***SEMIGROUPS OF ISOMETRIES, TOEPLITZ
ALGEBRAS AND TWISTED CROSSED PRODUCTS***

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DMS-630-IR

January 1993

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We study the C^* -algebras generated by projective isometric representations of semigroups, using a dilation theorem and the structure theory of twisted crossed products. These algebras include the Toeplitz algebras of noncommutative tori recently studied by Ji, and similar algebras associated to the twisted group algebras of other groups such as the integer Heisenberg group.

Suppose Γ is a subgroup of $\mathbf{R}$, viewed as a discrete group, and let $\Gamma^+ = \Gamma \cap [0, \infty)$. Douglas showed that the C^* -algebra $C^*(V_\gamma)$ generated by a representation V of Γ^+ as non-unitary isometries does not depend on the particular representation V [9, Theorem 1]. It follows quite easily that the commutator ideal $\mathcal{C}_\Gamma$ in $C^*(V_\gamma)$ is simple, and that the map $V_\gamma \rightarrow \gamma$ induces a short exact sequence of C^* -algebras

$$0 \longrightarrow \mathcal{C}_\Gamma \longrightarrow C^*(V_\gamma) \longrightarrow C(\widehat{\Gamma}) \cong C^*(\Gamma) \longrightarrow 0$$

[9, Corollary to Theorem 1]. These results were recently generalised by Ji to the C^* -algebras $C^*(V_\gamma, \sigma)$ generated by isometric σ -representations of Γ^+ , where $\sigma \in Z^2(\Gamma, \mathbf{T})$ is a fixed multiplier on Γ [11]: his exact sequence

$$0 \longrightarrow \mathcal{C}_\Gamma^\sigma \longrightarrow C^*(V_\gamma, \sigma) \longrightarrow C^*(\Gamma, \sigma) \longrightarrow 0$$

involves a “twisted commutator ideal” $\mathcal{C}_\Gamma^\sigma$ and the twisted group C^* -algebra $C^*(\Gamma, \sigma)$. Just as Douglas was motivated by problems involving Toeplitz operators with almost periodic symbol in $C^*(\Gamma) \cong C(\widehat{\Gamma}) \subset C_b(\mathbf{R})$, Ji was interested in a family of Toeplitz operators with symbol in the non-commutative torus $C^*(\Gamma, \sigma)$.

We shall discuss here an alternative approach to these results, in which we realise the C^* -algebras $C^*(V_\gamma, \sigma)$ generated by different isometric σ -representations V as different

representations of the same corner $p(B_\Gamma \times_{\tau, \sigma} \Gamma)p$ in a twisted crossed product C^* -algebra $B_\Gamma \times_{\tau, \sigma} \Gamma$. The simplicity of C_Γ^σ follows from standard properties of crossed products, and we can then deduce the other properties of $C^*(V_\gamma, \sigma)$ discussed above. Our version leads naturally to similar results about the isometric multiplier representations of some nonabelian groups, such as the integer Heisenberg group. Further, even in the case where C_Γ is not simple, we can view the corner $p(B_\Gamma \times_{\tau, \sigma} \Gamma)p$ as a universal Toeplitz algebra, and use the structure theory of crossed products to study it; indeed, we shall see that this can give interesting information about the Toeplitz algebras of [11].

That these problems are connected with crossed products is not a new observation: they were an essential tool in the calculation of the K -theory of C_Γ ([13], see also [22]), and in Murphy's study of Toeplitz algebras of more general ordered groups [15]; Ji himself noted the relationship of $C^*(V_\gamma, \sigma)$ to twisted crossed products in [11, p.156]; and the algebra B_Γ appears in Douglas's original paper [9], as well as [13, 15, 22]. However, we do feel that our systematic exploitation of the theory of crossed products gives the subject a different emphasis, as well as suggesting some interesting new applications. And our other main tool, which is a version of Ito's dilation theorem for isometric σ -representations, may have some independent interest.

