

NEW RESULTS INVOLVING A CERTAIN CLASS
OF q -ORTHOGONAL POLYNOMIALS

by

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A Christoffel–Darboux formula and a Brafman type generating function are given for the q –Racah (or hypergeometric ${}_4\Phi_3$) polynomials which were introduced recently by R. Askey and J. Wilson ([6] and [7]). The former result is then applied to deduce a q –extension of a class of finite summation formulas involving generalized double hypergeometric functions, which were considered earlier by H.M. Srivastava (see [14] and [15]). Finally, a bilinear generating function is proved for the q –Konhauser polynomials $Z_n^{(\alpha)}(x, k | q)$ which, for $k = 1$, reduce immediately to the familiar q –extension of the classical Laguerre polynomials.

1. INTRODUCTION, NOTATIONS, AND DEFINITIONS

For real or complex q , $|q| < 1$, put

$$(1.1) \quad (\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j)$$

and let $(\lambda; q)_\mu$ be defined by

$$(1.2) \quad (\lambda; q)_\mu = \frac{(\lambda; q)_\infty}{(\lambda q^\mu; q)_\infty}$$

for arbitrary parameters λ and μ , so that

$$(1.3) \quad (\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q) \cdots (1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, \dots\}. \end{cases}$$

Also define a basic (or q -) hypergeometric function by (cf. [13, Chapter 3]; see also [16, p. 347, Equation (272)])

$$(1.4) \quad {}_r\Phi_s \left[\begin{matrix} \alpha_1, \dots, \alpha_r; \\ \beta_1, \dots, \beta_s; \end{matrix} \middle| q, z \right] = \sum_{n=0}^{\infty} (-1)^{(1-r+s)n} q^{(1-r+s)n(n-1)/2} \frac{(\alpha_1, \dots, \alpha_r; q)_n}{(\beta_1, \dots, \beta_s; q)_n} \frac{z^n}{(q; q)_n},$$

where (and throughout this paper) we find it to be convenient to write

$$(1.5) \quad (\lambda_1, \dots, \lambda_{\kappa}; q)_{\mu} = (\lambda_1; q)_{\mu} \cdots (\lambda_{\kappa}; q)_{\mu},$$

and, for convergence of the infinite series in (1.4),

$$|q| < 1 \quad \text{and} \quad |z| < \infty \quad \text{when} \quad r \leq s,$$

or

$$\max\{|q|, |z|\} < 1 \quad \text{when} \quad r = s + 1,$$

provided that no zeros appear in the denominator.

Recently, Askey and Wilson ([6]; see also [7]) introduced the q -Racah polynomials:

$$(1.6) \quad R_n(\mu(x)) \equiv R_n(\mu(x); \alpha, \beta, \gamma, \delta; q) = {}_4\Phi_3 \left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1}; \\ q, q \\ \alpha q, \beta\delta q, \gamma q; \end{matrix} \right]$$

$$(\mu(x) \equiv q^{-x} + \gamma\delta q^{x+1})$$

which are orthogonal on the set $\{0, 1, \dots, N\}$ when (for example) one of

$$\alpha q, \beta\delta q \text{ or } \gamma q \text{ is } q^{-N},$$

and satisfy the following three-term recurrence relation (cf. [6, p. 1015, Equation (4.6)]):

$$(1.7) \quad \begin{aligned} & -(1-q^{-x})(1-\gamma\delta q^{x+1})R_n(\mu(x)) \\ & = A_n R_{n+1}(\mu(x)) - (A_n + C_n)R_n(\mu(x)) + C_n R_{n-1}(\mu(x)), \quad n \geq 0, \end{aligned}$$

where $R_{-1}(\mu(x)) = 0$, $R_0(\mu(x)) = 1$,

$$(1.8) \quad A_n = \frac{(1-\alpha\beta q^{n+1})(1-\alpha q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}$$

and

$$(1.9) \quad C_n = \frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}.$$

Many interesting special cases of the ${}_4\Phi_3$ polynomials (1.6) are scattered in the literature (cf., e.g., [6, p. 1015]). For this very general set of orthogonal polynomials, we

first give a Christoffel–Darboux formula and apply it to derive a class of finite summation formulas involving generalized double q -hypergeometric functions defined by (cf. [16, p. 349])

