GLOBAL EXISTENCE AND VALIDITY FOR THE
BBGKY HIERARCHY

By

A.E. HURD

DM-372-IR

JUNE 1985
REVISED: JUNE 1986

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§1 Introduction

In the rigorous theory of statistical mechanics the primary emphasis has been to work directly on a phase space of infinitely many particles. The equilibrium theory is now essentially complete, and accounts rigorously for many of the phenomena of phase transition. However, the nonequilibrium theory has proved to be much less tractable. The first problem was to prove the existence of a limiting dynamics on the infinite system phase space X resulting from given interparticle forces, and was solved by Lanford [16] for one-dimensional systems. An independent approach to the problem was developed by Sinai [28, 29]. The results were then extended to more general systems in a series of papers [9, 10, 17, 22, 25]. In all of these papers, the solution was defined only for almost all phase points with respect to any Gibbs state, but that is sufficient to establish a time evolution for any state which is absolutely continuous with respect to a Gibbs state.

The next obvious step is to investigate the time evolution of the correlation measures corresponding to the time evolving states obtained as described in the previous paragraph. These correlation measures give the probabilities of occurrence of configurations involving finitely many particles, and are important because the quantities involved in hydrodynamics are in principle defined in terms of them. Indeed, one of the fundamental aims in nonequilibrium statistical mechanics is to use the correlations to rigorously derive the equations of hydrodynamics from the underlying particle dynamics. Correlation measures can be defined for any state on X [18], but they are not

necessarily finite. A basic problem now occurs for the time evolutions of these correlations: if an initial state μ has finite correlations, does the time evolved state μ^t likewise have finite correlations? This is the problem of existence of evolutions.

In addition to the existence problem, we would like to know whether the correlation evolutions, if they exist, satisfy, at least weakly, the BBGKY hierarchy. This hierarchy was developed independently by Bogolyubov, Born, Green, Kirkwood, and Yvon (see [8]), and is widely regarded as a possible avenue of approach to the rigorous justification of the Boltzmann equation, from which would hopefully follow a rigorous analysis of the central enigma of the subject, the approach to equilibrium of systems in the gaseous phase.

There are two possible approaches to the BBGKY hierarchy. The first is to try to obtain existence and uniqueness theorems for the hierarchy in various spaces of functions. For work in this direction see the paper by Petrina and Gerasimenko [23].

The second approach, which we follow in this paper, is to try to prove that under reasonable assumptions there exist time-evolving states on the infinite system phase space which are weak limits of appropriate finite volume time evolving states, and whose correlation evolutions are finite and the weak limits of the corresponding finite volume correlation evolutions and also satisfy the BBGKY hierarchy. We call any theorem involving the second approach a validation of the BBGKY hierarchy since it deals with states which are obtained as limits of the physically meaningful finite volume states. The validation question has so far been addressed in only two papers, both restricted to one-dimensional systems. Using Lanford's result on infinite-dimensional dynamics mentioned above, Gallavotti, Lanford, and Lebowitz [11] established a validity result for weak solutions of the BBGKY hierarchy in one-dimensional systems with bounded,

finite-range pair interactions for a fairly restricted class of initial conditions. Then, using Sinai's construction of the infinite-dimensional dynamics, Sinai and Sukhov [30] established a similar result for one-dimensional systems with finite-range, hard-core potentials that are bounded below, and with initial conditions near to equilibrium. We should also mention the important results of Lanford [17, §7], which can be regarded as a local validity result for solutions of the so-called Boltzmann hierarchy for hard spheres (see also [15]).

The present paper addresses the validity question in general, and in particular is not restricted to one-dimensional systems nor to systems close to equilibrium. In §3, we present results indicating that the set of limiting time evolutions of finite volume time-evolving states and their corresponding (finite) limiting correlation evolutions in appropriate weak topologies is non-empty under very general conditions. The results in §4 show that, under general and natural assumptions on the pair-potential and the initial conditions and with no assumptions on the dimension, all limiting correlation evolutions mentioned above are weak solutions of the BBGKY hierarchy. Taken together these results settle the existence and validity problem for weak solutions of the BBGKY hierarchy.

As in [12], the methods are nonstandard. Roughly speaking, we extend to a nonstandard model the whole collection of finite volume structures, including states, correlation measures, and dynamics. The transfer principle yields an internal version of the hierarchy in the nonstandard model. The final step is to show that, with the assumptions mentioned above, the nonstandard hierarchy can be projected down onto the usual hierarchy for the correlation evolutions. The theory of Loeb integration is essential in this last step.

It should be emphasized that although the proofs are nonstandard, the final results are standard. But there is perhaps some merit in regarding the nonstandard structures as ends in themselves. Those familiar with nonstandard methods will be aware of the fact that the nonstandard infinite volume analogues of the finite volume structures can be regarded as representations of limiting finite volume structures. For example, though it is not possible to define a limiting infinite particle dynamics on all of the standard locally finite phase space X , the infinite volume dynamics is defined everywhere on the nonstandard extension *X of X . It can be argued that the natural "infinite particle" phase on which one should work is in fact the space *X equipped with its natural Loeb measurable structure. Such an approach is implicit (in a different context) in recent nonstandard existence theorems for the Boltzmann equation due to Arkeryd [4, 5, 6].

§2 Basic Definitions

Even though the ideas in this paper can be applied quite generally (in particular to lattice systems), we concern ourselves only with a formulation applicable to continuous models. For these models, the one particle phase space is $S \times S$ where $S = \mathbb{R}^d$, d an integer. Points in $S \times S$ are denoted, for example, by $x_i = (q_i, p_i)$ where q_i is the position and p_i is the momentum of the particle. We let $\mathcal{S}$ denote the collection of Borel subsets of S .

Throughout the paper we use $C(Y)$, $C_b(Y)$, and $C_0(Y)$ to denote the spaces of continuous functions, bounded continuous functions, and continuous functions with compact support, respectively, on a topological space Y .

The infinite system phase space $X(A)$ on a set $A \in \mathcal{S}$ is the set of all integer-valued measures of the form

$$(2.1) \quad x = \sum_n m_n \delta_{x_n} \quad (n \in \mathbb{N}),$$

where δ_{x_i} is a point mass at $x_i = (q_i, p_i) \in S \times S$ with $q_i \in A$, the integers m_i count repetitions (which are allowed in general) and the set of $x_n = (q_n, p_n)$ in x for which $q_n \in B$ is finite for any bounded set B . Points in $X(A)$ will be denoted (somewhat loosely considering the repetitions) by $x = \{x_1, x_2, \dots\}$. The infinite system phase space is the set $X = X(S)$; this is the standard locally finite space. If $A \subseteq S$ there is a projection $p_A: X \rightarrow X(A)$ where $p_A(x)$ is obtained from $x \in X$ by deleting those point masses δ_{x_n} , $x_n = (q_n, p_n)$, for which $q_n \notin A$. We let $p_A(x) = x_A$, and denote the cardinality of x_A for bounded A by $|x_A|$.

The spaces $X(A)$, $A \in \mathcal{A}$, are topologized with the vague topology on measures [7, p. 228]. With this topology, $X(A)$ is a Polish space and so the Baire and Borel structures induced on $X(A)$ coincide. Without further mention we regard X and $X(A)$, $A \in \mathcal{A}$, as measurable spaces with their Baire structures $\mathcal{B}$ and $\mathcal{B}(A)$.

Presently we will be considering singular interaction potentials, so it will be necessary to discuss subsets of our phase spaces in which points cannot come too close together. Thus for $\delta \geq 0$ in $\mathbb{R}$, we define the set $X^\delta(A)$, $A \in \mathcal{A}$, to be the set of points $x \in X(A)$ such that if $x_A = \{x_1, x_2, \dots, x_n\}$ for any $A \in \mathcal{A}$, and $x_i = (q_i, p_i)$, then $\inf\{\|q_i - q_j\|: 1 \leq i < j \leq n\} > \delta$, where $\|\cdot\|$ denotes the norm in $\mathbb{R}^d$. The set $X^\delta(A)$ inherits the Borel structures $\mathcal{B}^\delta(A)$ from $(X(A), \mathcal{B}(A))$.

