Optimal Harvesting of a Fishery subject

Random Catastrophic Collapse*

by

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Abstract

The socially optimal harvesting policy for a fishery subject to random catastrophic collapse is determined. It is shown how the problem can be converted to one of deterministic optimal control, and a solution obtained using the Pontryagin Maximum Principle. The optimal stock biomass satisfies a golden-rule equation containing extra terms reflecting the density-dependent and density-independent effects of the risk of collapse. These effects have opposite influences on the optimal biomass level. The optimal policy when there is the possibility of a recovery after a collapse is discussed. Economic interpretations of the results are given, and some numerical results presented.

Keywords Optimal harvesting; catastrophic collapse; hazard; golden rule equation; bioeconomics; maximum principle; stochastic dynamic programming.

1. Introduction

Fisheries management and fisheries science operate in a highly variable and uncertain world. On the one hand there is the presence of pure randomness in physical, biological and economic processes (such as for example, the process of catching fish; the dynamics of a fish population; and the market mechanism for determining fish prices). On the other hand there is uncertainty based on imprecise knowledge (for example with respect to the state of a stock; the values of various growth mortality and production parameters; and also with respect to fishermen's behaviour and the consequences of various management regulations). The bioeconomic theory of fisheries has moved to reflect this fact with the main thrust of research now involving stochastic and adaptive models, rather than the deterministic models so richly developed in the nineteen seventies (see e.g. Clark, 1976). The survey article Stochastic Bioeconomics of Andersen and Sutinen (1984) cites sixty-odd papers dealing with randomness (mainly in stock dynamics and fish prices). The books of Clark (1983, especially Chapt. 6), Walters (1986) and Mangel (1985) deal extensively with decision-theoretic models for handling uncertainty.

It is widely recognized that no analytic model can incorporate all sources of randomness and uncertainty simultaneously, and that a theoretical understanding of their effects can be obtained only through analyzing relatively simple models incorporating only one or two stochastic aspects. Nonetheless even with simple stochastic models the mathematics can be remarkably complex. It is the purpose of this paper to study analytically the effects of one source of randomness, viz. the possibility that a fish stock may undergo a sudden catastrophic collapse from a healthy level producing an apparently sustainable economic rent to

a level at which it becomes effectively economically moribund, at least for a substantial period of time.

Such sudden collapses in fisheries are well documented (see e.g. Murphy, 1977) especially among clupeid species (such as sardines, pilchards, herrings and anchovies). Perhaps the most spectacular crash of all was that of the Peruvian anchoveta fishery. At its peak, in 1971, the fishery was the largest in the world producing some 15%, by weight, of the total world marine fishery production. According to Clark (1976, p.33), by the end of March 1973, "the anchovy had all but disappeared from Peruvian waters". The cause of the collapse appears to have been a combination of overfishing and adverse environmental conditions caused by the incursion of the warm-water El Nino current. Other major collapses of clupeid fisheries include the North Sea herring during the mid-seventies (Saville and Bailey, 1980), the Southwest African pilchard also in the mid-seventies (Butterworth, 1980) and the California sardine in the early forties. Clark (1983, p.6) gives a list of nine once major fisheries which are now virtually moribund.

One explanation for the sudden collapse of schooling clupeid fish stocks is the inadequacy of stock assessment based on using catch per unit effort as a proxy for population size (see Clark 1983, pp. 42-56). However environmental factors also undoubtedly play an important role. Among them are changes in temperature and salinity, and strong vertical currents, which can all have disastrous effects on fish abundance, especially if they occur during critical development periods (Brongersma-Sanders, 1957, Sissenwine et al., 1983).

Optimal-management modelling for fisheries subject to sudden catastrophic declines seems to have been confined to the work of F. Hanson and his colleagues (Hanson and Tuckwell, 1981; Ryan and Hanson 1985, 1986;

Hanson and Ryan, 1988). This work uses Poisson jump processes to model large sudden changes in population size (disasters or bonanzas). The first two papers deal with exponential growth, while the last two consider logistic growth. Of these latter two papers, the first assumes that the amplitude and intensity of jumps is density independent while the second permits density dependence. Both papers rely on numerical solution of the Dynamic Programming (Bellman) equation to obtain a quantitative assessment of the effect of the possibility of sudden jumps in stock size on optimal fishing effort and the corresponding optimal value of the resource. In most cases the results seem to indicate that the effects on optimal effort are quite small but the effect on optimal value can be larger. The last aspect is more pronounced in the case of bonanzas than in the case of disasters.

The approach of this paper is somewhat different to that of Hanson and Ryan. In Section 2 it is assumed that disasters are sudden and random and completely destroy the fishery from an economic point of view. The probability of collapse (hazard rate) at any time is assumed to be density-dependent with higher probability of destruction occurring at lower stock levels. This is intended to reflect a greater susceptibility of a heavily-exploited stock to adverse environmental conditions, etc. Under these assumptions an analytic solution to the problem of determining the socially-optimal harvest policy is obtained. Indeed it is shown how the optimal control formulation and solution of Clark (1976) can be modified by including an extra state variable reflecting the probability of survival of the stock. This leads to a modification of the "golden-rule" equation (Clark 1976, p.43) for the optimal stock biomass.

In Section 3 some numerical results are given, which give a quantitative idea of the effects of risk on the optimal stock biomass and

sustained yield.

In Section 4 the assumption of the permanent extinction of the fishery is relaxed. It is assumed instead, that after a collapse to an economically non-viable level, the population can eventually recover. The recovery time is assumed to be random. It is shown how the possibility of recovery can be handled analytically through a simple modification to the hazard rate function.

In the Appendix the optimization problem is solved using Dynamic Programming rather than the Maximum Principle, as used in Section 2. This is partly for completeness, but also to establish certain formulas for the value function which are used in Section 2 in giving economic interpretations to the results.

Section 5 summarizes the results developed in the paper.

2. Harvesting model for a fishery subject to collapse

The basic continuous-time fishery model of Clark (1976) assumes that the dynamics of the population are governed by the differential equation

$$\frac{dX}{dt} = F(X) - u(t) \quad (1)$$

where $X(t)$ is the population biomass at time t , $F(X)$ is the (time) rate at which the population grows when at size X , and $u(t)$ is the (time) rate of harvesting at time t . It is also assumed that the unit cost of harvesting depends on the current population size, and can be represented by a decreasing function $c(X)$, and that there is a constant unit price for fish, p , corresponding to an infinitely elastic demand for fish.

