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Article

Applications of Higher-Order q -Derivative to Meromorphic q -Starlike Function Related to Janowski Function

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Abstract: By making use of a higher-order q -derivative operator, certain families of meromorphic q -starlike functions and meromorphic q -convex functions are introduced and studied. Several sufficient conditions and coefficient inequalities for functions in these subclasses are derived. The results presented in this article extend and generalize a number of previous results.

Keywords: q -derivative; analytic functions; differential subordination; meromorphic functions; meromorphic q -starlike functions; meromorphic q -convex functions

MSC: 30C45; 05A30



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1. Introduction

Recently, q -analysis has fascinated scholars due to various applications in many areas of physics and mathematics. The applications of q -analysis were first considered by Jackson [1,2]. In recent years, some scholars have written a number of papers [3–15] associated with q -starlike functions and the Janowski functions [16]. In particular, Srivastava [17,18] pointed out some applications and mathematical explanations of q -derivatives in GFT. In this paper, we consider several families of meromorphically multivalent q -starlike functions by making use of the Janowski function and the higher-order q -derivative. Certain sufficient conditions and coefficient inequalities for functions in these subclasses are derived. Moreover, several previous results are generalized.

Let $\Sigma(p)$ ($p \in \mathbb{N}$) denote the family of p -valent analytic functions in $U^* = \{z : 0 < |z| < 1\}$ which have the following form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}. \quad (1)$$

Furthermore, we let $\Sigma(1) = \Sigma$.

If $f \in \Sigma(p)$ satisfies the condition

$$\operatorname{Re} \left(-\frac{zf'(z)}{f(z)} \right) > 0, \quad (2)$$

then f is called a meromorphic p -valent starlike function. We note the family by $M\Sigma^*(p)$ and write $M\Sigma^*(1) = M\Sigma^*$. The class $M\Sigma^*$ was studied by Pommerenke [19].

If $f \in \Sigma(p)$ satisfies the condition

$$\operatorname{Re} \left(-\frac{(zf'(z))'}{f'(z)} \right) > 0, \quad (3)$$

then f is called a meromorphic p -valent convex function. We note this family by $MC(p)$ and write $MC(1) = MC$.

If a function ψ is analytic in $U = U^* \cup \{0\}$ and satisfies

$$\psi(z) = 1 + \sum_{k=1}^{\infty} \psi_k z^k \quad (4)$$

and

$$\operatorname{Re}(\psi(z)) > 0,$$

then ψ is said to be in the family P .

Let φ be analytic in U and $\varphi(0) = 1$. If φ satisfies

$$\varphi(z) \prec \frac{Az + 1}{Bz + 1} \quad (-1 \leq B < A \leq 1),$$

then φ is said to be in $P[A, B]$.

In [16], Janowski studied the family $P[A, B]$ and obtained that φ is in $P[A, B]$ if

$$\varphi(z) = \frac{(1-A) + (1+A)\psi(z)}{(1-B) + (1+B)\psi(z)} \quad (\psi \in P; -1 \leq B < A \leq 1).$$

Let f and g be analytic in U . If there exists w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, so that $f(z) = g(w(z))$, then we say that f is subordinate to g , written by $f \prec g$. Further, if the function g is analytic and univalent in U , then

$$f \prec g \quad (z \in U) \iff f(U) \subset g(U) \text{ and } f(0) = g(0).$$

Definition 1. Let $f \in \Sigma$. Then f is called to be in $M\Sigma^*[A, B]$ if

$$-\frac{zf'(z)}{f(z)} = \frac{(1-A) + (1+A)\psi(z)}{(1-B) + (1+B)\psi(z)} \quad (-1 \leq B < A \leq 1; \psi \in P). \quad (5)$$

In [20], Karunakaran studied the family $M\Sigma^*[A, B]$.

