

PROTECTING A FOREST AGAINST FIRE: OPTIMAL  
PROTECTION PATTERNS AND HARVEST POLICIES

By

WILLIAM J. REED

DM-404-IR

JANUARY 1986

PROTECTING A FOREST AGAINST FIRE:  
OPTIMAL PROTECTION PATTERNS AND HARVEST POLICIES\*

by

William J. Reed  
Department of Mathematics  
University of Victoria  
P.O. Box 1700  
Victoria, B.C. V8W 2Y2

\*Research supported by NSERC (grant no. A-7252).

### Abstract

A model for determining the optimal level of expenditure on fire protection for a forest, through time, is developed. The model requires the specification of a function relating the probability of fire at a given age to the current level of expenditure. Use of the Pontryagin Maximum Principle enables the determination of the optimal protection schedule for a single rotation with a fixed cutting age. The optimal cutting age and protection schedule for an ongoing succession of stands on a site can be determined numerically by an iterative technique.

KEYWORDS. Forest fire protection; hazard function; forest rotation; maximum principle.

### 1. Introduction

While it is usually assumed that it is economically worthwhile to protect commercial forests against the risk of fire, the question of how optimally to spend money on protection has received scant attention. Some work on the benefits of protection has been carried out (Martell, 1980, Reed, 1984) by comparing land expectation values under different fire probability scenarios. However, such analyses ignore the costs of protection and provide little help in answering many questions of practical importance. For example, is it better to concentrate protection efforts on older, more valuable trees, (which quite likely have a lower susceptibility to fire) than on younger, less valuable, but possibly more vulnerable, trees? If so, at what age, should protection begin and at what level should it be applied?

To answer questions such as these a model which incorporates a component relating the probability of destruction to the money spent on protection, is required. In this paper such a model is developed and optimal levels of protection, and the optimal rotation age are determined jointly. The model is for a single even-aged stand and follows the paradigm of the Faustmann (1849) model, adapted to allow for catastrophic loss along the lines of Reed (1984). Forest-level concerns such as those discussed in Reed and Errico (1986) are not considered.

In Section 2 the model is described and the optimization problem formulated. In Section 3 it is shown how a solution to the optimization problem, in the case of age-independent fire probabilities, can be obtained using the Pontryagin Maximum Principle. Section 4 discusses qualitative properties of the optimal

protection schedule. The results of a numerical example are presented in Section 5, while in Section 6 the case of age-dependent fire probabilities is discussed, and a numerical example given.

## 2. The Model and the Optimization Problem

Consider a succession of even-aged stands growing on a forest site each with an identical "value-at-age" relationship characterized by a function  $V(t)$ , where  $V$  is the stumpage value, net of harvesting costs, of timber in a stand of age  $t$ . Suppose that whenever a stand growing on the site is destroyed, either through fire or through clear-cut harvesting, a new stand is re-established without delay, and that the costs of re-establishment are  $\$C_1$  after a clear-cut harvest, and  $\$C_2$  after a fire.

Suppose that the probability of destruction through fire of a stand of age  $t$  is characterized by a hazard function  $h(t)$ . Specifically the hazard is the instantaneous conditional probability

$$(1) \quad h(t) = \lim_{\Delta \rightarrow 0} \left\{ P(\text{stand destroyed between ages } t \text{ and } (t+\Delta) \mid \text{stand has survived until age } t) / \Delta \right\}.$$

Related to the hazard function are the survivor function (see e.g. Kalbfleisch, 1979)

$$(2) \quad S(x) = \exp \left\{ - \int_0^x h(t) dt \right\}$$

which gives the probability that the stand has not been destroyed by age  $x$ , and the function

$$(3) \quad f(x) = h(x)S(x) = h(x) \exp \left\{ - \int_0^x h(t) dt \right\}$$

which is the probability density function (p.d.f.) of the age,  $X$  at which the stand is destroyed.

Consider now the situation in which money is spent on protecting the forest against fire. Suppose that when the stand is of age  $t$  money is spent on protection at the rate of  $p(t)$  dollars per year, and that, in consequence, the hazard is modified from  $h(t)$  to  $h_p(t)$ . In general we shall assume that

$$(4) \quad p(t) \geq 0,$$

and

$$(5) \quad 0 \leq h_p(t) \leq h(t).$$

With such protection in place the survivor function will be modified to

$$(6) \quad S_p(x) = \exp \left\{ - \int_0^x h_p(t) dt \right\},$$

and the life-time,  $X$ , of the stand will have p.d.f.

$$(7) \quad f_p(x) = h_p(x) S_p(x) = h_p(x) \exp \left\{ - \int_0^x h_p(t) dt \right\}.$$

If the stand is managed in such a way that it is clear-cut harvested whenever it reaches some pre-set rotation age,  $T$ , the time,  $X$ , until the destruction of the stand, either through fire or through harvesting, will have a distribution characterized by the cumulative distribution function (c.d.f.)

$$(8) \quad F(x) = P(X \leq x) = \begin{cases} 1 - S_p(x), & x < T \\ 1 & x \geq T \end{cases}$$

In other words the life-time,  $X$ , of the stand is a random variable with a distribution like that given by the p.d.f. (7), only truncated at  $x = T$ .

Consider now the revenue, net of costs of protection and re-establishment, earned over one cycle, i.e. from the time that one stand is established until the time that the following stand is established, whether that occurs after a fire or a clear-cut harvest. Clearly this net revenue will depend on whether the stand is destroyed by fire ( $X < T$ ) or by a harvest ( $X = T$ ). Explicitly we have that the net revenue has a present value (i.e. a value discounted to a value at the start of the cycle) equal to

$$(9) \quad \begin{cases} - \int_0^x p(t) e^{-\delta t} dt - C_2 e^{-\delta x}, & \text{if } X < T \\ \left[ V(T) - C_1 \right] e^{-\delta T} - \int_0^T p(t) e^{-\delta t} dt, & \text{if } X = T, \end{cases}$$

where  $\delta$  is the instantaneous rate of discounting (related to the per-annum discount rate  $i$  by the formula  $e^\delta = 1 + i$ ).

The expected present value of revenues net of costs,  $\pi$ , over one cycle is obtained by integrating (9) with respect to the distribution function given in (8), i.e.

$$(10) \quad \begin{aligned} \pi = & \int_0^T - \left[ \int_0^x p(t) e^{-\delta t} dt + C_2 e^{-\delta x} \right] f_p(x) dx \\ & + \left[ \left[ V(T) - C_1 \right] e^{-\delta T} - \int_0^T p(t) e^{-\delta t} dt \right] S_p(T), \end{aligned}$$

since for  $x < T$ ,  $dF(x) = f_p(x) dx$ , and for  $x = T$ ,  $dF(x) = S_p(T)$ .



After changing the order of integration in the first (double) integral, and observing that  $-S_p(x)$  is an anti-derivative of  $f_p(x)$ ; and after performing integration by parts on the second (single) integral and carrying out some simplification, (10) can be reduced to

$$(11) \quad \begin{aligned} \pi = & \left[ V(T) - C_1 \right] e^{-\delta T} S_p(T) - C_2 \left[ 1 - e^{-\delta T} S_p(T) \right] \\ & - \int_0^T \left[ p(t) - \delta C_2 \right] e^{-\delta t} S_p(t) dt. \end{aligned}$$

The expression  $\pi$  gives the expected present value of net revenue earned in a single cycle. Consider now the expected present value,  $J$ , of net revenue earned over an infinite number of cycles (discounted to a value at the start of the first cycle). Using conditional expectation, it can be expressed (see Reed, 1984) as

$$(12) \quad J = \sum_{n=1}^{\infty} E \left[ e^{-\delta [X_1 + \dots + X_{n-1}]} \right] \pi,$$

where  $X_1, X_2 \dots$  etc. represent the lengths of successive cycles, and are in fact independent random variables, all with the distribution given by the c.d.f. in (8). From the independence it follows that the expectation in (12) can be expressed as a product, giving

$$(13) \quad \begin{aligned} J &= \sum_{n=1}^{\infty} \prod_{i=1}^{n-1} E \left[ e^{-\delta X_i} \right] \cdot \pi \\ &= \pi / \left[ 1 - E \left[ e^{-\delta X} \right] \right] \end{aligned}$$

Now,

$$\begin{aligned}
 E\left[e^{-\delta X}\right] &= \int_0^{\infty} e^{-\delta x} dF(x) \\
 (14) \qquad &= \int_0^T e^{-\delta x} f_p(x) dx + e^{-\delta T} S_p(T)
 \end{aligned}$$

using (8), as before. Upon integrating by parts and carrying out some simplification

$$(15) \qquad E\left[e^{-\delta X}\right] = 1 - \int_0^T \delta e^{-\delta x} S_p(x) dx.$$

This expression and (11) can be substituted in (13) to give the expected present value of harvest revenues earned net of costs of protection and re-establishment, over an infinite number of cycles on the site (i.e. of a so-called "on-going forest"). Explicitly it is

$$(16) \qquad J = \frac{\left[V(T) - C_1\right] e^{-\delta T} S_p(T) - C_2 \left[1 - e^{-\delta T} S_p(T)\right] - \int_0^T \left[p(t) - \delta C_2\right] e^{-\delta t} S_p(t) dt}{\delta \int_0^T e^{-\delta t} S_p(t) dt}.$$

In order to manage the stand optimally we need to find a rotation (i.e. cutting) age  $T$ , and a protection schedule (function)  $p(t)$  to maximize (16). Of course the solution to this optimization problem will depend on the way in which the fire hazard is influenced by protection expenditure i.e. on the form of the function  $h_p(t)$ . Probably the simplest model for this is to assume that in the absence of protection, the fire-risk is age-independent (i.e. a constant hazard,

$h(t) = \rho$ ), but that it decreases with the rate at which money is spent on protection i.e.

$$(17) \quad h_p(t) = \varphi(p), \quad \text{for all } t \geq 0$$

where  $\varphi(p)$  is a decreasing, convex function with  $\varphi(0) = \rho$ , as illustrated in Fig. 1. In this case

$$(18) \quad S_p(x) = \exp \left\{ - \int_0^x \varphi(p(t)) dt \right\}.$$

With this model the optimization problem can be expressed as follows. Find a rotation age  $T$ , and a protection schedule  $p(t)$  to maximize (16), subject to the constraint (18) and subject to  $p(t) \geq 0$  for all  $t$ .