Using the methods of linear optimal control, it is easily shown that the socially-optimal harvest policy to maximize the present value of net revenues,

$$J = \int_0^{\infty} e^{-\delta t} [p - c(X)] u(t) dt, \quad (2)$$

(where $\delta > 0$ is the instantaneous discount rate), involves bang-bang-singular control, with the singular path given by $X(t) \equiv \hat{X}$, where the optimum population biomass level $\hat{X}$ is the (assumed unique) solution to the "golden rule" equation (Clark 1976, p.40)

$$F'(X) - \frac{c'(X)F(X)}{p - c(X)} = \delta \quad (3)$$

If the initial population level X_0 exceeds $\hat{X}$, the optimal policy involves harvesting at the maximum rate until the population level reaches $\hat{X}$. Subsequently it is maintained at $\hat{X}$ through harvesting at the rate $F(\hat{X})$. If initially $X_0 < \hat{X}$ then optimally there is a moratorium on harvesting until $X(t)$ has grown to $\hat{X}$.

Clark (1976, p.42, p.74 and p.86) gives two interpretations to the

equation (3). Firstly it can be interpreted in a capital-theoretic sense, with the l.h.s. of (3) representing the marginal productivity of "fish capital", where fish capital is represented by the "net cash value"

$$\int_{X_{\infty}}^X [p - c(X)] dx$$

of a fish stock of size X . The level X_{∞} is the zero-profit level at which $c(X_{\infty}) = p$. Thus equation (3) prescribes that optimally the population should be maintained at the level at which the marginal productivity of fish capital is equal to the marginal productivity, δ , of capital invested elsewhere in the economy.

The second interpretation is in terms of sustainable economic rent and involves re-writing (3) as

$$p - c(\hat{X}) = \{[p - c(\hat{X})]F'(\hat{X}) - c'(\hat{X})F(\hat{X})\} / \delta \quad (4)$$

The l.h.s. is the immediate gain through reducing the equilibrium stock size by one unit from $\hat{X}$ while the r.h.s., which can be written as

$$\frac{d}{dX} \int_0^{\infty} e^{-\delta t} [p - c(\hat{X})] F(\hat{X}) dt ,$$

represents the reduction in the present value of future revenues, brought about by such a marginal change.

We consider now the situation in which the population is subject to the possibility of a sudden catastrophic collapse. The probability of such collapse can be characterized by a hazard function (see e.g. Ross 1983, p.25)

$$h(t) = \lim_{\Delta \rightarrow 0} \{P(\text{collapse in } (t, t+\Delta] | \text{ no collapse in } [0, t]) / \Delta\}$$

We shall assume that the probability of collapse is density-dependent.

Specifically let

$$h(t) = \phi(X(t)) \quad (5)$$

where $\phi(X)$ is some decreasing positive function.

Let τ be a random variable denoting the time at which the population collapses. (Note τ will be finite with probability one if $\phi(X) > 0$ for every X). Using a harvest schedule $u(t)$, the expected present value (EPV) of net revenues is

$$E\left\{ \int_0^{\tau} e^{-\delta t} [p - c(X)] u(t) dt \right\},$$

where the expectation is with respect to τ . Here we are assuming that the fishery never recovers after a collapse. The possibility of a recovery is dealt with in Section 4.

Consider now the survivor function

$$S(t) = P(\text{no collapse in } [0, t]),$$

which is related to the hazard function by

$$S(t) = \exp\left\{ - \int_0^t h(s) ds \right\}$$

Clearly $S^C(t) = 1 - S(t)$ is the cumulative distribution function (c.d.f.) of the random variable τ . Using this fact the EPV can be written

$$\begin{aligned} J &= \int_0^{\infty} \int_0^{\tau} e^{-\delta t} [p - c(X)] u(t) dt dS^C(\tau) \\ &= \int_0^{\infty} e^{-\delta t} [p - c(X)] u(t) S(t) dt, \end{aligned} \quad (6)$$

after changing the order of integration and evaluating the inner integral.

Now let

$$y(t) = - \log S(t), \quad (7)$$

so that

$$\frac{dy}{dt} = \frac{\dot{S}(t)}{S(t)} = h(t) ,$$

(see Reed, 1987). The problem of determining the socially-optimal harvest schedule assuming risk neutrality can now be written as:

Maximize

$$J = \int_0^{\infty} e^{-\delta t - y(t)} [p - c(X)] u(t) dt \quad (8)$$

subject to

$$\frac{dX}{dt} = F(X) - u(t) \quad (9)$$

$$\frac{dy}{dt} = \phi(X) \quad (10)$$

$$X(0) = 0, \quad y(0) = 1; \quad \text{and}$$

$$0 \leq u(t) \leq u_{\max}$$

where $u_{\max}$ is the maximum rate of harvest possible, reflecting perhaps the capacity of the fishing fleet. Note that a new state variable y related to the survival probability has been introduced to the problem. (See Kamien & Schwartz (1971) for what is apparently the earliest application of this technique). However the optimal control problem is still linear in the control variable u . It can be solved by standard methods (see e.g. Kamien & Schwartz, 1981).

The Hamiltonian is

$$\begin{aligned} H &= e^{-\delta t - y} [p - c(X)] u + \lambda_1 [F(X) - u] + \lambda_2 \phi(X) \\ &= [e^{-\delta t - y} (p - c(X)) - \lambda_1] u + \lambda_1 F(X) + \lambda_2 \phi(X) . \end{aligned} \quad (11)$$

Optimal control will be bang-bang-singular with the switching function

$$\sigma(t) = e^{-\delta t - y} (p - c(X)) - \lambda_1 . \quad (12)$$

For singular control $\sigma(t) \equiv 0$ on an interval, so that

$$\lambda_1 = e^{-\delta t - y}(p - c(X)) \quad (13)$$

Furthermore on an interval of singular control $d\sigma/dt \equiv 0$, which implies that

$$\dot{\lambda}_1 = [-(\delta + \dot{y})(p - c(X)) - c'(X)\dot{X}]e^{-\delta t - y} \quad (14)$$

However the adjoint equation for λ_1 is

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial X} = e^{-\delta t - y}c'(X)u - \lambda_1 F'(X) - \lambda_2 \phi'(X) \quad (15)$$