We begin by establishing our conventions. We shall be concerned with a discrete group Γ , possibly nonabelian, which has a kind of total ordering pulled back along a homomorphism $\theta : \Gamma \rightarrow \mathbf{R}$; it need not be a partial ordering, because $\ker \theta$ can be non-zero, but every two elements are comparable. We have not tried to go beyond this generality — for example, by replacing $\mathbf{R}$ with other totally ordered groups (cf. [15]), or by axiomatising the necessary properties of the ordering — since we want to stress that our approach is quite straightforward. In §1, we discuss isometric multiplier representations of the semigroup $\Gamma^+ = \Gamma \cap \theta^{-1}([0, \infty))$, and how they are connected to the Toeplitz operators studied by Ji [11, 12] and Lesch [14]. In §2, we prove our dilation theorem, modifying Douglas's proof of Ito's theorem [8], which uses a construction rather like that of induced representations.

Our main results relating $C^*(V_\gamma, \sigma)$ to crossed products are in §3. In fact, we use two algebras B_Γ : $B_\Gamma(-\infty, \infty]$, which consists of functions on $\mathbf{R}$ with jump discontinuities at points of Γ and limits at ∞ , is used to study $C^*(V_\gamma, \sigma)$, and $B_\Gamma(\mathbf{R})$, the ideal of functions in $B_\Gamma(-\infty, \infty]$ which vanish at ∞ , to study C_Γ^σ . Theorem 3.5 shows how simplicity of $B_\Gamma(\mathbf{R}) \times_\sigma \Gamma$ implies the simplicity of C_Γ^σ , and hence gives the uniqueness of $C^*(V_\gamma, \sigma)$ and the exact sequence.

In our last section, we discuss some examples. We show that the results of Douglas and Ji follow from our theorem and classical criteria of Zeller-Meier for the simplicity of

twisted crossed products. Next, we consider the integer Heisenberg group Γ , for which there are many non-trivial multipliers σ , and many homomorphisms $\theta : \Gamma \rightarrow \mathbf{R}$. Again, the twisted commutator ideal is generically simple, and we have exact sequences and uniqueness theorems for $C^*(V_\gamma, \sigma)$ like those for the case $\Gamma \subset \mathbf{R}$. When $\sigma = 1$, the algebra $B_\Gamma(\mathbf{R}) \times_\sigma \Gamma$ is not simple, so our theorem does not apply; however, by a theorem of Williams [21], the crossed product $B_\Gamma(\infty, \infty] \times_\sigma \Gamma$ is given by a C^* -bundle over $\mathbf{T}$ whose fibres are the Toeplitz algebras associated to the different non-commutative 2-tori.

Notation. We use the notation of [17, 18] for twisted crossed products, although we do not need separability hypotheses because our groups are discrete. Thus a twisted action of a group G on a C^* -algebra A is a pair $\alpha : G \rightarrow \text{Aut } A$, $u : G \times G \rightarrow UM(A)$, such that u is a 2-cocycle and $\alpha_s \alpha_t = \text{Ad } u(s, t) \alpha_{st}$. Usually, u will take values in the centre of $M(A)$; then α is an ordinary action, and the theory of [23] also applies. We write $A \times_{\alpha, u} G$ for the crossed product, and i_A, i_G for the canonical embeddings of A, G in $M(A \times_{\alpha, u} G)$ (see [17]).

Acknowledgements. This research was supported by the Natural Sciences and Engineering Research Council of Canada and the Australian Research Council. We thank Chris Bose for help with the arguments in §4(b).

1. Semigroups of isometries and Toeplitz operators.

Let Γ be a discrete group, θ a homomorphism of Γ into $\mathbf{R}$, and let Γ^+ denote the semigroup $\Gamma \cap \theta^{-1}([0, \infty))$. In particular, Γ could be a dense subgroup of $\mathbf{R}$, such as $\mathbf{Z} + \phi\mathbf{Z}$ for some $\phi \in \mathbf{R} \setminus \mathbf{Q}$, and θ the inclusion map. Or Γ could be the integer Heisenberg group, which can be viewed as $\mathbf{Z}^3$ with the product

$$(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2 + p_1 n_2, n_1 + n_2, p_1 + p_2)$$

(see §4(b)), and $\theta : \Gamma \rightarrow \mathbf{R}$ given by $\theta(m, n, p) = \alpha n + \beta p$.