In Section 6 we shall consider the possibility of an age-dependent fire risk (non-constant hazard), but for now we restrict attention to the model (17).

The maximization problem posed above does not lend itself readily to solution in its present form. However we show in the next section, how it can be solved iteratively by considering a single cycle of the succession (the so-called "once-and-for-all forest") and including a payoff value at the end of the cycle corresponding the land value of the site.

### 3. Solution to the Optimization Problem

For a given rotation age  $T$ , and protection schedule  $p(t)$ , the expected present value,  $J$ , of revenues net of costs over an infinite time horizon can be expressed as,

$$(19) \quad J = \pi(T, p) + E \left[ e^{-\delta X} \right] J$$

where  $\pi(T, p)$  is given by (11), and represents the expected present value of revenues net of costs over a single cycle, and  $X$  is a random variable with distribution given by (8). Equation (19) can be derived purely formally from (13), by using the fact that the right hand side (r.h.s.) of that equation is the sum of a geometric series. However, in the case when optimal policies are employed equation (19) also has an important and well-known economic interpretation (see e.g. Samuelson, 1976). Suppose that the optimal policy involves a protection schedule  $p^*(t)$  and a rotation age  $T^*$ , and that the resulting maximum expected net present value is  $J^*$ . In this case (19) gives

$$(20) \quad J^* = \pi(T^*, p^*) + E \left[ e^{-\delta X} \right] J^*.$$

In a perfect market the selling price of a newly established site would be exactly equal to the maximum expected present value of the net revenue that could be derived from it, i.e. equal to  $J^*$ . For this reason  $J^*$  is known as the site value, or the land expectation value (see e.g. Samuelson op. cit). In equation (20) the first term on the r.h.s. represents the expected present value of revenues net of costs over the first cycle. The second term represents the

expected discounted selling price of the site at the end of the first cycle, regardless of whether that end comes about through a fire ( $X < T^*$ ) or through a clear-cut harvest ( $X = T^*$ ). Thus the optimum protection schedule  $p^*$  and the optimal cutting age  $T^*$  must be chosen so as to maximize the expected net revenues that can be earned over one cycle, along with the revenue that can be earned through selling the site in a perfect market, at the end of the cycle.

The relationship (19) can be used to obtain a solution to the optimization problem discussed in Section 2. Consider first a fixed rotation age  $T$ . The maximum value,  $J_T$ , of the expected net present value using this rotation age satisfies

$$(21) \quad J_T = \max_{p(t)} \left\{ \pi(T, p) + E \left[ e^{-\delta X} \right] J_T \right\}.$$

Equation (21) could be solved iteratively for  $J_T$  (using, for example the Secant Method (see e.g. Atkinson, 1978)) provided that the maximization problem:

$$(22) \quad \begin{aligned} &\text{maximize} \\ &\pi(T, p) + E \left[ e^{-\delta X} \right] L \\ &\text{over protection schedules } p(t) \text{ subject to (17) and (4)} \end{aligned}$$

could be solved for any constant  $L$ . If this were the case the optimal protection schedule for a given rotation age could be determined. The final step of finding the optimal rotation age would involve simply maximizing  $J_T$  over  $T$ . Any direct-search optimization routine which does not involve the use of derivatives could be used to accomplish this. A simpler but adequate method would be simply to evaluate  $J_T$  for various  $T$  and determine the maximum graphically by plotting  $J_T$  vs.  $T$ .

Solution to the optimization problem thus rests upon being able to maximize (22) over harvest schedules  $p(t)$ , and subject to constraints (17) and (4). One can think of this maximization problem as involving only a single cycle of the process - i.e. for a so-called "once-and-for-all forest" - with a terminal payoff of value  $L$  being awarded following the destruction of a stand. One can think of  $L$  as being a land-value. The objective in (21) can be determined explicitly from (11) by simply replacing  $-C_1$  and  $-C_2$  by  $L-C_1$  and  $L-C_2$  respectively.

The optimization problem we shall consider is thus:

maximize

$$(23) \quad Q = \left[ v(T) - C_1 + C_2 \right] e^{-\delta T} S_p(T) + (L - C_2) \\ - \int_0^T \left[ p(t) + \delta(L - C_2) \right] e^{-\delta t} S_p(t) dt$$

over the control  $p(t) \geq 0$ , and subject to the constraint

$$(24) \quad S_p(t) = \exp \left\{ - \int_0^t \mu(p(x)) dx \right\}$$

for a fixed  $T$  and a fixed  $L$ .

The objective (23) involves two components. The first (which comprises the first two terms of (23)) depends on the state  $S_p(T)$  at the end of the control period. The second depends on the control  $p(t)$  and the state  $S_p(t)$ , at each time point throughout the entire control period ( $0 \leq t \leq T$ ).

As it stands the above optimization problem looks somewhat unusual. However it can be converted into a more standard optimal dynamic control problem by using a simple transformation. Let

$$(25) \quad y(t) = -\log S_p(t),$$

so that (24) becomes

$$(26) \quad y(t) = \int_0^t \phi(p(x)) dx$$

which implies

$$(27) \quad \frac{dy}{dt} = \phi(p(t)).$$

Note that this is just another way of writing (17), because  $dy/dt = -S'_p(t)/S_p(t) = f_p(t)/S_p(t) = h_p(t)$  (from (6) and (7)).

Upon replacing  $S_p(t)$  by  $e^{-y(t)}$  in the objective (23) we arrive at the following optimal dynamic control problem, with state variable  $y$  and control variable  $p$ :

maximize

$$(28) \quad Q(p) = \left[ V(T) - C_1 + C_2 \right] e^{-\delta T - y(T)} + L - C_2 \\ - \int_0^T \left[ p(t) - \delta [L - C_2] \right] e^{-\delta t - y(t)} dt$$

subject to the dynamic equation

$$(29) \quad \frac{dy}{dt} = \phi(p),$$

and

$$(30) \quad p(t) \geq 0, \quad \text{for } 0 \leq t \leq T.$$

Put in this form the problem lends itself readily to solution using the Pontryagin Maximum Principle (see e.g. Clark 1976). To use the maximum principle we introduce an adjoint variable,  $\lambda(t)$  and a Hamiltonian function

$$(31) \quad H_t = - \left[ p(t) + \delta(L - C_2) \right] e^{-\delta t - y(t)} + \lambda \phi(p(t)).$$

Necessary conditions for an optimum are that the variables  $p$ ,  $y$  and  $\lambda$  satisfy the so-called adjoint equation

$$(32) \quad \frac{d\lambda}{dt} = - \frac{\partial H_t}{\partial y} = - \left[ p(t) + \delta(L - C_2) \right] e^{-\delta t - y(t)};$$

that the so-called transversality condition

$$\lambda(T) = \frac{\partial}{\partial y} \left[ \left[ v(T) - C_1 + C_2 \right] e^{-\delta T - y(T)} + L - C_2 \right],$$

i.e.

$$(33) \quad \lambda(T) = -e^{-\delta T - y(T)} \left[ v(T) - C_1 + C_2 \right]$$

holds; that the initial condition

$$(34) \quad y(0) = 0$$

holds; and finally that at every time  $t$ ,  $0 \leq t \leq T$ , the control  $p(t)$  maximizes the Hamiltonian  $H_t$ . From the convexity of  $H_t$  (in  $p$ ) it follows that this last condition will be met if  $p(t)$  is the solution to  $\partial H_t / \partial p = 0$  if that solution is  $\geq 0$ , otherwise  $p(t) = 0$ .