Thus on the singular path, from (14) and (15)

$$(\delta + \phi(X))(p - c(X)) + c'(X)\dot{X} = -c'(X)u + [\lambda_1 F'(X) + \lambda_2 \phi'(X)]e^{\delta t + y}$$

which on using (9) and (13) gives

$$e^{\delta t + y}\lambda_2 = \frac{p - c(X)}{\phi'(X)} \left\{ (\delta + \phi(X)) + \frac{c'(X)F(X)}{p - c(X)} - F'(X) \right\} \quad (16)$$

$$= R(X), \text{ say.}$$

Differentiating this equation with respect to t , and using the adjoint equation for λ_2 ,

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = e^{-\delta t - y}(p - c(X))u \quad (17)$$

and equations (14), (9) and (10) gives the following equation for the singular path:

$$[R'(X) + p - c(X)] \frac{dX}{dt} = (p - c(X))F(X) + (\delta + \phi(X))R(X) \quad (18)$$

We claim that the singular path is of the form $X(t) \equiv \text{constant}$. This seems fairly obvious from the autonomous nature of the problem. A formal proof based on Dynamic Programming is given in the Appendix¹. Thus on the singular path, the r.h.s. of (18) equals zero, and the population level X^* , on the singular path satisfies

$$(p - c(X^*)) F(X^*) + (\delta + \phi(X^*)) R(X^*) = 0 \quad (19)$$

It follows using (16) that X^* is the solution to:

$$F'(X) - \frac{c'(X)F(X)}{p - c(X)} = \delta + \phi(X) + \frac{\phi'(X)F(X)}{\delta + \phi(X)} \quad (20)$$

The optimal policy involves harvesting initially at the maximum rate until the population is reduced to the level X^* (or refraining from harvesting until the population reaches X^* , if initially $X_0 < X^*$) and subsequently maintaining the population at the level X^* .

Equation (20) shows how the golden rule equation (3) must be modified to account for a density-dependent risk of collapse.

We shall discuss the additional terms on the r.h.s. of the golden rule equation later but first we make some comments on interpretations of the equation parallel to those discussed by Clark (1976), and outlined earlier in this section, for the no-risk case.

Analogous to equation (4), equation (20) can be re-expressed as:

$$p - c(X^*) = \frac{p - c(X^*)}{\delta + \phi(X^*)} \left\{ F'(X^*) - \frac{c'(X^*)F(X^*)}{p - c(X^*)} - \frac{\phi'(X^*)F(X^*)}{\phi(X^*) + \delta} \right\} \quad (21)$$

The l.h.s. represents the gain in immediate revenue obtained through reducing the equilibrium stock size by one unit from X^* . The r.h.s. represents the corresponding marginal loss in the expected present value of future sustained-yield revenues, being the derivative, with respect to

X^* , of

$$V(X^*) = \int_0^{\infty} e^{-\delta t - \phi(X^*)t} [p - c(X^*)] F(X^*) dt = \frac{p - c(X^*)}{\delta + \phi(X^*)} F(X^*) \quad (22)$$

This is entirely analogous to the no-risk case. The capital-theoretic interpretation given by Clark (1976, p.74) in the no-risk case however does not appear to carry over directly to the case when the risk of collapse is present.

Following Clark (1976, p.86) let

$$U(X) = \int_{X_{\infty}}^X [p - c(z)] dz$$

denote the net cash value of the resource. The expected productivity of this "fish capital" is

$$E(dU) = (U + [p - c(X)] F(X) dt) (1 - \phi(X) dt) + 0 \cdot \phi(X) dt - U + o(dt)$$

so that

$$\lim_{dt \rightarrow 0} \left\{ \frac{E(dU)}{dt} \right\} = (p - c(X)F(X) - U\phi(X)) = G(U), \text{ say}$$

The marginal expected productivity is

$$\frac{d}{dU} G(U) = F'(X) - \frac{c'(X)F(X)}{p - c(X)} - \phi(X) - \frac{\phi'(X)U}{p - c(X)} \quad (23)$$

Using this in (20), we see that at the optimal stock size X^* , the marginal expected productivity of "fish capital" must be equal to

$$\delta + \phi'(X^*) \left\{ \frac{F(X^*)}{\delta + \phi(X^*)} - \frac{U(X^*)}{p - c(X^*)} \right\}.$$

Since

$$U(X^*) < V(X^*) = \frac{p - c(X^*)}{\delta + \phi(X^*)} F(X^*)$$

(otherwise it would be more profitable to immediately harvest the stock

rather than garner a sustained yield from it), and since $\phi'(X) < 0$, it follows that at the optimal stock size, the marginal expected productivity of "fish capital" U is less than that of capital invested elsewhere.

However if one thinks of the market value of a fish stock rather than its net cash value as a measure of "fish capital" then the above apparently paradoxical situation does not prevail. The market value $V(X)$ of a fish stock of size X is the capitalized value of future returns using optimal management. i.e.

$$V(X) = \sup_{u(t)} \left\{ E \left[\int_0^{\tau} e^{-\delta t} (p-c(X))u(t)dt \mid X(0) = X \right] \right\} \quad (24)$$

Conditional on there being no collapse during the interval $(t, t+dt]$, the natural productivity of this capital, at any time t is:

$$\frac{dV}{dt} = V'(X) \frac{dX}{dt} = V'(X)F(X) \quad (25)$$

It follows that the expected productivity of the capital is:

$$\lim_{dt \rightarrow 0} \left\{ \frac{E(dV)}{dt} \right\} = V'(X)F(X) - V(X)\phi(X) \quad (26)$$

Explicit expressions for $V(X)$ are derived in the Appendix (Equations (A.6) and (A.7)). Furthermore it is shown that for $X \leq X^*$

$$V'(X) - \frac{\delta + \phi(X)}{F(X)} V(X) = 0 \quad (27)$$

and for $X > X^*$

$$V'(X) + \frac{\delta + \phi(X)}{u_{\max} - F(X)} V(X) = \frac{u_{\max}(p-c(X))}{u_{\max} - F(X)} \quad (28)$$

Using (27) in the r.h.s. of (26) it can be seen that for $X \leq X^*$ the expected productivity of the capital stock V is δV . On the other hand for $X > X^*$, using (28) and the fact that $V(X) > (p-c(X))F(X)/(\delta + \phi(X))$, it can be seen that the expected productivity is less than δV . Thus optimally one

harvests only when the capital (market) value of the fish stock is growing at an expected rate lower than that of capital invested elsewhere in the economy.