$$\begin{aligned}
 (1.10) \quad & \Phi_{s:h;v}^{r:p;u} \left[\begin{matrix} \lambda_1, \dots, \lambda_r; \alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_u; q; x, y \\ \mu_1, \dots, \mu_s; \beta_1, \dots, \beta_h; \delta_1, \dots, \delta_v; i, j, k \end{matrix} \right] \\
 &= \sum_{\ell, m=0}^{\infty} \frac{(\lambda_1, \dots, \lambda_r; q)_{\ell+m} (\alpha_1, \dots, \alpha_p; q)_{\ell} (\gamma_1, \dots, \gamma_u; q)_m}{(\mu_1, \dots, \mu_s; q)_{\ell+m} (\beta_1, \dots, \beta_h; q)_{\ell} (\delta_1, \dots, \delta_v; q)_m} \\
 &\quad \cdot q^{i\ell(\ell-1)/2 + jm(m-1)/2 + k\ell m} \frac{x^{\ell}}{(q; q)_{\ell}} \frac{y^m}{(q; q)_m},
 \end{aligned}$$

which, for $i = j = k = 0$, may be written simply as

$$(1.11) \quad \Phi_{s:h;v}^{r:p;u} \left[\begin{matrix} \lambda_1, \dots, \lambda_r; \alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_u; \\ q; x, y \\ \mu_1, \dots, \mu_s; \beta_1, \dots, \beta_h; \delta_1, \dots, \delta_v; \end{matrix} \right].$$

We then proceed to derive a general Brafman type generating function for the q -Racah polynomials (1.6) and show how this general result would yield a number of known or new generating functions for such other q -orthogonal polynomials as the little q -Jacobi polynomials (cf. [9, p. 29] and [3, p. 48, Equation (3.34)])

$$(1.12) \quad p_n^{(\alpha, \beta)}(x; q) = \frac{(\alpha q; q)_n}{(q; q)_n} {}_2\Phi_1 \left[\begin{matrix} q^{-n}, \alpha \beta q^{n+1}; \\ q, qx \\ \alpha q; \end{matrix} \right]$$

and the big q -Jacobi polynomials [3, p. 47, Equation (3.28)]

$$(1.13) \quad P_n^{(\alpha, \beta)}(x; \gamma, \delta; q) = \frac{(\alpha q, -\delta \alpha q / \gamma; q)_n}{(q, -q; q)_n} \left[\frac{\gamma}{\alpha q} \right]^n \\ \cdot {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha \beta q^{n+1}, \alpha x q / \gamma; \\ \alpha q, -\delta \alpha q / \gamma; \end{matrix} \quad q, q \right].$$

Finally, in Section 4 we prove a bilinear generating function for the q -Konhauser polynomials (cf. [1, p. 3, Equation (3.1)])

$$(1.14) \quad Z_n^{(\alpha)}(x, k | q) = \frac{(\alpha q; q)_{nk}}{(q^k; q^k)_n} \sum_{j=0}^n \frac{(q^{-nk}; q^k)_j}{(\alpha q; q)_{kj}} q^{kj(n+1)+kj(kj-1)/2} \\ \cdot \frac{(\alpha x)^{kj}}{(q^k; q^k)_j},$$

which reduces, when $k = 1$, to the familiar q -Laguerre polynomials.

2. A CHRISTOFFEL-DARBOUX FORMULA FOR q -RACAH POLYNOMIALS

With a view to applying it in the next section, we begin here by stating the following Christoffel-Darboux formula (cf. [18, p. 43]) for the q -Racah polynomials (1.6):

$$(2.1) \quad \sum_{k=0}^n g_k^{-1} R_k^*(\mu(x)) R_k^*(\mu(y))$$

$$= \frac{h_n}{g_n h_{n+1}} \{f(x,y;q)\}^{-1} \left\{ R_{n+1}^*(\mu(x)) R_n^*(\mu(y)) - R_n^*(\mu(x)) R_{n+1}^*(\mu(y)) \right\},$$

where, for convenience,

$$(2.2) \quad R_n^*(\mu(x)) = \frac{(\alpha q; q)_n}{(q; q)_n} R_n(\mu(x)),$$

$$(2.3) \quad f(x,y;q) = (q^{-x} - q^{-y})(1 - \gamma \delta q^{x+y+1}),$$

$$(2.4) \quad g_n = \frac{(\gamma \delta q)^n (\alpha q, \beta q, \alpha \beta q / \gamma, \alpha q / \delta; q)_n}{(1 - \alpha \beta q^{2n+1})(q, \alpha \beta q, \gamma q, \beta \delta q; q)_n},$$

and

$$(2.5) \quad h_n = \frac{(\alpha \beta q; q)_{2n}}{(q, \alpha \beta q, \gamma q, \beta \delta q; q)_n}.$$

