

***THE ROLE OF STOCHASTIC MONOTONICITY IN  
THE DECISION TO CONSERVE OR HARVEST  
OLD-GROWTH FOREST***

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## Abstract

The problem of when, if ever, a stand of old-growth forest should be harvested is formulated as an optimal stopping problem, and a decision rule to maximize the expected present value of amenity services plus timber benefits is found analytically. This solution can be thought of as providing the “correct” way in which cost-benefit analysis should be carried out. Future values of amenity services provided by the standing forest and of timber are considered to be uncertain, and are modelled by *Geometric Poisson Jump* (GPJ) processes. This specification avoids the ambiguity which arises with *Geometric Brownian Motion* (GBM) models, as to which form of stochastic integral (Itô or Stratonovich) would be employed, but more importantly allows for monotonic (yet stochastic) processes. It is shown that monotonicity (or lack of it) in the value of amenity services relative to timber values plays an important part in the solution. If amenity values never go down (or never go up) relative to timber values then the certain-equivalence cost-benefit procedure provides the optimal solution, and there is no *option value*. It is only to the extent that the relative valuations can change direction that the certainty-equivalence procedure becomes sub-optimal and option value arises.

## 1. Introduction

Natural (or virgin or “old-growth”) forests are essentially exhaustible resources, at least when viewed from the ecological point of view. Their destruction can mean the disappearance of ecosystems which have taken many thousands of years to evolve. The major cause of extinction of species, which is now taking place at a rate unprecedented in the history of the planet, is habitat destruction especially the destruction of natural forests. Many would claim that from the point of view of mankind as a whole the continued destruction of natural forests is a folly on a scale perhaps not seen before in human history. The reasons behind the rapid “consumption” of this increasingly scarce exhaustible resource are manifold. On the one hand in many poor tropical countries there is an ever growing demand for agricultural land, driven by a rapidly growing population. Rapid population growth in poorer countries also drives the demand for fuelwood. It has been estimated

(Dasgupta [1982]) that 50% of the trees cut down annually are for fuel use. The demand for timber is clearly another reason for the destruction of forests, in both tropical and temperate regions. In spite of being a renewable resource, timber is one of the few natural resources whose price has shown a real increase over the last one hundred years or so. Old growth forests, containing the accumulated biological growth of centuries, are regarded by the timber industry as a source of cheap, high quality wood. Thus there are many specific and direct benefits to individuals and companies associated with the destruction of natural forests.

On the other hand many of the costs of harvesting are not incurred directly by those doing the logging. The ecological costs involved with flooding, erosion and the silting of rivers and estuaries, the destruction of salmon spawning habitat etc., are usually borne by others, sometimes at a great distance from the logging activities. For example Dasgupta (*op. cit*) attributes the severity of monsoon floods in Northern India in 1978 to deforestation that had taken place in the upper reaches of the Ganges, hundreds of miles away. If global climate change occurs through an accumulation of carbon dioxide and other gases in the upper atmosphere, the consequences will be borne unevenly throughout the world. Certainly from the point of view of those involved in deforestation, such costs are external, and as such are largely ignored.

As with costs, so with the benefits of conservation. The services provided by standing old-growth forests are essentially public goods. The benefits of preserving biodiversity, of regulating water flow, of storing carbon not to mention the opportunities for outdoor recreation and touristic activities are all public goods, not readily accessible to valuation. Thus with respect to the utilization of old-growth forests, since so many of the benefits and costs are diffuse and external the “invisible hand” of Adam Smith cannot be expected to guide the many actions of individuals towards the common good of all of mankind. In fact because of the common property aspect of the natural environment the contrary situation described by Hermann Daly [1980] seems to prevail – viz that rather than there being an invisible hand guiding human activities towards the best of all worlds there is operative “an invisible foot which kicks the commons to pieces.”

While the economic mechanisms behind the destruction of the world’s forests are

now well understood the question of how society should make decisions concerning the conservation or destruction of natural forests is still a vexing one. The standard tool of public policy decisions of this type is cost-benefit analysis and much progress has been made over the last twenty years or so in empirically evaluating the benefits provided by standing forest, and in theoretically incorporating amenity and existence values into the cost-benefit procedure. However as Kneese & Schulze [1985] point out in their discussion of the philosophical and ethical issues of environmental decision making, cost-benefit analysis has become increasingly strained when applied to large-scale environmental issues. Dealing in aggregate benefits and costs the methodology ignores distributional and equity issues – in particular it ignores the question of who gets to reap the benefits and who has to pay the costs. For example cost-benefit methodology cannot effectively address the issue of how to trade off the immediate needs of a poor villager for firewood with those of mankind for the preservation of biodiversity and carbon storage; or how to trade off the needs of loggers and logging communities for continued employment against those of recreationalists and others who enjoy the services provided by old-growth forests; or how to strike a balance between the benefits enjoyed by people now living with the rights and needs of generations as yet unborn. These and other issues stretch beyond the limits the applicability of cost-benefit methodology. However in spite of these serious shortcomings, cost-benefit methodology still has a role to play, if only as a starting point for a discussion of the political, moral, social and environmental aspects of decisions of this kind. Indeed Kneese & Schultze (*op. cit*) suggest that environmental issues are perhaps best dealt with using a utilitarian criterion (cost-benefit methodology) constrained by libertarian and egalitarian considerations.

A technical difficulty with cost-benefit methodology in relation to irreversible decisions such as the destruction of natural forest and wilderness arises because of *uncertainty* in future valuations of wilderness services. Arrow & Fisher [1974] and Henry [1974] showed using simple two-period models (henceforth referred to as the *AFH formulation*) that if future uncertain quantities (random variables) are replaced by their expected values, the corresponding deterministic decision rule which maximizes net benefits does not necessarily maximize the expected net benefit in the stochastic setting, and may prescribe taking the irreversible action prematurely. They introduced the idea of an *option value* (*quasi*

*option value* in the Arrow-Fisher paper) associated with not taking the irreversible action, and showed that option value increased with increased uncertainty. Simply put there is some value associated with keeping open the option of taking the irreversible action at some later date when conditions are more propitious. In mathematical terms the option value arises because in stochastic optimal control problems the *certainty-equivalence* control (obtained by replacing random variables by their expected values and applying the resulting deterministic optimal control in a feedback manner) is not necessarily optimal in the stochastic setting. The idea of option value, while easily understood in a two-period setting is more difficult to apply in more realistic problems involving possibly an infinite time horizon. Clarke & Reed [1990] looked at irreversible land development problems over an infinite time horizon and attempted to reconcile ideas of option value etc. with those of stochastic optimal control.

More recently Reed [1993] considered the problem of when, if ever, a stand of old-growth forest should be harvested. The problem was treated as an optimal stopping problem, with the objective of maximizing the expected present value of amenity services derived from the forest up until the time of its destruction (whether by fire or other natural catastrophe or by logging) plus that of timber revenues if a harvest takes place (in other words finding the “correct” stochastic optimal policy, in the utilitarian sense). Future uncertainty in the valuation of amenity services and in the price of timber was modelled by assuming that both followed observable, possibly correlated, continuous-time Markov processes, specifically *Geometric Brownian Motions* (GBMs). In the paper it was shown analytically that the optimal harvest rule was of a particularly simple form, *viz* to harvest when (if ever) the current timber value exceeded the expected present value of future amenity services foregone by the harvest by a factor bigger than one. Thus the optimal harvest policy was shown to be *more conservative* than that prescribed by a deterministic cost-benefit analysis. Furthermore it was shown that the optimal policy becomes more conservative as the future uncertainty in either amenity service values or timber values becomes larger. These results concurred with earlier results obtained for the AFH formulation using the simple two-period model.

In Reed’s paper the assumption was made that future values of timber and amenity ser-

vices followed the stochastic differential equations of *Geometric Brownian Motion* (GBM). While it is of course necessary to assume some model in order to carry out analysis, the assumption of a stochastic differential equation (SDE) model has some important implications. One is that a process modelled by an SDE is implicitly excluded from being monotonic. Now a very good case could be made that the valuation of amenity services provided by old-growth forest is likely to increase monotonically in the future (*i.e.* the valuation will never go down in time), the only uncertainty being by how much that valuation will increase, and how that increase will be distributed over time. After all as already discussed old-growth forest is an exhaustible resource, and it is already approaching levels of absolute scarcity as the current conflicts over its use testify. From the modelling point of view the critical question is whether or not this implicit assumption has important consequences. In other words how much do the results of the analysis depend on this assumption? This is one of the questions which we address in this article.

Another important issue which must be addressed when using SDE models is what form of the stochastic integral should be used in integrating such equations. Reed, following what has now become almost standard practice in the economics and finance literature, used the Itô calculus. This certainly makes the analysis cleaner, but from a modelling point of view the question must be asked, of how much the results depend on the use of the Itô integral? Would they change for instance if the Stratonovich integral had been used in its place? This is an issue which has been addressed at some length in other areas where stochastic mathematical models are applied (see for instance Turelli [1977] concerning biological and ecological applications, and Mortensen [1969] concerning physical and engineering applications) but has, by and large, been ignored in economics (although see Sethi & Lehoczky [1981], for a discussion of applications in finance and Clarke & Reed [1988, 1989], Reed & Clarke [1990] for a discussion relating to resource and urban economics). This question is particularly important for studying the effects of uncertainty, because when the variance parameter is changed in a non-linear SDE the effect on the mean rate can depend on which form of the stochastic integral is used. Thus conclusions regarding the effects of uncertainty on the conservatism of an optimal policy, or on the size of an option value should be scrutinized with some care.