Returning now to the golden-rule equation (20), one can see on comparison with the no-risk equation (3), that the effect of the risk of catastrophic collapse is to add two additional terms to the discount rate on the r.h.s. The two terms depend respectively on $\phi(X)$ and its derivative $\phi'(X)$. The $\phi(X)$ term would be present even for a constant (density-independent) risk of collapse. Its effect is to increase the effective discount rate and thereby reduce the optimal equilibrium biomass level. The phenomenon of an exogenously determined risk of catastrophe having the same effect on optimal policy as an increase in discount rate is well-known in other areas of resource economics (see e.g. Dasgupta and Heal, 1981; Reed, 1984).

The second additional term, $\phi'(X)F(X)/(\delta+\phi(X))$ on the r.h.s. of (20) occurs only in the case of a density-dependent probability of collapse, and would not be present if $\phi(X) \equiv \text{constant}$. Since $\phi'(X) < 0$, its effect is the opposite to that of the $\phi(X)$ term. i.e. it reduces the effective discount rate, and thereby has the effect of increasing the optimal equilibrium population size. It is this term that reflects the intuitive notion that, with a density-dependent probability of collapse present, one would want to avoid low population levels with their associated high risk of collapse. The $\phi(X)$ term on the other hand arises from recognition of the fact that collapse will eventually occur, and so future revenues, being uncertain, should be discounted more strongly. Which of the two terms will dominate depends on the nature of the functions, ϕ , F , and the value of the discount rate δ . It is possible that the effect of the risk of collapse could either increase or decrease

the optimal population level, as the numerical examples in the next section will illustrate.

One final point about equation (20) is that its r.h.s. can be written as

$$\delta + \phi(X) + \frac{d}{dt} \ln[\delta + \phi(X)]$$

where the derivative represents the natural rate of increase (no harvesting) since in that case $\dot{X} = F(X)$. Thus one can think of the effective discount rate under optimal management as being obtained by adding the risk premium $\phi(X)$ to δ , but then making the appropriate downward adjustment due to the fact in the absence of harvesting $\delta + \phi(X)$ will decrease over time. The above formula extends the result of Clark (1985, p.140, equation 3.145) to the case where the discount rate depends on stock size X .

3. Numerical Results

To illustrate the effect of the risk of catastrophic collapse on optimal population size and sustained yield, we give some numerical results based on data for the aggregated Antarctic baleen whale fishery (Clark and Lamberson, 1982). Following Clark and Lamberson we use the Gordon-Schaefer model:

$$F(X) = rX(1 - X/k)$$

$$c(X) = cq/X$$

where the parameters r , k , c and q represent respectively the intrinsic growth rate, and environmental carrying capacity for the population, the unit cost of effort and the catchability coefficient. Note that we ignore fixed capital costs. Values of these parameters used are:

$$r = .05 \text{ per annum, } k = 400,000 \text{ blue whale units (BWU),}$$

$$q = 1.3 \times 10^{-5} \text{ per catcher day, } c = \$5000 \text{ per catcher day.}$$

The unit price is assumed to be $p = \$7500$ per BWU.

We consider two forms of the risk function $\phi(X)$, viz.

$$(A) \quad \phi(X) = \frac{a}{X}$$

$$(B) \quad \phi(X) = \frac{b}{X^2}$$

Table 1 displays the optimal biomass, X^* , and the corresponding optimum sustained yield for the model (A) with $a = 10^3$. With this model and parameter value the hazards when the population size is respectively at 100%, 50%, 10% and 1% of the carrying capacity, are .0025, .005, .025 and .25 per year. Also shown in ^{Table} Fig. 1 are the risk-free optimal biomass and optimum sustained yield and the discount rate, which in a risk-free environment would result in the biomass X^* being optimal (i.e. the r.h.s. of

(20)). It can be seen that for discount rates of $2\frac{1}{2}\%$ or higher the effect of the risk of collapse is to cause the optimal biomass to be slightly lower than that in a risk-free environment, with a correspondingly lower optimum sustained yield. In this case the density-independent term on the r.h.s. of (20) dominates over the density-dependent term. This condition is reversed for very low discount rates. For example with a zero discount rate the optimal biomass X^* exceeds the risk-free optimal biomass $\hat{X}$. The optimal sustained yield however is still lower in the risky case because both X^* and $\hat{X}$ exceed the MSY level, $k/2$. The density-dependent term on the r.h.s. of (20) now dominates over the density-independent term, and the overall effect of risk is equivalent to having a negative discount rate in a risk-free environment.

Table 2 displays corresponding results for Model (B) with $b = 15 \times 10^8$. For this model the hazards when the population is respectively at 100%, 50%, 10% and 1% of carrying capacity are .00094, .0038, .094 and 9.4 per annum. The hazard is lower than that for the model of Table 1, for population sizes in excess of 37.5% of carrying capacity, but higher for smaller sizes. Indeed it is extremely high for small population sizes. It can be seen that for discount rates of $7\frac{1}{2}\%$ and higher the overall effect of risk is to reduce the optimal biomass level and optimum sustained yield (the density-independent term dominating), but for discount rates of 5% or less the overall effect of risk is to increase the optimal biomass level (with the density-dependent term dominating). Again there is a corresponding reduction in sustained yield, except for very small discount rates where even the risk-free optimal biomass exceeds the MSY level.

The results of this section have demonstrated that the overall effect of the risk of collapse on the optimal policy can be to either

increase or decrease the optimal biomass level. However in most cases the effect on optimal biomass level and on the corresponding optimum sustained yield is fairly small. It would appear that it has a significant effect on optimal policy only when the risk of destruction is very high or the discount rate very low. Ryan and Hanson (1986) and Hanson and Ryan (1988) observed for their model, that the effect of the possibility of jumps in biomass was greater on the expected present value than on the optimal harvest effort. This is true also for the model of this paper, the reason being that the risk of collapse changes the effective discount rate, and present value is very sensitive to the value of the discount rate. Broadly speaking, the numerical results of this section seem to concur with those of Hanson and Ryan, although the model used is somewhat different.

4. The Possibility of Recovery

A collapse of a fish stock is unlikely to result in the total extinction of the population. A more typical situation, experienced by many clupeid fisheries, is that following a crash, the stock remains at low levels with low productivity for a number of years, but eventually recovers provided a moratorium or at least severe limitations are placed on its exploitation.