Let $0 < q < 1$ and define $[\tau]_q$ as the following:

$$[\tau]_q = \begin{cases} \frac{q^\tau - 1}{q - 1} & (\tau \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1} & (\tau = n \in \mathbb{N}). \end{cases}$$

Let $0 < q < 1$. The q -factorial $[m]_q!$ is defined by

$$[m]_q! = \begin{cases} 1 & (m = 0) \\ \prod_{k=1}^m [k]_q & (m \in \mathbb{N}). \end{cases}$$

Let $\gamma \in \mathbb{N}$. We define q -Pochhammer symbol $[\gamma]_{q,n}$ by (see [21])

$$[\gamma]_{q,n} = \begin{cases} 1 & (n = 0) \\ \prod_{k=\gamma}^{\gamma+n-1} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Specifically, we write $[0]_{q,n} = 0$.

Next, we define the q -derivative D_q ($0 < q < 1$) for $f \in \Sigma(p)$ by

$$\begin{aligned} (D_q f)(z) &= \frac{f(zq) - f(z)}{z(q-1)} \\ &= -\sum_{k=1}^p \left(\frac{[k]_{q,1}}{q^k} \right) a_{-k} z^{-1-k} + \sum_{k=1}^{\infty} [k]_q a_k z^{-1+k}, \end{aligned} \quad (6)$$

where $a_{-p} = 1$.

From (6), we can see that

$$\lim_{q \rightarrow 1} (D_q f)(z) = \lim_{q \rightarrow 1} \frac{f(zq) - f(z)}{z(q-1)} = f'(z).$$

Further, one can find that

$$(D_q^{(2)} f)(z) = \sum_{k=1}^p \left(\frac{[k]_{q,2}}{q^{2k+1}} \right) a_{-k} z^{-k-2} + \sum_{k=2}^{\infty} [k-1]_{q,2} a_k z^{k-2}, \quad (7)$$

$$(D_q^{(3)} f)(z) = - \sum_{k=1}^p \left(\frac{[k]_{q,3}}{q^{3k+3}} \right) a_{-k} z^{-k-3} + \sum_{k=3}^{\infty} [k-2]_{q,3} a_k z^{k-3}, \quad (8)$$

.....

$$(D_q^{(p)} f)(z) = (-1)^p \sum_{k=1}^p \left(\frac{[k]_{q,p}}{q^{p[k+\frac{1}{2}(p-1)]}} \right) a_{-k} z^{-k-p} + \sum_{k=p}^{\infty} [k-p+1]_{q,p} a_k z^{k-p}, \quad (9)$$

where $a_{-p} = 1$ and $D_q^{(p)}$ is called p th order q -derivatives.

Definition 2. Let $f \in \Sigma$. If f satisfies

$$\left| \frac{qz D_q f(z)}{f(z)} + \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad (10)$$

then f is called to be in the meromorphic q -starlike function family $M\Sigma_q^*$.

It is easily seen that, when $q \rightarrow 1^-$, the disk given by (10) becomes

$$\operatorname{Re} \left(-\frac{zf'(z)}{f(z)} \right) > 0.$$

Thus, the class $M\Sigma_q^*$ reduces to the meromorphic starlike function family $M\Sigma^*$ (see [19]). Furthermore, we can rewrite (10) as the following:

$$-\frac{qz D_q f(z)}{f(z)} \prec \hat{h}(z) \quad \text{where} \quad \hat{h}(z) = \frac{z+1}{1-qz}.$$

Further, the meromorphic q -convex function family MC_q could be derived by

$$f(z) \in MC_q \Leftrightarrow -qz D_q f(z) \in M\Sigma_q^*.$$

Definition 3. If a function $f \in \Sigma(p)$ satisfies

$$-\frac{q^{2p-1} z (D_q^{(p)} f)(z)}{[2p-1]_q (D_q^{(p-1)} f)(z)} \prec \frac{(A+1)\hat{h}(z) + (1-A)}{(B+1)\hat{h}(z) + (1-B)} \quad \left(\hat{h}(z) = \frac{z+1}{1-qz} - 1; \leq B < A \leq 1 \right),$$

or, equivalently,

$$-\frac{q^{2p-1} z (D_q^{(p)} f)(z)}{[2p-1]_q (D_q^{(p-1)} f)(z)} \prec s(z), \quad (11)$$

where

$$s(z) = \frac{z(1+A) - zq(1-A) + 2}{z(1+B) - zq(1-B) + 2} \quad (0 < q < 1; -1 \leq B < A \leq 1), \quad (12)$$

then f is in the family $M\Sigma_q^*[p, A, B]$.