If $A \in \mathcal{A}$ we can write $X(A) = \bigcup_n X_n(A)$ ($n \geq 0$), where $X_n(A) = \{x \in X(A): |x_A| = n\}$. Each $X_n(A)$ is naturally related to $\pi_n(A) = (A \times S)^n$ (n -fold cartesian product), where we define $\pi_0(A)$ to be a single point. Indeed, we can define an equivalence relation $\sim$ on $\pi_n(A)$ by putting

$(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ if there exists a permutation ξ of $\{1, 2, \dots, n\}$ with $x_i = y_{\xi(i)}$ for $1 \leq i \leq n$. The quotient spaces of $\pi_n(A)$ under $\sim$ can be identified with $X_n(A)$, with a resulting map $\xi_{A,n}: \pi_n(A) \rightarrow X_n(A)$. If we define $\pi(A) = \bigcup \pi_n(A)$ ($n \geq 0$), then there is a map $\xi_A: \pi(A) \rightarrow X(A)$ which reduces to $\xi_{A,n}$ on each summand. In the same way, if we define $\pi_n^\delta(A) = \{x \in \pi_n(A): \inf\{\|q_i - q_j\|: 1 \leq i < j \leq n\} > \delta\}$, and $\pi^\delta(A) = \bigcup \pi_n^\delta(A)$ ($n \geq 0$), then ξ_A maps $\pi^\delta(A)$ onto $X^\delta(A)$.

We use the constructions of the previous paragraph to define measures on $X(A)$. The natural measure on $\pi(A)$ is $\lambda_A = \sum \lambda_A^n / n!$ ($n \geq 0$), where λ_A^n is the n -fold product measure on $(A \times S)^n$ of Lebesgue measure λ_n on A (λ_A^0 being point mass on the single point $\pi_0(A)$). Since ξ_A is Borel measurable, we may put $\omega_A = \lambda_A \circ \xi_A^{-1}$, the measure induced on $X(A)$ by ξ_A . Also, ω_A^n is ω_A restricted to $X_n(A)$.

To each state (probability measure) μ in X we want to associate a measure ρ , called the correlation measure associated with μ , on the space $X_0 = \{x \in X: x(S) < \infty\}$ of finite configurations. To define ρ we need first to define a map $E: C_0(X_0) \rightarrow C(X)$. Note that $X_0 = \bigcup (S \times S)^n$ ($n \in \mathbb{N}$) and a set $K \subset X_0$ is compact if it is of the form $K = \bigcup K_n$ ($1 \leq n \leq m$) for some finite integer m , where each K_n is compact in $(S \times S)^n$. We define (see [19], where E is denoted by S)

$$(2.2) \quad (E\phi)(x) = \sum \phi(x_{i_1}, x_{i_2}, \dots, x_{i_n}),$$

where the points of $x \in X$ are $x_1, x_2, \dots$ and the sum is extended over all finite sequences $(i_1, i_2, \dots, i_n)$ of distinct positive integers (a finite sum by compactness of the support of ϕ).

The important properties of E are the following, which can be found in [19]:

- 2.1 (a) If ϕ is continuous, then $E\phi$ is continuous
- 2.2 (b) E is linear and positive (i.e. $\phi \geq 0$ implies $E\phi \geq 0$)
- 2.3 (c) $|E\phi(x)| \leq M|x_B|^m$ for each function $\phi \in C_c(X_0)$, where M, m , and the bounded set B depend only on the support of ϕ .

We say that a state μ has a finite correlation measure if $E\phi$ is integrable with respect to μ for each $\phi \in C_o(X_0)$. By 2.1(a) this is the case if $|x_A|^m$ is μ -integrable for each $m \in \mathbb{N}$ and each bounded set A . If this condition is satisfied, the functional ρ on $C_o(X_0)$ defined by

$$(2.3) \quad \rho(\phi) = \int (E\phi) d\mu, \quad \phi \in C_o(X_0)$$

is a positive linear functional and, by the Riesz representation theorem, defines a Borel measure on X_0 which we also denote by ρ . The measure ρ is called the correlation measure associated with μ . Similarly we can define the n^{th} correlation measure associated with μ by

$$(2.4) \quad \rho_n(\phi) = \int (E\phi) d\mu, \quad \phi \in C_o\left[(S \times S)^n\right].$$

We are concerned with time-evolving states on X which are limits of "finite volume" time-evolving states. To be specific, let $\mathcal{C}$ be a collection of bounded domains contained in S . We suppose that $\mathcal{C}$ is directed upwards by the containment relation $\subseteq$ and contains arbitrarily large sets; i.e., given

any bounded Borel subset $\Lambda \in \mathcal{V}$, there is an $A \in \mathcal{C}$ with $\Lambda \subseteq A$ (in this case, $\mathcal{C}$ is said to be cofinal in the collection of bounded subsets in $\mathcal{V}$).

Following [12], we suppose that a specification $\mathcal{V} = \{\mu_\Lambda; \Lambda \in \mathcal{C}\}$ and a flow specification $\mathcal{D} = \{T_\Lambda^t; t \in \mathbb{R}, \Lambda \in \mathcal{C}\}$ are given. Here the μ_Λ are "finite volume" states on X , and the $\{T_\Lambda^t; t \in \mathbb{R}\}$ form a group of "finite volume" measurable mappings $T_\Lambda^t: X \rightarrow X$ for each $\Lambda \in \mathcal{C}$ (for examples see [12]). The T^t need only be defined a.e. but without loss of generality we assume they are defined everywhere. To $\mathcal{V}$ and $\mathcal{D}$ is associated a set $\{\mu_\Lambda^t; \Lambda \in \mathcal{C}\}$ of "finite volume" time evolutions of the states μ_Λ , where μ_Λ^t is defined by $m_\Lambda^t(F) = \mu_\Lambda[T_\Lambda^{-t}(F)]$ for $F \in \mathcal{B}$, $\Lambda \in \mathcal{C}$, and $t \in \mathbb{R}$.

In the following, a state evolution will be a map from $\mathbb{R}$ into the space of states on X and will be denoted by $\langle \mu^t; t \in \mathbb{R} \rangle$ or simply by $\langle \mu^t \rangle$. We assume that any state evolution $\langle \mu^t \rangle$ is such that $\int \varphi(t, x) d\mu^t(x)$ is measurable for any bounded continuous function $\varphi(t, x)$. If a state evolution $\langle \mu^t \rangle$ is such that μ^t has a finite correlation measure ρ^t for each $t \in \mathbb{R}$, we can associate with $\langle \mu^t \rangle$ a correlation evolution $\langle \rho^t \rangle$, i.e., a map from $\mathbb{R}$ into the space of correlatins on X_0 .

§3 Existence of Correlation Evolutions

In this section we state the basic results to be proved on the existence question. Our objective is to investigate the existence of state evolutions $\langle \mu^t \rangle$ which are cluster points, in certain topologies to be defined, of the finite volume state evolutions $\langle \mu_\Lambda^t \rangle$, and which have correlation evolutions which are cluster points of the correlation evolutions $\langle \rho_\Lambda^t \rangle$ of the $\langle \mu_\Lambda^t \rangle$.

The topologies on the state and correlation evolutions are defined as follows:

3.1 Definition

Let $\mathcal{E}(Y)$ denote the space of evolutions of states on a space $(Y, \mathcal{A})$. A neighborhood subbase about $\langle \hat{\nu}^t \rangle \in \mathcal{E}(Y)$ in the pointwise weak (resp. vague) topology is given by sets of the form

$$N_{\epsilon, t, \varphi}[\langle \hat{\nu}^t \rangle] = \left\{ \langle \nu^t \rangle \in \mathcal{E}(Y) : \left| \int \varphi(y) d\nu^t(y) - \int \varphi(y) d\hat{\nu}^t(y) \right| < \epsilon \right\}$$

for $t \in \mathbb{R}$ and $\varphi \in C_b(Y)$ (resp. $C_0(Y)$).

A neighborhood subbase about $\langle \hat{\nu}^t \rangle$ in the weak (resp. vague) topology is given by sets of the form

$$M_{\epsilon, \Psi}[\langle \hat{\nu}^t \rangle] = \left\{ \langle \nu \rangle \in \mathcal{E}(Y) : \left| \int \left[\int \Psi(t, y) d\nu^t(y) \right] d\lambda(t) - \int \left[\int \Psi(t, y) d\hat{\nu}^t(y) \right] d\lambda(t) \right| < \epsilon \right\}$$

where λ is Lebesgue measure on $\mathbb{R}$ and $\Psi: \mathbb{R} \times Y \rightarrow \mathbb{R}$ is continuous, bounded, and vanishes for t outside a bounded interval (respectively, has compact support).

Theorem 3.4 below states that, under certain conditions, the finite volume state evolutions $\langle \mu_A^t \rangle$ converge pointwise weakly and weakly to a state evolution $\langle \mu^t \rangle$, and their corresponding correlation evolutions $\langle \rho_A^t \rangle$ converge pointwise vaguely and vaguely to the correlation evolution $\langle \rho^t \rangle$ of $\langle \mu^t \rangle$. In general we cannot expect uniqueness, so we show that $\langle \mu^t \rangle$ and $\langle \rho^t \rangle$ are cluster evolutions, in the corresponding topologies, of the $\langle \mu_A^t \rangle$ and $\langle \rho_A^t \rangle$.