To model this phenomenon suppose that after a collapse the time required for the biomass of the population to recover to a level X is a random variable T_X with

$$E(e^{-\delta T_X}) = m(X) \quad (29)$$

($m(X)$ is the moment-generating function (Laplace transform) of the random variable T_X , evaluated at δ). We shall assume $m'(\bar{X}) < 0$ reflecting the fact that "probabilistically" it will take longer for the stock to recover to higher biomass levels.

Consider now the problem of determining the optimal harvest pattern, when collapse followed by eventual recovery can occur. Suppose that the optimal policy involves bang-bang-singular harvesting with the singular path of the form $X \equiv \text{constant}, \bar{X}$ (the optimality of such a policy under certain conditions is proved formally in the Appendix). Then the population biomass $X(t)$ will follow a trajectory like that displayed in Figure 1. Here $\tau_1, \tau_2, \dots$ are independent random variables with the same exponential distribution with mean $[\phi(\bar{X})]^{-1}$, representing the durations of the periods in each cycle during which the fishery is active, and $T_1, T_2, \dots$

are independent random variables representing the recovery periods in each cycle.

The present value of earnings over one harvest period, conditional on the length of the period τ is

$$[p - c(\bar{X})]F(\bar{X}) \int_0^\tau e^{-\delta t} dt = \frac{(p - c(\bar{X}))F(\bar{X})}{\delta} (1 - e^{-\delta\tau})$$

Thus the expected present value over one harvest period is

$$\frac{(p - c(\bar{X}))F(\bar{X})}{\delta} E(1 - e^{-\delta\tau}) = \frac{(p - c(\bar{X}))F(\bar{X})}{\delta + \phi(\bar{X})}$$

since there are no earnings over a recovery period (and since we are ignoring fixed capital costs), the expected present value of net revenues over an infinite time horizon, given an initial stock biomass level X_0 and assuming impulse controls possible² (i.e. $u_{\max} = \infty$) is

$$\begin{aligned} \Pi &= \int_{\bar{X}}^{X_0} (p - c(z)) dz + E \left\{ \frac{(p - c(\bar{X}))F(\bar{X})}{\delta + \phi(\bar{X})} [1 + e^{-\delta(\tau_1 + T_1)} + e^{-\delta(\tau_1 + \tau_2 + T_1 + T_2)} + \dots] \right\} \\ &= \int_{\bar{X}}^{X_0} (p - c(z)) dz + \frac{(p - c(\bar{X}))F(\bar{X})}{\delta + \phi(\bar{X})} \left[1 + \frac{m(\bar{X})\phi(\bar{X})}{\delta + \phi(\bar{X})} + \left(\frac{m(\bar{X})\phi(\bar{X})}{\delta + \phi(\bar{X})} \right)^2 + \dots \right] \\ &= \int_{\bar{X}}^{X_0} (p - c(z)) dz + \frac{(p - c(\bar{X}))F(\bar{X})}{\delta + \phi(\bar{X})} \left(1 - \frac{m(\bar{X})\phi(\bar{X})}{\delta + \phi(\bar{X})} \right)^{-1} \\ &= \int_{\bar{X}}^{X_0} (p - c(z)) dz + \frac{(p - c(\bar{X}))F(\bar{X})}{\delta + \phi(\bar{X})(1 - m(\bar{X}))} \end{aligned} \quad (30)$$

We note that this is exactly like the EPV formula for a single cycle (no recovery) only with the risk function $\phi(X)$ replaced by $\phi(X)(1 - m(X))$, c.f. (A9). Indeed if we let the recovery time tend to infinity in probability so that $m(\bar{X}) \rightarrow 0$, then (30) reduces to the single cycle formula.

Conversely if the recovery time tends to zero in probability, so that $m(X) \rightarrow 1$, then (30) reduces to

$$\int_{\bar{X}}^{X_0} (p-c(z))dz + \frac{(p-c(\bar{X}))F(\bar{X})}{\delta},$$

which is the well-known present value formula for a deterministic fishery with impulse controls possible.

To determine the optimal biomass level we need only set the derivative of (30) w.r.t. $\bar{X}$ equal to zero. This gives the following "golden rule" equation

$$F'(\bar{X}) - \frac{c'(\bar{X})F(\bar{X})}{p-c(\bar{X})} = \delta + \phi(\bar{X})(1-m(\bar{X})) + \frac{\frac{d}{d\bar{X}} [\phi(\bar{X})(1-m(\bar{X}))]F(\bar{X})}{\delta + \phi(\bar{X})(1-m(\bar{X}))}. \quad (31)$$

Economic interpretations of this equation entirely analogous to those given to the single-cycle (no recovery) case in the last section can be derived. Also in a similar way the second and third terms on the r.h.s. of (31) can be split up into density-independent and density-dependent components. The density-independent component is

$$\phi(\bar{X})(1-m(\bar{X})) - \frac{m'(\bar{X})\phi(\bar{X})F(\bar{X})}{\delta + \phi(\bar{X})(1-m(\bar{X}))} \quad (32)$$

and the density-dependent component

$$\frac{\phi'(\bar{X})(1-m(\bar{X}))F(\bar{X})}{\delta + \phi(\bar{X})(1-m(\bar{X}))} \quad (33)$$

Clearly since $0 \leq m(\bar{X}) \leq 1$, the density-dependent component, at a given level $\bar{X}$, is of less importance (has smaller absolute value) than in the case when no recovery is possible. Whether this is likewise true for the density-independent component depends on the behaviour of $m(X)$. As an example consider the case when all continuous growth is deterministic and the randomness in the recovery time results only from the randomness in the level X_c to which the population biomass falls in a collapse. Assume

also that X_c is independent of the pre-collapse level and that further collapses can occur while the population is recovering from an earlier one. In this case $m(X)$ satisfies the Riccati-type differential equation

$$m'(X) = -\frac{m(X)}{F(X)} \{\delta + \phi(X)(1-m(X))\} \quad (34)$$

To see this let $T(a,b)$ denote the random time taken for the population to reach the biomass level b from the lower level a . Then

$$m(X) = E \left\{ e^{-T(X_c, X)} \right\} \quad (35)$$

and

$$m(X+dX) = E \left\{ e^{-T(X_c, X)} e^{-T(X, X+dX)} \right\},$$

From independence one can write

$$m(X+dX) = m(X) \left[e^{-\delta dX/F(X)} \left(1 - \frac{\phi(X)}{F(X)} dX\right) + \frac{\phi(X)}{F(X)} dX m(X+dX) \right],$$

since if there are no collapses it takes a time $dX/F(X)$ to grow from X to $X+dX$. Dividing by dX and passing to the limit results in (34).