In this article we eschew use of SDE models entirely. Instead we assume that changes in valuations of amenity services and in timber prices occur in discrete jumps at random times. Specifically it is assumed that the flow of amenity services and the price of timber follow *Geometric Poisson Jump* (GPJ)<sup>1</sup> processes. In doing this we finesse the problem of choosing a stochastic integral, while at the same time allowing for the possibility of monotone stochastic processes for future valuations. Apart from the use of the GPJ specification in place of the GBM specification the model assumptions are identical to those of Reed [1993]. While different methods are required to solve the resulting optimal stopping problem, it turns out that the optimal harvest rule has a form analagous to the one obtained earlier for the GBM specification.

The main new results of this paper concern the reason for the conservatism of the optimal policy and the existence of option value. It is shown that while it is uncertainty that is the ultimate cause, it is a special kind of uncertainty. Specifically, the underlying cause of option value is *the possibility of a reversal in the direction of the relative valuations of wilderness amenity benefits and timber benefits*. If the ratio of wilderness valuation to timber valuation increases monotonically, even though it might be stochastic, then the certainty-equivalence (cost-benefit) harvest rule is optimal, and there is no option value. A similar situation prevails in the much less likely case of a monotonically decreasing process for this ratio.

In Section 2 the model is developed and the optimal harvesting decision problem formulated as an optimal stopping problem. The solution (which is derived in detail in Appendix 1) is given along with a formula for the expected present value of timber and amenity benefits under optimal management as a function of current values of timber benefits and amenity flows. In Section 3 it is shown how the optimal harvest rule depends on the degree to which the relative valuations of amenity and timber benefits can change direction. Also some discussion of the appropriate definition of option value and how it depends on the monotonicity or lack thereof of the ratio of amenity values to timber values is given.



## 2. Formulation of a Model for the Harvesting Decision Problem

As in Reed [1993] we consider an area of old-growth virgin forest which can either be preserved *in toto* or clear-cut harvested. The harvest action is irreversible. Once the virgin forest is cut it cannot be restored, at least not within a human time frame.

Suppose that if the forest is harvested at time  $t$  a net revenue of size  $V(t)$  is realized. This includes revenues from the sale of timber net of harvest costs plus the value of the bare land on which future rotations of trees may be grown, or which may possibly be put to some other productive use. Rather than assuming that  $\{V(t)\}$  is a Geometric Brownian Motion (GBM) process (as in Reed [1993]), we shall assume that it is an observable *Geometric Poisson Jump (GPJ) process*, i.e. that  $\{p(t)\}$ , where

$$(1) \quad p(t) = \ln V(t)$$

is a point process, which can take jumps up or down of size  $\epsilon$ , i.e.

$$(2) \quad p(t+dt) = \begin{cases} p(t) + \epsilon & \text{with probability } \lambda_1^u dt + o(dt) \\ p(t) & \text{with probability } 1 - (\lambda_1^u + \lambda_1^d) dt + o(dt) \\ p(t) - \epsilon & \text{with probability } \lambda_1^d dt + o(dt) \end{cases}$$

The parameters  $\lambda_1^u$  and  $\lambda_1^d$  are the intensities for jumps up and down respectively. The expected time between upward jumps is  $(\lambda_1^u)^{-1}$  and between downward jumps is  $(\lambda_1^d)^{-1}$ , while the expected time between jumps of any kind is  $(\lambda_1^u + \lambda_1^d)^{-1}$ .

For this specification it is straightforward to show that

$$(3) \quad E(V(t) | V(0) = V_0) = V_0 \exp[\mu_1 t]$$

where

$$(4) \quad \mu_1 = \lambda_1^u \theta + \lambda_1^d \theta^{-1} - (\lambda_1^u + \lambda_1^d) = \lambda_1^u (\theta - 1) + \lambda_1^d (\theta^{-1} - 1)$$

with  $\theta = e^\epsilon$ . Thus  $\mu_1$  is the *mean growth rate* of GPJ process for timber price.

The variance of  $V(t)$  is

$$(5) \quad \text{var}(V(t) | V(0) = V_0) = V_0^2 e^{2\mu_1 t} [e^{v_1 t} - 1]$$

where

$$(6) \quad v_1 = \lambda_1^u \theta(\theta - 2) + \lambda_1^d \theta^{-1}(\theta^{-1} - 2) + (\lambda_1^u + \lambda_1^d).$$

We shall refer to  $v_1$  as the instantaneous variance of the GPJ process  $\{V(t)\}$  since

$$(7) \quad \text{var}(V(dt) | V(0) = V_0) = V_0^2 v_1 dt + o(dt).$$

Before proceeding we note that the parameters  $\mu_1$  and  $v_1$  in the GPJ process play similar roles to the mean drift and variance parameters  $b$  and  $\sigma^2$  in the geometric Brownian motion process governed by the Itô S.D.E.

$$(8) \quad dx = bx dt + \sigma x dw,$$

where  $\{w(t)\}$  is a standard Wiener process, since for this GBM

$$(9) \quad E(x(t) | x(0) = x_0) = x_0 e^{bt}; \quad \text{var}(x(t) | x(0) = x_0) = x_0^2 e^{2bt}(e^{\sigma^2 t} - 1).$$

In assuming a ‘geometric’ process for timber value we are assuming that price can jump up or down by a fixed *proportional* rate at any time. Such an assumption is common in the economics and finance literature, where GBM specifications are widely used. The difference between a GBM process and a GPJ process is that in the former changes are taking place continuously (at all times) while in the latter changes occur only at distinct points in time.

For simplicity we have assumed here that jumps are always of the same proportional size. A more general and perhaps more realistic model specification would allow the jumps  $\epsilon$  in (2) to be random variables. However this is not necessary for demonstrating the phenomenon we wish to discuss; for this the simpler model is adequate.

We turn now to the amenity service benefits provided by the standing forest. Suppose that at time  $t$  the *flow* of amenity service benefits is at the rate  $A(t)$  value units<sup>2</sup> per unit

time. We shall assume that the current value of the flow is known exactly (*i.e.* that  $A(t)$  is observable) but that future values of this flow are uncertain, and can be modelled by a GPJ stochastic process. Explicitly if

$$(10) \quad q(t) = \ln A(t)$$

we assume

$$(11) \quad q(t+dt) = \begin{cases} q(t) + \epsilon & \text{with probability } \lambda_2^u dt + o(dt) \\ q(t) & \text{with probability } 1 - (\lambda_2^u + \lambda_2^d) dt + o(dt) \\ q(t) - \epsilon & \text{with probability } \lambda_2^d dt + o(dt) \end{cases}$$

The parameters  $\lambda_2^u$  and  $\lambda_2^d$  are intensities or jump rates as before. The *mean growth rate* for the flow of amenity services  $A(t)$  is

$$(12) \quad \mu_2 = \lambda_2^u \theta + \lambda_2^d \theta^{-1} - (\lambda_2^u + \lambda_2^d) = \lambda_2^u (\theta - 1) + \lambda_2^d (\theta^{-1} - 1)$$

with *instantaneous variance*

$$(13) \quad v_2 = \lambda_2^u \theta (\theta - 2) + \lambda_2^d \theta^{-1} (\theta^{-1} - 2) + (\lambda_2^u + \lambda_2^d).$$

The present value of the flow of amenity services earned from time zero to time  $\tau$  is the random variable

$$(14) \quad \int_0^\tau e^{-\delta t} A(t) dt$$

where  $\delta$  is the *instantaneous discount rate*. In this calculation of the present value of the ongoing flow of amenity benefits it is assumed that, given no harvest, the forest will survive until time  $\tau$ . It does not include the possibility of catastrophic destruction through fire or other loss agent. This can be included in the model via the use of an indicator random variable,

$$(15) \quad I(t) = \begin{cases} 1 & \text{if no fire has occurred by time } t \\ 0 & \text{if a destructive fire has occurred in } [0, t]. \end{cases}$$

The present value can now be written

$$(16) \quad \int_0^\tau e^{-\delta t} I(t) A(t) dt.$$

The harvest rule which will maximize the expected present value of amenity benefits and timber benefits can be found by solving the *optimal stopping problem* of maximizing (over stopping times  $\tau$ ) the objective

$$(17) \quad E \left\{ \int_0^\tau e^{-\delta t} I(t) A(t) dt + e^{-\delta \tau} I(\tau) V(\tau) \right\}$$

(a *stopping time* can depend on current and past values of the state variables  $V$ ,  $A$  and  $I$ , but not on future values).

In order to proceed further we need to make some assumptions about the probability of catastrophic destruction. This can be done via the use of a *hazard-rate function* (e.g. Thompson [1988])

$$(18) \quad h(t) = \lim_{\Delta \rightarrow 0} \{P[\text{stand catastrophically destroyed in } (t, t + \Delta) | \text{'alive' at } t] / \Delta\}$$

Thus if the stand has not suffered catastrophic destruction by time  $t$ , (*i.e.*  $I(t) = 1$ ) the probability that it will be destroyed in the interval  $(t, t + dt]$  (*i.e.* that  $I(t + dt) = 0$ ) is  $h(t)dt + o(dt)$ , while the probability that it will survive until time  $t + dt$  (*i.e.* that  $I(t + dt) = 1$ ) is  $1 - h(t)dt + o(dt)$ .