Derivation of Formula (2.1). In view of the definitions (2.2), (2.4), and (2.5), the recurrence relation (1.7) can be written in the form:

$$(2.6) \quad \begin{aligned} & -(1 - q^{-x})(1 - \gamma \delta q^{x+1}) g_k^{-1} R_k^*(\mu(x)) \\ &= \frac{h_k}{g_k h_{k+1}} R_{k+1}^*(\mu(x)) + \frac{h_{k-1}}{g_{k-1} h_k} R_{k-1}^*(\mu(x)) \\ & \quad - \left\{ \frac{(1 - \alpha q^{k+1}) h_k}{(1 - q^{k+1}) g_k h_{k+1}} + \frac{(1 - q^k) h_{k-1}}{(1 - \alpha q^k) g_{k-1} h_k} \right\} R_k^*(\mu(x)), \quad k \geq 1, \end{aligned}$$

which, upon some manipulations, yields

$$(2.7) \quad g_k^{-1} f(x,y;q) R_k^*(\mu(x)) R_k^*(\mu(y)) = t_k T_k - t_{k-1} T_{k-1} ,$$

where $f(x,y;q)$ is given by (2.3),

$$(2.8) \quad t_k = \frac{h_k}{g_k h_{k+1}} , \text{ and } T_k = R_{k+1}^*(\mu(x)) R_k^*(\mu(y)) - R_k^*(\mu(x)) R_{k+1}^*(\mu(y)) .$$

Summing each member of (2.7) from $k = 1$ to $k = n$, we readily find that

$$(2.9) \quad f(x,y;q) \sum_{k=1}^n g_k^{-1} R_k^*(\mu(x)) R_k^*(\mu(y)) = t_n T_n - t_0 T_0 ,$$

which leads us at once to the Christoffel–Darboux formula (2.1), since

$$(2.10) \quad t_0 T_0 = f(x,y;q) g_0^{-1} R_0^*(\mu(x)) R_0^*(\mu(y)) .$$

For $y = 0$, (2.1) reduces easily to the following connection (or summation) formula for the q -Racah polynomials (1.6):

$$(2.11) \quad \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(\alpha q, \gamma q, \alpha\beta q, \beta\delta q; q)_k}{(1-\alpha\beta q)(\gamma\delta q)^k (q, \beta q, \alpha\beta q/\gamma, \alpha q/\delta; q)_k} R_k(\mu(x); \alpha, \beta, \gamma, \delta; q) \\ = \frac{(\alpha q^2, \gamma q^2, \alpha\beta q^2, \beta\delta q^2; q)_n}{(q, \beta q, \alpha\beta q/\gamma, \alpha q/\delta; q)_n} (\gamma\delta q^2)^{-n} R_n(\mu(x-1); \alpha q, \beta, \gamma q, \delta q; q),$$

which was given recently by Chihara [8, p. 193, Equation (2.11)].

3. APPLICATIONS OF THE CHRISTOFFEL–DARBOUX FORMULA (2.1)

By specializing the various parameters involved, the Christoffel–Darboux formula (2.1) can indeed be applied to derive the corresponding results for several simpler classes of q -orthogonal polynomials (see, e.g., [6, p. 1015]; see also [7] and [10]). In particular, for the little q -Jacobi polynomials (1.12) which may be recovered from $R_n^*(\mu(x))$ upon setting $\gamma q = q^{-N}$, replacing x by $N - x$, letting $N \rightarrow \infty$ and $\delta \rightarrow 0$, and then writing x for q^x , (2.1) yields the Christoffel–Darboux formula:

$$\begin{aligned}
 (3.1) \quad & \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(q, \alpha\beta q; q)_k}{(1-\alpha\beta q)(\alpha q)^k(\alpha q, \beta q; q)_k} p_k^{(\alpha, \beta)}(x; q) p_k^{(\alpha, \beta)}(y; q) \\
 &= \frac{(q; q)_{n+1}(\alpha\beta q^2; q)_n (x-y)^{-1}}{(1-\alpha\beta q^{2n+2})\alpha^n(\alpha q, \beta q; q)_n} \\
 &\quad \cdot \left\{ p_{n+1}^{(\alpha, \beta)}(x; q) p_n^{(\alpha, \beta)}(y; q) - p_n^{(\alpha, \beta)}(x; q) p_{n+1}^{(\alpha, \beta)}(y; q) \right\}.
 \end{aligned}$$