Upon substituting (34) into (32) it can be seen that for this model, (32) is equal to $\phi(X)$. i.e. that the presence of the possibility of recovery has no effect on the density-independent component. Thus the r.h.s. of (31) is greater than or equal to

$$\delta + \phi(\bar{X}) + \frac{\phi'(\bar{X})F(\bar{X})}{\delta + \phi(\bar{X})},$$

and therefore the optimal biomass level, $\bar{X}$, when recovery is possible is less than or equal to the optimal biomass level X^* when no recovery is possible. The reason for this is solely due to a decrease in the density-dependent effect (i.e. a decrease in the upward influence on the optimal biomass level). The density-independent effect is unchanged by the

presence of the possibility of recovery. Indeed for a density-independent hazard, $\phi(X) \equiv \text{const}$, the optimal biomass level remains unchanged by the possibility of a recovery ($\bar{X} = X^*$).

Furthermore if one considers what happens as $m(X)$ decreases (i.e. recovery time increases in a probabilistic sense, because, for example, the collapsed level X_c becomes on average lower) it can be seen that $\bar{X}$ increases towards the no-recovery level X^* .

While the model of this section, involving recovery, bears some resemblance to the model studied numerically by Hanson and Ryan (1988) it differs from it in several important ways. Firstly in the Hanson-Ryan model it is assumed that drops in biomass are of a magnitude deterministically related to the pre-collapse level. In the model of this section the jumps are of a size which, while dependent on the pre-collapse level, also involve a random component. Specifically the level to which the population biomass collapses is assumed to be a random variable independent of the pre-collapse level. Furthermore growth back to viable levels can include random (diffusion-like) components. Secondly the Hanson-Ryan model assumes that cost and harvest rate are respectively quadratically and linearly related to effort. In contrast here cost is assumed to be linearly related to effort, while the harvest rate can be related in any way to effort, as is reflected in the general cost function $c(X)$ (See Clark 1976, p.37 for a discussion of these issues). Finally the possibility of impulse controls, allowing instantaneous discontinuous adjustment of biomass size is assumed in this paper. In the Hanson-Ryan paper effort is assumed to be bounded above.

5. Summary and Conclusions

The socially-optimal harvest policy for a fishery subject to random catastrophic collapse has been determined, using methods of optimal control. It has been shown how the risk of destruction can be incorporated by including an extra state variable corresponding to the survival probability. The optimal policy is shown to be one of constant escapement (bang-bang-singular control) with the optimal biomass level given by a modified form of the well-known "golden rule" equation of deterministic theory. The modification includes the addition of extra terms corresponding respectively to the density-independent and the density-dependent nature of the hazard. The former, which would be present even for a constant (density-independent) hazard, has the same effect as an increase in the discount rate; this causes a decrease in the optimal biomass level. The latter depends on how the hazard increases as biomass level decreases. Its effect is to increase the optimal biomass level. Which of these opposing tendencies dominates depends on the particular values employed. Numerical examples are given in which the overall effect of the presence of risk is both to increase and decrease optimal biomass levels. From the limited cases investigated numerically it would appear that the effect of the risk of collapse is not large either on the optimum harvest or on the optimal biomass level. It will be greatest when the discount rate is small and the risk of collapse high.

The reason for the two opposing tendencies on optimal policy is as follows. On the one hand, since there is a risk of the fishery collapsing in the future, it pays the resource owner to exploit it more heavily in the present. Conservation is of less value since the fishery may collapse before its benefits can be reaped. On the other hand if the risk of

collapse increases as stock biomass decreases, then it pays the resource owner to maintain the stock at a higher level than he otherwise would, in order to reduce the probability of collapse.

The possibility of a recovery after a collapse can be handled fairly easily, at least when impulse controls are permitted. In this case it can be shown that the effect of the possibility of recovery on optimal policy is the same as that of a modification of the hazard function. Specifically one multiplies the hazard function by the complement of a characteristic function of recovery time. When growth is deterministic and the randomness in recovery time arises only through randomness in the level to which the population drops after a collapse, it is shown that the density-independent effect of the risk of collapse is unchanged by the possibility of recovery, while the density-dependent effect is reduced resulting in an overall decrease in the optimal biomass level.

Economic interpretations of the "golden rule" equation for determining the optimal stock biomass are given. It is shown that under optimal management harvesting takes place only when the market value of the asset (the capitalized value of maximized future returns) is growing in expectation at a rate lower than the discount rate. In other words if capital invested elsewhere in the economy can grow faster than the capital value of the fishery, then optimally it pays to disinvest in the fishery (i.e. reduce the biomass level) until its growth rate has increased to that of capital invested elsewhere.

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Appendix

Solution to the Optimization Problem using Dynamic Programming

Let $V(X)$ denote the expected present value of future net revenues using optimal management and given $X_0 = X$. i.e.

$$V(X) = \sup_{u(t)} \left\{ E \left[\int_0^{\tau} e^{-\delta t} (p-c(X))u(t) dt \mid X(0) = X \right] \right\} \quad (A1)$$

Clearly from the Principle of Optimality (e.g. Kamien & Schwartz, 1981, p.238), V satisfies:

$$V(X) = \max_{0 \leq u \leq u_{\max}} \left\{ u(p-c(X))dt + e^{-\delta dt} [(1-\phi(X)dt)V(X+(F(X)-u)dt) + \phi(X)dt \cdot 0] + o(dt) \right\}$$

since with probability $\phi(X)dt$ the fishery collapses to a level of zero value in an increment of time of length dt . After expanding in powers of dt , dividing by dt and passing to the limit, one arrives at the following functional equation:

$$[\delta + \phi(X)]V(X) - F(X)V'(X) = \max_{0 \leq u \leq u_{\max}} \left\{ u[p-c(X) - V'(X)] \right\} \quad (A2)$$

This is the Bellman equation of Dynamic Programming. It is obvious from (A2) that the optimal control u^* must be of bang-bang-singular form, with:

$$u^*(t) = \begin{cases} 0 & \text{if } p-c(X) < V'(X) \\ u_{\max} & \text{if } p-c(X) > V'(X) \end{cases}$$

Singular control can only occur when $p-c(X) \equiv V'(X)$ on some interval of time.