In general the hazard rate function could be of any form, but for simplicity, and in the absence of any compelling reason to the contrary, we shall assume that  $h(t)$  is a constant for all  $t$ ; *i.e.*

$$(19) \quad h(t) \equiv h \geq 0.$$

A solution to the optimal stopping problem above is given in Appendix 1. It should be noted that the problem has no solution in some cases which are of considerable interest. They are:

- (a)  $\mu_1 \geq \delta + h$ . In this case the mean growth rate in timber values (of  $\{V(t)\}$ ) exceeds the ‘*risk-adjusted*’ discount rate  $\Delta = \delta + h$ , and it would always

pay to defer a harvest since the expected growth in discounted timber price would always be positive, *i.e.*

$$E(e^{-\delta dt} I(t+dt)V(t+dt)) = V(t)(\mu_1 - \Delta) dt + o(dt) \geq 0.$$

In the technical parlance of optimal stopping the problem is not *stable*<sup>3</sup> in this case (see *e.g.* Ross [1983, pp. 51-54]).

(b)  $\mu_2 \geq \delta + h$ . In this case amenity service values are growing at a faster rate than the risk-adjusted discount rate. Again it would never be optimal to harvest since this would foreclose the ‘earnings’ from amenity services which, even when discounted, would be growing at a positive rate. Once again the stopping problem is not stable.

Conditions (a) and (b) provide conditions under which it would never be optimal to harvest the stand. If the discount rate and the hazard rate are low either could be met, although since amenity services values are likely to grow at a faster rate than timber values ( $\mu_2 > \mu_1$ ) the condition (b) is more likely to be met than (a).

In Appendix 1, the optimal stopping problem is solved assuming that (a) and (b) do not hold *i.e.* that the risk-adjusted discount rate  $\Delta$  exceeds the mean growth rates of timber values and amenity service values. The optimal harvest policy involves harvesting only when the ratio of the current timber value,  $V(t)$  to the current flow of amenity service benefits,  $A(t)$  exceeds some critical level, *i.e.* only when

$$(20) \quad \frac{V(t)}{A(t)} > M^*.$$

In a fashion analagous to the result in Reed [1993] this can be expressed in terms of the ratio of the *current timber value*  $V(t)$  to the *expected present value of amenity services foregone through harvesting*

$$(21) \quad \begin{aligned} \bar{A}_f(t) &= E \left\{ \int_0^\infty e^{-\delta s} I(s+t)A(s+t)ds \mid A(t), I(t) = 1 \right\} \\ &= \frac{A(t)}{\Delta - \mu_2}. \end{aligned}$$

One can think of the ratio  $V(t)/\bar{A}_f(t)$  as a benefit-cost ratio, where the benefit of harvesting is in terms of immediate revenues, while the costs are in terms of amenity services foregone through harvesting. The optimal policy involves harvesting only when

$$(22) \quad \frac{V(t)}{\bar{A}_f(t)} > r^* = \frac{\theta - \beta}{\theta(1 - \beta)} \geq 1$$

where  $\beta$  is the smaller of two roots of a quadratic equation (see (A.19), Appendix 1), always lying in the interval  $[0,1)$ .

Thus optimally harvesting will take place only when the immediate benefits of harvesting exceed the costs by a factor at least as large as one. This result is analagous to the one obtained using a GBM specification (Reed [1993]). In the next section the behaviour of the critical ratio  $r^*$  is discussed and the implications examined.

Also in Appendix 1 an explicit expression for the expected present value (EPV) of amenity services plus timber benefits, when the optimal policy is employed is derived. It is shown that if the current state is  $(V(t), A(t))$  the optimal EPV,  $W$  is

$$(23) \quad W = \begin{cases} V(t), & \text{if } V(t) > \frac{r^* A(t)}{\Delta - \mu_2} \\ V(t) \left(1 - \frac{1}{r^*}\right) \beta^{-\frac{1}{\epsilon} \ln[(\Delta - \mu_2)V(t)/r^* A(t)]} \\ \quad + \frac{A(t)}{\Delta - \mu_2}, & \text{if } V(t) \leq \frac{r^* A(t)}{\Delta - \mu_2}. \end{cases}$$

In terms of the current timber value  $V(t)$  and the EPV,  $\bar{A}_f(t)$ , of amenity services which would be foregone if a harvest took place, the optimal EPV is

$$(24) \quad W = \begin{cases} V(t) & \text{if } V(t) > r^* \bar{A}_f(t) \\ V(t) \left(1 - \frac{1}{r^*}\right) \beta^{-\frac{1}{\epsilon} \ln[V(t)/r^* \bar{A}_f(t)]} \\ \quad + \bar{A}_f(t) & \text{if } V(t) \leq r^* \bar{A}_f(t). \end{cases}$$

### 3. Qualitative Behaviour of the Optimal Development Rule and of the Associated Option Value

In Appendix 1, it is shown that the optimal development rule involves harvesting only when the ratio  $V(t)/\bar{A}_f(t)$  of immediate (timber) benefits to the cost (of amenity services foregone) exceeds the critical level

$$(25) \quad r^* = \frac{\theta - \beta}{\theta(1 - \beta)}$$

where  $\beta$  is the smaller root of the characteristic equation

$$(26) \quad \bar{\lambda} \bar{\rho} z^2 - (\bar{\lambda} + \bar{\delta}) z + \bar{\lambda}(1 - \bar{\rho}) = 0.$$

The parameters  $\bar{\lambda}$ ,  $\bar{\rho}$  and  $\bar{\delta}$  are defined in (A.9)-(A.12). For convenience we repeat the definitions here.

$$(27) \quad \bar{\lambda} = \Lambda + \mu_1$$

where  $\Lambda = \lambda_1^u + \lambda_1^d + \lambda_2^u + \lambda_2^d$ , is the rate at which jumps of any kind occur in the ratio  $V(t)/A(t)$  ( $\Lambda^{-1}$  is the expected time between jumps), and  $\mu_1$  is the mean growth rate of  $V(t)$ ;

$$(28) \quad \bar{\rho} = (\lambda_2^d + \theta^{-1} \lambda_1^d) / \bar{\lambda}$$

$$(29) \quad 1 - \bar{\rho} = (\lambda_2^d + \theta \lambda_1^u) / \bar{\lambda}$$

and

$$(30) \quad \bar{\delta} = \Delta - \mu_1.$$

The parameter  $\bar{\delta}$  is a discount rate adjusted for risk and for the (expected) exponential growth in timber values  $V(t)$ . In Appendix 1, it is shown that the two assumptions  $\Delta > \mu_1$  and  $\Delta > \mu_2$  required for a solution to the optimal harvesting problem to exist are equivalent to the condition

$$(31a) \quad \bar{\delta} > 0$$

$$(31b) \quad \bar{\delta} > \bar{\mu} = \mu_2 - \mu_1$$

Thus we assume that the adjusted discount rate is positive and exceeds the mean growth rate,  $\bar{\mu}$ , of amenity service values relative to timber values, *i.e.* of  $A(t)/V(t)$ .

The parameter  $\bar{\rho}$  reflects the tendency for the ratio of amenity values to timber values to go up rather than down. Thus when  $\bar{\rho} = 1$ ,  $\lambda_2^d = \lambda_1^u = 0$  and  $A(t)/V(t)$  is monotonic, with only upward jumps possible. In contrast when  $\bar{\rho} = 0$ ,  $\lambda_2^u = \lambda_1^d = 0$  and  $A(t)/V(t)$  can only take jumps downwards. With  $\bar{\rho}$  near to one, the probability of a downward jump is small. As the intensity (mean jump rate) of either jumps up in  $A(t)$  or jumps down in  $V(t)$  increases, so  $\bar{\rho}$  would increase (except in the case  $\lambda_2^d = \lambda_1^u = 0$ , when  $\bar{\rho}$  would be always equal to one). Conversely as the intensity of jumps down in  $A(t)$  or jumps up in  $V(t)$  increases, so  $\bar{\rho}$  would decrease. We shall examine the behaviour of  $r^*$  as  $\bar{\rho}$  changes.

We consider firstly the polar case  $\bar{\rho} = 1$ , in which the ratio  $A(t)/V(t)$  can only experience upward jumps. In this case the smaller root  $\beta$  of (26) becomes zero and the critical ratio  $r^* = 1$ . Thus in this case the optimal harvest rule involves harvesting as soon as the timber value  $V(t)$  exceeds the EPV of amenity benefit foregone through harvesting  $\bar{A}_f(t)$ . But this rule is simply the *certainty-equivalence* harvest rule which would arise if the stochastic (uncertain) variables were replaced by their expected values, and the resulting deterministic optimization problem solved (see Reed [1993, App. 5]). In other words *if cost-benefit analysis were applied in the usual way using the expected values of uncertain future variables, it would in this case produce the optimal policy.*<sup>4</sup>

The fact that the certainty-equivalence cost-benefit procedure produces the optimal harvest rule when  $A(t)/V(t)$  is non-decreasing, implies that in this case *there is no option value* (Arrow & Fisher [1974], Hanneman [1989]) associated with the decision not to harvest.

In the other polar case,  $\bar{\rho} = 0$  when  $A(t)/V(t)$  is non-increasing (can only experience downward jumps) the smaller root  $\beta$  of the characteristic equation (26) assumes a value  $\bar{\lambda}/(\bar{\lambda} + \bar{\delta})$  (see Appendix 2, (A.51)) and the critical ratio  $r^*$  reduces, after some algebra to

$$(32) \quad r^* = \frac{\Delta - \mu_2}{\Delta - \mu_1}.$$



Note in this case  $\mu_2 < 0$  and  $\mu_1 > 0$ , so that  $r^* > 1$ . The optimal harvest rule (20) in this case can be expressed as harvest only when

$$(33) \quad V(t) > \frac{A(t)}{\Delta - \mu_1}.$$

Again this turns out to be the certainty-equivalence, cost-benefit harvest rule,<sup>5</sup> so once again there is no option value associated with the decision not to harvest. Thus *in the two cases where  $A(t)/V(t)$  is monotone (non-decreasing or non-increasing) the certainty-equivalence cost-benefit procedure provides the optimal harvest rule and there is no option value*, even though there is uncertainty in future timber and amenity values. We shall see however that when  $0 < \bar{\rho} < 1$ , *i.e.* when there is the possibility of the ratio  $A(t)/V(t)$  changing direction, that the optimal policy is more conservative than the certain-equivalence cost-benefit procedure, and that there is in these cases an option value associated with the decision not to harvest.