In view of the limit relationship:

$$(3.2) \quad L_n^{(\alpha)}(x; q) = \frac{(\alpha q; q)_n}{(q; q)_n} {}_1\Phi_1 \left[\begin{matrix} q^{-n}; \\ \alpha q; \end{matrix} q, -xq^n \right] = \lim_{\beta \rightarrow \infty} \left\{ p_n^{(\alpha, \beta)} \left[-\frac{x}{\alpha\beta q^2}; q \right] \right\},$$

it is not difficult to further specialize the Christoffel–Darboux formula (3.1) to the case of the q -Laguerre polynomials $L_n^{(\alpha)}(x;q)$.

Now replace x and y in (3.1) by xt and yt , respectively, multiply both sides by

$$t^{\lambda-1}(tq;q)_{\mu-\lambda-1} ,$$

and q -integrate with respect to t , using the familiar result (cf., e.g., [4, p. 135, Equation (5.7)]; see also [5, p. 257, Equation (1.9)]):

$$(3.3) \quad \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(tq;q)_{\beta-1} d(t;q) ,$$

where the (Jackson's) q -Gamma function $\Gamma_q(z)$ is defined by

$$(3.4) \quad \Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z} ,$$

and the measure $d(t;q)$ is given by (Jackson's) q -integrals (cf. [4] and [5]):

$$(3.5) \quad \int_0^a f(t) d(t;q) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n ,$$

$$(3.6a) \quad \int_a^\infty f(t) d(t;q) = a(1-q) \sum_{n=1}^{\infty} f(aq^{-n}) q^{-n} ,$$

and

$$(3.6b) \quad \int_0^\infty f(t) d(t;q) = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n .$$

Upon writing λ for q^λ , and μ for q^μ , we thus find from (3.1) that

$$\begin{aligned}
 (3.7) \quad & \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(\alpha q, \alpha\beta q; q)_k}{(1-\alpha\beta q)(\alpha q)^k (q, \beta q; q)_k} \\
 & \cdot \Phi_{1:1;1}^{1:2;2} \left[\begin{matrix} \lambda: q^{-k}, \alpha\beta q^{k+1}; q^{-k}, \alpha\beta q^{k+1}; \\ \mu: \alpha q; \alpha q; \end{matrix} \quad q; x, y \right] \\
 & = \frac{(1-\mu q^{-1})(\alpha q; q)_{n+1}(\alpha\beta q^2; q)_n (x-y)^{-1}}{(1-\lambda q^{-1})(1-\alpha\beta q^{2n+2})\alpha^n (q, \beta q; q)_n} \\
 & \cdot \Phi_{1:1;1}^{1:2;2} \left[\begin{matrix} \lambda q^{-1}: q^{-n}, \alpha\beta q^{n+1}; q^{-n-1}, \alpha\beta q^{n+2}; \\ \mu q^{-1}: \alpha q; \alpha q; \end{matrix} \quad q; x, y \right] + x \leftrightarrow y,
 \end{aligned}$$

where $x \leftrightarrow y$ indicates the presence of a second term which originates from the first by interchanging x and y .

For $\beta = 0$, (3.7) immediately reduces to the form:

$$\begin{aligned}
 (3.8) \quad & \sum_{k=0}^n \frac{(\alpha q; q)_k}{(\alpha q)^k (q; q)_k} \Phi_{1:1;1}^{1:1;1} \left[\begin{matrix} \lambda: q^{-k}; q^{-k}; \\ \mu: \alpha q; \alpha q; \end{matrix} \quad q; x, y \right] \\
 & = \frac{(1-\mu q^{-1})(\alpha q; q)_{n+1}}{(1-\lambda q^{-1})\alpha^n (q; q)_n} (x-y)^{-1} \Phi_{1:1;1}^{1:1;1} \left[\begin{matrix} \lambda q^{-1}: q^{-n}; q^{-n-1}; \\ \mu q^{-1}: \alpha q; \alpha q; \end{matrix} \quad q; x, y \right] + x \leftrightarrow y.
 \end{aligned}$$