Now from (31)  $\partial H_t / \partial p = 0$  is equivalent to

$$(35) \quad -e^{-\delta t - y(t)} + \lambda \phi'(p) = 0$$

i.e.

$$(36) \quad \lambda = \frac{e^{-\delta t - y(t)}}{\phi'(p)}.$$



Differentiating (36) with respect to  $t$  and using the adjoint equation (32) we get that the optimal value of  $p(t)$  will be equal to the solution of

$$(37) \quad \left[ p(t) + \delta(L - C_2) \right] = \frac{\phi'(p) \left[ \delta + dy/dt \right] + \frac{d}{dt} \phi'(p)}{[\phi'(p)]^2}$$

if that solution is  $\geq 0$ , otherwise it will be zero. Substituting for  $dy/dt$  from the dynamic equation (27), equation (37) can be expressed:

$$(38) \quad \frac{dp}{dt} = \frac{[\phi'(p)]^2 [p + \delta(L - C_2)] - \phi'(p) [\delta + \phi(p)]}{\phi''(p)}$$

A boundary (terminal) condition is given by the transversality condition (33) which using (36) can be expressed as

$$(39) \quad \phi'(p(T)) = - \left[ V(T) - C_1 + C_2 \right]^{-1}.$$

Because the differential equation (38) is autonomous it follows that any solution trajectory crosses the axis  $p = 0$  at most once. It follows that the optimal protection schedule can be determined by solving (in the backward sense) the differential equation

$$(40) \quad \frac{dp}{dt} = \begin{cases} \frac{[\phi'(p)]^2 [p + \delta(L - C_2)] - \phi'(p) [\delta + \phi(p)]}{\phi''(p)}, & \text{if } p > 0 \\ 0, & \text{if } p \leq 0 \end{cases}$$

subject to the terminal condition (39). Note that if (39) has no solution, i.e. if  $\phi'(0) > \left[ v(T) - C_1 + C_2 \right]^{-1}$ , then the optimal protection schedule is  $p(t) \equiv 0$ .

To obtain the optimal value of the objective  $Q$ , one must first solve (29), using the optimal  $p$ , and then evaluate the integral in (28).

If one assumes a particular functional form for the response function  $\phi(p)$  and numerical values for the various parameters, solution of (40) and (29) and the integration in (28) are fairly easy to accomplish numerically. The Secant Method or others convenient numerical method can then be employed to determine the optimal value  $J_T$  of the resource using a fixed rotation period  $T$  (i.e. to solve (21)). By performing this procedure for various values of  $T$ , both the optimal rotation age  $T^*$  and the optimal protection schedule  $p^*(t)$  can be determined. An example is given in Section 5. Before presenting this example however we shall discuss in the next section some qualitative properties of the optimal protection schedule.

#### 4. Qualitative Properties of the Optimal Protection Schedule

In this section we investigate analytically some properties of the solution to the differential equation (40) with terminal condition (39). Note that this solution gives the optimal protection schedule for a "once-and-for-all" forest with a fixed rotation age  $T$  and a fixed land-value  $L$ . In itself it does not give the optimal protection schedule for an "ongoing forest". As mentioned in the previous section, to find that optimum optimorum, an iterative procedure must be employed.

We firstly give a "marginal" economic interpretation to the terminal (transversality) condition (39). We shall consider the rate (per unit time) at which the value of the asset is growing at the cutting age  $T$ .

Consider an infinitesimal time interval  $[T, T+\Delta]$ . The probability that the stand is destroyed during this period is, to order  $o(\Delta)$ , given by

$$(41) \quad h_p(T)\Delta = \phi(p(T))\Delta$$

The expected growth in the value of the stand through delaying the cutting age from  $T$  to  $T + \Delta$  is, using conditional expectation,

$$(42) \quad \left[ V(T+\Delta) + L - C_1 \right] \left[ 1 - \phi(p(T))\Delta \right] + (L - C_2)\phi(p(T)) - \left[ V(T) - L + C_1 \right] + o(\Delta)$$

which can be written

$$\left\{ V'(T) - \phi(p(T)) \left[ V(T) - C_1 + C_2 \right] \right\} \Delta + o(\Delta)$$

Thus the expected rate of growth in the value of the stand at age  $T$  is

$$(43) \quad V'(T) - \phi(p(T)) \left[ V(T) - C_1 + C_2 \right]$$

The derivative of this with respect to  $p(T)$  is

$$(44) \quad - \left[ V(T) - C_1 + C_2 \right] \phi'(p(T))$$

The terminal condition (39) says that this derivative should be equal to 1, or in other words that at the cutting age  $T$ , the resource should be protected at such a level that the marginal change in the expected rate of growth of its value, corresponding to a one dollar per annum increase in the rate of spending on protection, should be exactly equal to one dollar per annum.

An analogous marginal interpretation can be given to the maximum principle for times  $t < T$ , when the optimum expenditure  $p^*(t)$  is an interior point i.e. when  $p^*(t) > 0$ . In this case, however, the value of the stand at age  $t$  is not simply its immediately realizable net stumpage value, as it is at age  $T$ . Rather it is the present value of expected revenue to be earned  $(T-t)$  years later net of the expected costs incurred, with optimal protection, during the remaining life-time of the stand. In the Appendix an explicit expression for the expected growth in the value of the stand at ages  $t < T$  is given. It is then shown how the maximum principle condition  $\partial H_t / \partial p = 0$  implies that the marginal increase in the expected rate of growth of the present value of the asset (the stand) given that it is alive at age  $t$ , corresponding to a one dollar per annum increase in the rate of protection expenditure at age  $t$ , must be exactly equal to one dollar per annum.

In the case when the Hamiltonian is maximized at  $p = 0$ , the marginal expected return on an expenditure of \$1 per annum on protection is less than one dollar per annum, and the expenditure cannot in consequence be justified under optimal management.

Since there is no future for the stand beyond the cutting age  $T$ , under optimal management, the last dollar spent on protection at that age must have the immediate effect of increasing the expected revenue through stumpage by one dollar. However at younger ages the effect of increasing the flow of investment on protection is felt into the future in that it increases the probability of survival to all subsequent ages, and in consequence changes the expected future revenues and expenditures. The overall effect of a change in the flow of protection dollars must be evaluated through its effect on the expected rate of growth of the present value of the asset as the above marginal interpretation indicates.

We now look at some properties of the solution trajectory for (40). It has been assumed that  $\phi$  is a convex decreasing function (see Fig. 1). Thus  $\phi'(p) < 0$  and  $\phi''(p) > 0$ . In consequence the r.h.s. of (40) will be positive for  $p > 0$ . (We can assume  $L > C_2$  otherwise it would never be worth re-establishing a stand after a fire). It follows that the solution trajectory will be increasing whenever  $p \neq 0$ , and thus will be of either of one of the forms shown in Fig. 2(a) and (b).

We now investigate the effect of changes in parameter values on the solution trajectory. The time  $t_o$ , at which protection starts, is given by

$$(45) \quad t_o = \max \left\{ 0, T - \int_0^{p^*} \frac{dp}{H(p)} \right\}$$

where  $H(p)$  is given by the first line of (40) (when  $p > 0$ ), and  $p^* = p^*(T)$  is the solution to the terminal condition (39).

For a fixed  $T$ , as the value of  $V(T)$  increases, (or as the cost  $C_1$ , of re-establishment after a cut, decreases) the terminal value  $p^*$  increases, but  $H(p)$  stays unchanged. In consequence protection starts earlier ( $t_0$  decreases) and also rises to higher levels.

As the cost  $C_2$  (for re-establishment after a fire) increases, both  $p^*$  increases and  $H(p)$  decreases. In consequence protection again starts earlier and also rises to higher levels. Note that the effects of an increase in the cost of re-establishment after a cut ( $C_1$ ) and after a fire ( $C_2$ ) operate in opposite directions. An increase in  $C_1$  in effect reduces the value of a harvest and has the effect of reducing protection. An increase in  $C_2$  increases the penalty incurred through a fire, and results in higher levels of protection to reduce the probability of paying that penalty.

As the discount rate  $\delta$  increases  $H(p)$  increases, but  $p^*$  remains unchanged. The result is that protection starts earlier and remains higher than with a lower discount rate. Eventually at the cutting age  $T$  the protection levels converge for all discount rates.

Suppose now that the response function  $\varphi(p)$  can be expressed

$$(46) \quad \varphi(p) = \rho \Psi(p)$$

where  $\Psi(p)$  is a convex decreasing function with  $\Psi(0) = 1$ . The parameter  $\rho$  then represents the probability of fire with no protection. In this case the function  $H(p)$  in (45) can be expressed

$$(47) \quad H(p) = \frac{\Psi'(p)}{\Psi''(p)} \left\{ \rho \left[ \Psi'(p) \left[ p + \delta(L - C_2) \right] - \Psi(p) \right] - \delta \right\}$$

which is increasing in  $\rho$ . Also the terminal condition (39) can be expressed as

$$(48) \quad \varphi'(p(T)) = -\left[\rho\left[V(T)-C_1+C_2\right]\right]^{-1}$$

As the unprotected probability of fire,  $\rho$ , increases the terminal value  $p^*$  increases. However since  $H(p)$  also increases the effect on the time  $t_o$ , at which protection begins, is ambiguous, depending on the functions  $V$  and  $\varphi$  etc.