The theorem will be proved under the assumptions to follow. First we assume that the initial states μ_A are absolutely continuous with respect to a

$\mathcal{X}$ -invariant specification $\mathcal{G} = \{\sigma_A; A \in \mathcal{C}\}$, i.e., $\mu_A = f_A \sigma_A$ for a $\mathcal{X}$ -measurable f_A , $A \in \mathcal{C}$ and $\sigma^t = \sigma_A$ for all $t \in \mathbb{R}$, $A \in \mathcal{C}$.

We also suppose that $\mathcal{G}$ is tight in the following sense. If $T \subset X$ we write $f \prec T$ if $f \in C(X)$, $\sup|f| \leq 1$, and $f(x) = 0$ for $x \notin T$.

3.2 Definition

(i) The state μ on $(X, \mathcal{B})$ is tight if given $\epsilon > 0$ in $\mathbb{R}$ there exists a compact set $K \subset X$ so that $\left| \int f d\mu \right| < \epsilon$ for all $f \prec K^c$ (the complement of K).

(ii) The specification $\mathcal{G}$ is tight if given $\epsilon > 0$ in $\mathbb{R}$ there is a compact set $K \subset X$ and $B \in \mathcal{C}$ so that $\int f d\sigma_A < \epsilon$ for all $f \prec K^c$ and all $A \supseteq B$ in $\mathcal{C}$.

Finally we need an assumption on the f_A , $A \in \mathcal{C}$, which will be stated using the notion of uniform integrability.

3.3 Definition

A family $\{g_A; A \in \mathcal{C}\}$ of measurable functions on X is uniformly p -integrable with respect to $\mathcal{G} = \{\sigma_A; A \in \mathcal{C}\}$ ($1 \leq p < \infty$) if, given $\epsilon > 0$ in $\mathbb{R}$, there exists a $k \in \mathbb{N}$ and a $B \in \mathcal{C}$ so that $\int_{|g_A| \geq k} |g_A|^p d\sigma_A < \epsilon$ for all $A \supseteq B$ in $\mathcal{C}$.

Under the conditions of Definition 3.3 it is clear that each $g_A \in L_p(\sigma_A)$.

We can now state the main theorem of this section.

3.4 Theorem

Let $\mathcal{V} = \{\mu_A: A \in \mathcal{C}\}$ and $\mathcal{D} = \{T_A^t: A \in \mathcal{C}, t \in \mathbb{R}\}$ be a given specification and flow specification respectively, and suppose that $\mathcal{G} = \{\sigma_A: A \in \mathcal{C}\}$ is a tight, $\mathcal{D}$ -invariant specification. Also assume that $\mu_A = f_A \sigma_A$ for each $A \in \mathcal{C}$.

(a) Suppose that $\{f_A: A \in \mathcal{C}\}$ is uniformly 1-integrable. Then there is a set $\{\langle \hat{\mu}_A^t: A \in {}^*\mathcal{C}_\infty\}$ of state evolutions which is precisely the set of all cluster evolutions of the net $\{\langle \mu_A^t: A \in \mathcal{C}\}$ in the pointwise weak and weak topologies on measure evolutions. Furthermore, each $\hat{\mu}_A^t$ is tight and $\hat{\mu}_A^t = \hat{f}_A^t \sigma_A$, $A \in {}^*\mathcal{C}$, where $\hat{f}_A^t$ is a measurable function on X , and $\hat{\sigma}_A$ is a cluster state of the net $\{\sigma_A: A \in \mathcal{C}\}$ in the weak topology.

(b) If $\{f_A: A \in \mathcal{C}\}$ is uniformly p -integrable for some $p > 1$ and $\{|x_B|^n\}$ is uniformly 1-integrable with respect to $\mathcal{G}$ for any $n \in \mathbb{N}$ and any bounded set $B \subset S$, then each $\hat{\mu}_A^t$, $A \in \mathcal{C}$, $t \in \mathbb{R}$, has a finite correlation measure $\hat{\rho}_A^t$ and $\{\langle \hat{\rho}_A^t: A \in {}^*\mathcal{C}_\infty\}$ is precisely the set of all cluster evolutions of the net $\{\langle \rho_A^t: A \in \mathcal{C}\}$ of finite volume correlation evolutions in the pointwise vague and vague topologies. The same result holds for n^{th} correlation measures.

We end this section by showing that the assumptions we make on the time invariant specification $\mathcal{G} = \{\sigma_A: A \in \mathcal{C}\}$ are valid under very general conditions. Suppose that the σ_A are Gibbs ensembles $\pi_A(0, \cdot)$ arising from a superstable, lower regular potential with zero boundary conditions (here we use the notation of Preston [24]). As on p. 107 of [24], we let A_n be the cube with side $2n$, $n \in \mathbb{N}$, centered at the origin, $\beta_n = (2n)^d - (2n-2)^d$ be the volume of $V_n - V_{n-1}$, and $U_{n,m} = \{x \in X: x(A_n - A_{n-1}) \leq m\beta_n\}$. Then the proof of Lemma 6.8 in [24] shows that for any $\epsilon > 0$, there is an $m \geq 1$ so that

$\pi_A(0, \tilde{U}_m) \geq 1 - \epsilon$ for all $A \in \mathcal{C}$, where $\tilde{U}_m = \cap U_{n,m}$ ($n \in \mathbb{N}$). It is easy to see that the $\tilde{U}_m$ are compact in X , thus establishing the tightness of $\{\pi_A(0, \cdot) : A \in \mathcal{C}\}$.

It is a little more difficult to show that $\{|x_B|^n\}$ is uniformly 1-integrable with respect to $\{\pi_A(0, \cdot) : A \in \mathcal{C}\}$ for any bounded set B and any $n \in \mathbb{N}$. We may suppose without loss of generality that B is a cube of side $2L$. By dividing $A_n - A_{n-1}$ up into β_n unit cubes C_j as in the proof of Lemma 6.8 in [24], we have $\pi_A(0, E_{n,m}) < \exp\{-(pm^2 - q)\beta_n\}$ for some constants $p > 0$ and $q \geq 0$, where $E_{n,m} = \left\{x \in X : \sum_{j=1}^{\beta_n} [x(C_j)]^2 \geq m^2 \beta_n\right\}$, (here the basic estimates of Ruelle for superstable interactions [27] are used). Now

$$F_{n,m} \subseteq E_{n,m} \quad \text{where} \quad F_{n,m} = \{x \in X : x(A_n - A_{n-1}) \geq m^2 \beta_n\} \quad \text{since} \quad \sum_{j=1}^{\beta_n} x(C_j) \leq \sum_{j=1}^{\beta_n} [x(C_j)]^2, \quad \text{so that} \quad \pi_A(0, F_{n,m}) < \exp\{-(pm^2 - q)\beta_n\}. \quad \text{If} \quad x(A_n - A_{n-1}) < m^2 \beta_n$$

for $1 \leq n \leq L$ then $x(B) < \sum_{j=1}^L m^2 \beta_n < Lm^2 \beta_L$. Thus $G_m = \{x \in X : x(B) \geq Km^2\} \subseteq \bigcup_{n=1}^L F_{n,m}$, where K is an integer $\geq L\beta_L$, and so

$$\pi_A(0, G_m) \leq \sum_{n=1}^L \exp\{-(pm^2 - q)\beta_n\} \leq L \exp\{-(pm^2 - q)\beta_0\}$$

for m large enough that $pm^2 - q > 0$.

Now if m is the largest integer $\leq \sqrt{K/K}$,

$$\begin{aligned}
\int_{|x_B|^n \geq k} |x_B|^n d\pi_A(0, \cdot) &\leq \sum_{j=Km^2}^{\infty} j^n \pi_A(0, \{|x_B| = j\}) \\
&\leq \sum_{j=0}^{\infty} \sum_{\ell=K(m+j)^2}^{K(m+j+1)^2-1} j^n \pi_A(0, \{|x_B| \geq j\}) \\
&\leq \sum_{j=0}^{\infty} \sum_{\ell=K(m+j)^2}^{K(m+j+1)^2} K^{n(m+j+1)^{2n}} \pi_A(0, G_{m+j}) \\
&\leq \sum_{j=0}^{\infty} LK^{n+1}(m+j+1)^{2(n+1)} \exp\{-[p(m+j)^2-q]\beta_0\}
\end{aligned}$$

and the right-hand side is $< \epsilon$ for m , and hence k , sufficiently large, and independent of A . This shows that the $\{|x_B|^n\}$ are uniformly 1-integrable with respect to $\{\pi_A(0, \cdot): A \in \mathcal{C}\}$.