Formulas (3.7) and (3.8) provide q -extensions of certain finite summation formulas

due to Srivastava [14, p. 5, Equation (3.4); p. 4, Equation (2.4)]. More generally, by repeating the aforementioned process of q -integration using (3.3) and some manipulations, we can prove the q -summation formula:

$$\begin{aligned}
 (3.9) \quad & \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(\alpha q, \alpha\beta q; q)_k}{(1-\alpha\beta q)(\alpha q)^k (q, \beta q; q)_k} \\
 & \cdot \Phi_{s:1;1}^{r:2;2} \left[\begin{matrix} \lambda_1, \dots, \lambda_r: q^{-k}, \alpha\beta q^{k+1}; q^{-k}, \alpha\beta q^{k+1}; \\ \mu_1, \dots, \mu_s: \alpha q; \alpha q; \end{matrix} \quad q; x, y \right] \\
 & = \frac{(1-\mu_1 q^{-1}) \cdots (1-\mu_s q^{-1})(\alpha q; q)_{n+1} (\alpha\beta q^2; q)_n (x-y)^{-1}}{(1-\lambda_1 q^{-1}) \cdots (1-\lambda_r q^{-1})(1-\alpha\beta q^{2n+2}) \alpha^n (q, \beta q; q)_n} \\
 & \cdot \Phi_{s:1;1}^{r:2;2} \left[\begin{matrix} \lambda_1 q^{-1}, \dots, \lambda_r q^{-1}: q^{-n}, \alpha\beta q^{n+1}; q^{-n-1}, \alpha\beta q^{n+2}; \\ \mu_1 q^{-1}, \dots, \mu_s q^{-1}: \alpha q; \alpha q; \end{matrix} \quad q; x, y \right] + x \mapsto y,
 \end{aligned}$$

which reduces to (3.7) when $r = s = 1$, and (for $\beta = 0$) yields a similar generalization of (3.8).

Finite summation formulas involving (ordinary) double hypergeometric functions, corresponding to (3.9) and its aforementioned special case when $\beta = 0$, were also given by Srivastava [14, p. 6, Equation (3.6); p. 3, Equation (2.1)]. As a matter of fact, it is not difficult to prove q -extensions of Srivastava's finite summation formulas [15, p. 315, Equations (2.1) and (2.2)] involving (ordinary) triple hypergeometric functions.

It may be of interest to remark that the various q -summation formulas stemming from the Christoffel–Darboux formula for the q -Laguerre polynomials (3.2) will naturally

involve double q -hypergeometric functions defined by (1.10) with nonzero values of i and j .

4. GENERATING FUNCTIONS FOR q -RACAH POLYNOMIALS

Corresponding to a bounded complex sequence $\{\Omega_n\}_{n=0}^{\infty}$, we prove the following class of generating functions for the q -Racah polynomials defined by (2.2) and (1.6):

$$\begin{aligned}
 (4.1) \quad & \sum_{n \geq 0} \frac{(\gamma q, \beta \delta q; q)_n}{(\alpha q, \beta q; q)_n} \Omega_n R_n^*(\mu(x); \alpha, \beta, \gamma, \delta; q) t^n \\
 &= \sum_{\ell, m \geq 0} \Omega_{\ell+m} \frac{(\alpha q^{x+1}, \gamma \delta q^{x+1}; q)_{\ell} (\beta q^{-x} / \gamma, q^{-x} / \delta; q)_m}{(\alpha q; q)_{\ell} (\beta q; q)_m} \frac{(t q^{-x})_{\ell}}{(q; q)_{\ell}} \frac{(\gamma \delta t q^{x+1})_m}{(q; q)_m},
 \end{aligned}$$

where (and in what follows) the parameters α, β, γ , and δ , and the variable t , are so constrained that both sides exist.

Formula (4.1) and its numerous special or limiting cases would provide q -extensions of a considerably large number of known generating functions for classical orthogonal polynomials, which are presented systematically by Srivastava and Manocha [17, Chapter 2 et seq.]. In particular, if in (4.1) we set

$$\Omega_n = \frac{(\lambda_1, \dots, \lambda_r; q)_n}{(\mu_1, \dots, \mu_s; q)_n} \quad (n = 0, 1, 2, \dots),$$

we shall readily obtain (cf. [17, p. 145, Equation (31)])