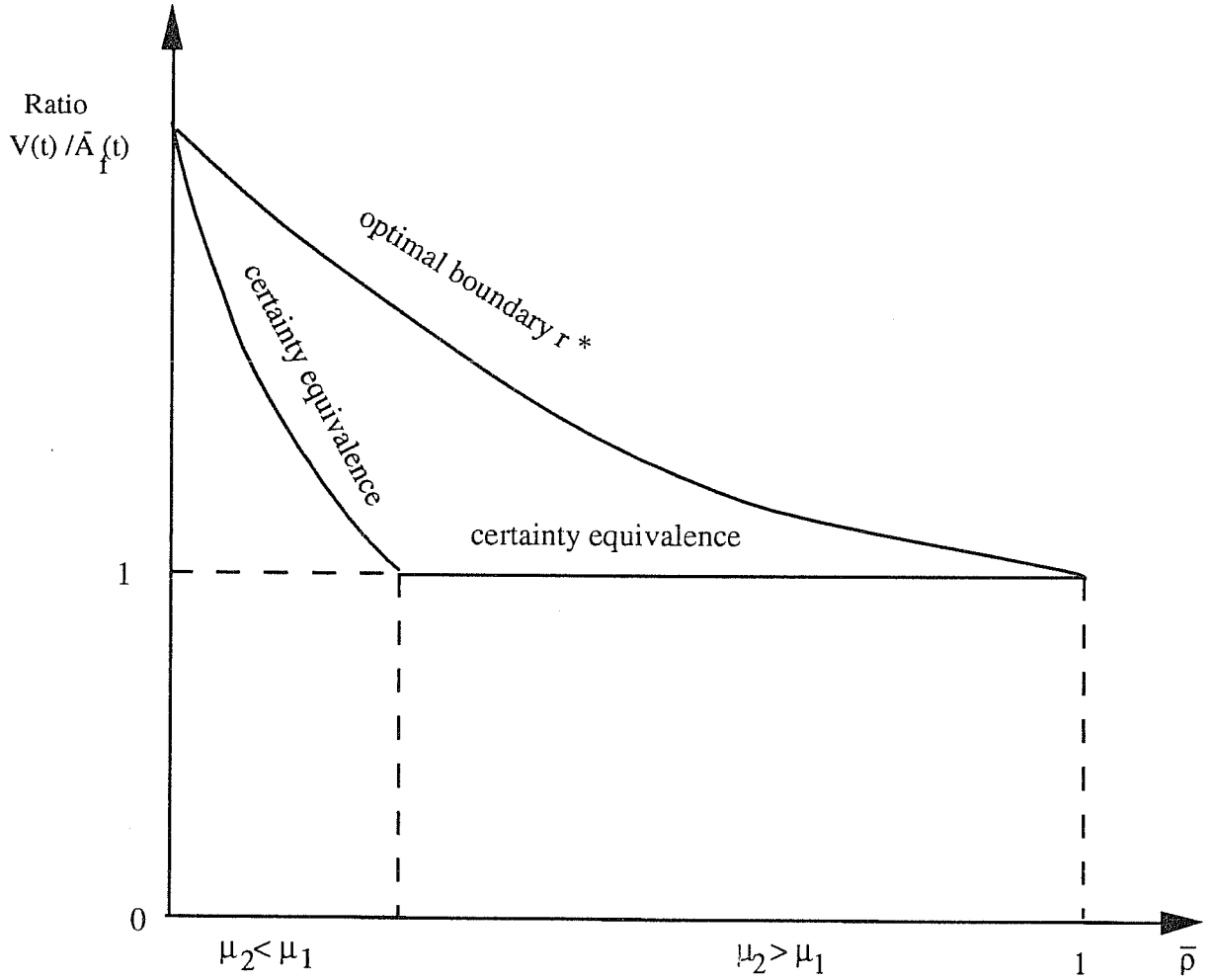
The qualitative behaviour of the critical ratio  $r^*$  as the parameter  $\bar{\rho}$  changes can be examined by finding the derivative  $\partial\beta/\partial\bar{\rho}$ , since

$$(34) \quad \frac{\partial r^*}{\partial \bar{\rho}} = \frac{\theta - 1}{\theta(1 - \beta)^2} \frac{\partial \beta}{\partial \bar{\rho}}$$

which on using the result (A.53) gives

$$\frac{\partial r^*}{\partial \bar{\rho}} = \frac{\bar{\lambda}}{\bar{\rho}} \frac{(1 + \beta)}{(\beta - \alpha)(1 - \beta)} \frac{\theta - 1}{\theta} < 0.$$

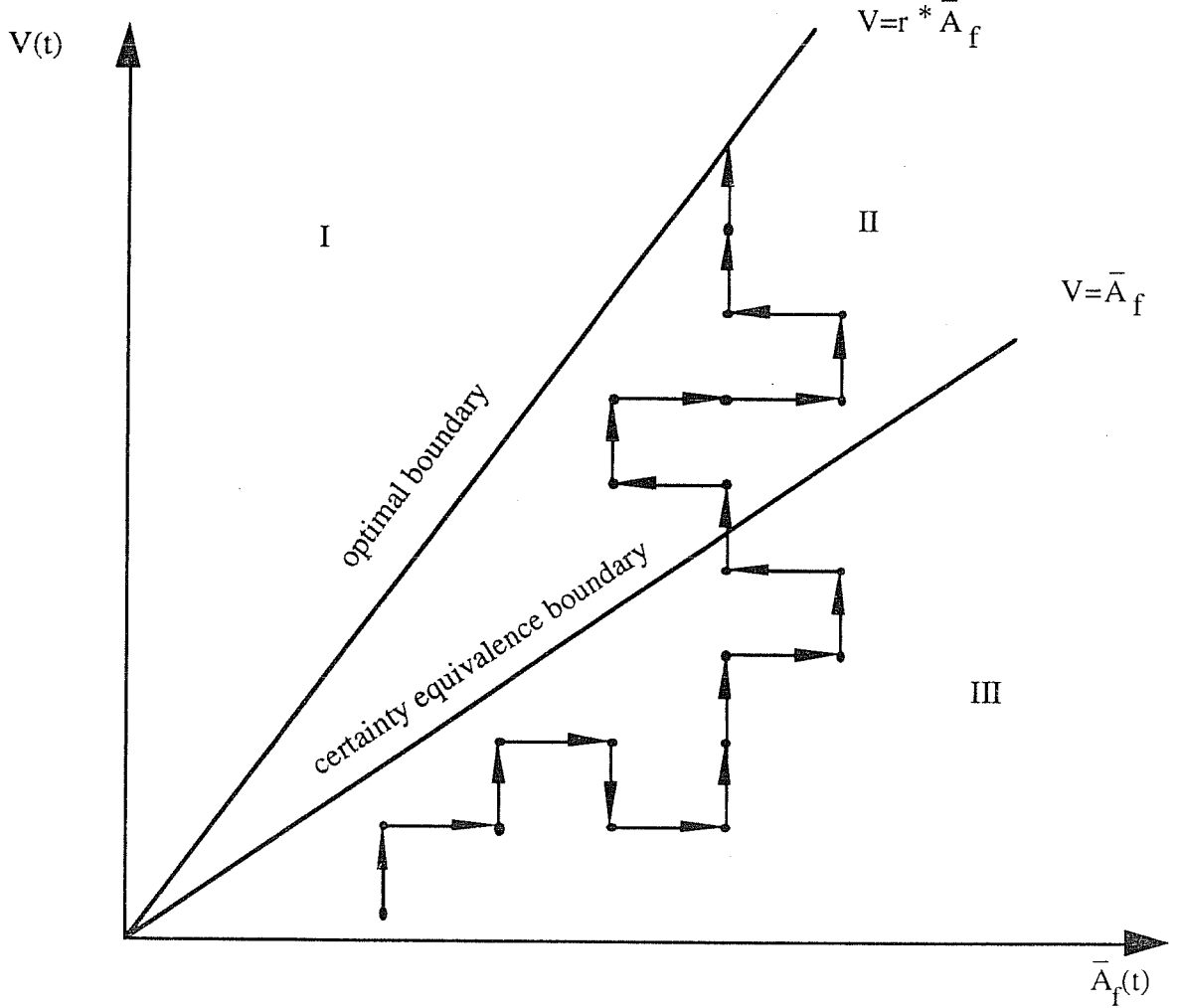
Thus the greater the probability of the ratio  $A(t)/V(t)$  dropping in value (the smaller the value  $\bar{\rho}$  assumes), so the larger the critical barrier  $r^*$  becomes.<sup>6</sup> Furthermore at least for all the time that  $\mu_2 > \mu_1$ , the greater the difference between the optimal harvest rule and the certainty equivalence cost-benefit harvest rule. Qualitatively the optimal harvest rule and the certainty equivalence cost-benefit rule change with  $\bar{\rho}$  in the way depicted in Fig. 1.



**Figure 1.** The optimal harvest boundary and the certainty-equivalence cost-benefit harvest boundary as functions of the parameter  $\bar{\rho}$ . The respective harvest rules prescribe a harvest when the ratio  $V(t)/\bar{A}_f(t)$  is above the given boundary. Note how in the two monotone cases ( $\bar{\rho} = 1$  &  $\bar{\rho} = 0$ ) the optimal boundary coincides with the certainty-equivalence cost-benefit boundary. In other cases the optimal harvest rule is more conservative.