Remark 1. We write the following special cases:

- (i) $M\Sigma_q^*[1, A, B] = M\Sigma_q^*[A, B]$, when $p = 1$.
- (ii) $\lim_{q \rightarrow 1^-} M\Sigma_q^*[1, A, B] = M\Sigma^*[A, B]$, when $p = 1$.
- (iii) $M\Sigma_q^*[p, A, B] = M\Sigma_q^*[\alpha]$, when $p = 1$, $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$. In [19], Pommerenke considered the family $M\Sigma_q^*[\alpha]$.

Now we define the meromorphic q -convex function family $MC_q[p, A, B]$ by

$$f \in MC_q[p, A, B] \iff \frac{(-1)^p q^{\frac{1}{2}p(3p-1)}}{[p]_{q,p}} z^p D_q^{(p)} f \in M\Sigma_q^*[p, A, B].$$

In particular, we write $MC_q[p, A, B] = MC_q[A, B]$ when $p = 1$.

Lemma 1 ([22]). Let $\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \dots$ belong to the family P . Then

$$|\psi_2 - \nu \psi_1^2| \leq \begin{cases} 4\nu - 2 & (\nu > 1) \\ 2 & (0 \leq \nu \leq 1) \\ -4\nu + 2 & (\nu < 0). \end{cases} \quad (13)$$

Lemma 2 ([23]). Let $h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$ be analytic in U . Furthermore, let $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$ be univalent convex in U . If $h(z) \prec H(z)$, then

$$|h_k| \leq |C_1| \quad (k \geq 1).$$

2. Main Results

Theorem 1. If

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k} z^{-p+k} \in M\Sigma_q^*[p, A, B] \quad (p \geq 2),$$

then

$$|a_{2-p} - \mu a_{1-p}^2| \leq \begin{cases} \left(\frac{A-B}{2} \right) \left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}} \right) \Lambda(q) & (\mu > \sigma_1) \\ \left(\frac{A-B}{2} \right) \left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}} \right) & (\sigma_2 \leq \mu \leq \sigma_1) \\ \left(\frac{B-A}{2} \right) \left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}} \right) \Lambda(q) & (\mu < \sigma_2), \end{cases}$$

where

$$\Lambda(q) = \frac{\left\{ \begin{aligned} & \{ (1 + Bq[2p-2]_q - A[2p-1]_q)(q+1) - 2 \} [p-1]_q [2p-3]_q \\ & + \mu(q+1)^2 (A-B)[2p-2]_{q,2} [p-2]_q \end{aligned} \right\}}{2[p-1]_q [2p-3]_q},$$

$$\sigma_1 = \frac{[p-1]_q [2p-3]_q \{ 4 + (q+1)(A[2p-1]_q - qB[2p-2]_q - 1) \}}{(q+1)^2 (A-B)[p-2]_q [2p-2]_{q,2}}$$

and

$$\sigma_2 = \frac{[p-1]_q [2p-3]_q \{ (q+1)(A[2p-1]_q - 1 - qB[2p-2]_q) \}}{4(q+1)^2 (A-B)[p-2]_q [2p-2]_{q,2}}.$$

Proof. From the assumption of the theorem, we obtain

$$-\frac{q^{2p-1} z (D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \prec \phi(z),$$

where

$$\phi(z) = \frac{2 - q(1 - A)z + (1 + A)z}{2 - q(1 - B)z + (1 + B)z}.$$

This gives that

$$-\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} = \phi(w(z)),$$

where $w(z)$ is a Schwarz function. Now a function $h(z)$ is defined as follows:

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \sum_{n=1}^{\infty} h_n z^n \in P.$$

□

Furthermore, one can see that

$$\begin{aligned} \phi(w(z)) &= \frac{2(h(z) + 1 - q(h(z) - 1)) + (1 + A)(q + 1)(h(z) - 1)}{2(h(z) + 1 - q(h(z) - 1)) + (1 + B)(q + 1)(h(z) - 1)} \\ &= 1 + \frac{1}{4}(1 + q)(A - B)h_1 z + \frac{1}{16}(1 + q)(A - B)\{4h_2 - (B(1 + q) - q + 3)h_1^2\}z^2 + \dots \end{aligned}$$

Similarly, we find that

$$\begin{aligned} -\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} &= 1 - \frac{[p-1]_q}{[2p-2]_{q,2}}q^{p-1}a_{1-p}z \\ &+ \frac{[p-1]_q}{[2(p-1)]_{q,2}}q^{2(p-1)}\left(\frac{[p-1]_q}{[2(p-1)]_q}a_{1-p}^2 - \frac{[p-2]_q}{[2p-3]_q}(1 + q)a_{2-p}\right)z^2 + \dots \end{aligned}$$

for $p \geq 2$. Therefore, for $p \geq 2$, we obtain

$$a_{1-p} = -\frac{(q + 1)(A - B)[2(p - 1)]_{q,2}}{4q^{p-1}[p - 1]_q}h_1 \quad (14)$$

and

$$a_{2-p} = \frac{(A - B)[2p - 3]_{q,3}}{q^{2p-2}[p - 2]_{q,2}}\left(\frac{1}{16}k_1(q)h_1^2 - \frac{1}{4}h_2\right), \quad (15)$$

where

$$k_1(q) = (1 + q)\{A[2p - 1]_q - B([2p - 1]_q - 1) - 1\} + 4. \quad (16)$$

Hence we obtain for $p \geq 2$ that

$$|a_{2-p} - \mu a_{1-p}^2| = \left(\frac{A - B}{4}\right)\left(\frac{[2p - 3]_{q,3}}{q^{2p-2}[p - 2]_{q,2}}\right)|h_2 - k_2 h_1^2|, \quad (17)$$

where

$$k_2 = \frac{[p - 1]_q[2p - 3]_q k_1(q) - \mu[2(p - 1)]_{q,2}[p - 2]_q(A - B)(q + 1)^2}{4[2p - 3]_q[p - 1]_q}$$

with $k_1(q)$ given by (16).

Now we can see that the conditions $\mu > \sigma_1$, $\sigma_2 \leq \mu \leq \sigma_1$ and $\mu < \sigma_2$ in Theorem 1 imply that $k_2 < 0$, $0 \leq k_2 \leq 1$ and $k_2 > 1$, respectively. By applying Lemma 1 in (17), the desired result is obtained. This proves Theorem 1.

Applying the same method as in the proof of Theorem 1, we obtain the following theorem for the case $p = 1$.

Theorem 2. *If*

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in M\Sigma_q^*[A, B],$$

$$|a_1 - \mu a_0^2| \leq \begin{cases} \left(\frac{A-B}{4}\right)[(1-A)q + \mu(q+1)^2(A-B) - A - 1] & \left(\mu > \frac{(A-1)q+A+3}{(A-B)(q+1)^2}\right) \\ \frac{A-B}{2} & \left(\frac{A-1}{(A-B)(q+1)} \leq \mu \leq \frac{(A-1)q+A+3}{(A-B)(q+1)^2}\right) \\ \left(\frac{B-A}{4}\right)[(1-A)q + \mu(q+1)^2(A-B) - A - 1] & \left(\mu < \frac{A-1}{(A-B)(q+1)}\right). \end{cases}$$

Letting $q \rightarrow 1^-$, $A = 1$ and $B = -1$ in Theorem 2, we obtain a result of the known family $M\Sigma^*$.