§4 Validity of the BBGKY hierarchy

In order to discuss the BBGKY hierarchy we must be more specific about how the flow specifications and finite volume states of §3 are defined. To define the dynamics and hence the flow specifications on $X(A)$, we must first consider the interaction forces between two particles and between a particles and the boundary. We suppose for simplicity that the two particle interaction is given by a fixed pair potential $\Phi: S \rightarrow \mathbb{R} \cup \{\infty\}$ satisfying the following assumptions:

$$4.1 \quad (a) \quad \Phi(q) = \Phi(-q);$$

- (b) If Φ takes the value ∞ (in which case it is called singular), then $\{q \in S: \Phi(q) = \infty\} = \{q \in S: \|q\| \leq \delta\}$ for some real $\delta \geq 0$, where we assume that $\Phi(q) \rightarrow \infty$ as $\|q\| \rightarrow \delta$. If $\delta > 0$, then we say that Φ has a hard core.
- (c) Wherever Φ does not take the value ∞ , it is a C^1 function;
- (d) Φ is lower regular and superstable [27]

The interaction between a particle and the boundary ∂A of a domain A is specified by a potential $\Phi_A: A \rightarrow \mathbb{R}^+ \cup \{\infty\}$ satisfying

4.2 (a) Φ_A is C^1 if $|q - \partial A| > \delta/2$, where $|q - \partial A|$ denotes the distance from q to ∂A .

(b) There is a $\beta > \delta/2$ so that $\Phi_A(q) = 0$ if $|q - \partial A| > \beta$.

(c) $\Phi_A(q) \rightarrow \infty$ as $|q - \partial A| \rightarrow \delta/2$ and $\Phi_A(q) = \infty$ if $|q - \partial A| < \delta/2$.

This potential is used to ensure that the particles never leave A .

The Hamiltonian associated with Φ and Φ_A is a function on $\pi(A)$ which has the form

$$(4.1) \quad H'(x) = \sum H_1^i(x_i) \quad (1 \leq i \leq n) + \sum \Phi(q_i, q_j) \quad (1 \leq i < j \leq n)$$

for $x = (x_1, \dots, x_n) \in \pi_n(A)$, where $H_1^i(x_i) = \frac{1}{2m} p_i^2 + \Phi_A(q_i)$, $p_i^2 = \|p_i\|^2$, and m is the mass of a particle. H' can also be regarded as a function on $X(A)$ because of the symmetry condition 4.1(a).

Because we will use test functions which vanish whenever a particle is in the support of Φ_A , the potential Φ_A will not enter into the BBGKY hierarchy, and in the BBGKY hierarchy we will only need to consider the reduced Hamiltonian

$$(4.2) \quad H(x) = \sum H_i(x_i) \quad (1 \leq i \leq n) + \sum \Phi(q_i - q_j) \quad (1 \leq i < j \leq n)$$

where $H_i(x_i) = \frac{1}{2m} p_i^2$.

We now define a map $\tilde{T}_A^t: \pi(A) \rightarrow \pi(A)$ for any $t \in \mathbb{R}$ as follows. We denote the restriction of $\tilde{T}_A^t$ to $\pi_n(A)$ by $\tilde{T}_{A,n}^t$. For $x = \{x_1, \dots, x_n\} \in X_n(A)$, $\tilde{T}_{A,n}^t(x) = \{x_1(t), \dots, x_n(t)\} = x(t)$ is obtained by solving Newton's equations

$$(4.3)(a) \quad \begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i} \\ x_i(t) &= (q_i(t), p_i(t)), \quad 1 \leq i \leq n, \\ \frac{dp_i}{dt} &= - \frac{\partial H}{\partial q_i} \end{aligned}$$

together with the initial conditions

$$(4.3)(b) \quad x(0) = x.$$

In the case of singular potentials, the map $\tilde{T}_A^t$ is not defined if $x \in X(A) - X^\delta(A)$. For $x \in X(A) - X^\delta(A)$ we put $\tilde{T}_A^t(x) = x$. Because of the symmetry condition (4.1)(a), $\tilde{T}_A^t$ defines a map $T_A^t: X(A) \rightarrow X(A)$. We may then define a map $T_A^t: X \rightarrow X$ by putting $T_A^t(x) = x_{S-A} \times \tilde{T}_A^t(x_A)$. That is, we keep the particles outside A fixed and move particles inside A using $\tilde{T}_A^t$. We have now prescribed a flow specification $\mathfrak{D} = \{T_A^t\}$ in the sense of [12].

For any measurable function f on $X(\mathcal{A})$ (or $\pi(\mathcal{A})$) we define the measurable function f^t by $f^t(x) = f \circ \tilde{T}_{\mathcal{A}}^{-t}(x)$.

Next we consider a prescription for defining finite volume states $\mu_{\mathcal{A}}$ on X for any $\mathcal{A} \in \mathcal{C}$ which is appropriate to the BBGKY hierarchy. Let the measurable function $h_{\mathcal{A}}$ be the density of a state on $X(\mathcal{A})$. Thus we assume that

- 4.3 (a) $h_{\mathcal{A}} \geq 0$,
 (b) $\int h_{\mathcal{A}} d\omega_{\mathcal{A}} = 1$,
 (c) In the case of singular potentials, $h_{\mathcal{A}}(x) = 0$ for $x \in X(\mathcal{A}) - X^{\delta}(\mathcal{A})$.

Using $h_{\mathcal{A}}$ we may now define a state $\mu_{\mathcal{A}}$ on X by

$$(4.4) \quad \mu_{\mathcal{A}}(F) = \int_{F_{\mathcal{A}}} h_{\mathcal{A}} d\omega_{\mathcal{A}}, \quad \text{where } F_{\mathcal{A}} = \{y_{\mathcal{A}} : y \in F\}.$$

The state $\mu_{\mathcal{A}}$ can also be written as

$$(4.5) \quad \mu_{\mathcal{A}}(F) = \int_{\tilde{F}_{\mathcal{A}}} \tilde{h}_{\mathcal{A}} d\lambda_{\mathcal{A}} = \sum 1/n! \int_{\tilde{F}_{\mathcal{A},n}} \tilde{h}_{\mathcal{A},n} dx^n \quad (n \geq 0)$$

where $\tilde{h}_{\mathcal{A}} = h_{\mathcal{A}} \circ \xi_{\mathcal{A}}$, $\tilde{F}_{\mathcal{A}} = \xi_{\mathcal{A}}^{-1}(F_{\mathcal{A}})$, $\tilde{h}_{\mathcal{A},n} = \tilde{h}_{\mathcal{A}} \mid \pi_n(\mathcal{A})$, $\tilde{F}_{\mathcal{A},n} = \tilde{F}_{\mathcal{A}} \cap \pi_n(\mathcal{A})$, and $dx^n = dx_1 \dots dx_n$ signifies Lebesgue integration. Using this representation, the n^{th} correlation measure $\rho_{\mathcal{A},n}$ of $\mu_{\mathcal{A}}$ has a density which takes the form [18]

$$(4.6) \quad \rho_{\mathcal{A},n}(x^n) = \sum 1/(m-n)! \int \tilde{h}_{\mathcal{A},m}(x^m) dx_{n+1} \dots dx_m \quad (m \geq n)$$

when $x^n = (x_1, \dots, x_n) \in \pi_n(A)$; the integration is over $\pi_{m-n}(A)$.

Clearly $\mu_A^t(F) = \int \tilde{h}_A^t d\lambda_A$, and the correlation measure $\rho_{A,n}^t$ of μ_A^t is given by

$$(4.7) \quad \rho_{A,n}^t(x^n) = \sum 1/(m-n)! \int \tilde{h}_{A,m}^t(x^m) dx_{n+1} \dots dx_m \quad (m \geq n)$$

This defines the sets $\{\langle \mu_A^t \rangle : A \in \mathcal{C}\}$ and $\{\langle \rho_A^t \rangle : A \in \mathcal{C}\}$ of finite volume state and correlation evolutions respectively.

The finite volume Gibbs ensembles, which we denote by σ_A , occur in the special case that $h_A(x)$ is replaced by $g_A(x) = Z_A^{-1} \exp[-H'(x)]$, where Z_A is the usual normalizing constant. It is important to note that $\sigma_A^t = \sigma_A$, i.e., that the specification $\mathcal{G} = \{\sigma_A : A \in \mathcal{C}\}$ is invariant under $\mathcal{D}$. It follows from 4.1(d) that $\mathcal{G}$ is tight. We now augment our assumptions on the density h_A as follows:

- 4.3 (d) The density h_A is of the form $h_A = f_A g_A$, where g_A is the Gibbs density introduced above, and the set $\{f_A : A \in \mathcal{C}\}$ is uniformly p -integrable with respect to $\{\sigma_A : A \in \mathcal{C}\}$ for some $p > 1$.