It is tempting to conclude that the option value increases as  $\bar{\rho}$  decreases. While this has obvious intuitive appeal, some care has to be taken in attempting to make a precise statement along these lines, because a satisfactory definition of option value is not obvious. Adopting the spirit of the original definition of Arrow and Fisher [1974]

one could define the (quasi) option value as the difference between the EPV of timber and amenity benefits using the optimal rule, and the corresponding EPV using the certainty-equivalence cost-benefit rule. Under this definition the option value would depend on the initial state  $(V_0, A_0)$ , or equivalently on  $V_0$  and  $\bar{A}_f(0)$ . This is shown graphically in Fig. 2.



**Fig. 2.** The optimal and certainty-equivalence cost-benefit harvest (stopping) regions in the case  $\mu_2 > \mu_1$  and  $0 < \bar{\rho} < 1$ . The optimal harvest region is the sector I, while the certainty equivalence harvest region is the union of sectors I and II. A possible sample path is illustrated.

For an initial state in sector I, both the optimal and certainty-equivalence

procedures prescribe an immediate harvest, so there is no option value. For an initial state in sector II the certainty-equivalence procedure prescribes an immediate harvest while the optimal procedure does not. In this case the option value can be computed exactly (using (24)) as

$$(35) \quad OV = V_0 \left( 1 - \frac{1}{r^*} \right) \beta^{-\frac{1}{\epsilon} \ln[V_0/r^* \bar{A}_f(0)]} + \bar{A}_f(0) - V_0$$

which clearly depends on  $V_0$  and  $A_0$ . To examine the dependence of the option value on  $\bar{\rho}$  one can differentiate it totally with respect to  $\bar{\rho}$ . This is done in Appendix 3 where it is shown that the option value as defined in (35) decreases with increases in  $\bar{\rho}$  *i.e.* that option value increases as the probability of  $A(t)/V(t)$  taking jumps downward gets larger.

For an initial state in sector III of Fig. 2 the option value can be expressed as

$$(36) \quad OV = \beta^{\lceil \frac{\ln r^*}{\epsilon} \rceil} \left( 1 - \frac{1}{r^*} \right) E \left\{ e^{-\delta T} \tilde{V} \mid V_0, \bar{A}_f(0) \right\}$$

where  $T$  is the (random) time when  $V(t)/\bar{A}_f(t)$  first crosses the certainty-equivalence boundary, and  $\tilde{V}$  is the (random variable) value  $V(T)$  when this occurs (since the amenity benefits in time  $[0, T]$  are the same for both the certainty-equivalence and optimal policies). Since the expectation in (36) does not depend on  $r^*$  or  $\bar{\rho}$  one can use the same arguments (Appendix 3) to show that as before option value decreases as  $\bar{\rho}$  increases. Thus we have that *the option value* (if thought of as the difference in EPV of amenity and timber benefits between the optimal procedure and the certainty-equivalence cost-benefit procedure) *depends on the initial timber and amenity service values,  $V_0$  and  $A_0$ , but is always non-negative. It is zero for initial values for which the optimal policy prescribes an immediate harvest, but for other initial values is zero only when  $A(t)/V(t)$  is monotone increasing or decreasing. In the more likely case of amenity service values growing faster in expectation than timber values ( $\mu_2 > \mu_1$ ) the option value grows in size as the probability of  $A(t)/V(t)$  taking downward jumps increases.*<sup>7</sup>

Thus the cause of option value seems to be not so much uncertainty in general (there can be uncertainty in future values of  $V(t)$  and  $A(t)$  when  $\bar{\rho} = 1$ ) but rather the possibility of reversals in direction of the ratio  $A(t)/V(t)$ .

The above definition of option value does not satisfy the property of the Arrow-Fisher concept of representing the conditional expected value of information given the conservation decision is taken (Conrad [1980], Hanneman [1989]). It seems unlikely that any definition for an infinite time horizon problem will meet this requirement. Nor does the above definition satisfy the condition of being equivalent to a tax imposed on development (harvesting) which would cause the certainty-equivalence procedure to produce the socially optimal harvest policy (Hanneman [1989]). Again for the current problem it seems that no lump-sum tax will produce this result. However one could consider an *ad valorem* tax which reduced the timber benefits from  $V(t)$  to  $V(t)/r^*$  i.e. a tax at the rate  $100(1 - \frac{1}{r^*})\%$  on timber revenues. This would cause the certainty-equivalence cost-benefit procedure to produce the optimal harvest rule. Thus one could conceive of an option value proportional to  $1 - \frac{1}{r^*}$ . This idea is very similar to the concept of an ‘adjustment’ or ‘bias’ factor put forward by Henry [1974] and Hodge [1984], which would give extra weight to the benefits of conservation. Since the function  $1 - \frac{1}{r^*}$  is increasing in  $r^*$  similar qualitative results to those described above for option value thought of as the difference in EPVs, pertain when option value is thought of as *ad valorem* tax on development benefits – viz that there is no option value when  $V(t)/A(t)$  is monotone; and that the option value would increase with the probability of  $A(t)/V(t)$  experiencing reversals in direction.

#### 4. Conclusions

In this paper we have examined the decision problem of when if ever a stand of old-growth or natural forest should be harvested. In essence we have adopted a very narrow utilitarian standpoint of seeking the policy which will maximize

the expected present value of amenity and timber benefits from the stand. We do not claim that the “optimal” policy derived herein is the way in which conservation decisions *should* be made. Rather we have sought to address a specific issue with respect to cost-benefit analysis, *viz* to what extent is the certainty-equivalence application of cost-benefit methodology sub-optimal when there is uncertainty with respect to future values of amenity services and timber prices in addition to the risk of catastrophic loss. The model is completely analagous to an earlier model used for this problem by Reed [1993]; the only difference lies in the specification of the stochastic processes describing future amenity service and timber values. Rather than use Geometric Brownian Motion processes, which by their very nature must experience both increases and decreases, we have used Geometric Poisson Jump processes, in which changes occur only at distinct points in time. The advantage of the GPJ formulation is that with a suitable choice of parameter values the processes can be made to be monotonic while still stochastic. It turns out that the degree of sub-optimality of the certainty-equivalence procedure depends on the degree to which the ratio of amenity service values to timber value can change direction. If the ratio of these values is monotonic (even though stochastic), then the certainty-equivalence procedure turns out to be optimal and there is no option value; if on the other hand reversals in direction of this ratio are possible then the certainty-equivalence procedure is sub-optimal, and there is a positive option value. Thus it appears that the source of option value is not, as perhaps previously belived, uncertainty by itself, but rather a special kind of uncertainty, *viz* the possibility of reversals in direction of the relative valuations. With simple two-period models this kind of distinction between types of uncertainty is not apparent; with continuous-time GBM models it is impossible. The adoption of a jump process model has not only avoided the thorny technical issue which arises with the use of SDE models (*viz* the adoption of the Itô or Stratonovich calculus), it has also permitted the investigation of the effects of monotonicity or the lack thereof.

While the results of the paper may indicate that in certain circumstances, option value is less than might previously have been believed, it should be emphasized that the results are of a theoretical nature only, and should not be interpreted as lending weight to the arguments in favour of logging old-growth forest rather than preserving it. All of the caveats with respect to the application of cost-benefit methodology discussed in the introduction must be taken into account, along with awareness of the fact that in the model of this article risk neutrality has been assumed. Clearly a risk-averse objective would lead to more conservative actions. Also the employment of a discount rate to render future costs and benefits comparable to current ones, is a construct heavily laden with implicit assumptions. It is highly debatable, when dealing with environmental issues which could have momentous bearing on the future of the biosphere, whether discounting of the future is at all appropriate. We have not attempted to address these issues; rather we have simply addressed a very specific technical issue in the theory of irreversible environmental decisions as applied to forest conservation, and hopefully have added something to the understanding of the provenance of option value in such situations.

## Appendix 1

### Solution of the Optimal Stopping Problem

To solve the optimal stopping problem we shall use stochastic dynamic programming (SDP). To this end we define the *value function*

$$(A.1) \quad W(q, p, I) = \max_{\tau} \left\{ E \left[ \int_0^{\tau} e^{-\delta t} I(t) A(t) dt + e^{-\delta \tau} I(\tau) V(\tau) \mid \right. \right. \\ \left. \left. A(0) = e^q, V(0) = e^p, I(0) = 1 \right] \right\}$$

where the maximum (or more strictly the supremum) is taken over all stopping times  $\tau$ . Equivalently

$$(A.2) \quad W(q, p, I) = \max_{\tau} \left\{ E \left[ \int_0^{\tau} e^{-\delta t + q(t)} I(t) dt + e^{-\delta \tau + p(\tau)} I(\tau) \mid \right. \right. \\ \left. \left. q(0) = q, \quad p(0) = p, \quad I(0) = 1 \right] \right\}$$

Clearly for any  $p, q$ , if  $I = 0$  then  $W \equiv 0$  i.e.  $W(q, p, 0) \equiv 0$ , since if  $I(0) = 0$  then  $I(t) \equiv 0$  for  $t > 0$ ; in other words once the stand is destroyed it stays destroyed for ever and is worthless.