Corollary 1. *If*

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in M\Sigma^*,$$

then

$$|a_1 - \mu a_0^2| \leq \begin{cases} 4\mu - 1 & (\mu > \frac{1}{2}) \\ 1 & (0 \leq \mu \leq \frac{1}{2}) \\ 1 - 4\mu & (\mu < 0). \end{cases}$$

Theorem 3. *Let $p \geq 2$. If*

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k} z^{-p+k} \in M\Sigma_q^*[p, A, B],$$

then

$$|a_{-p+k}| \leq \prod_{j=1}^k \frac{2[j-1]_q + [2p-1]_q(q+1)(A-B)}{2[j]_q q^{p-1}[p-j]_q} \quad (18)$$

for $1 \leq k \leq p-1$.

Proof. If g belongs to $M\Sigma_q^*[p, A, B]$, then

$$\psi(z) := -\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} \prec \phi(z), \quad (19)$$

where

$$\phi(z) = \frac{z(1+A) - zq(1-A) + 2}{z(1+B) - zq(1-B) + 2} = 1 - \frac{1}{2}(1+q)(B-A)z + \frac{1}{4}(1+q)(B-A)\{1+B(1+q)-q\}z^2 + \dots$$

□

Let

$$\psi(z) = 1 + \sum_{k=1}^{\infty} \psi_k z^k.$$

Applying Lemma 2, we obtain

$$|\psi_k| \leq \frac{1}{2}(q+1)(A-B) \quad (k \geq 1). \quad (20)$$

Furthermore, from (19), we have

$$-q^{2p-1}z(D_q^{(p)}g)(z) = \{[2p-1]_q(D_q^{(p-1)}g)(z)\}\psi(z),$$

which implies that

$$\begin{aligned} & -q^{2p-1} \left(\sum_{k=1}^p \frac{(-1)^p [k]_{q,p}}{q^{p[k+\frac{1}{2}(p-1)]}} a_{-k} z^{-p-k+1} + \sum_{k=p}^{\infty} [-p+k+1]_{q,p} a_k z^{-p+k+1} \right) \\ & = [2p-1]_q \left(1 + \sum_{k=1}^{\infty} \psi_k z^k \right) \left(\sum_{k=1}^p \frac{(-1)^{p-1} [k]_{q,p-1}}{q^{[k+\frac{1}{2}(p-2)](p-1)}} a_{-k} z^{-p-k+1} \right. \\ & \quad \left. + \sum_{k=p-1}^{\infty} [2+k-p]_{q,p-1} a_k z^{-p+k+1} \right), \end{aligned}$$

where $a_{-p} = 1$.

It is easily seen from the above formula that

$$|a_{-p+k}| \leq \frac{(A-B)(q+1)[2p-1]_q q^{\frac{1}{2}(p-1)(3p-2k-2)}}{2[k]_q [p-k]_{q,p-1}} \sum_{l=1}^k \frac{[p+l-k]_{q,p-1}}{q^{\frac{1}{2}(3p-2k+2l-2)(p-1)}} |a_{k-p-l}| \quad (21)$$

for $1 \leq k \leq p-1$.

Now

$$\begin{aligned} |a_{1-p}| & \leq \frac{[2p-2]_{q,2}(1+q)(A-B)}{2q^{p-1}[p-1]_q}, \\ |a_{2-p}| & \leq \frac{q^{\frac{1}{2}(p-1)(3p-6)}[2p-1]_q(1+q)(A-B)}{2[p-2]_{q,p-1}[2]_q} \left\{ \frac{[p]_{q,p-1}}{q^{\frac{1}{2}(3p-2)(p-1)}} + \frac{[p-1]_{q,p-1}}{q^{\frac{1}{2}(3p-4)(p-1)}} |a_{1-p}| \right\} \\ & = \frac{(1+q)(A-B)[2p-2]_{q,2}}{2q^{p-1}[p-1]_q} \cdot \frac{\{2+[2p-1]_q(1+q)(A-B)\}[2p-3]_q}{2q^{p-1}[2]_q[p-2]_q}, \\ & \quad \dots\dots\dots \\ |a_{k-p}| & \leq \prod_{j=1}^k \frac{2[j-1]_q + (1+q)(A-B)[2p-1]_q}{2q^{p-1}[j]_q[p-j]_q} \end{aligned}$$