$$\begin{aligned}
(4.2) \quad & \sum_{n \geq 0} \frac{(\lambda_1, \dots, \lambda_r, \gamma q, \beta \delta q; q)_n}{(\mu_1, \dots, \mu_s, \alpha q, \beta q; q)_n} R_n^*(\mu(x); \alpha, \beta, \gamma, \delta; q) t^n \\
&= \Phi_{s:1;1}^{r:2;2} \left[\begin{matrix} \lambda_1, \dots, \lambda_r: & \alpha q^{x+1}, \gamma \delta q^{x+1}; & \beta q^{-x}/\gamma, q^{-x}/\delta; \\ & & q; & t q^{-x}, \gamma \delta t q^{x+1} \\ \mu_1, \dots, \mu_s: & \alpha q; & \beta q; \end{matrix} \right],
\end{aligned}$$

which, for

$$\lambda_j = 0 \quad (j = 3, \dots, r) \quad \text{and} \quad \mu_j = 0 \quad (j = 1, \dots, s),$$

would immediately yield a Brafman type generating function [17, p. 109, Equation (20)] for q -Racah polynomials. Lastly, in (4.1) we set

$$\Omega_n = 1 \quad (n = 0, 1, 2, \dots),$$

and obtain the following Bateman type generating function [17, p. 282, Problem 33] for q -Racah polynomials (cf. [11, p. 45, Equation (1.9)]):

$$\begin{aligned}
(4.3) \quad & \sum_{n \geq 0} \frac{(\gamma q, \beta \delta q; q)_n}{(\alpha q, \beta q; q)_n} R_n^*(\mu(x); \alpha, \beta, \gamma, \delta; q) t^n \\
&= {}_2\Phi_1 \left[\begin{matrix} \alpha q^{x+1}, \gamma \delta q^{x+1}; \\ & q, t q^{-x} \\ \alpha q; \end{matrix} \right] {}_2\Phi_1 \left[\begin{matrix} \beta q^{-x}/\gamma, q^{-x}/\delta; \\ & q, \gamma \delta t q^{x+1} \\ \beta q; \end{matrix} \right],
\end{aligned}$$

which obviously follows also from (4.2) when

$$\lambda_j = 0 \quad (j = 1, \dots, r) \quad \text{and} \quad \mu_j = 0 \quad (j = 1, \dots, s).$$

Derivation of the q-Generating Function (4.1). Denoting, for convenience, the left-hand side of (4.1) by S , and transforming the ${}_4\Phi_3$ polynomials by means of Sears's formula [12, p. 167, Equation (8.3)]:

$$(4.4) \quad {}_4\Phi_3 \left[\begin{matrix} a, & b, & c, & q^{-n}; \\ & d, & e, & g; \end{matrix} \middle| q, q \right] = \frac{(e/c, de/ab; q)_n}{(e, de/abc; q)_n} \cdot {}_4\Phi_3 \left[\begin{matrix} d/a, & d/b, & c, & q^{-n}; \\ d, & cq^{1-n}/e, & cq^{1-n}/g; \end{matrix} \middle| q, q \right],$$

$$abc = q^{n-1} ged \quad (n = 0, 1, 2, \dots),$$

we find that

$$(4.5) \quad S = \sum_{n \geq 0} \Omega_n \frac{(\beta q^{-x}/\gamma, q^{-x}/\delta; q)_n}{(q, \beta q; q)_n} (\gamma \delta q^{x+1})^n \cdot {}_4\Phi_3 \left[\begin{matrix} q^{-n}, & q^{-n}/\beta, & \alpha q^{x+1}, & \gamma \delta q^{x+1}; \\ & \alpha q, & \gamma q^{x-n+1}/\beta, & \delta q^{x-n+1}; \end{matrix} \middle| q, q \right].$$

The q-generating function (4.1) now follows on simplifying the second member of (4.5) as a double series.

Each of the q-generating functions (4.1), (4.2), and (4.3), and their special cases indicated above, can be suitably applied to deduce the corresponding result for such classes of q-orthogonal polynomials as the little as well as big q-Jacobi polynomials, q-Laguerre polynomials, q-Bessel polynomials, q-Hahn polynomials, q-Meixner polynomials, and

q -Krawtchouk polynomials. For instance, (4.3) with $\delta \rightarrow 0$ and q^{-x} replaced by x would yield a known generating function (due to Ismail and Wilson [11, p. 46, Equation (1.16)]) for the big q -Jacobi polynomials [11, p. 45, Equation (1.14)] which are normalized slightly differently from (1.13).