Finally we consider the effect of changing the cutting age  $T$ . Clearly the final level of protection  $p^*$  increases. However the effect on the time of onset of protection  $t_o$  is not so clear. If  $t_o > 0$ , from (45)

$$(49) \quad \frac{dt_o}{dT} = 1 - \frac{dp^*}{dT} / H(p^*),$$

and from (39)

$$(50) \quad \varphi''(p^*) \frac{dp^*}{dT} = \frac{d}{dT} \left[ -\left[V(T)-C_1+C_2\right]^{-1} \right]$$

Using (40), (50) and (49) it follows that  $dt_o/dT$  will be positive if and only if

$$(51) \quad \frac{d}{dT} \left[ -\left[V(T)-C_1+C_2\right]^{-1} \right] < \varphi'(p^*) \left\{ \varphi'(p^*) \left[ p^* + \delta(L-C_2) \right] - \left[ \delta + \varphi(p^*) \right] \right\}$$

Whether this condition holds or not will depend on the functions  $V$ ,  $\varphi$  etc.

While the effect of changing  $T$  on the onset of protection may be ambiguous, the effect on the duration of protection can be determined. Let

$$R = T - t_o = \min \left[ T, \int_0^{p^*} dp/H(p) \right] \quad \text{denote the duration of protection. Using (49),}$$

(50) and (40) one can show

$$(52) \quad \frac{d}{dT} \int_0^{p^*} \frac{dp}{H(p)} = \left[ V(T) - C_1 + C_2 \right]^{-2} V'(T) / \left[ \phi''(p^*) H(p^*) \right].$$

Since both the numerator and denominator are positive, it follows that the derivative is itself positive, and thus since  $R$  is the minimum of two increasing functions that the duration of protection,  $R$ , increases with the cutting age  $T$ .

It should be re-emphasized that the above results are for a "once-and-for-all" forest only and not for an "ongoing" forest. For an ongoing forest, a change in any one of the parameters, of the value-age function  $V$ , or of the response function  $\phi$ , or of the other parameters  $\delta$ ,  $C_1$ ,  $C_2$ , etc., will result in a change in the optimal rotation age  $T$  and in the land expectation value  $L$ . To examine the effect of a change in one of these parameters on the optimal protection schedule for an ongoing forest, one must not only examine the effects of the parameter change in (39) and (40), as we have done above, but also take into account the effects of the resulting changes in  $L$  and  $T$ . It does not appear possible to perform this analytically. However a sensitivity analysis for an ongoing forest can be carried out numerically, simply by solving the problem with different parameter values.



### 5. A Numerical Example

To illustrate the procedures discussed in Section 3, the results of a numerical example are presented in this section. The value-at-age function  $V$  used is as given in Table 1, and represents the estimated per hectare values of pure spruce (*Picea glauca* Moench Voss) stands on sites of site index 130+m (reference age 100) of medium accessibility to mills in the Fort Nelson Timber Supply Area of northeastern British Columbia (see Anonymous 1982). Costs (per hectare) of re-establishment after a cut and after a fire were assumed to be  $C_1 = \$10$  and  $C_2 = \$20$ , and the discount rate was assumed to be  $\delta = .03$ . It was assumed that, in the absence of protection the (age independent) hazard is  $\rho = .01$  (i.e. on average one fire per 100 years on a given site). This is fairly close to the historical per annum probability of fire (before protection measures were employed) which has been estimated at 0.013 (Murphy, 1982).

It was assumed in the example that, with protection, the hazard is reduced to

$$(53) \quad h_p(t) = \rho(p) = \rho e^{-\beta p}$$

where  $p$  is the rate (\$ per annum per hectare) at which money is spend on protection, and  $\beta$  is a constant reflecting the effectiveness of protection expenditure. Two different cases were considered:  $\beta = 1.0$  and  $\beta = 10.0$  which we shall call respectively "expensive protection" and "cheap protection". With "expensive" protection it would cost \$0.69 per ha. per yr. to reduce the hazard by 50 percent, whilst with "cheap" protection the corresponding cost would be only \$0.07 per ha. per yr.

Solution to the differential equation (40) and evaluation of the objective (28) was performed using NAG (Anonymous, 1984) Fortran subroutines. The Secant Method (e.g. Atkinson, 1978) was then used to determine the optimal value  $J_T$  of the resource (see (21)) using a fixed rotation age  $T$ . In almost all cases, with a judicious choice of starting values, convergence occurred after a single iteration. Since the secant method requires two starting values, this meant that the maximum value of the objective (28) had to be determined for three different values of  $L$ . In a few cases a second iteration of the secant method was required, necessitating a fourth evaluation of the maximum value of (28).

Figure 3 shows the optimal value of  $J_T$  for various values of the rotation age  $T$ . The lower curve (d) is with no protection while the next two curves are for "expensive" protection ( $\beta = 1.0$ , curve (c)) and "cheap" protection ( $\beta = 10.0$ , curve (b)). The top curve (a) is for the case of no fire risk ( $\rho = 0$ ). As is well-known (Reed and Errico, 1985) the loss in land expectation value (i.e. in the present value of harvests under optimal management) due to a fire hazard of  $\rho = .01$ , is considerable when there is no protection in place. From Fig. 3 (curves (d) and (a)) it can be seen that the loss is of the order of 63 percent (from \$32.3 per ha. to \$12.0 per ha.). Also the presence of the risk of fire has the effect of shortening the optimal rotation age (from about 81 years to 76 years). When protection is possible some of this loss in land expectation value can be avoided. For example with "cheap protection" in place the land expectation value is increased by about 95 percent over the no-protection situation (from \$12.0 per hectare to \$23.5 per hectare) while with "expensive protection" the corresponding increase is only about 6 percent (from \$12.0 to \$12.7 per hectare). The optimal rotation age increases with protection. For "cheap" protection it is 80.8 years (almost the same as the no-fire level),

while for "expensive" protection it is 78.3 years, closer to that for no protection.

When protection is "expensive", it does not commence optimally, until time  $t_0 = 46.6$  years. From that point the rate of expenditure increases almost linearly until at the optimal rotation age of 78.3 years it is at the rate of \$1.19 per ha. per yr. (see Fig. 4(a)). The corresponding decrease in the hazard is from .01 for ages less than 46.6 years to .003 at the rotation age (see Fig. 5(a)).

When protection is "cheap", it is in place optimally throughout the life of the stand. It starts with expenditure at the rate of \$0.07 per ha. per yr. and increases to \$0.36 per ha. per yr. at the optimal rotation age of 80.8 years (Fig. 4(b)). The hazard, in consequence, decreases from an initial value of .005 to a final value of .0003 (see Fig. 5(b)).

It should be noted that in the above example it has been assumed that in the absence of protection the hazard is constant i.e. that there is an age-independent probability of fire. A consequence of this is that the optimal expenditure on fire protection increases (or at least does not decrease) with the age of the stand. This may not necessarily be the case for an age-dependent hazard, as we shall see in the next section.

## 6. Age-Dependent Hazards

Consider now the case in which, in the absence of protection, the hazard function  $h(x)$  is non-constant. Two such typical hazard functions are illustrated in Fig. 6. A model for the influence of protection expenditure on the hazard is required. Perhaps the simplest is a "separable" model in which the hazard at age  $x$ , when protection expenditure is at a level  $p$  is

$$(54) \quad h_p(x) = \Psi(p)h(x)$$

where  $\Psi(p)$  is convex and decreasing with  $\Psi(0) = 1$ . We shall assume that this model holds in what follows. The modifications for a more general model of the form  $h_p(x) = h(p, x)$  are obvious and will not be mentioned explicitly.

The objective in the single cycle optimization problem (once-and-for-all forest) is the same as in the age-independent hazard case, and is given by (28). The constraint (30) is as before but the dynamic equation (29) must be modified to

$$(55) \quad \frac{dy}{dt} = \Psi(p)h(t).$$

The Hamiltonian for this maximization problem is

$$(56) \quad H_t = - \left[ p(t) + \delta(L - C_2) \right] e^{-\delta t - y} + \lambda \Psi(p)h(t).$$

The adjoint equation is, as before,

$$(57) \quad \frac{d\lambda}{dt} = -\frac{\partial H_t}{\partial y} = -\left[p(t) + \delta(L - C_2)\right] e^{-\delta t - y(t)}.$$

The transversality condition is

$$(58) \quad \lambda(T) = -e^{-\delta T - y(T)} \left[v(T) - C_1 + C_2\right],$$

and the initial condition is

$$(59) \quad y(0) = 0.$$

The maximum principle says that at each time  $t$ , the optimal control  $p(t)$  maximizes the Hamiltonian (56). As before we have that the optimal  $p(t)$  satisfies  $\partial H_t / \partial p = 0$  if that solution is  $\geq 0$ , otherwise it is zero.