The conditions 4.1(d) and 4.3(d) are all that is necessary for the results of §3 to hold.

We also need to make the following further assumption about Φ .

- 4.4 The images under the map Ξ of the functions

$$\{H, \mathcal{P}\}(x_1, \dots, x_n) = \sum \left[\frac{\partial H}{\partial p_i} \frac{\partial \mathcal{P}}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial \mathcal{P}}{\partial p_i} \right] \quad (1 \leq i \leq n)$$

and

$$\varphi_1(x_1, \dots, x_{n+1}) = \sum \frac{\partial \Phi(q_i - q_{n+1})}{\partial q_i} \frac{\partial \varphi}{\partial p_i} \quad (1 \leq i \leq n)$$

are uniformly q -integrable with respect to $\mathcal{G}$ for any fixed function $\varphi \in C_0^1(\pi_n(\Lambda))$ and any $\Lambda \in \mathcal{E}$, where q is conjugate to the p of 4.3(d).

Condition 4.3 will be used to establish the validity of the limiting BBGKY hierarchy. We have stated it in this somewhat arcane form in order to make the proof easier to follow; in the remarks after the statement of Theorem 4.5 we present a simple sufficient condition for it to hold.

We are now in a position to formulate the main theorem of this section. Given the specification $\nu = \{\mu_\Lambda; \Lambda \in \mathcal{E}\}$, the flow specification $\mathcal{D} = \{T_\Lambda^t; \Lambda \in \mathcal{E}, t \in \mathbb{R}\}$, and the tight $\mathcal{D}$ -invariant specification $\mathcal{G} = \{\sigma_\Lambda; \Lambda \in \mathcal{E}\}$, we can construct the finite volume state and correlation evolutions $\{\langle \mu_\Lambda^t \rangle; \Lambda \in \mathcal{E}\}$ and $\{\langle \rho_\Lambda^t \rangle; \Lambda \in \mathcal{E}\}$ as indicated above. The results of §3 hold under assumptions 4.1 and 4.2. Then the set $\{\hat{\mu}_\Lambda^t; \Lambda \in {}^*\mathcal{E}_\infty\}$ of all cluster evolutions of the net $\{\langle \mu_\Lambda^t \rangle; \Lambda \in \mathcal{E}\}$ in both the pointwise weak and the weak topologies is non-empty. Likewise, the corresponding set of all cluster correlation evolutions $\{\hat{\rho}_\Lambda^t; \Lambda \in {}^*\mathcal{E}_\infty\}$ of the net $\{\langle \rho_\Lambda^t \rangle; \Lambda \in \mathcal{E}\}$ in the pointwise vague and vague topologies is non-empty.

4.5 Theorem

Each correlation evolution $\hat{\rho}^t \in \{\hat{\rho}_\Lambda^t; \Lambda \in {}^*\mathcal{E}\}$ satisfies the weak BBGKY hierarchy, i.e.

$$(4.8) \quad \hat{\rho}_n^t(\varphi) = \hat{\rho}_n(\varphi) + \int_0^t [\hat{\rho}_n^s(\varphi_t + \{H, \varphi\}) - \hat{\rho}_{n+1}^s(\varphi_1)] ds, \quad n = 1, 2, \dots,$$

for any $\varphi \in C_0^1(\mathbb{R}^+ \times X_n(S))$, where $\hat{\rho}_n^t(f) = \int f(t, x) \hat{\rho}_A^t(x) dx^n$.

The infinite hierarchy of equations (4.8) is called the BBGKY hierarchy.

We end this section with some remarks on the assumptions we have made. The assumption 4.1 on the interaction potential, and in particular the assumption of superstability, is indispensable in any discussion of limiting behavior and need not be discussed further.

The assumptions 4.3 on the initial states, in particular 4.3(d) which is essentially the requirement that the initial state is in L^p with respect to a Gibbs state, is likewise a very mild restriction. In [11] this is true with $p = 2$.

There remains the assumption 4.4, which we will now consider. First we note that from the linearity of $\mathcal{E}$ we need only verify the uniform q -integrability of the respective summands. Similarly, because $\varphi \in C_0^1(\pi_n(A))$ we need only consider terms of the form $\partial\Phi(q_1 - q_j)/\partial q_j$. Using the fact that φ has compact support, it is then easy to see that 4.4 results from the following finiteness condition:

$$(4.9) \quad \int_{A \times A} \left| \frac{\partial\Phi}{\partial q_1}(q_1 - q_2) \right|^q e^{-\Phi(q_1 - q_2)} dq_1 dq_2 < \infty.$$

for all $A \in \mathcal{C}$. In the case of singular potentials, the condition (4.9) is a mild restriction on the rate of growth of $|\partial\Phi/\partial q_1|$ near the singularity and at infinity.

§5 Proofs of the Theorems

As in [12] the proofs use nonstandard analysis for which the reader is referred to [13, 26, 31]. The analysis will be carried out in a κ -saturated enlargement ${}^*\mathcal{M} = V({}^*\mathcal{R})$ of the superstructure $V(\mathcal{R})$ over $\mathcal{R}$ (which contains X), where the topology of X has a base of cardinality $< \kappa$ and $\kappa > \aleph_0$. We write $r \simeq s$ if r and s in ${}^*\mathcal{R}$ are infinitesimally close. The standard part of $r \in {}^*\mathcal{R}$ is denoted by $\circ r$ or $\text{st}(r)$, with the convention that $\circ r = +\infty$ or $-\infty$ if r is positive or negative infinite. A set $A \in {}^*\mathcal{C}$ is infinite if ${}^*\mathcal{A} \subset A$ for all $\mathcal{A} \in \mathcal{C}$. The existence of infinite sets in ${}^*\mathcal{C}$ follows from the fact that ${}^*\mathcal{M}$ is an enlargement. The infinite elements of ${}^*\mathcal{C}$ will be denoted by ${}^*\mathcal{C}_\infty$.

The nonstandard extension *X of X consists of all ${}^*\mathbb{N}$ -valued, finitely additive measures on ${}^*\mathcal{S} \times {}^*\mathcal{S}$ of the form $x = \sum^* m_n \delta_{x_n}$ ($n \in {}^*\mathbb{N}$), where $\sum^*$ is the extension of standard summation, $m_n \in {}^*\mathbb{N}$, and $x_n \in {}^*\mathcal{S} \times {}^*\mathcal{S}$. The monad of $x \in X$ is $m(x) = \bigcap^* U(x \in U, U \text{ open in } X)$. From the definition of the topology on X it is easy to see that if $x = \sum m_n \delta_{x_n}$ ($n \in \mathbb{N}$) $\subset X$, then $m(x)$ consists of all $x = \sum^* m_n \delta_{x_n}$ ($n \in {}^*\mathbb{N}$) for which $m_n = m_n$ and $x_n \simeq x_n$ for all $n \in \mathbb{N}$ (here $\simeq$ is the relation of nearness in the Euclidean topology on ${}^*\mathcal{S} \times {}^*\mathcal{S}$). We regard *X as a measurable space, using the σ -algebra $\sigma({}^*\mathcal{B})$ generated by the collection ${}^*\mathcal{B}$ of internal Baire subsets of *X .

By the fundamental result of Peter Loeb [20], any internal, finitely additive, ${}^*\mathbb{R}$ -valued measure μ on an internal algebra $\mathcal{A}$ of subsets of an internal set Y (we say that $(Y, \mathcal{A})$ is an internal measurable space) induces a measure $\circ\mu$ on $\sigma(\mathcal{A})$ defined by $\circ\mu(A) = \text{st}(\mu(A))$ for all $A \in \mathcal{A}$. The measure $\circ\mu$ can be extended to the completion (with respect to $\circ\mu$) $L_\mu(\mathcal{A})$ of $\sigma(\mathcal{A})$. The complete measure (also denoted by $\circ\mu$) so obtained is called the

Loeb measure associated with μ . We will be constructing limiting states using the states $\circ\mu_A$ on $({}^*X, {}^*\mathcal{B})$, where $\mu_A \in {}^*\mathcal{V}$ and $A \in {}^*\mathcal{C}_\infty$.

By transfer, the maps $T_A^t: {}^*X \rightarrow {}^*X$, $A \in {}^*\mathcal{C}$, in ${}^*\mathcal{D}$ are ${}^*\mathcal{B}$ -measurable and hence $\sigma({}^*\mathcal{B})$ -measurable [7, Theorem 1.7.2]. Also the T_A^t ($t \in {}^*\mathbb{R}$) form a group parameterized by ${}^*\mathbb{R}$ for any $A \in {}^*\mathcal{C}$. These facts will be used without explicit mention in the rest of this section.