5. A BILINEAR GENERATING FUNCTION FOR THE q -KONHAUSER POLYNOMIALS (1.14)

For the q -Konhauser polynomials defined by (1.14), we first prove a linear generating function in the form:

$$\begin{aligned}
 (5.1) \quad & \sum_{n=0}^{\infty} \frac{(q^k; q^k)_{m+n}}{(\alpha q; q)_{(m+n)k} (q^k; q^k)_n} Z_{m+n}^{(\alpha)}(x, k | q) t^n \\
 &= \frac{1}{(t; q^k)_{\infty}} \sum_{r,s=0}^{\infty} \frac{(-1)^s (t; q^k)_r (\alpha x)^{(r+s)k} q^{mkr + \theta(k;r,s)}}{(\alpha q; q)_{(r+s)k} (q^k; q^k)_r (q^k; q^k)_s}, \\
 &\theta(k;r,s) = \frac{1}{2} (r+s)k \{ (r+s)k + 1 \} + \frac{1}{2} ks(s-1),
 \end{aligned}$$

which, for $m = 0$, reduces to a known result [1, p. 5, Equation (4.1)].

Proof of Formula (5.1). Consider the sum

$$(5.2) \quad \frac{(q; q)_{m+n}}{(q; q)_n (q; q)_{m+n-r}} = \sum_{s=0}^{\min(m,r)} \frac{(q^{-m}, q^{-r}; q)_s}{(q; q)_s (q; q)_{n-r+s}} q^{s+m},$$

which follows readily from a q -extension of the familiar Chu-Vandermonde summation

theorem [13, p. 247, Equation (IV.1)].

Now multiply both sides of (5.2) by t^n and sum the resulting expression on each side with respect to n from 0 to ∞ . Since

$$\begin{aligned}
 & \sum_{n=0}^{\infty} t^n \sum_{s=0}^{\min(m,r)} \frac{(q^{-m}, q^{-r}; q)_s}{(q; q)_s (q; q)_{n-r+s}} q^{s+mr} \\
 &= \sum_{s=0}^{\min(m,r)} \frac{(q^{-m}, q^{-r}; q)_s}{(q; q)_s} q^{s+mr} \sum_{n=r-s}^{\infty} \frac{t^n}{(q; q)_{n-r+s}} \\
 &= \sum_{s=0}^{\min(m,r)} \frac{(q^{-m}, q^{-r}; q)_s}{(q; q)_s} q^{s+mr} t^{r-s} {}_0\Phi_0 \left[\begin{matrix} \text{---}; \\ \text{---}; \end{matrix} q, t \right] \\
 &= \frac{1}{(t; q)_{\infty}} \sum_{s=0}^{\min(m,r)} \frac{(q^{-m}, q^{-r}; q)_s}{(q; q)_s} q^{s+mr} t^{r-s},
 \end{aligned}$$

we thus find that

$$\begin{aligned}
 (5.3) \quad & (-1)^r q^{(m+n)r-r(r-1)/2} \sum_{n=0}^{\infty} (q^{-m-n}; q)_r \frac{t^n}{(q; q)_n} \\
 &= \frac{(tq^m)_r}{(t; q)_{\infty}} {}_2\Phi_1 \left[\begin{matrix} q^{-m}, q^{-r}; \\ q, q/t \end{matrix} \right]_0.
 \end{aligned}$$

If we multiply each member of (5.3) by

$$C_r q^{r(r-1)/2} \frac{(-x)^r}{(q;q)_r},$$

and sum the resulting expressions on both sides with respect to r from 0 to ∞ , we obtain

$$(5.4) \quad \Delta \equiv \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{r=0}^{m+n} \frac{(q^{-m-n};q)_r}{(q;q)_r} C_r (xq^{m+n})^r$$

$$= \frac{1}{(t;q)_{\infty}} \sum_{r=0}^{\infty} C_r q^{mr+r(r-1)/2} \frac{(-xt)^r}{(q;q)_r} {}_2\Phi_1 \left[\begin{matrix} q^{-m}, q^{-r}; \\ q, q/t; \\ 0; \end{matrix} \right],$$

for a bounded complex sequence $\{C_n\}_{n=0}^{\infty}$. Transforming this last ${}_2\Phi_1$ series in (5.4) by using a particular case ($c = 0$) of the following iterate of the familiar Heine's transformation:

$$(5.5) \quad {}_2\Phi_1 \left[\begin{matrix} a, b; \\ q, z; \\ c; \end{matrix} \right] = \frac{(bz, c/b; q)_{\infty}}{(z, c; q)_{\infty}} {}_2\Phi_1 \left[\begin{matrix} abz/c, b; \\ q, c/b; \\ bz; \end{matrix} \right],$$

we find from (5.4) that

$$(5.6) \quad \Delta = \frac{1}{(t;q)_{\infty}} \sum_{r,s=0}^{\infty} C_{r+s} q^{mr+s(s-1)/2} (t;q)_r \frac{x^r}{(q;q)_r} \frac{(-x)^s}{(q;q)_s}$$

for every bounded complex sequence $\{C_n\}_{n=0}^{\infty}$.