At any point in time there are only two choices (actions) open to the decision maker - stop (harvest) or continue (don't harvest). Thus  $W(q, p, 1)$  can be expressed via the *principle of optimality* of SDP as the maximum of the expected present values resulting from each of these actions. This leads to the *dynamic programming* (D.P.) *equation* (or Hamilton-Jacobi-Bellman, HJB equation)

$$(A.3) \quad W(q, p, 1) = \max \left\{ \left[ e^q dt + e^{-\delta dt} E(W(q(t+dt), p(t+dt), I(t+dt))) \right], \right. \\ \left. [e^p] \right\}$$

(Dempster and Ye [1993]), which on expanding the exponentials and the expectation, gives (since  $dt$  is arbitrarily small and positive)

$$(A.4) \quad 0 = \max \left\{ \left[ e^q + W(q + \epsilon, p, 1) \lambda_2^u + W(q - \epsilon, p, 1) \lambda_2^d + W(q, p + \epsilon) \lambda_1^u \right. \right. \\ \left. \left. + W(q, p - \epsilon) \lambda_1^d - (\delta + h + \lambda_1^u + \lambda_1^d + \lambda_2^u + \lambda_2^d) W(q, p, 1) \right], \right. \\ \left. [e^p - W(q, p, 1)] \right\}.$$

Before proceeding we note from the definition (A.2) of the value function  $W$  that

$$(A.5) \quad W(q, p + \epsilon, q) = e^{\epsilon} W(q - \epsilon, p, 1) = \theta W(q - \epsilon, p, 1)$$

and

$$(A.6) \quad W(q, p - \epsilon, 1) = e^{-\epsilon} W(q + \epsilon, p, 1) = \theta^{-1} W(q + \epsilon, p, 1)$$



since adding a constant  $\epsilon$  to  $p(0)$  is equivalent to multiplying  $e^{p(t)}$  by  $\theta = e^\epsilon$ , which then can be factored out of the expression (A.2). Thus (A.4) can be expressed

$$(A.7) \quad 0 = \max \left\{ \left[ e^q + (\lambda_2^u + \theta^{-1} \lambda_1^d) W(q + \epsilon, p, 1) + (\lambda_2^d + \theta \lambda_1^u) W(q - \epsilon, p, 1) \right. \right. \\ \left. \left. (\delta + h + \Lambda) W(q, p, 1) \right], \quad [e^p - W(q, p, q)] \right\}$$

where

$$\Lambda = \lambda_1^u + \lambda_1^d + \lambda_2^u + \lambda_2^d$$

is the rate at which jumps of any kind occur in the process  $\{V(t)/A(t)\}$ .

From this equation it can be seen that the state space can be divided into two regions: a *continuation region*  $\Gamma$  on which the first term (in square brackets) on the right hand side of (A.7) is equal to zero, and a *stopping region*  $\Gamma'$  on which the second term (in square brackets) is equal to zero.

Before attempting to solve the D.P. equation (A.7) we shall re-express the state variables in discrete form, and also make some parameter transformations. Firstly we note that a state  $q$  must be of the form  $q = q_0 + j\epsilon$  ( $j = 0, \pm 1, \pm 2, \dots$ ). Thus we shall refer to the state of  $q(t)$  or  $A(t)$  as an integer  $j$  (where  $j = (q - q_0)/\epsilon$ ). Also since  $p$  enters the first term on the right hand side of (A.7) only through  $W$  we can re-express  $W(q, p, 1)$  as simply

$$(A.8) \quad W_j = W(q_0 + j\epsilon, p, 1)$$

where  $p$  is understood as a parameter of  $W_j$ .

A convenient re-parameterization is obtained by letting

$$(A.9) \quad \bar{\lambda} = \lambda_2^u + \theta^{-1} \lambda_1^d + \lambda_2^d + \theta \lambda_1^u = \Lambda + \mu_1$$

$$(A.10) \quad \bar{\rho} = (\lambda_2^u + \theta^{-1} \lambda_1^d) / \bar{\lambda},$$

$$(A.11) \quad 1 - \bar{\rho} = (\lambda_2^d + \theta \lambda_1^u) / \bar{\lambda},$$

and

$$\begin{aligned}
\bar{\delta} &= \delta + h - (\lambda_1^u (\theta - 1) + \lambda_1^d (\theta^{-1} - 1)) \\
&= \Delta - \mu_1
\end{aligned}
\tag{A.12}$$

where  $\mu_1$  is the mean growth rate of the  $\{V(t)\}$  process and  $\Delta$  is the risk-adjusted discount rate.

We shall assume that the conditions

$$\Delta = \delta + h > \mu_1 \tag{A.13a}$$

$$\Delta = \delta + h > \mu_2 \tag{A.13b}$$

are met (as discussed in Section 2).

In terms of the new parameters these conditions are

$$\bar{\delta} > 0 \tag{A.14a}$$

$$\bar{\delta} > \mu_2 - \mu_1 = \bar{\mu}, \tag{A.14b}$$

say, where  $\bar{\mu}$  is the *mean growth rate of the ratio  $A(t)/V(t)$ , of amenity service values to timber values*.

In terms of the function  $W_j$  and the new parameters the D.P. equation is

$$\begin{aligned}
0 = \max \Big\{ & [\bar{\lambda} \bar{\rho} W_{j+1} - (\bar{\lambda} + \bar{\delta}) W_j + \bar{\lambda}(1 - \bar{\rho}) W_{j-1} + A_0 \theta^j], \\
& [e^p - W_j] \Big\}
\end{aligned}
\tag{A.15}$$

or more simply in terms of the second-order difference operator  $\mathcal{L}$ , defined by

$$\mathcal{L} u_j = \bar{\lambda} \bar{\rho} u_{j+1} - (\bar{\lambda} + \bar{\delta}) u_j + \bar{\lambda}(1 - \bar{\rho}) u_{j-1}, \tag{A.16}$$

as

$$0 = \max \{ [\mathcal{L} W_j + A_0 \theta^j], [e^p - W_j] \} \tag{A.17}$$

On the interior of the continuation region  $\Gamma$ , we have

$$(A.18) \quad \mathcal{L} W_j = -A_0 \theta^j$$

and  $W_j \geq e^p$ . The non-homogeneous difference equation (A.18) is readily solved. The solution can be expressed in terms of the roots,  $\alpha$ ,  $\beta$  of the characteristic equation

$$(A.19) \quad f(z) = \bar{\lambda} \bar{\rho} z^2 - (\bar{\lambda} + \bar{\delta}) z + \bar{\lambda}(1 - \bar{\rho}) = 0.$$

In Appendix 2 it is shown that the roots are both real with

$$(A.20) \quad \alpha > \theta > 1 > \beta \geq 0.$$

The solution to (A.18) is

$$(A.21) \quad W_j = C_1 \alpha^j + C_2 \beta^j - A_0 \theta^j / [f(\theta)/\theta]$$

or (see Appendix 2) as

$$(A.22) \quad W_j = C_1 \alpha^j + C_2 \beta^j + \frac{A_0 \theta^j}{\bar{\delta} - \bar{\mu}},$$

where  $C_1$  and  $C_2$  are constants yet to be determined.

The constant  $C_1$  can be determined from the behaviour of  $W_j$  as  $j \rightarrow \infty$  (*i.e.* as amenity service values grow unboundedly large). The expected present value (EPV) or amenity service benefits if there is never a harvest is

$$(A.23) \quad E \left\{ \int_0^\infty e^{-\delta t} I(t) A(t) dt \mid A_0 \right\}$$

where the expectation is with respect to  $\{I(t), A(t)\}$ . This can be re-expressed

$$(A.24) \quad A_0 \int_0^\infty e^{-(\delta+h)t} E \left( \exp (N^+(t)\epsilon - N^-(t)\epsilon) \right) dt$$

where  $\{N^+(t)\}$  and  $\{N^-(t)\}$  are Poisson processes with rates  $\lambda_2^u$  and  $\lambda_2^d$  representing numbers of jumps up and down respectively. Using well-known results

on the moment generating function of a Poisson distribution the EPV can be expressed as

$$(A.25) \quad A_0 \int_0^\infty e^{-(\delta+h)t} e^{\mu_2 t} dt = \frac{A_0}{\Delta - \mu_2} = \frac{A_0}{\bar{\delta} - \bar{\mu}}$$

If the current state is  $j$  ( $A = A_0 \theta^j$ ) then the EPV, assuming no harvest, is  $A_0 \theta^j / (\bar{\delta} - \bar{\mu})$ . Thus as  $j \rightarrow \infty$ , asymptotically  $W_j$  will behave like  $A_0 \theta^j / (\bar{\delta} - \bar{\mu})$  *i.e.* (from (A.22))

$$(A.26) \quad \frac{W_j}{\theta^j} = C_1 \left( \frac{\alpha}{\theta} \right)^j + C_2 \left( \frac{\beta}{\theta} \right)^j + \frac{A_0}{\bar{\delta} - \bar{\mu}} \rightarrow \frac{A_0}{\bar{\delta} - \bar{\mu}}.$$

Since  $\alpha > \theta$  it follows that  $C_1 = 0$  and thus that on the interior of  $\Gamma$

$$(A.27) \quad W_j = C_2 \beta^j + \frac{A_0 \theta^j}{\bar{\delta} - \bar{\mu}}.$$

To eliminate the second constant  $C_2$  we consider a state  $j^*$  on the boundary  $\partial\Gamma$  of  $\Gamma$  *i.e.* for which  $j^* + 1 \in \Gamma$  with  $j^* - 1 \in \Gamma'$ . From the continuity of  $W$ <sup>8</sup> (as a function (A.2) of  $p$  and  $q$ ) we have

$$(A.28) \quad W_{j^*} = C_2 \beta^{j^*} + \frac{A_0 \theta^{j^*}}{\bar{\delta} - \bar{\mu}}$$

also since  $j^* + 1 \in \Gamma$

$$(A.29) \quad W_{j^*+1} = C_2 \beta^{j^*+1} + \frac{A_0 \theta^{j^*+1}}{\bar{\delta} - \bar{\mu}}.$$