Consider now the policy of maintaining the population biomass at some level $\tilde{X}$, (constant-escapement policy). The value function $W(X)$ (denoting the expected present value of net returns through using this policy, conditional on $X(0)=X$) satisfies (using an argument similar to that used above to establish the Bellman equation):

$$[\delta + \phi(X)]W(X) - F(X)W'(X) = 0, \quad X < \tilde{X}$$

$$[\delta + \phi(X)]W(X) - [F(X) - u_{\max}]W'(X) = u_{\max}(p - c(X)), \quad X > \tilde{X}.$$

i.e. on $(0, \tilde{X})$ W satisfies the homogeneous equation

$$W'(X) - \frac{\delta + \phi(X)}{F(X)} W(X) = 0 \quad (A3)$$

and on $(\tilde{X}, \infty)$ W satisfies the non-homogeneous equation

$$W'(X) + \frac{\delta + \phi(X)}{u_{\max} - F(X)} W(X) = \frac{u_{\max}(p - c(X))}{u_{\max} - F(X)} \quad (A4)$$

At $X = \tilde{X}$,

$$\begin{aligned} W(\tilde{X}) &= E\left\{ \int_0^{\tau} e^{-\delta t} (p - c(\tilde{X})) F(\tilde{X}) dt \right\} \\ &= (p - c(\tilde{X})) F(\tilde{X}) E\{(1 - e^{-\delta \tau}) / \delta\} \\ &= \frac{p - c(\tilde{X})}{\delta + \phi(\tilde{X})} \cdot F(\tilde{X}) \end{aligned} \quad (A5)$$

since when $X(t) \equiv \tilde{X}$, the survival time τ has an exponential distribution with mean $1/\phi(\tilde{X})$. To determine $W(X)$ explicitly the differential equations (A3) and (A4) can be solved using (A5) as a boundary condition. This yields

$$W(X) = \frac{p - c(\tilde{X})}{\delta + \phi(\tilde{X})} F(\tilde{X}) \exp \left\{ - \int_X^{\tilde{X}} \frac{\delta + \phi(z)}{F(z)} dz \right\} \quad \text{for } 0 \leq X \leq \tilde{X}; \quad (A6)$$

and on using the method of variation of constants (e.g. Collatz p.15), the solution to (A4), is for $X > \tilde{X}$

$$W(X) = K(X) \exp \left\{ - \int_{\tilde{X}}^X \frac{\delta + \phi(z)}{u_{\max} - F(z)} dz \right\} \quad (A7)$$

where

$$K(X) = \frac{p-c(\tilde{X})}{\delta+\phi(\tilde{X})} F(\tilde{X}) + \int_{\tilde{X}}^X \frac{u_{\max}(p-c(t))}{u_{\max}-F(t)} \exp \left\{ \int_{\tilde{X}}^t \frac{\delta+\phi(z)}{u_{\max}-F(z)} dz \right\} dt \quad (A8)$$

Note that as $U_{\max} \rightarrow \infty$, the exponential terms in (A7) and (A8) vanish and

$$W(X) \rightarrow \frac{p-c(\tilde{X})}{\delta+\phi(\tilde{X})} F(\tilde{X}) + \int_{\tilde{X}}^X [p - c(t)] dt \quad (A9)$$

for $X > \tilde{X}$.

Consider now the problem of determining the best constant escapement level. If $dW(X)/d\tilde{X} = 0$ has a unique solution for all X , then clearly this solution will provide it. Direct differentiation of (A6) and (A7) w.r.t. $\tilde{X}$ will confirm that $dW(X)/d\tilde{X} \equiv 0$ for all X only for values of $\tilde{X}$ which satisfy

$$\frac{d}{d\tilde{X}} \left[\frac{p-c(\tilde{X})}{\delta+\phi(\tilde{X})} F(\tilde{X}) \right] = p - c(\tilde{X}) . \quad (A10)$$

Carrying out the differentiation on the l.h.s. reveals that (A10) is simply the golden-rule equation (20). We shall assume that the various functions involved satisfy the necessary concavity/convexity conditions in order for there to be a unique solution, $\tilde{X}^*$, to this equation, with the l.h.s. exceeding the r.h.s. for $X < \tilde{X}^*$ and vice-versa for $X > \tilde{X}^*$. In this case $\tilde{X}^*$ as defined by the golden-rule equation will provide the best constant escapement level.

It remains to be proved that this constant escapement policy is the best among all policies. To do this we shall show that the associated value function $W^*(X)$ satisfied the Bellman equation (A2).

From (A2), (A3) and (A4) it can be seen that this will be the case provided

$$W^{*'}(X) > p - c(X) \quad \text{for } X < X^*$$

$$W^{*'}(X) < p - c(X) \quad \text{for } X > X^*$$

$$\text{and } W^{*'}(X) = p - c(X) \quad \text{for } X = X^* .$$

To establish this we note firstly that for $X \neq X^*$

$$W^*(X) > \frac{p - c(X)}{\delta + \phi(X)} F(X)$$

since constant escapement at X^* is a superior policy to constant escapement at X . Thus from (A3) for $X < X^*$

$$W^{*'}(X) = \frac{\delta + \phi(X)}{F(X)} W^*(X) > p - c(X)$$

and from (A4) for $X > X^*$

$$\begin{aligned} W^{*'}(X) &= \frac{u_{\max}(p - c(X))}{u_{\max} - F(X)} - \frac{\delta + \phi(X)}{u_{\max} - F(X)} W^*(X) \\ &< p - c(X) . \end{aligned}$$

Finally from (A3)

$$W^{*'}(X^*) = \frac{\delta + \phi(X^*)}{F(X^*)} W^*(X^*) = p - c(X^*)$$

and from (A4)

$$\begin{aligned} W^{*'}(X^*) &= \frac{u_{\max}(p - c(X^*))}{u_{\max} - F(X^*)} - \frac{p - c(X^*)}{u_{\max} - F(X^*)} F(X^*) \\ &= p - c(X^*) \end{aligned}$$

This completes the proof that $W^*(X)$ satisfies the Bellman equation. Thus constant escapement at X^* is the optimal policy, and the expected present

value of the asset $V(X)$ is given by $W^*(X)$. i.e. an explicit expression for $V(X)$ is given by equations (A6) and (A7) with $\tilde{X}$ replaced by X^* .