Replacing q by q^k , and x by x^k , and setting

$$C_n = \frac{q^{nk(nk+1)/2} \alpha^{nk}}{(\alpha q; q)_{nk}} \quad (n = 0, 1, 2, \dots),$$

(5.6) yields the generating function (5.1).

If, in our generating function (5.1), we replace m by $m + j$, multiply both sides by

$$\frac{(-t)^j (\beta y)^{kj} q^{k(k+1)j^2/2}}{(q^k; q^k)_j (\beta q; q)_{kj}},$$

and sum the resulting expression on each side with respect to j from 0 to ∞ , we shall obtain the following bilinear generating function for the q -Konhauser polynomials (1.14):

$$\begin{aligned}
 (5.7) \quad & \sum_{n=0}^{\infty} \frac{(q^k; q^k)_{m+n}}{(\alpha q; q)_{(m+n)k} (\beta q; q)_{nk}} Z_{m+n}^{(\alpha)}(x, k|q) Z_n^{(\beta)}(y, k|q) t^n \\
 &= \frac{1}{(t; q^k)_{\infty}} \sum_{r=0}^{\infty} \frac{(t; q^k)_r}{(\alpha q; q)_{kr}} q^{kr(2m+kr+1)/2} \frac{(\alpha x)^{kr}}{(q^k; q^k)_r} \\
 &\quad \cdot \left[\sum_{s=0}^{\infty} \frac{(-1)^s q^{ks(2kr+ks+s)/2}}{(\alpha q^{kr+1}; q)_{ks}} \frac{(\alpha x)^{ks}}{(q^k; q^k)_s} \right] \\
 &\quad \cdot \left[\sum_{j=0}^{\infty} \frac{(-t)^j q^{kj(kr+j-2)/2}}{(\beta q; q)_{kj}} \frac{(\beta y)^{kj}}{(q^k; q^k)_j} \right],
 \end{aligned}$$

in which each of the inner sums may be written in terms of q -hypergeometric ${}_0\Phi_k$ functions with base q^k for every positive integer k .

Our method of derivation of this last result (5.7) can indeed be applied mutatis

mutandis in order to prove a multilinear generating function for the q -Konhauser polynomials (1.14).

We conclude by remarking that another pair of q -Konhauser biorthogonal polynomials may be defined by

$$(5.8) \quad z_n^{(\alpha)}(x, k | q) = \frac{(\alpha q; q)_n}{(q^k; q^k)_n} \sum_{j=0}^n \frac{(q^{-nk}; q^k)_j}{(\alpha q; q)_{kj}} \frac{(xq)^{kj}}{(q^k; q^k)_j}$$

and

$$(5.9) \quad y_n^{(\alpha)}(x, k | q) = \frac{1}{(q; q)_n} \sum_{j=0}^n \frac{(xq)^j}{(q; q)_j} \sum_{\ell=0}^j \frac{(q^{-j}; q)_\ell (\alpha q^{\ell+1}; q^k)_n}{(q; q)_\ell} q^{(j-n)\ell}.$$

Although these new q -Konhauser polynomials are motivated by a particular case ($\beta = 0$) of a system of biorthogonal polynomials studied by Al-Salam and Verma [2, p. 275, Equations (2.1) and (2.2)], it may be of interest to construct independent (and much more simple) proofs of many of their important properties including, for example, their biorthogonality relation. The proposed definition (5.8) corresponds to the q -Konhauser polynomials (1.14), whereas (5.9) provides an alternative (and simpler) version of the q -Konhauser polynomials defined by (cf. [1, p. 3, Equation (3.2)])

$$(5.10) \quad Y_n^{(\alpha)}(x, k | q) = \frac{1}{(q; q)_n} \sum_{j=0}^n q^{j(j-1)/2} \frac{x^j}{(q; q)_j} \sum_{\ell=0}^j \frac{(q^{-j}; q)_\ell (\alpha q^{\ell+1}; q^k)_n}{(q; q)_\ell} q^\ell.$$

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