Also from the D.P. equation (A.17) we have

$$(A.30) \quad W_{j^*-1} = e^p$$

$$(A.31) \quad W_{j^*} = e^p$$

and

$$(A.32) \quad \mathcal{L} W_{j^*} + A_0 \theta^{j^*} = 0.$$

This last equation can be expressed (using (A.28), (A.29) and (A.30)) as

$$\begin{aligned}
C_2 \beta^{j^*-1} f(\beta) + \frac{A_0 \theta^{j^*-1}}{\bar{\delta} - \bar{\mu}} f(\theta) \\
+ \bar{\lambda}(1 - \bar{\rho}) \left[ e^p - C_2 \beta^{j^*-1} - \frac{A_0 \theta^{j^*-1}}{\bar{\delta} - \bar{\mu}} \right] \\
+ A_0 \theta^{j^*} = 0
\end{aligned}$$

which since  $f(\beta) = 0$  and  $\frac{f(\theta)}{\theta} = -(\bar{\delta} - \bar{\mu})$  (see Appendix 2) can be expressed as

$$(A.33) \quad C_2 \beta^{j^*-1} = e^p - \frac{A_0 \theta^{j^*-1}}{\bar{\delta} - \bar{\mu}}.$$

Also from (A.28) and (A.31)

$$(A.34) \quad C_2 \beta^{j^*} = e^p - \frac{A_0 \theta^{j^*}}{\bar{\delta} - \bar{\mu}}.$$

Eliminating  $C_2$  we get the following equation for the value  $j^*$  at the boundary  $\partial\Gamma$

$$(A.35) \quad A_0 \theta^{j^*} = e^p(\bar{\delta} - \bar{\mu}) \theta(1 - \beta)/(\theta - \beta).$$

In other words the optimal harvesting rule is (using (A.12) and the definition of  $\bar{\mu}$ ) to harvest only when the ratio of  $V(t)/A(t)$  ( $= e^p/A_0 \theta^j$ ) exceeds

$$(A.36) \quad M^* = \frac{(\theta - \beta)}{\theta(1 - \beta)(\Delta - \mu_2)}.$$

Another way of expressing this optimal harvest rule is in terms of the ratio of current timber value  $V(t)$  (the benefit from harvesting) to the cost of harvesting, *i.e.* the EPV of amenity benefits foregone  $\bar{A}_f(t) = A(t)/(\Delta - \mu_2)$ , (see (21)). Optimally one harvests only when

$$(A.37) \quad \frac{V(t)}{\bar{A}_f(t)} > \frac{\theta - \beta}{\theta(1 - \beta)} = r^*$$

say.

An explicit expression for the value function  $W_j$  (or  $W(q, p, I)$ ) can be found by evaluating the constant  $C_2$ . From (A.28) and (A.31) we have that at the boundary ( $j^*$ ) of the continuation region

$$(A.38) \quad W_{j^*} = C_2 \beta^{j^*} + \frac{A_0 \theta^{j^*}}{\bar{\delta} - \bar{\mu}} = e^p.$$

Also from (A.35)

$$(A.39) \quad \frac{A_0 \theta^{j^*}}{\bar{\delta} - \bar{\mu}} = \frac{1}{r^*} e^p.$$

It follows from these two equations that

$$(A.40) \quad C_2 \beta^{j^*} = e^p \left( 1 - \frac{1}{r^*} \right),$$

and

$$(A.41) \quad j^* = \frac{1}{\epsilon} \ln \left[ \frac{(\bar{\delta} - \bar{\mu}) e^p}{A_0 r^*} \right].$$

Thus

$$(A.42) \quad \begin{aligned} C_2 \beta^j &= e^p \left( 1 - \frac{1}{r^*} \right) \beta^{j - \frac{1}{\epsilon} \ln[(\bar{\delta} - \bar{\mu}) e^p / A_0 r^*]} \\ &= e^p \left( 1 - \frac{1}{r^*} \right) \beta^{-\frac{1}{\epsilon} \ln[(\bar{\delta} - \bar{\mu}) e^p / r^* A_0 \theta^j]} \end{aligned}$$

and

$$(A.43) \quad W_j = e^p \left( 1 - \frac{1}{r^*} \right) \beta^{-\frac{1}{\epsilon} \ln[(\bar{\delta} - \bar{\mu}) e^p / r^* A_0 \theta^j]} + \frac{A_0 \theta^j}{\bar{\delta} - \bar{\mu}}$$

or in terms of the current timber price  $V(t)$  and the EPV of amenity services foregone,  $\bar{A}_f(t)$  on the continuation region (where  $V(t) < r^* \bar{A}_f(t)$ )

$$(A.44) \quad W = V(t) \left( 1 - \frac{1}{r^*} \right) \beta^{-\frac{1}{\epsilon} \ln[V(t) / r^* \bar{A}_f(t)]} + \bar{A}_f(t).$$

The second term is the EPV of amenity services if the stand is never harvested given the current state; the first term represents the extra net EPV that could be earned through harvesting (when conditions are propitious). If the current

state is far from the optimal development boundary (*i.e.*  $V(t) \ll r^* \bar{A}_f(t)$ ) then the exponent of  $\beta$  will be a large positive number, and since  $\beta$  is less than one, the first term will be small, *i.e.*, the optimal EPV will be close to that which can be earned through receiving amenity services alone with no harvest. If, on the other hand, the current state is near the optimal development boundary ( $V(t)$  only slightly less than  $r^* \bar{A}_f(t)$ ) then the exponent of  $\beta$  will be close to zero, and  $W$  will be close to  $V(t)$ , the value to be earned through harvesting.

We conclude this Appendix by noting the close connection between the solution of the optimal stopping problem here, and that for the analagous problem using a GBM formulation (Reed [1993]).

The second-order difference equation (A.18) is analagous to the second-order differential equation on the continuation region for the GBM problem, while the boundary conditions (A.30) and (A.31) correspond to the continuity and ‘smooth-pasting’ or ‘high-contact’ conditions (Brekke & Oskendal [1991]) for the GBM optimal stopping problem, since subtracting (A.30) from (A.31) yields the condition that the first difference of  $W_j$  is zero at the boundary.

## Appendix 2

### Properties of the Characteristic Equation (A.19)

As defined in Appendix 1 the characteristic polynomial of the difference equation  $\mathcal{L} W_j = 0$  is

$$(A.45) \quad f(z) = \bar{\lambda} \bar{\rho} z^2 - (\bar{\lambda} + \bar{\delta}) z + \bar{\lambda}(1 - \bar{\rho}) = 0$$

where  $\bar{\lambda} > 0$ ,  $0 \leq \bar{\rho} \leq 1$ ,  $\bar{\delta} > 0$ .

We firstly show that the roots of this equation are both real and non-negative. Since  $0 \leq \bar{\rho} \leq 1$ , we have  $\bar{\rho}(1 - \bar{\rho}) \leq \frac{1}{4}$  so that  $4\bar{\lambda}^2 \bar{\rho}(1 - \bar{\rho}) \leq \bar{\lambda}^2 \leq (\bar{\lambda} + \bar{\delta})^2$ ; thus both roots are real. That they are non-negative follows from the fact that their sum  $(\bar{\lambda} + \bar{\delta})/\bar{\lambda}$  and their product  $(1 - \bar{\rho})/\bar{\rho}$  are both non-negative.

We next show that

$$(A.46) \quad \frac{f(\theta)}{\theta} = -(\bar{\delta} - \bar{\mu}) = -(\Delta - \mu_2) < 0.$$

By substitution into (A.45), using (A.9)-(A.12)

$$\begin{aligned}
\frac{f(\theta)}{\theta} &= \bar{\lambda} \bar{\rho} \theta - (\bar{\lambda} + \bar{\delta}) + \bar{\lambda}(1 - \bar{\rho}) \theta^{-1} \\
&= (\lambda_2^u + \theta^{-1} \lambda_1^d) \theta + (\lambda_2^d + \theta \lambda_1^u) \theta^{-1} - (\Lambda - \Delta) \\
&= \mu_2 - \Delta < 0.
\end{aligned}
\tag{A.47}$$

Thus the roots  $\alpha, \beta$  of  $f(z) = 0$  lie on either side of  $\theta$ .