Now from (57)  $\partial H_t / \partial p = 0$  is equivalent to

$$(60) \quad \lambda = \frac{e^{-\delta t - y(t)}}{\varphi'(p)h(t)}.$$

Differentiating this with respect to  $t$  and using the adjoint equation (57), we see that the solution to  $\partial H_t / \partial p = 0$  satisfies

$$(61) \quad p(t) + \delta(L - C_2) = \frac{\varphi'(p)h(t)(\delta + dy/dt) + \frac{d}{dt}[\varphi'(p)h(t)]}{[\varphi'(p)h(t)]^2}.$$

Carrying out the differentiation, and using (55), this gives

$$(62) \quad \begin{aligned} \varphi''(p) \frac{dp}{dt} = & [\varphi'(p)]^2 \left[ p(t) + \delta(L - C_2) \right] h(t) \\ & - \varphi'(p) [\delta + \varphi(p)h(t)] - \frac{h'(t)}{h(t)} \varphi'(p) \end{aligned}$$

which is the analogue of (38). Using (60) the transversality condition (58) can be expressed

$$(63) \quad h(T)\Psi'(p(T)) = -\left[V(T)-C_1+C_2\right]^{-1},$$

provided that the solution  $p(T)$  is positive.

If the solution to the differential equation (62) on  $0 \leq t \leq T$  subject to the terminal condition (63) is always positive it will provide the optimal protection schedule. Unlike in the time-independent hazard case, the equation (62) is not autonomous, and thus the solution may cross the axis  $p = 0$  more than once. In such a case the optimal protection schedule may involve intervals when protection expenditure is zero. The times of switching on and off the control cannot be determined exclusively from (62) and must be determined by other means. We shall not discuss this problem further here, but consider only the case when the solution to (62) is positive on  $0 \leq t \leq T$ .

While the interpretation of the maximum principle given in Section 4 still holds in the age-dependent hazard case, the other qualitative properties of the optimal solution given in that section do not in general remain true. For example for a hazard function which is decreasing the last term on the r.h.s. of (62) will be negative. In consequence it cannot be asserted that the optimal solution trajectory for  $p(t)$  is increasing.

The fact that (62) is non-autonomous presents no new problems for its numerical solution, and the same numerical procedures as outlined in Section 5 can be used to solve the optimization problem, both for a once-and-for-all forest and for an ongoing forest.

As an example we have considered a decreasing hazard of the form

$$(64) \quad h(t) = \alpha \exp(-t^2/\gamma),$$

with two different sets of parameter values. In case (A)  $\alpha = 0.0167$  and  $\gamma = 5000$ , and in case (B)  $\alpha = 0.0267$  and  $\gamma = 1800$ . In both cases the average hazard over a 100 year period is 0.01. In case (A) the hazard decreases fairly gently from a value of .0167 at  $t = 0$  to .0023 at  $t = 100$ . In case (B) the decline is steeper, from an initial value of 0.0267 to 0.0001 at  $t = 100$  (see Fig. 6). A decreasing hazard with a shoulder such as the ones depicted here is probably fairly realistic for the forest fire situation. Before the crowns of trees close at age 30 to 40 years the fire hazard is relatively high because of the fact that litter on the forest floor can become very dry. At later ages with the crowns closed the litter is able to maintain a higher moisture content and is in consequence less flammable.

The effect of protection expenditure is given by the function  $\Psi(p)$  which in the example was assumed to be of the form

$$(65) \quad \Psi(p) = e^{-\beta p}$$

with  $\beta = 10.0$  corresponding to "cheap protection" in Section 5.

In order to make comparisons with the constant (age-independent) hazard case, the optimal protection schedules for the age-dependent hazard functions, using a rotation age of  $T = 80.8$  years (the optimal rotation period when  $h(t) \equiv 0.01$ ) were determined. They are illustrated in Fig. 7 along with the optimal protection schedule for the age-independent hazard. It can be seen that the effect of moving from a constant hazard to a decreasing hazard is to put

increased emphasis on protection at younger ages at the expense of protection at older ages. For a gently decreasing hazard (case (A)) the optimal expenditure on protection still increases with age, but for a more steeply decreasing hazard (case (B)), the optimal protection expenditure at first increases, but later decreases. Young trees, while they are less valuable than older ones, are at the same time more vulnerable to destruction. It is the interaction between the decreasing hazard function and the increasing value function which determines the precise form of the optimal protection schedule.



## 7. Conclusions

The paper has addressed the problem of jointly determining the optimal pattern of fire protection expenditure and the optimal rotation age for a forest subject to the risk of destruction through fire. In determining the optimal protection schedule it has been assumed that there is a known relationship between the rate at which money is spent on protection, and the resulting fire hazard.

It has been shown in the paper how the joint optimization problem for an "ongoing" forest can be reduced to the somewhat simpler problem of determining the optimal protection schedule for a single cycle of the process, using a fixed rotation age and with a fixed value for the land at the end of the cycle i.e. reduced to the problem of finding the optimal protection schedule for a "once-and-for-all" forest. This problem can be tackled using the Pontryagin Maximum Principle, and it is shown how the optimal solution can be obtained by solving a straightforward first order differential equation. In the case when the fire risk is age-independent this differential equation is autonomous, and qualitative properties of its solution can be determined quite easily. In the case of an age-dependent fire probability the resulting differential equation is non-autonomous. In both cases numerical solution of the differential equation, to determine the optimum protection schedule, is easily accomplished.

To determine the optimal rotation age a numerical iterative procedure must be employed.

The optimal management policies depend on the complex interactions between the growth curve, the hazard function and the response function (which determines how the hazard responds to protection expenditure) along with other economic

parameters. While it is unfortunate that a complete analytic solution is not possible, the methods discussed in the paper at least indicate how optimal policies can be determined in specific instances. The methods developed should help in providing a guide to priorities in forest-fire protection, and be of use in the development benefit-cost analyses of protection measures.

### References

1. Anonymous, [1982]. Fort Nelson TSA yield analysis technical report. B.C. Ministry of Forests. 1982.
2. Anonymous, [1984]. NAG Fortran Library Manual Mark 11. Numeriical Algorithms Group, Oxford.
3. K.E. Atkinson, [1978]. An introduction to numerical analysis. J. Wiley and Sons, New York.
4. C.W. Clark, [1976]. Mathematical Bioeconomics. J. Wiley and Sons, New York.
5. M. Faustmann, [1849]. Berechnung des werthes, welchen waldboden sowie nacht  
nicht haubare holzbestande fur die weldwirtschaft besitzen, Allgemeine  
Forst-und Jagn- 25, 441-455.
6. J.G. Kalbfliesch, [1979]. Probability and statistical inference. Springer-Verlag. New York.
7. D. Martell, [1980]. The optimal rotation of a flammable forest stand. Can. J. For. Res. 10, 30-34.
8. P.J. Murphy, [1982]. Describing the occurrence of fire in the forest - a review of methods. Unpub. rep.
9. W.J. Reed, [1984]. The effects of the risk of fire on the optimal rotation of a forest. J. Environ. Econ. Manag. 11, 180-190.
10. W.J. Reed and D. Errico, [1985]. Assessing the long-run yield of a forest stand subject to the risk of fire. Can. J. For. Res. 15, 680-687.
11. W.J. Reed and D. Errico, [1986]. Optimal harvest scheduling at the forest level in the presence of the risk of fire. Can. J. For. Res.
12. P.A. Samuelson, [1976]. Economics of forestry in an evolving society. Econ. Inquiry 14, 466-492.

Table 1

Value-age relationship for interior spruce (*Picea* sp.) (site index,  $\geq 30\text{m}$ ; reference age 100 years) in the fort Nelson Timber supply area.

Age (years)	Value (\$/ha)
30	0
40	0
50	32
60	102
70	214
80	334
90	434
100	504
110	550
120	578
130	596
140	606
150	612

## Appendix

### Economic Interpretation of the Maximum Principle

In Section 3 it was shown how the optimal protection expenditure flow at time  $t$  is given by the value  $p^*(t)$  which maximizes the Hamiltonian,  $H_t$  defined in (31). This is known as the maximum principle. Because of the convexity of  $H_t$  in  $p$ , it follows that  $p^*(t)$  is given by the solution to  $\partial H_t / \partial p = 0$  if that solution is positive, otherwise it is given by  $p^*(t) = 0$ . In this appendix we give an economic interpretation to the condition  $\partial H_t / \partial p = 0$ .