Again by transfer, if $\mathcal{F} = \{g_A: A \in \mathcal{C}\}$ is a collection of measurable functions on X , then the ${}^*\mathbb{R}$ -valued internal functions $g_A \in {}^*\mathcal{F}$ are internally measurable. Furthermore, the standard parts $\circ g_A$ of these functions, defined by $\circ g_A(x) = \text{st}(g_A(x))$, $x \in {}^*X$, are $\sigma({}^*\mathcal{B})$ -measurable. Even if each g_A is integrable with respect to a measure μ_A for each $A \in \mathcal{C}$, it is not necessarily true that the $\circ g_A$ are integrable with respect to $\circ\mu_A$ for a $A \in {}^*\mathcal{C}$. The relevant concept here is that of S-integrability.

5.1 Definition

Let $(Y, \mathcal{A})$ be an internal measurable space. An internally measurable, ${}^*\mathbb{R}$ -valued function g on Y is S-integrable with respect to an internal measure μ on $\mathcal{A}$ if $\circ g$ is integrable with respect to $\circ\mu$ and $\circ \int |g| d\mu = \int \circ |g| d\circ\mu$.

The following two conditions are each equivalent to S-integrability of a ${}^*\mathcal{B}$ -measurable g with respect to an internal state μ :

$$(5.1) \quad \circ \int |g| d\mu < \infty \quad \text{and} \quad \int_A g d\mu \simeq 0 \quad \text{if} \quad A \in {}^*\mathcal{B} \quad \text{and} \quad \mu(A) \simeq 0 \quad [1].$$

$$(5.2) \quad \int_{|g| \geq \omega} |g| d\mu \simeq 0 \quad \text{for all infinite } \omega \in {}^*N \quad [21].$$

The next lemma is part of Theorem 6.5 in [3].

5.2 Lemma

If the family $\{g_A : A \in \mathcal{C}\}$ of measurable functions on X is uniformly p -integrable ($1 \leq p < \infty$) with respect to $\mathcal{G} = \{\sigma_A : A \in \mathcal{C}\}$ then $|g_A|^p \in {}^*\mathcal{G}$ is S -integrable with respect to σ_A for any $A \in {}^*\mathcal{C}_\infty$.

Proof. For any $\epsilon > 0$ there exists a $k \in N$ and a $B \in \mathcal{C}$ such that

$$\int_{|g_A|^p \geq k} |g_A|^p < \epsilon \quad \text{for all } m \geq k \text{ in } N \text{ and all } A \supseteq B \text{ in } \mathcal{C}. \quad \text{By transfer}$$

$$\int_{|g_A|^p \geq \omega} |g_A|^p d\sigma_A < \epsilon \quad \text{for any infinite } \omega \in {}^*N \text{ and any } A \in {}^*\mathcal{C}_\infty. \quad \text{This is true}$$

$$\text{for any real } \epsilon > 0 \text{ and so } \int_{|g_A|^p \geq \omega} |g_A|^p d\sigma_A \simeq 0 \text{ and } |g_A|^p \text{ is } S\text{-integrable}$$

by (5.2). $\blacksquare$

The set $X^\#$ of near-standard points in *X consists of those $x \in {}^*X$ for which there exists a (unique because X is Hausdorff) point $x \in X$ so that $x \in m(x)$. We then write $st(x) = x$, thus defining a map $st: X^\# \rightarrow X$. (The set $X^\#$ and map st here correspond to the set $X^\#$ and map π defined in [12].)

The basic technique of the proof is to show that internal measures are "near-standardly concentrated" and then to project down to X using st . The following lemma of Anderson and Rashid [2] provides the technical tool we need.

Let $(Y, \mathcal{A})$ be a Baire space. An internal Baire state μ on $(Y, {}^*\mathcal{A})$ is S-tight if for each $\epsilon > 0$ in $\mathbb{R}$, there is a compact set $K \subseteq Y$ such that $f \prec K'$ (the complement of K) implies $\left| \int {}^*f d\mu \right| < \epsilon$. The state μ is near-standardly concentrated if $\circ\mu({}^*Y - Y^\#) = 0$, where $Y^\#$ denotes the set of near-standard points in *Y . If μ is a near-standardly concentrated state, then $st^{-1}(A) \in L_\mu({}^*\mathcal{A})$ for each $A \in \mathcal{A}$, and so $\circ\mu$ induces a state $\hat{\mu}$ on $(Y, \mathcal{A})$ defined by $\hat{\mu}(A) = \circ\mu[st^{-1}(A)]$, $A \in \mathcal{A}$.

5.3 Lemma [2]

Let μ be an internal state on $({}^*Y, {}^*\mathcal{A})$,

- (a) If μ is S-tight, then it is near-standardly concentrated.
- (b) If μ is near-standardly concentrated, then $\hat{\mu}$ is tight and

$$(5.3) \quad \int_{{}^*Y} {}^*\psi d\mu = \int_{{}^*Y} \circ({}^*\psi) d\circ\mu = \int_{Y^\#} \circ({}^*\psi) d\circ\mu = \int_{Y^\#} \psi(\circ y) d\circ\mu(y) = \int_Y \psi d\hat{\mu},$$

for each $\psi \in C(Y)$ for which ${}^*\psi$ is S-integrable with respect to μ .

Part (b) of Lemma 5.3 is a slight modification of Part (iii) of Lemma 2 in [2], where the equalities are stated only for bounded continuous function ψ .

To begin the proof of Theorem 3.4(a), we first show that $\sigma_A \in {}^*\mathcal{G}$ is S-tight for any $A \in {}^*\mathcal{C}_\infty$. Let $\epsilon > 0$ in $\mathbb{R}$ be given. Then there is a compact set $K \subset X$ and a $B \in \mathcal{C}$ so that if $A \in \mathcal{C}$ and $A \supseteq B$, then $\left| \int {}^*f d\sigma_A \right| < \epsilon$ for all $f \prec K'$ (the complement of K). By transfer, $\left| \int {}^*f d\sigma_A \right| < \epsilon$ for any $f \prec K'$ and any $A \in {}^*\mathcal{C}_\infty$, so that σ_A is S-tight. We conclude from Lemma 5.3(a) that σ_A is near-standardly concentrated for any $A \in {}^*\mathcal{C}_\infty$.

We next show that the states μ_A^t are near-standardly concentrated for any $t \in {}^*\mathbb{R}$ and any $A \in {}^*\mathcal{C}_\infty$. In the following we let $f_A \circ T_A^{-t}(x) = f_A^t(x)$. First

note that

$$(5.4) \quad \mu_A^t = f_A^t \sigma_A^t = f_A^t \sigma_A$$

for any $A \in \mathcal{C}$ by transfer from [7, Theorem 2.10.3] and the invariance of σ_A . The family $\{f_A: A \in \mathcal{C}\}$ is uniformly 1-integrable with respect to $\mathcal{C}$. It follows from Lemma 5.2 and (3.2) that

$$(5.5) \quad \int_{|f_A| \geq \omega} |f_A| d\sigma_A \simeq 0$$

for $\omega \in {}^*N_\infty$ and $A \in {}^*\mathcal{C}_\infty$. Now note that if g is ${}^*\mathcal{B}$ -measurable and $A \in {}^*\mathcal{B}$, then by the invariance of σ_A , $A \in {}^*\mathcal{C}$, we have

$$(5.6) \quad \int_A (g \circ T_A^{-t}) d\sigma_A = \int_A (g \circ T_A^{-t}) d\sigma_A^t = \int_{T_A^{-t}(A)} g d\sigma_A$$

for any $t \in {}^*\mathbb{R}$ (here we have used the transfer of identities in the proof of [7, Theorem 2.10.3]). Applying (5.6) with $g = |f_A|$ and $A = \{x \in {}^*X: |f_A^t(x)| \geq \omega\} = T_A^t\{x \in {}^*X: |f_A| \geq \omega\}$ for $\omega \in {}^*N_\infty$ and $A \in {}^*\mathcal{C}_\infty$, we see that

$$(5.7) \quad \int_{|f_A^t| \geq \omega} |f_A^t| d\sigma_A \simeq 0,$$

so that $|f_A|$ is S-integrable with respect to σ_A for any $A \in {}^*\mathcal{C}_\infty$ and $t \in {}^*\mathbb{R}$. It follows that $\circ f_A^t$ is integrable with respect to σ_A for any

$\Lambda \in {}^*\mathcal{C}_\infty$ and $t \in {}^*\mathbb{R}$. From (5.4) we have

$$(5.8) \quad \circ\mu_\Lambda^t = \circ f_\Lambda^t \circ \sigma_\Lambda$$

for $\Lambda \in {}^*\mathcal{C}_\infty$ and $t \in {}^*\mathbb{R}$. Thus ${}^*X - X^\#$ is a set of measure zero with respect to $\circ\mu_\Lambda^t$ since σ_Λ^t is near-standardly concentrated and $\circ f_\Lambda^t$ is integrable, and so μ_Λ^t is near-standardly concentrated for $\Lambda \in {}^*\mathcal{C}_\infty$ and $t \in {}^*\mathbb{R}$.