We next take a multiplier σ on Γ ; that is, a 2-cocycle $\sigma \in Z^2(\Gamma, \mathbf{T})$. We suppose that σ has been normalised so that $\sigma(e, \gamma) = \sigma(\gamma, e) = 1$, but *not* necessarily that $\sigma(\gamma, \gamma^{-1}) = 1$. (If Γ is a subgroup of $\mathbf{R}$, we could equally well start with a multiplier on Γ^+ , since Dinh has shown that they all extend to multipliers on Γ [6].) An **isometric σ -representation** of Γ^+ on a Hilbert space H is a map $\gamma \rightarrow V_\gamma$ of Γ^+ into $B(H)$ such that V_γ is an isometry for all $\gamma \in \Gamma^+$, and $V_e = 1$. Notice straight away that when $\theta(\gamma) = 0$, the equation $V_\gamma V_{\gamma^{-1}} = \sigma(\gamma, \gamma^{-1}) V_e$ implies that V_γ is surjective, and hence unitary.

If V is an isometric σ -representation of Γ^+ on H , we let $C^*(V_\gamma, \sigma)$ be the C^* -algebra generated by $\{V_\gamma \mid \gamma \in \Gamma^+\}$, and $\mathcal{C}(V_\gamma, \sigma)$ the ideal in $C^*(V_\gamma, \sigma)$ generated by

$$\{V_\gamma V_\chi - \sigma(\gamma, \chi) V_{\gamma\chi} \mid \gamma, \chi \in \Gamma\},$$

where, if $\theta(\gamma) < 0$, V_γ is by definition $\sigma(\gamma, \gamma^{-1})V_{\gamma^{-1}}^*$. (Note that, while the σ in $C^*(V_\gamma, \sigma)$ is there purely for emphasis, it is actually used in the definition of $\mathcal{C}(V_\gamma, \sigma)$.) In this first section, we just want to show that these are familiar objects.

We first consider the regular σ -representation of Γ^+ on $l^2(\Gamma^+)$ defined by

$$(V_\gamma \xi)(\chi) = \begin{cases} \sigma(\gamma, \gamma^{-1}\chi)\xi(\gamma^{-1}\chi) & \text{if } \theta(\chi) \geq \theta(\gamma) \\ 0 & \text{if } \theta(\chi) < \theta(\gamma). \end{cases}$$

If $\Gamma \subset \mathbf{R}$, and the cocycle satisfies $\sigma(\gamma, \gamma^{-1}) = 1$, this is the representation T studied by Ji in [11, p.153]. The algebra $C^*(V_\gamma, \sigma)$ is there denoted $\mathcal{T}_\Gamma^\sigma$, and the ideal $\mathcal{C}(V_\gamma, \sigma)$ coincides with Ji's twisted commutator ideal $\mathcal{C}_\Gamma^\sigma$. This last is true in general: for any V , $\mathcal{C}(V_\gamma, \sigma)$ is also generated by the twisted commutators

$$[V_\gamma, V_\chi]_\sigma = V_\gamma V_\chi - \sigma(\gamma, \chi)\sigma(\chi, \chi^{-1}\gamma\chi)^{-1}V_\chi V_{\chi^{-1}\gamma\chi},$$

although proving this will be easier once we have the dilation theorem (see Remark 3.3(2) below).