The adjoint equation (32) is

$$(A1) \quad \frac{d\lambda}{dt} = \left[ p(t) + \delta(L - C_2) \right] e^{-\delta t - y(t)}$$

Integrating this and using the transversality condition (33) gives

$$(A2) \quad \lambda(t) = -e^{-\delta T - y(T)} \left[ V(T) - C_1 + C_2 \right] + \int_t^T \left[ p(x) + \delta(L - C_2) \right] e^{-\delta x - y(x)} dx$$

for  $0 \leq t < T$ . Now the maximum principle  $\partial H_t / \partial p = 0$  implies (equation (36)) that

$$(A3) \quad \lambda(t) = \frac{e^{-\delta t - y(t)}}{\phi'(p)}$$

From (A2) and (A3)

$$(A4) \quad -\phi'(p) \frac{e^{-\delta T - y(T)}}{e^{-\delta t - y(t)}} \left[ V(T) - C_1 + C_2 \right] + \frac{\phi'(p)}{e^{-\delta t - y(t)}} \int_t^T \left[ p(x) + \delta(L - C_2) \right] e^{-\delta x - y(x)} dx = 1$$

Now from integration by parts

$$(A5) \quad \int_t^T \delta e^{-\delta x - y(x)} dx = - \int_t^T e^{-\delta x} f_p(x) dx - \left[ e^{-\delta T} S_p(T) - e^{-\delta t} S_p(t) \right]$$

since  $-f_p(x)$  is the derivative of  $S_p(x) = e^{-y(x)}$ . In consequence (A4) can be re-expressed

$$(A6) \quad -\phi'(p) \left\{ e^{-\delta(T-t)} \cdot \frac{S_p(T)}{S_p(t)} \left[ V(T) - C_1 + L \right] - \int_t^T p(x) e^{-\delta(x-t)} \frac{S_p(x)}{S_p(t)} dx + \int_t^T (L - C_2) e^{-\delta(x-t)} \frac{f_p(x)}{S_p(t)} dx - (L - C_2) \right\} = 1$$

Now for  $x > t$ ,  $S_p(x)/S_p(t)$  is the conditional survivor function given that the stand survives until age  $t$  i.e.

$$\frac{S_p(x)}{S_p(t)} = P(\text{stand survives to age } x \mid \text{survives to age } t),$$

and  $f_p(x)/S_p(t)$  is the conditional p.d.f. of the time of destruction  $X$ , given that the stand survives until age  $t$  (i.e. given  $X > t$ ). Thus equation (A6) can be written in the form

$$(A7) \quad -\phi'(p) \left\{ \pi^*(t) - (L - C_2) \right\} = 1$$

where  $\pi^*(t)$  is the expected present (time  $t$ ) value of the stand given that it is alive at age  $t$ . Explicitly

$$(A8) \quad \pi^*(t) = \int_t^T \left[ - \int_t^x p(z) e^{-\delta(z-t)} dz + (L-C_2) e^{-\delta(x-t)} \right] \frac{f_p(x)}{S_p(x)} dx \\ + \left[ \left[ V(T)-C_1+L \right] e^{-\delta(T-t)} - \int_t^T p(z) e^{-\delta(z-t)} dz \right] \frac{S_p(T)}{S_p(t)}$$

(see equation (10)). This can be reduced to the form (see (11))

$$(A9) \quad \pi^*(t) = \left[ V(T)-C_1+L \right] e^{-\delta(T-t)} \frac{S_p(T)}{S_p(t)} + \int_t^T (L-C_2) e^{-\delta(x-t)} \frac{f_p(x)}{S_p(t)} dx \\ - \int_t^T p(x) e^{-\delta(x-t)} \frac{S_p(x)}{S_p(t)} dx$$

which occurs in (A6).

Consider now a small time interval  $[t, t+\Delta]$ . The expected increase in the value of the stand over this interval is

$$(A10) \quad \pi^*(t+\Delta) \cdot P(\text{survives over } (t, t+\Delta)) + (L-C_2) \cdot P(\text{destroyed in } (t, t+\Delta)) - \pi^*(t)$$

since a revenue  $L - C_2$  is realized whenever the stand is destroyed. Since the probability of destruction is  $h_p(t)\Delta + o(\Delta)$ , the expected increase in value can be written

$$(A11) \quad [\pi^*(t) + \Delta d\pi^*/dt](1 - h_p(t)\Delta) + (L - C_2)h_p(t)\Delta - \pi^*(t) + o(\Delta)$$

i.e.

$$(A12) \quad \frac{d\pi^*}{dt} \Delta - [\pi^*(t) - (L - C_2)]h_p(t)\Delta + o(\Delta)$$

The expected rate of growth in the value of the stand at time  $t$  is thus

$$(A13) \quad \frac{d\pi^*}{dt} - [\pi^*(t) - (L - C_2)]h_p(t)$$

Since  $h_p(t) = \phi(p)$ , we have that the derivative of (A13) with respect to  $p$  is

$$(A14) \quad - [\pi^*(t) - (L - C_2)]\phi'(p)$$

We have seen (A7) that the maximum principle requires that this derivative be equal to 1. In other words the maximum principle requires that at every time  $t$ , the marginal increase in the expected rate of growth of the present value of the stand, given that it is alive at time  $t$ , corresponding to a one dollar per annum increase in the rate of protection expenditure at time  $t$ , must be equal to one dollar per annum. In other words the expected marginal return (in terms of the increase in the rate of growth in present value) corresponding to an increase in protection expenditure, must be equal to the marginal cost.

The marginal interpretation of the transversality condition (39) given in Section 3 is a special case of the above since  $\pi^*(T) = [V(T) - C_1 + L]$ .



Figure Captions

- Figure 1. A typical "response function" showing how the instantaneous probability of fire or hazard (vertical axis) is assumed to depend on the rate of expenditure on protection (horizontal axis).
- Figure 2. Two possible forms for the optimal protection schedule for a once-and-for-all forest when, in the absence of protection, the probability of fire is age-independent.
- Figure 3. Land expectation value as a function of rotation age. Curve (a) corresponds to no fire risk, while curve (d) corresponds to the case of no fire protection, but with an age-independent probability of fire (hazard) of 0.01 per annum. Curves (b) and (c) correspond respectively to the cases of "cheap" protection ( $\beta = 10.0$ ) and "expensive" protection ( $\beta = 1.0$ ) when there is an age-independent per annum probability of fire of 0.01 present. Units of rotation age are years, while units of land expectation value are dollars per hectare.
- Figure 4. Optimal protection schedules corresponding to "expensive" protection (curve (a)) and "cheap" protection (curve (b)), when there is an age-independent probability of fire of 0.01 per annum present. Units of protection expenditure are dollars per hectare per year, while units of age are years.

Figure 5. The hazard functions that result from optimal protection. Curve (a) corresponds to "expensive" protection and curve (b) corresponds to "cheap" protection. In the absence of protection the hazard would be constant (an age-independent probability) at 0.01. Units of age are years, while units of hazard are probability per annum.

Figure 6. Two age-dependent hazard functions. Curves (a) and (b) correspond to hazard functions (A) and (B) discussed in text. In both cases the probability of destruction decreases with age. Also shown is the constant hazard at level 0.01. Over the time span 0 to 100 years both hazard functions (A) and (B) have an average value of 0.01. Units of age are years, while units of hazard are probability per annum.

Figure 7. Optimal protection schedules for "cheap" protection using a rotation age of 80.8 years and corresponding to the decreasing hazard functions (A) and (B) in text (curves (a) and (b)), and to a constant hazard of 0.01 (curve (c)). Note how for the sharply declining hazard function (B), the optimal protection expenditure at first increases, but later decreases. Units of age are years, while units of protection expenditure are dollars per hectare per year.