Using Lemma 5.3, we see that the state $\hat{\mu}_\Lambda^t$ ($\Lambda \in {}^*\mathcal{C}_\infty$, $t \in \mathbb{R}$) induced on $(X, \mathfrak{B})$ by $\circ\mu_\Lambda^t$ is tight and

$$(5.9) \quad \int_{{}^*X} \phi \, d\mu_\Lambda^t = \int_{{}^*X} \phi(\circ x) \, d\circ\mu_\Lambda^t = \int_X \phi \, d\hat{\mu}_\Lambda^t \text{ if } \phi \in C_b(X), \Lambda \in {}^*\mathcal{C}_\infty, t \in \mathbb{R}.$$

This shows that the evolution $\langle \mu_\Lambda^t \rangle$ is in the monad of $\langle \hat{\mu}_\Lambda^t \rangle$ for $\Lambda \in {}^*\mathcal{C}_\infty$ in the pointwise weak topology. By Theorem 4.2.6 in [26], the set $\{\langle \hat{\mu}_\Lambda^t \rangle : \Lambda \in {}^*\mathcal{C}_\infty\}$ is precisely the set of all cluster evolutions of the net $\{\langle \mu_\Lambda^t \rangle : \Lambda \in \mathcal{C}\}$ of finite volume evolutions in the pointwise weak topology.

By 10.2 in [7], there is, for any $t \in {}^*\mathbb{R}$, a function $\hat{f}_\Lambda^t(x)$ on X so that $E^{st}(\circ f_\Lambda^t(x)) = \hat{f}_\Lambda^t(\circ x)$, where E^{st} denotes conditional expectation with respect to the σ -subalgebra generated by $st: X^\# \rightarrow X$. Then we have $\hat{\mu}_\Lambda^t = \hat{f}_\Lambda^t \circ d\sigma_\Lambda$ [7, Theorem 10.2.2] and

$$(5.10) \quad \begin{aligned} \int_{{}^*X^\#} \psi(\circ x) \, d\circ\mu_\Lambda^t(x) &= \int_{{}^*X^\#} \psi(\circ x) \circ f_\Lambda^t(\circ x) \, d\circ\sigma_\Lambda(x) = \int_X \psi(x) \hat{f}_\Lambda^t(x) \, d\sigma_\Lambda(x) \\ &= \int_X \psi(x) \, d\hat{\mu}_\Lambda^t(x) \end{aligned}$$

for any $\hat{\mu}_\Lambda^t$ -integrable function ψ for which $\psi(\circ x)$ is $\circ\mu_\Lambda^t$ integrable on X by [7, Lemma 10.1.3].

We next consider the result of Theorem 3.4(a) for the weak topology. First note that the last equation in (5.3) holds if ψ is integrable with respect to $\circ\mu$ on $ns(^*Y)$ [7, Corollary 2.10.2]. This fact will be used below without explicit mention. We need to show that $\langle \mu_{\lambda}^t \rangle$ is in the monad of $\langle \hat{\mu}_{\lambda}^t \rangle$ in the weak topology, and for this it suffices to show that

$$(5.11) \quad \int_{^*R} \left[\int_{^*X} ^*\psi(t, x) d\mu_{\lambda}^t \right] d^*\lambda = \int_R \left[\int_X \psi(t, x) d\hat{\mu}_{\lambda}^t \right] d\lambda$$

for $\lambda \in {}^*\mathcal{C}_{\infty}$ and ψ as described in Definition 3.1. Note that $^*\psi(x, t)$ is finitely bounded and hence S-integrable [21, Theorem 4, p. 130] for any $t \in {}^*R$. Denoting $\circ(^*\psi)$ by $\hat{\psi}$, it follows that $\int ^*\psi(t, x) d\mu_{\lambda}^t = \int \psi(t, *) d\hat{\mu}_{\lambda}^t$ is finite for any $t \in {}^*R$ and vanishes outside a finite interval $[a, b]$ since the same is true of $^*\psi(t, x)$. Thus $\int_{^*X} ^*\psi(t, x) d\mu_{\lambda}^t$ is S-integrable with respect to $^*\lambda$ as a function of t , and, with $ns(^*R) = R^{\#}$,

$$(5.12) \quad \begin{aligned} \int_{^*R} \left[\int_{^*X} ^*\psi(t, x) d\mu_{\lambda}^t \right] d^*\lambda &= \int_{R^{\#}} \left[\int_{^*X} ^*\psi(t, x) d\mu_{\lambda}^t(x) \right] d\circ(^*\lambda)(t) \\ &= \int_{R^{\#}} \left[\int_{X^{\#}} \circ\psi(t, x) d\circ\mu^t(x) \right] d\circ(^*\lambda)(t) \\ &= \int_{R^{\#}} \left[\int_{X^{\#}} \psi(\circ x, \circ t) \circ f_{\lambda}^t(x) d\circ\sigma_{\lambda}(x) \right] d\circ(^*\lambda)(t) \\ &= \int_{R^{\#} \times X^{\#}} \left[\psi(\circ x, \circ t) \circ f_{\lambda}^t(x) \right] d\circ\sigma_{\lambda}(x) \bigotimes d\circ(^*\lambda)(t), \end{aligned}$$

using the continuity of ψ and Keislers Fubini Theorem [14]. We now let $\bar{f}_{\lambda}^t(x)$

be the measurable function on $R \times X$ satisfying $E^{ST}[\circ f_A^t(x)] = \bar{f}_A^{\circ t}(\circ x)$, where E^{ST} is the conditional expectation with respect to the standard part map $ST: R^\# \times X^\# \rightarrow R \times X$. Then

$$\begin{aligned}
 (5.13) \quad & \int_{R^\# \times X^\#} \left[\Psi(\circ x, \circ t) \circ f_A^t(x) \right] d\circ\sigma_A(x) \otimes d\circ(^*\lambda)(t) \\
 &= \int_{R^\# \times X^\#} \left[\Psi(\circ x, \circ t) \bar{f}_A^{\circ t}(\circ x) \right] d\circ\sigma_A(x) \otimes d\circ(^*\lambda)(t).
 \end{aligned}$$

To complete the argument we note that

$$\begin{aligned}
 (5.14) \quad & \int_{R^\#} \left[\int_{X^\#} \Psi(\circ x, \circ t) \circ f_A^t(x) d\circ\sigma_A(x) \right] d\circ(^*\lambda)(t) \\
 &= \int_{R^\#} \left[\int_{X^\#} \Psi(\circ x, \circ t) \hat{f}_A^t(\circ x) d\circ\sigma_A(x) \right] d\circ(^*\lambda)(t) \\
 &= \int_{R^\#} \left[\int_{X^\#} \Psi(\circ x, \circ t) \bar{f}_A^{\circ t}(\circ x) d\circ\sigma_A(x) \right] d\circ(^*\lambda)(t)
 \end{aligned}$$

by a similar argument, so that

$$(5.15) \quad \int_{X^\#} \Psi(\circ x, \circ t) \bar{f}_A^{\circ t}(\circ x) d\circ\sigma_A(x) = \int_{X^\#} \Psi(\circ x, \circ t) \hat{f}_A^t(\circ x) d\circ\sigma_A(x)$$

for a.e. $t \in R^\#$ (with respect to $\circ(^*\lambda)$ measure). Hence

$$(5.16) \quad \int_{R^\#} \left[\int_{X^\#} \Psi(\circ x, \circ t) \bar{f}_A^{\circ t}(\circ x) d\circ\sigma_A(x) \right] d\circ(^*\lambda)(t)$$

$$\begin{aligned}
&= \int_{R^\#} \left[\int_{X^\#} \Psi(\circ x, \circ t) \hat{f}^{\circ t}(\circ x) d\sigma_A(x) \right] d\sigma^*(\lambda)(t) \\
&= \int_R \left[\int_X \Psi(x, t) \hat{f}_A^t(x) d\sigma_A(x) \right] d\lambda(t) = \int_R \left[\int_X \Psi(x, t) d\mu_A^t \right] d\lambda
\end{aligned}$$

as desired. This completes the proof of Theorem 3.4(a).

