 of [BD]), if ϵ_1 is small enough, we have normal elements $y' \in M_{2k-1}(B)$ and $y'' \in M_{2k}(B)$ such that

$$\|y \oplus y' - y''\| < \epsilon/4$$

and $sp(y') \subset X_\delta = \{\lambda : dis(\lambda, X) < \delta\}$.

Now, if δ is small enough, since B is finite dimesional and $sp(y') \subset X_\delta$, we may assume that there are mutually orthognal projections $q_i \in M_{2k-1}(B)$, $i = 1, 2, ..., s_1$ and mutually orthogonal projections $q'_j \in M_{2k}(B)$ such that

$$\|y \oplus \sum_{i=1}^{s_1} \lambda_i q_i - \sum_{j=1}^{s_2} \alpha_j q'_j\| < \epsilon/2,$$

where $0 \leq s_1 \leq n$. Since $[p_k] > 2(k+1)[1-\sum_{i=1}^n p_i]$, we may write

$$p_k = p_k^{(1)} \oplus p_k^{(2)}, \; k = 1, 2, ..., s_1$$

such that there are unitaries $v_k \in A$ such that $v_k p_k^{(1)} v_k^* = q_k$, $k = 1, 2, ..., s_1$. Therefore, there is a unitary $v \in M_{4k}(A)$ such that

$$\|(\sum_{i=1}^{s_1} \lambda_i p_i^{(1)} \oplus y) - (\sum_{j=1}^{s_2} \alpha_j v^* q_j' v)\| < \epsilon/2.$$

Thus we conclude that

$$\|x - (\sum_{i=s_1+1}^{n} \lambda_i p_i \oplus \sum_{i=1}^{s_1} \lambda_i p_i^{(2)} \oplus \sum_{j=1}^{s_2} \alpha_j v^* q_j' v)\| < \epsilon.$$

This completes the proof for the case that A is unital.

Now we assume that A is not unital. Then $0 \in sp(x)$. Let h be a continuous function define on the unit disk D such that $\|h\| \leq 1$, $h(\zeta) = \zeta$ if $|\zeta| > \epsilon/2$ and $h(\zeta) = 0$ if $|\zeta| < \epsilon/4$. Then

$$\|h(x) - x\| < \epsilon/2.$$

Now let p be the spectral projection of x in A^{**} corresponding to the open subset $\{\zeta \in D : |\zeta| > \epsilon/16\}$ and q be the spectral projection of x corresponding to the closed subset $\{\zeta \in D : |\zeta| \geq \epsilon/8\}$. Then p is an open projection in A^{**} and q is a closed projection in A^{**}. Moreover, $q \leq p$. Suppose that g is a continuous function defined on D such that $\|g\| \leq 1$, $g(\zeta) = 1$ if $|\zeta| \geq \epsilon/8$ and $g(\zeta) = 0$ if $|\zeta| > \epsilon/16$. Then $g(x) \in A$ and $g(x) \geq q$. So q is compact. It follows from [Bn] that there is a projection $e \in A$ such that

$$q \leq e \leq p.$$

Clearly,

$$h(x)q = qh(x) = h(x).$$

So $h(x) \in eAe$. Since eAe is a unital simple AF-algebra with the same compact convex space T, from what we have established, there is a normal element $z \in eAe$ with finite spectrum contained in $sp(x)$ such that

$$\|h(x) - z\| < \epsilon/2.$$

Therefore

$$\|x - z\| < \epsilon.$$

Q.E.D.

Proof of Theorem E

By [Bl2, Prop.4.1], we know that if p and q are two projections in A and for every (lower semi-continuous and semifinite) trace τ on A $\tau(p) > \tau(q)$ then $[p] \geq [q]$. So the proof is exactly the same as that of Theorem A.
Q.E.D.

Acknowlegdements This work was done when the author was in the University of Victoria and supported by grants from Natural Sciences and Engineering Research Council of Canada. The author is very grateful to both Professor John Phillips and Ian Putnam for their support and hospitality. He would also like to thank Ken Davidson for a helpful e-mail correspondence and Chris Phillips for some comments.

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