Fig 1

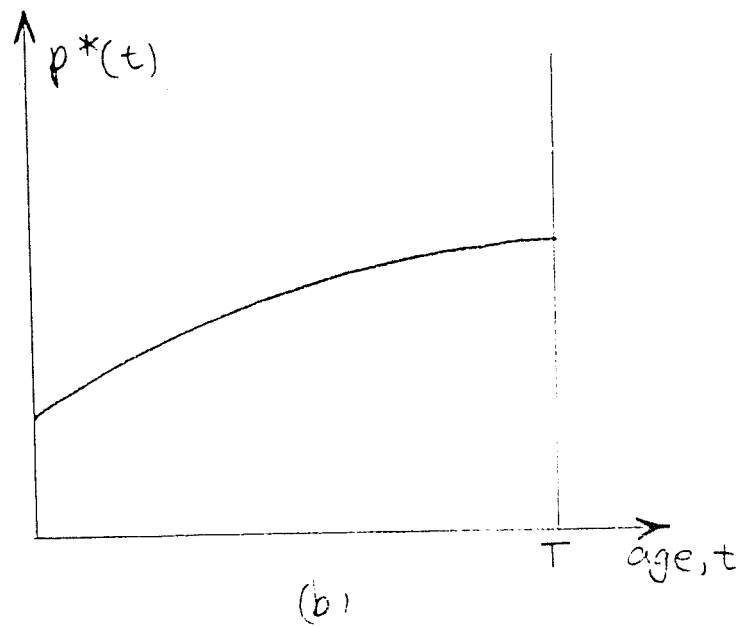
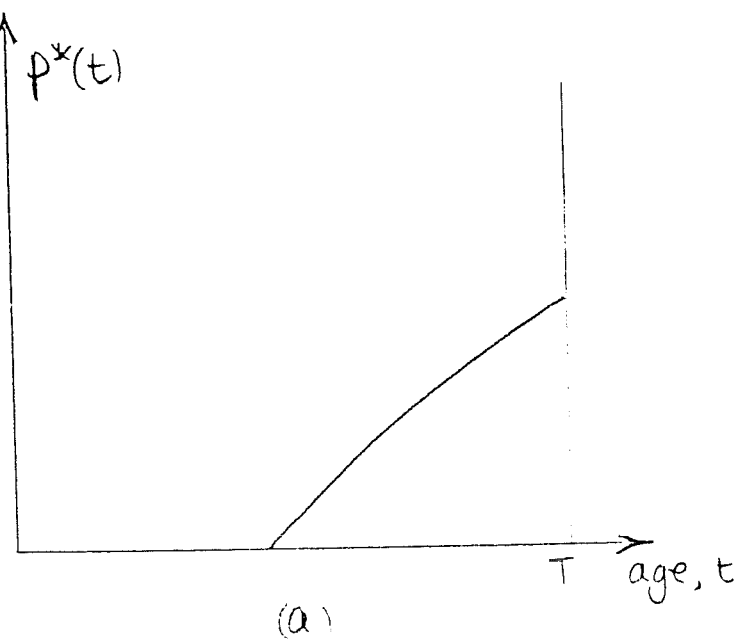
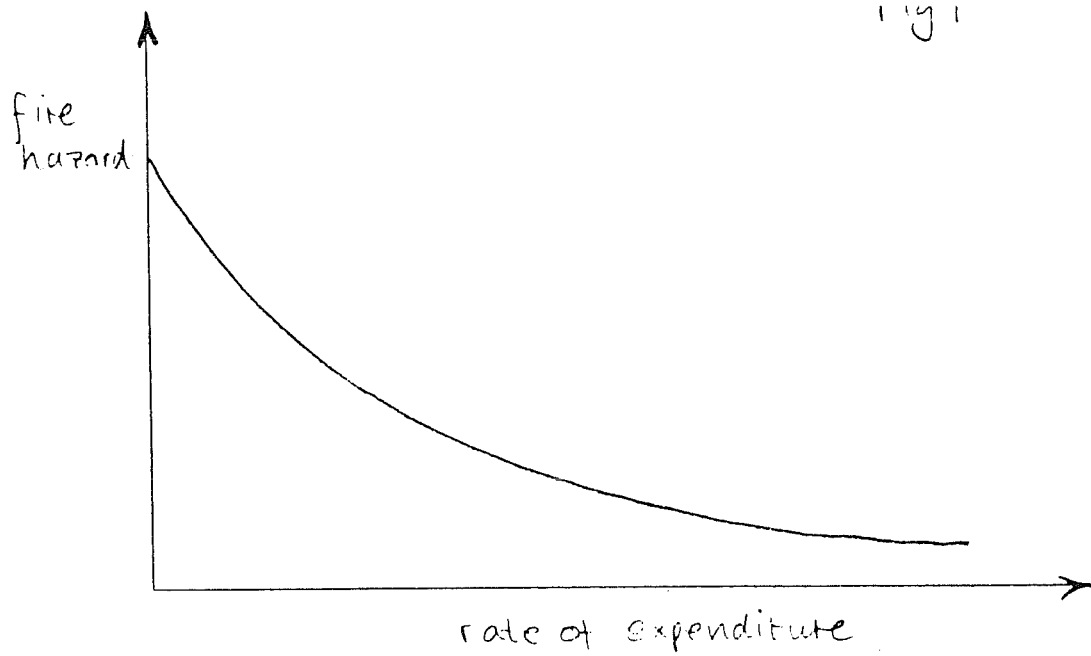
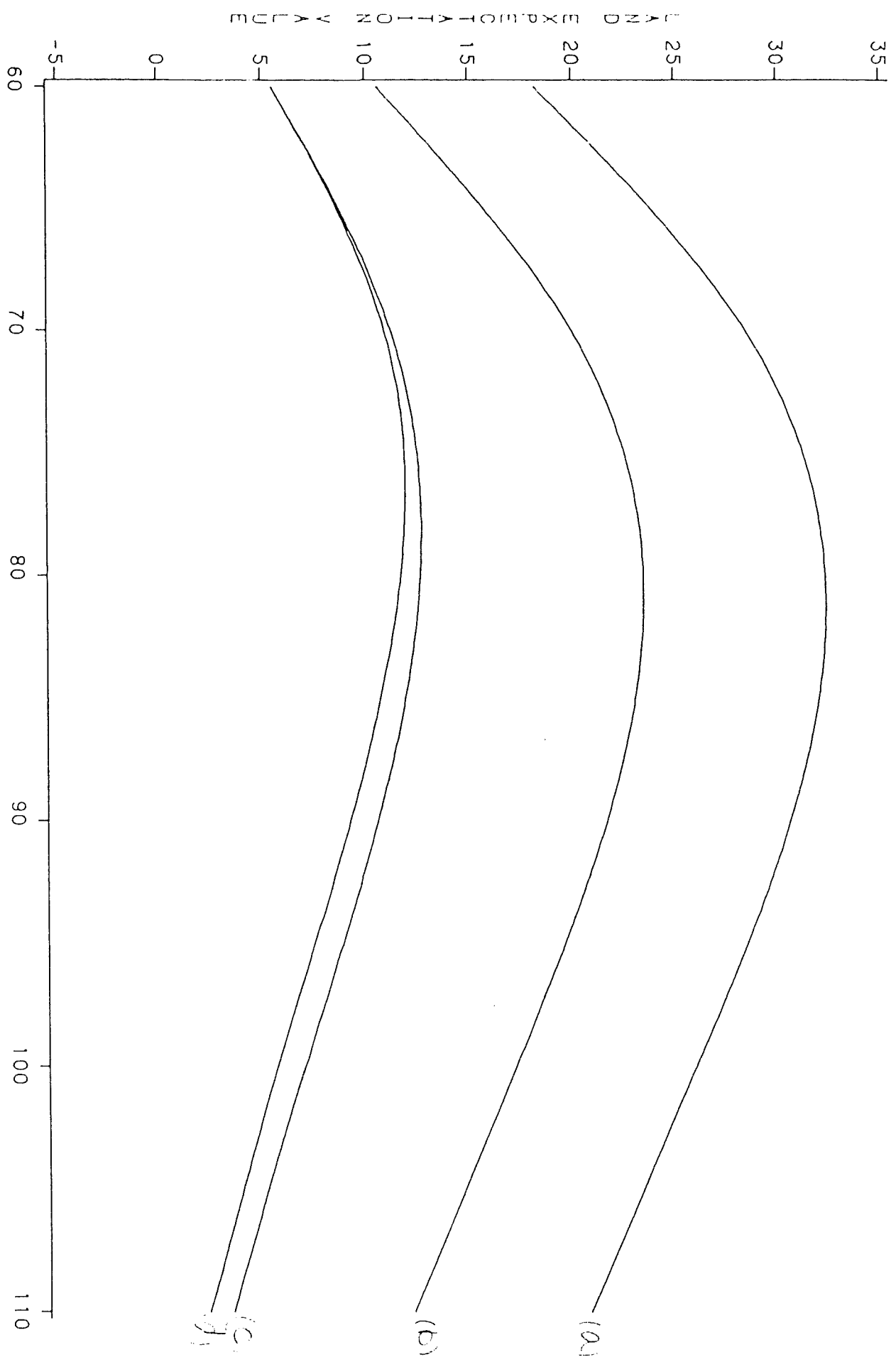
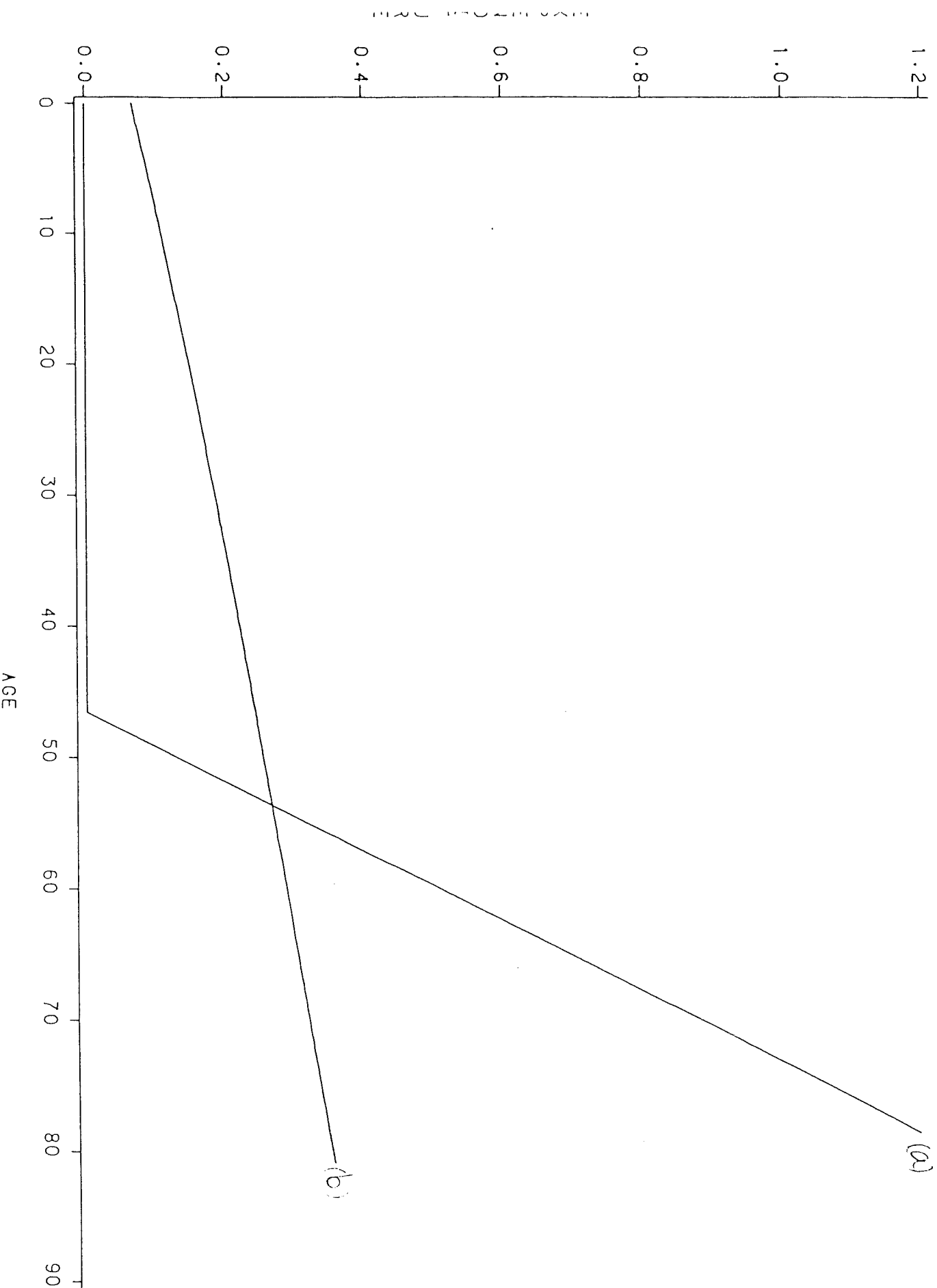