for $1 \leq k \leq p-1$. This proves Theorem 3.

Applying the same methods as in the proof of Theorem 3, we obtain the following Theorems 4 and 5.

Theorem 4. Let $p \geq 2$. If $g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k} z^{-p+k}$ belongs to $MC_q[p, A, B]$, then

$$|a_{-p+k}| \leq \frac{[p]_{q,p}}{q^{pk}[p-k]_{q,p}} \prod_{n=1}^k \frac{[2p-1]_q(1+q)(A-B) + 2[n-1]_q}{2[n]_q q^{p-1}[p-n]_q}$$

for $1 \leq k \leq p-1$.

Theorem 5. Let $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1}$ belong to $M\Sigma_q^*[A, B]$. Then

$$|a_{k-1}| \leq \prod_{n=1}^k \frac{(1+q)(A-B) + 2[n-1]_q}{2[n]_q}$$

for $k \geq 2$.

Letting $q \rightarrow 1-$, $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$ in Theorem 5, we have a result of the known family $M\Sigma^*(\alpha)$.

Corollary 2. Let $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in M\Sigma^*(\alpha)$. Then

$$|a_{k-1}| \leq \prod_{j=1}^k \frac{j-2\alpha+1}{j}$$

for $k \geq 2$.

The following equivalence could help us to study the family $M\Sigma_q^*[p, A, B]$:

$$g \in M\Sigma_q^*[p, A, B] \iff \left| \frac{(1-B) \left\{ \frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} \right\} + (1-A)}{(1+B) \left(-\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} \right) - (1+A)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Theorem 6. If a function $g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k}z^{-p+k} \in \Sigma(p)$ satisfies

$$\begin{aligned} & \sum_{k=1}^p \left(\frac{[k]_{q,p-1}}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}} \right) \left\{ \left| [2p-1]_q(1+A) - [p+k-1]_q q^{p-k}(1+B) \right| + 2[p-k]_q \right\} |a_{-k}| \\ & + \sum_{k=p}^{\infty} [k-p]_{q,p-1} \left\{ \left| [2p-1]_q(1+A) - [p+k-1]_q q^{2p-1}(1+B) \right| + 2[k+p]_q \right\} |a_k| \\ & + [2p-1]_q [p-1]_q! (2+A) |a_{p-1}| < \frac{(A-B)[p]_{q,p}}{q^{\frac{1}{2}(3p-2)(p-1)}}, \end{aligned} \quad (22)$$

then $g \in M\Sigma_q^*[p, A, B]$.