We now consider Theorem 3.4(b). First we show that if $\phi \in C_0(X_0)$ then ${}^*(\Xi\phi)$ is S-integrable with respect to μ_A^t for any $t \in {}^*R$ if $A \in {}^*C_\infty$. Note that $|{}^*(\Xi\phi)| \leq M|x_{*B}|^m$ by transfer from 2.1(c). It is also easy to see that $|f_A^t|^p$ is S-integrable with respect to σ_A for any $t \in {}^*R$ (see the previous argument for the case $p = 1$). Using the transfer of Hölder's inequality, we have

$$\begin{aligned}
(5.17) \quad \int_A |{}^*(\Xi\phi)| d\mu_A^t &= \int_A |{}^*(\Xi\phi)| f_A^t d\sigma_A \\
&\leq M \left[\int_A |x_{*B}|^{mq} d\sigma_A \right]^{1/q} \left[\int_{*X} |f_A^t|^p d\sigma_A \right]^{1/p}, \quad \frac{1}{p} + \frac{1}{q} = 1, \\
&\leq M' \left[\int_{\tilde{A}} |x_{*B}|^{mq'} d\sigma_A \right]^{1/q}
\end{aligned}$$

where q' is an integer $\geq q$, $M' = M \left[\int_{*X} |f_A^t|^p \right]^{\frac{1}{p}}$ is finite, $A = \{x \in {}^*X:$

$|{}^*(\Xi\phi)| \geq \omega\}$, $\tilde{A} = \{x \in {}^*X: |x_{*B}| \geq \frac{\omega}{B}\}$ and $\omega \in {}^*N_\infty$, the set of positive

infinite integers. Now $\int_{\tilde{A}} |x_{*B}|^{mq'} d\sigma_A \approx 0$ by Lemma 5.2 since $\{|x_B|^n\}$ is

uniformly 1-integrable, showing that ${}^*(\Xi\phi)$ is S-integrable with respect to

μ_A^t . Using Lemma 5.3(b) we see that $\int ({}^*(\Xi\phi)) d\mu_A^t \approx \int {}^*(\Xi\phi) d\mu_A^t < \infty$, for any $t \in {}^*R$.

This shows that $\hat{\mu}_A^t$ has a finite correlation measure $\hat{\rho}_A^t$ for $t \in \mathbb{R}$, and

$$(5.18) \quad \hat{\rho}_A^t(\varphi) = \int (\Xi \varphi) d\hat{\mu}_A^t \simeq \int^* (\Xi \varphi) d\mu_A^t = \int^* \Xi(\varphi^*) d\mu_A^t = \rho_A^t(\varphi^*)$$

for any $\varphi \in C_0(X_0)$ and any $t \in \mathbb{R}$. Equation (5.18) shows that $\langle \hat{\rho}_A^t \rangle$ is in the monad of $\langle \hat{\rho}_A^t \rangle$, and hence that $\left\{ \langle \hat{\rho}_A^t \rangle : A \in {}^*e_\infty \right\}$ is the set of all cluster correlation evolutions of the net $\{ \langle \hat{\rho}_A^t \rangle : A \in \mathcal{C} \}$ in the pointwise vague topology.

The proof of the corresponding result for the vague topology on correlation evolutions is similar to the previous proof for the states. The proof for the n^{th} correlation measures is identical. This completes the proof of Theorem 3.4.

Finally we come to the proof of Theorem 4.4. First for completeness we derive the weak form of the BBGKY hierarchy for finite volume correlation functions. Suppose that $\psi: \mathbb{R} \times \pi_n(A) \rightarrow \mathbb{R}$ is in $C_0^1(\mathbb{R}^+ \times \pi_n(A))$, and, if $x^n = (x_1, \dots, x_n)$ with $x_i = (q_i, p_i)$, then $\psi(t, x^n) = 0$ whenever $q_i \in \text{supp } \Phi_A$ for some i , $1 \leq i \leq n$. Denote $\int \psi(t, x) \rho_{A,n}^t(x) dx^n$ by $\rho_{A,n}^t(\psi)$. In the following calculation, we encounter integrals of the form $\int \psi(t, x^n) \tilde{h}_{A,m}^t(x^m) dx^m$, in which we may regard ψ as a function, denoted again by $\psi(t, x^m)$ of t and x^m where $\frac{\partial \psi}{\partial x_i} = 0$ if $n+1 \leq i \leq m$. We use the fact that

$$\begin{aligned} \int \psi(t, x^m) \tilde{h}_{A,m}^t(x^m) dx^m &= \int \psi(t, x^m) \tilde{h}_{A,m}^t \left[T_{A,m}^{-t}(x^m) \right] dx^m \\ &= \int \psi(t, x^m(t)) \tilde{h}_{A,m}(x^m) dx, \end{aligned}$$

where $x^m(t) = \left[x_1(t), \dots, x_m(t) \right] = T_{A,m}^t(x^m)$. Using the symmetry of $\tilde{h}_{A,m}$ and interchanging summation and integration, we have

$$\begin{aligned}
(5.20) \quad \frac{d}{dt} \rho_{A,n}^t(\varphi) &= \frac{d}{dt} \sum_{m=0}^n \frac{1}{(m-n)!} \int \varphi \left[t, x^m(t) \right] \tilde{h}_{A,m}(x^m) dx^m \\
&= \sum_{m=n}^{\infty} \frac{1}{(m-n)!} \int \left[\frac{\partial \varphi}{\partial t} + \sum_{i=1}^{\infty} \left[\frac{\partial \varphi}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \varphi}{\partial p_i} \frac{dp_i}{dt} \right] \right] \tilde{h}_{A,m}(x^m) dx^m \\
&= \rho_{A,n}^t(\varphi_t + \{H, \varphi\}) - \rho_{A,n+1}^t(\varphi_1)
\end{aligned}$$

from which we immediately deduce the equation

$$(5.21) \quad \rho_{A,n}^t(\varphi) = \rho_{A,n}(\varphi) + \int_0^t [\rho_{A,n}^s(\varphi_t + \{H, \varphi\}) - \rho_{A,n+1}^s(\varphi_1)] ds, \quad n \geq 0.$$

The usual derivation of the BBGKY hierarchy [8, 17] involves integrating down from the Liouville equation using integration by parts and is more complicated. We call (5.21) the weak BBGKY hierarchy for the finite volume correlation measures. The only delicate point in the calculation is the interchange of differentiation and summation. This is justified by standard theorems on term by term differentiation of series after invoking assumption 4.3 and using "Holders inequality. By transfer from (5.21), we see that

$$(5.22) \quad \rho_{A,n}^t(\varphi) = \int_0^t \left[\rho_{A,n}^s(*F_s) - \rho_{A,n+1}^s(*\varphi_1) \right] * ds$$

for any $A \in {}^* \mathcal{C}$ and $n \in {}^* \mathbb{N}$, where $F_s(x) = \varphi_t(s, x) + \{H, \varphi\}(x, s)$. By the transfer of "Holders inequality,

$$(5.23) \quad \int_A |^*(\Xi F_s)| d\mu_A^S \leq \left[\int_A |^*(\Xi F_s)|^q d\sigma_A \right]^{1/q} \left[\int |f_A^S|^p d\sigma_A \right]^{1/p}.$$

Putting $A = \{x \in {}^*X: |^*(\Xi F)| \geq \omega\}$, and noticing that $\int_A |^*(\Xi F)|^q d\sigma_A \approx 0$ by 4.3 and Lemma 5.2, and that $\int |f_A^S|^p d\sigma_A$ is finite since $|\hat{f}_A^S|^p$ is S -integrable with respect to σ_A , we see that ${}^*(\Xi F)$ is S -integrable with respect to μ_A^S . Hence $\rho_{A,n}^S({}^*F)$ is finite and thus S -integrable with respect to *ds for $s \in {}^*[0, t]$ since $\rho_{A,n}^S({}^*F) = \int {}^*(\Xi F) d\mu_A^S$. The same result holds for $\rho_{A,n+1}^S({}^*F_1)$. From (5.22) we obtain

$$(5.24) \quad \begin{aligned} \circ\rho_{A,n}^t(\tilde{F}(t)) &= \circ\rho_{A,n}(\tilde{F}(t)) + \int_0^t \left[\circ\rho_{A,n}^S(\tilde{F}_s) \right] \circ ds - \\ &\quad - \int_0^t \left[\circ\rho_{A,n+1}(\tilde{F}_1) \right] \circ ds. \end{aligned}$$

The desired result is now obtained by projecting down to $\pi(S)$ by the standard part map as in the proof of Theorem 3.4, in particular the details concerning the weak topology.

The research reported here was supported by the Natural Science and Engineering Research Council of Canada under Grant A-8198.

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