To show that the smaller root (which we'll call  $\beta$ ) is less than one, suppose it is not. This implies that

$$\bar{\lambda} + \bar{\delta} - \sqrt{(\bar{\lambda} + \bar{\delta})^2 - 4\bar{\lambda}^2 \bar{\rho}(1 - \bar{\rho})} \geq 2 \bar{\lambda} \bar{\rho}
\tag{A.48}$$

and hence that  $\bar{\lambda} + \bar{\delta} - 2\bar{\lambda} \bar{\rho} \geq 0$ . On moving the square root to the right hand side and  $2\bar{\lambda} \bar{\rho}$  to the left hand side of (A.48) and squaring, one gets a simplification that  $\bar{\delta} \leq 0$  which is a contradiction. We can thus conclude that

$$0 \leq \beta < 1 < \theta < \alpha.
\tag{A.49}$$

The limiting behaviour of the roots  $\alpha$  and  $\beta$  as  $\bar{\rho} \rightarrow 1$  and  $\bar{\rho} \rightarrow 0$  is readily shown to be

$$\beta \rightarrow 0, \quad \alpha \rightarrow \frac{\bar{\lambda} + \bar{\delta}}{\bar{\lambda}} \quad (\bar{\rho} \rightarrow 1)
\tag{A.50}$$

and

$$\beta \rightarrow \frac{\bar{\lambda}}{\bar{\lambda} + \bar{\delta}}, \quad \alpha \rightarrow \infty \quad (\bar{\rho} \rightarrow 0).
\tag{A.51}$$

The derivative  $\partial\beta/\partial\bar{\rho}$  can be found by implicitly differentiating

$$f(\beta) = \bar{\lambda} \bar{\rho} \beta^2 - (\bar{\lambda} + \bar{\delta}) \beta + \bar{\lambda}(1 - \bar{\rho}) = 0
\tag{A.52}$$

to get

$$\bar{\lambda} \beta^2 + 2\bar{\lambda} \bar{\rho} \beta \frac{\partial\beta}{\partial\bar{\rho}} - (\bar{\lambda} + \bar{\delta}) \frac{\partial\beta}{\partial\bar{\rho}} - \bar{\lambda} = 0.$$



Using the fact that the sum of the root of (A.45) is  $\alpha + \beta = (\bar{\lambda} + \bar{\delta})/\bar{\lambda}\bar{\rho}$  one gets

$$(A.53) \quad \frac{\partial \beta}{\partial \bar{\rho}} = \frac{\bar{\lambda}(1 - \beta^2)}{\rho(\beta - \alpha)} < 0.$$

### Appendix 3

#### Dependence of Option Value on the Parameter $\bar{\rho}$

If option value is thought of as the difference between the total EPVs obtained using the optimal harvest policy and the certainty-equivalence cost-benefit policy, then for an initial state  $A_0, V_0$  in sector II of Fig. 2

$$(A.54) \quad OV = V_0 \left(1 - \frac{1}{r^*}\right) \beta^{-\frac{1}{\epsilon} \ln[V_0/r^* \bar{A}_f(0)]} + \bar{A}_f(0) - V_0$$

(see (A.44)), which can be written as

$$(A.55) \quad OV = V_0 \left(1 - \frac{1}{r^*}\right) \exp \left\{ \frac{\ln \beta}{\ln \theta} [\ln r^* - C] + \bar{A}_f(0) - V_0 \right\},$$

where

$$(A.56) \quad C = \ln (\bar{A}_f(0)/V_0)$$

so that  $0 < C < \ln r^*$ .

In the case when the initial value  $(A_0, V_0)$  falls in sector III the option value is (from (36))

$$(A.57) \quad \begin{aligned} OV &= \beta^{\frac{\ln r^*}{\epsilon}} \left(1 - \frac{1}{r^*}\right) E \left\{ e^{-\delta T} \tilde{V} \middle| V_0, A_0 \right\} \\ &= K_0 \left(1 - \frac{1}{r^*}\right) \exp \left\{ \frac{\ln \beta}{\ln \theta} \cdot \ln r^* \right\} \end{aligned}$$

where the constant  $K_0$  depends on  $A_0$  and  $V_0$  but not on  $r^*$  or  $\beta$ .

To investigate how  $OV$  changes with changes in  $\bar{\rho}$  we need only look at how

$$(A.58) \quad F(r^*, \beta) = \left(1 - \frac{1}{r^*}\right) \exp \left\{ \frac{\ln \beta}{\ln \theta} [\ln r^* - C] \right\}$$

changes with  $\bar{\rho}$ , where  $C$  is a constant, with  $0 \leq C < \ln r^*$ ;

$$(A.59) \quad \frac{dF}{d\bar{\rho}} = \frac{\partial F}{\partial r^*} \frac{\partial r^*}{\partial \bar{\rho}} + \frac{\partial F}{\partial \beta} \frac{\partial \beta}{\partial \bar{\rho}}.$$

From Appendix 2 we have  $\frac{\partial \beta}{\partial \bar{\rho}} < 0$  and from (34) we have  $\frac{\partial r^*}{\partial \bar{\rho}} < 0$ . Also  $\frac{\partial F}{\partial \beta} > 0$ , so that to show  $\frac{dF}{d\bar{\rho}} < 0$  we need only show that  $\frac{\partial F}{\partial r^*} > 0$ . Now

$$(A.60) \quad \begin{aligned} \frac{\partial F}{\partial r^*} &= \left[ \frac{1}{r^{*2}} + \left( 1 - \frac{1}{r^*} \right) \frac{1}{r^*} \frac{\ln \beta}{\ln \theta} \right] \exp \left\{ \frac{\ln \beta}{\ln \theta} [\ln r^* - C] \right\} \\ &= \frac{1}{r^{*2}} \left[ 1 + (r^* - 1) \frac{\ln \beta}{\ln \theta} \right] \exp \left\{ \frac{\ln \beta}{\ln \theta} [\ln r^* - C] \right\}. \end{aligned}$$

To determine the sign of  $\frac{\partial F}{\partial r^*}$  we need only determine the sign of

$$(A.61) \quad 1 + (r^* - 1) \frac{\ln \beta}{\ln \theta} = \frac{1}{\ln \theta} [\ln \theta + (r^* - 1) \ln \beta]$$

which, on using (22) has the same sign as

$$(A.62) \quad \ln \theta + \frac{(\theta - 1)\beta}{\theta(1 - \beta)} \ln \beta,$$

or as

$$(A.63) \quad \frac{\theta}{\theta - 1} \ln \theta - \frac{\beta}{\beta - 1} \ln \beta.$$

That (A.63) is positive follows from the monotonicity of the function  $\frac{x}{x-1} \ln x$ , and the fact that  $\theta > 1 > \beta$ . Thus we have shown that  $\frac{\partial F}{\partial r^*} > 0$  and  $\frac{dF}{d\bar{\rho}} < 0$ , or in other words that as the probability of  $A(t)/V(t)$  taking downward jumps increases ( $\bar{\rho}$  moves backwards from 1) so the option value increases.

## Footnotes

1. The Geometric Poisson Jump (GPJ) process is a simple special case of a Piecewise Deterministic Markov Process (PDP). PDPs, and the optimal control of systems governed by PDPs, form an area of current active research (see e.g. Dempster & Ye [1993]).
2. The units of  $A(t)$  are the units of  $V(t)$  per unit time. Thus if  $V(t)$  is measured in dollars (\$),  $A(t)$  would be measured in dollars per year (\$/yr.) or dollars per month, etc.
3. In this case the stopping problem resembles the famous St. Petersburg Paradox (see *e.g.* Smith, 1988, pp. 23). While at each point in time it pays, in expectation, to defer the harvest, the eventual return in timber benefits will be zero, since if the hazard rate  $h$  is positive, then with probability one the stand will eventually be destroyed.
4. This fact could be derived by an alternative route. It is well-known (*e.g.* Ross, 1970, p. 188) that for optimal stopping problems for which the myopic (or infinitesimal) look-ahead (MLA) stopping region is closed, (in the sense that once the process enters it can never again leave), that the MLA stopping region is optimal. One can show without a great deal of difficulty that when  $\bar{\rho} = 1$ , the MLA stopping region involves stopping once  $V(t)/\bar{A}_f(t)$  exceeds one. The monotonicity of  $\{V(t)/A(t)\}$  ensures the closedness of this region, and hence its optimality.
5. It can be shown (see Reed [1993, App. 5]) that the form of the certainty-equivalence, cost-benefit harvest rule differs depending on whether amenity values are growing faster in expectation than timber values, or not (*i.e.* whether  $\mu_2 > \mu_1$  or otherwise). In the former case the certainty-equivalence rule is to harvest only when  $V(t)/\bar{A}_f(t) > 1$ , while in the latter case it is to harvest only when  $V(t)/\bar{A}_f(t) > \frac{\Delta - \mu_2}{\Delta - \mu_1}$ , *i.e.* when  $V(t) > A(t)/(\Delta - \mu_1)$ . When  $\mu_1 = \mu_2$  the two forms are equivalent. In general one can think of

the certainty-equivalence procedure as involving a comparison of immediate timber benefits  $V(t)$  with the EPV of amenity benefits foregone where one calculates this EPV assuming that amenity values  $A(t)$  have a mean growth rate equal to the maximum of  $\mu_1, \mu_2$ .

6. Here in using the partial derivative of  $\bar{\rho}$  we are assuming that  $\bar{\lambda}$ ,  $\bar{\lambda} + \bar{\delta}$ , and  $\theta$  remain fixed while  $\bar{\rho}$  changes. One way in which this could occur is through letting  $\lambda_2^u = r \lambda_2$  and  $\lambda_2^d = (1 - r) \lambda_2$ , (where  $\lambda_2$  is a constant reflecting the rate of jumps of any kind in  $A(t)$ ), and varying  $r$  between 0 and 1. This would change the mean growth rate  $\mu_2$  of  $A(t)$  but leave the other parameters unchanged.
7. Here we have assumed  $\mu_2 \geq \mu_1$ . When  $\mu_2 < \mu_1$  the certainty-equivalence harvest rule involves harvesting when  $V(t)/\bar{A}_f(t) > \frac{\Delta - \mu_2}{\Delta - \mu_1}$ , which depends on  $\bar{\rho}$ . When  $\bar{\rho} = 0$ ,  $r^* = \frac{\Delta - \mu_2}{\Delta - \mu_1}$  so there is no option value, but when  $\bar{\rho} > 0$ ,  $r^* > \frac{\Delta - \mu_2}{\Delta - \mu_1}$  so there is a positive option value. Thus again as  $A(t)/V(t)$  changes from a monotone process, (which can only experience jumps downward) to one in which there is a small probability of upward jumps, so a positive option value arises. Since it seems unlikely that timber values would grow faster in expectation than amenity values (*i.e.* that  $\mu_1 > \mu_2$ ) we have not pursued all of the details of this case.
8. The Lipschitz continuity of the value function follows from a much more general theorem for Piecewise Deterministic Markov Processes (PDPs) (Dempster and Ye [1993]).

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