Proof. By a simple calculation, we obtain

$$\begin{aligned} & \left| \frac{(1-B) \left\{ \frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} \right\} + (1-A)}{(1+B) \left\{ -\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} \right\} - (1+A)} - \frac{1}{1-q} \right| \\ & \leq \left| \frac{\{-q^{2p-1}z(D_q^{(p)}g)(z)\}(1-B) - [2p-1]_q(1-A)(D_q^{(p-1)}g)(z)}{\{-q^{2p-1}z(D_q^{(p)}g)(z)\}(1+B) - [2p-1]_q(1+A)(D_q^{(p-1)}g)(z)} + 1 \right| + \frac{q}{1-q} \\ & = 2 \left| \frac{\left\{ \sum_{k=1}^p \frac{(-1)^{p-1}[k]_{q,p-1}[2p-1]_q}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}} a_{-k} z^{p-k} + \sum_{k=p-1}^{\infty} [k+2-p]_{q,p} [2p-1]_q a_k z^{k+p} \right\}}{\left\{ (A-B) \frac{(-1)^{p-1}[p]_{q,p}}{q^{\frac{1}{2}(p-1)(3p-2)}} + (A+1)[p-1]_q! [2p-1]_q a_{p-1} z^{2p-1} \right.} \right. \\ & \quad \left. \left. + \sum_{k=1}^{p-1} \frac{(-1)^{p-1}[k]_{q,p-1}}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}} \{ [2p-1]_q(A+1) - (1+B)q^{p-k}[p-1+k]_q \} a_{-k} z^{p-k} \right. \right. \\ & \quad \left. \left. + \sum_{k=p}^{\infty} \{ [2p-1]_q(A+1) - (B+1)q^{2p-1}[p-1+k]_q \} [k-p]_{q,p+1} a_k z^{k+p} \right\}} \right| \\ & \quad + \frac{q}{1-q} \\ & \leq 2 \left(\frac{\sum_{k=1}^{p-1} \frac{[k]_{q,p-1}[p-k]_q}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}} |a_{-k}| + [p-1]_q! [2p-1]_q |a_{p-1}| + \sum_{k=p}^{\infty} [k-p]_{q,p+1} [k+p]_q |a_k|}{\left(\frac{[p]_{q,p}}{q^{\frac{1}{2}(p-1)(3p-2)}} (A-B) - (1+A)[p-1]_q! [2p-1]_q |a_{p-1}| \right)} \right) + \frac{q}{1-q}. \end{aligned} \quad (23)$$

□

Now, it follows from (22) that the last expression in (23) is less than $\frac{1}{1-q}$. This proves Theorem 6.

Theorem 7. If a function $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1}z^{k-1} \in \Sigma$ satisfies

$$\sum_{k=1}^{\infty} \{2[k]_q + |q[k-1]_q(1+B) - (1-A)|\} |a_{k-1}| < A-B, \quad (24)$$

then $g \in M\Sigma_q^*[A, B]$.

Proof. By simple calculation, we have

$$\begin{aligned} & \left| \frac{(1-B) \left(\frac{qz(D_q g)(z)}{g(z)} \right) + (1-A)}{(1+B) \left(-\frac{qz(D_q g)(z)}{g(z)} \right) - (1+A)} - \frac{1}{1-q} \right| \\ & \leq \frac{q}{1-q} + 2 \left| \frac{g(z) + qz(D_q g)(z)}{q(1+B)z(D_q g)(z) + (1+A)} \right| \\ & = \frac{q}{1-q} + 2 \left| \frac{\sum_{k=1}^{\infty} [k]_q |a_{k-1}|}{(A-B) - \sum_{k=1}^{\infty} [q(1+B)[k-1]_q + (1+A)] |a_{k-1}|} \right|. \end{aligned} \quad (25)$$

From (24) we can see that (25) is less than $\frac{1}{1-q}$. This proves Theorem 7. \square

Letting $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$), $B = -1$ and $q \rightarrow 1^-$ in Theorem 7, we obtain a result of the known family $M\Sigma^*(\alpha)$.

Corollary 3. If a function $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1}z^{k-1} \in \Sigma$ satisfies

$$\sum_{k=1}^{\infty} (1+k-\alpha) |a_{k-1}| < 1-\alpha \quad (0 \leq \alpha < 1),$$

then $g \in M\Sigma^*(\alpha)$.

Applying the same method as in the proof of Theorem 7, we obtain the following theorem.

Theorem 8. If a function $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1}z^{k-1} \in \Sigma$ satisfies

$$\sum_{k=1}^{\infty} \{2[k]_q + |q[k-1]_q(1+B) - (1-A)|\} [k-1]_q |a_{k-1}| < A-B,$$

then $g \in MC_q[A, B]$.

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