Fig 2

Fig 3





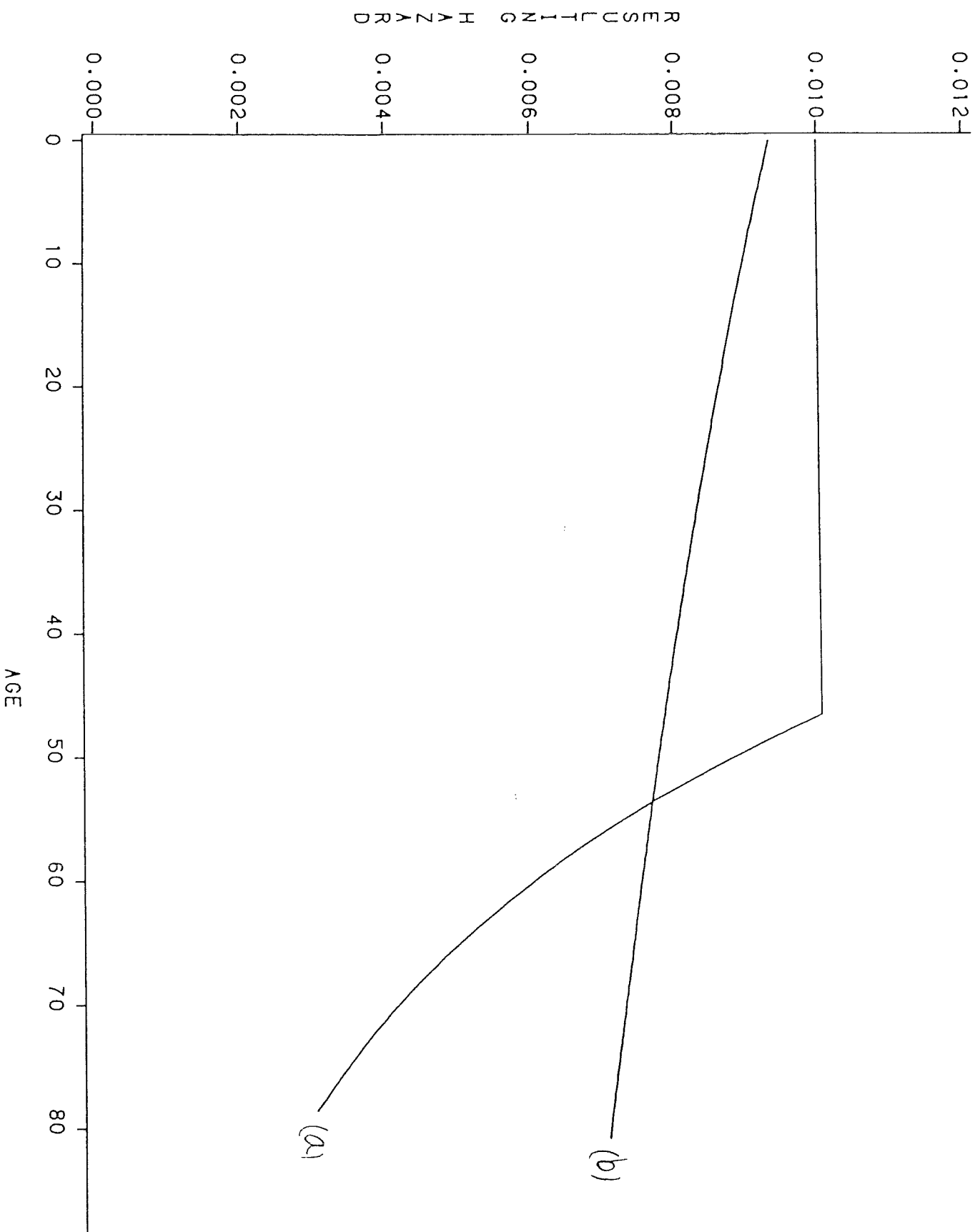


Fig 5

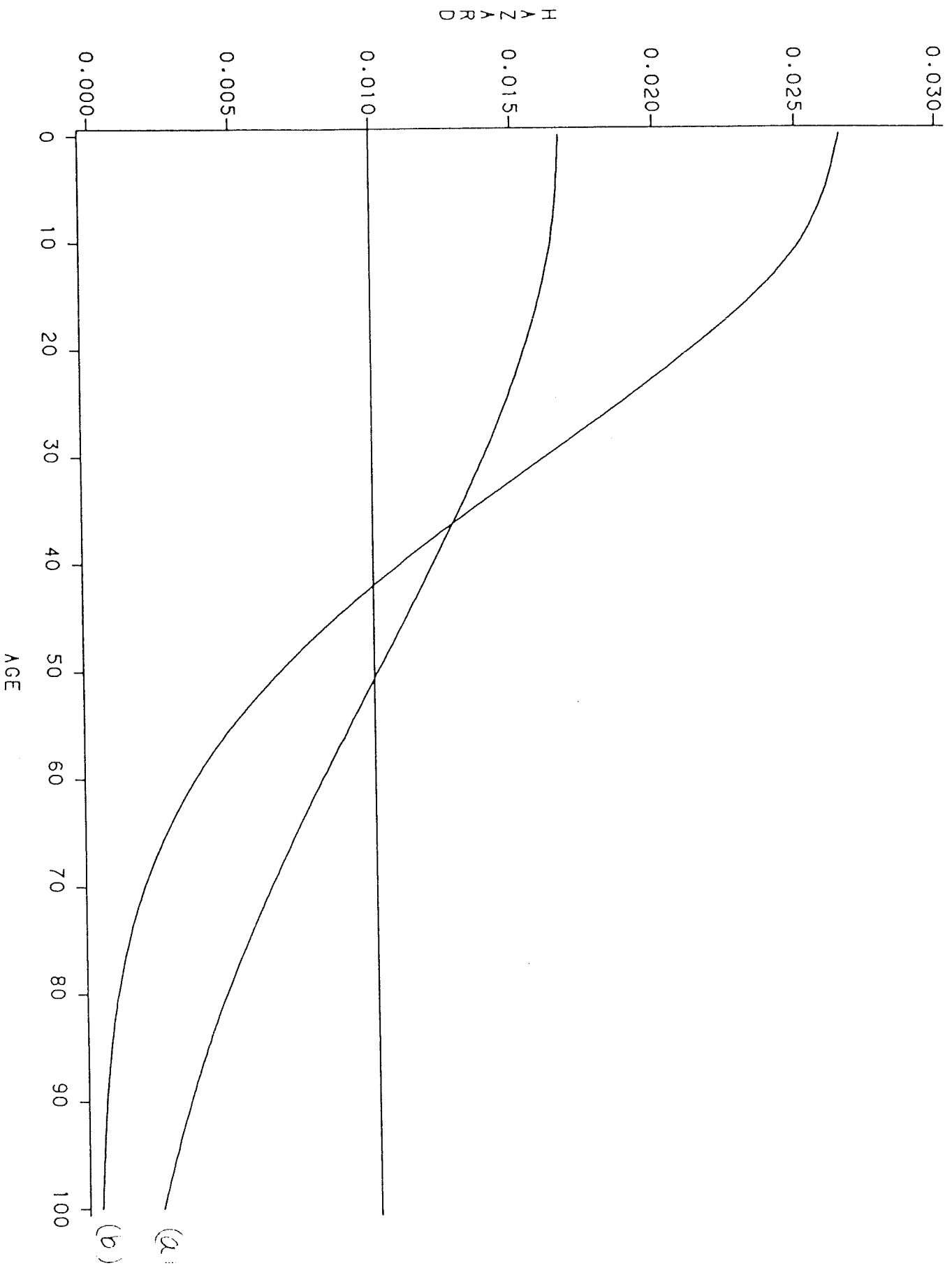


Fig 7