The Case when Recovery is Possible

Suppose that after a collapse the biomass is reduced to some random level X_c . Then from the Principle of Optimality the value function $V(X)$ satisfies:

$$V(X) = \sup_u \left\{ u(p-c(X))dt + e^{-\delta dt} [(1-\phi(X)dt)V(X+(F(X)-u)dt) + \phi(X)dt E(V(X_c)) + o(dt)] \right\}$$

which gives the following Bellman equation

$$\delta V(X) + \phi(X)[V(X) - \bar{V}(X_c)] - F(X)V'(X) = \max_u \left\{ u[p-c(X) - V'(X)] \right\}, \quad (A11)$$

where $\bar{V}(X_c) = E(V(X_c))$.

Now as before consider a policy of constant escapement at a level $\tilde{X}$, and let $W(X)$ denote the corresponding value function. If it takes a random time T_X to recover from the collapsed level to the level X , then clearly for $X \leq \tilde{X}$

$$W(X_c) = E(W(X_c)) = E \left\{ e^{-T_X} W(X) \right\} = m(X)W(X) \quad (A12)$$

It follows that on $(0, \tilde{X})$ $W(X)$ satisfies

$$W'(X) - \frac{\delta + \phi(X)(1-m(X))}{F(X)} W(X) = 0 \quad (A13)$$

and for $X > \tilde{X}$,

$$W'(X) + \frac{\delta + \phi(X)}{u_{\max} - F(X)} W(X) = [u_{\max}(p-c(X)) + \phi(X)m(\tilde{X})W(\tilde{X})] / (u_{\max} - F(X)) \quad (A14)$$

with

$$W(\tilde{X}) = \frac{p - c(\tilde{X})}{\delta + \phi(\tilde{X})(1 - m(\tilde{X}))} F(\tilde{X}) \quad (A15)$$

As before (A13) and (A14) can be solved using (A15) as an initial condition. Denote now by $\tilde{X}$ the solution to the golden-rule equation (29). It can be confirmed (after solving A13) that, as before, $dW(X)/d\tilde{X} \equiv 0$ for all $X < \tilde{X}$ provided $\tilde{X} = \bar{X}$. However if $u_{\max} < \infty$, there is no unique value of $\tilde{X}$ which will make $dW(X)/d\tilde{X} \equiv 0$, for all $X > \tilde{X}$. This suggests that if there is an upper bound to the rate of harvesting the optimal policy is not one of constant escapement.

The reason for this is that with $u_{\max} < \infty$, the transition time from an initial state to a target escapement level is non-zero. During this time a collapse can occur. Furthermore, unlike in the situation in which no recovery is possible, the value of a collapsed stock depends on the escapement level used. This is reflected in the fact that $\tilde{X}$ appears in the differential equation (A14) (in the case when recovery is possible), but does not appear in the corresponding differential equation (A4) (for the case when no recovery is possible).

When impulse controls are permitted ($u_{\max} = \infty$), however, (A14) reduces to

$$W'(X) = p - c(X)$$

for $X > \tilde{X}$. In this case $dW(X)/d\tilde{X} \equiv 0$ for all $X > \tilde{X}$ when $\tilde{X} = \bar{X}$. Thus in this case $\tilde{X}$ is unambiguously the best constant-escapement level. The overall optimality of this policy can be confirmed as before by showing that the corresponding value function $\tilde{W}(X)$ satisfies the Bellman equation (A11).

1. In fact a complete solution to the problem using Dynamic Programming is given in the Appendix. We have chosen, however, in the main body of the paper to convert the problem to one of deterministic control, and use the Maximum Principle to solve it because this approach reveals in a more obvious way the parallels with the no-risk problem (Clark 1976).
2. If there is a finite upper bound ($u_{\max} < \infty$) to the rate of harvesting the expression for the expected present value is more complicated. This is because the fishery can collapse during the transition phase from the initial level to the constant escapement level. It turns out that in this case a policy of constant escapement may not be optimal. (For a discussion based on Dynamic Programming, see the Appendix). In order to avoid needless mathematical complications we shall assume in this section that instantaneous adjustment, through impulse controls, is possible (i.e. $u_{\max} = \infty$).

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Discount Rate δ (% per annum)	Optimal Biomass X^* (BWU)	Optimal Risk-free Biomass $\hat{X}$ (BWU)	Optimum Sustained Yield (BWU per annum)	Optimum Risk-free Sustained Yield (BWU per annum)	Equivalent Risk-free Discount Rate (% per annum)
0	267,200	225,600	4,440	4,920	-1.29
2.5	157,600	158,100	4,780	4,780	2.52
5.0	110,700	114,900	4,000	4,090	5.35
7.5	88,700	92,300	3,450	3,550	8.12
10.0	77,900	80,500	3,130	3,210	10.83

Table 1. Optimal biomass and optimum sustained yield when the risk function is of the form $\phi(X) = a/X$, with $a = 1000$. Also shown are the risk-free optimal biomass and optimum sustained yield, and the risk-free discount rate which would result in the optimal biomass X^* .

Discount Rate δ (% per annum)	Optimal Biomass X^* (BWU)	Optimal Risk-free Biomass $\hat{X}$ (BWU)	Optimum		Equivalent	
			Sustained Yield (BWU per annum)	Risk-free Sustained Yield (BWU per annum)	Risk-free Discount Rate (% per annum)	Risk-free Discount Rate (% per annum)
0	301,700	225,600	3,710	4,920	-2.29	
2.5	169,200	158,100	4,880	4,780	2.02	
5.0	117,100	114,900	4,140	4,090	4.82	
7.5	90,400	92,400	3,500	3,550	7.81	
10.0	77,700	80,500	3,130	3,210	10.88	

Table 2. Optimal biomass and optimal sustained yield when the risk function is of the form $\phi(X) = b/X^2$, with $b = 1.5 \times 10^8$. Also shown are the risk-free optimal biomass and optimum sustained yield, and the risk-free discount rate which would result in the optimal biomass X^* .

Figure 1 A possible time-path of the stock biomass under a constant-escapement policy at the level $\bar{X}$. An initial impulse harvest is followed by a period of sustained yield harvesting, until the stock collapses at the random time τ_1 . No harvesting is carried out during the (random) time T_1 required for the biomass to recover to the level $\bar{X}$. Subsequently periods of sustained yield harvesting and recovery alternate.