Next, we show that the Toeplitz algebras associated to non-commutative tori [11, 14] have the form $C^*(V_\gamma, \sigma)$. Let $C^*(\Gamma, \sigma)$ be the twisted group C^* -algebra, and $i : \Gamma \rightarrow UC^*(\Gamma, \sigma)$ the canonical embedding: thus we have $i(\gamma)i(\chi) = \sigma(\gamma, \chi)i(\gamma\chi)$, and each representation $\pi : C^*(\Gamma, \sigma) \rightarrow B(K)$ is given by a unitary σ -representation $W : \Gamma \rightarrow U(K)$ via the formula

$$\pi(f) = \sum_{\gamma \in \Gamma} f(\gamma)W_\gamma \quad \text{for } f \in l^1(\Gamma) \subset C^*(\Gamma, \sigma). \quad (1.1)$$

The homomorphism $\theta : \Gamma \rightarrow \mathbf{R}$ induces an action α of $\mathbf{R}$ on $C^*(\Gamma, \sigma)$, characterised by

$$\alpha_s(i(\gamma)) = e^{2\pi i s \theta(\gamma)} i(\gamma) \quad \text{for } \gamma \in \Gamma, s \in \mathbf{R}. \quad (1.2)$$

Now associated to a covariant representation (π, U) of a system $(A, \mathbf{R}, \alpha)$ is a family of Toeplitz operators with symbol in A (see [14], and also [5, 11, 12] for the case where (π, U) is induced from a representation of A). To define them, we first Fourier transform U to give a representation ρ of $C_0(\mathbf{R})$ such that

$$\rho(\tilde{f}) = U(f) = \int_{\mathbf{R}} f(s)U_s ds \quad \text{for } f \in C_c(\mathbf{R}),$$

where $\tilde{f}$ is the inverse Fourier transform

$$\tilde{f}(t) = \int_{\mathbf{R}} f(s)e^{2\pi i s t} ds.$$

Next we extend ρ to the algebra $\mathcal{B}(\mathbf{R})$ of bounded Borel functions, and let $P = \rho(1_{[0,\infty)})$ (see Remark 1.1). The Toeplitz operator T_a with symbol $a \in A$ is the operator $P\pi(a)P$ on $P(H_\pi)$, and the Toeplitz algebra is the C^* -algebra $\mathcal{T}(\pi, U)$ generated by $\{T_a \mid a \in A\}$. We shall show that, when $A = C^*(\Gamma, \sigma)$ and α is given by (1.2), $\mathcal{T}(\pi, U)$ is also generated by an isometric σ -representation, namely $\gamma \rightarrow T_{i(\gamma)}$.

Remark 1.1. It may not be obvious that $P = \rho(1_{[0,\infty)})$ is the projection used in [14], which was the spectral projection $1_{[0,\infty)}(D)$ of the infinitesimal generator D of the one-parameter group U_t . But if E is the spectral measure for D , so that $U_t = \int e^{2\pi i t x} dE(x)$, then for $f \in C_c(\mathbf{R})$ we have

$$\begin{aligned} \rho(\tilde{f}) &= \int f(t) \int e^{2\pi i t x} dE(x) dt \\ &= \iint f(t) e^{2\pi i t x} dt dE(x) \\ &= \tilde{f}(D). \end{aligned}$$

(To justify the application of Fubini's Theorem, fix $h, k \in H_\pi$ and calculate $(\rho(\tilde{f})h|k)$ instead.) Since both sides respect increasing limits in $\mathcal{B}(\mathbf{R})$, it follows that $P = 1_{[0,\infty)}(D)$.

Proposition 1.2. Suppose θ is a homomorphism of a discrete group Γ into $\mathbf{R}$, $\sigma \in Z^2(\Gamma, \mathbf{T})$, and α is the action of $\mathbf{R}$ on $C^*(\Gamma, \sigma)$ characterised by (1.2). If (π, U) is a covariant representation of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$, then $\gamma \rightarrow T_{i(\gamma)}$ is an isometric σ -representation of Γ^+ on $P(H_\pi)$, which generates the Toeplitz algebra $\mathcal{T}(\pi, U)$. Further, $\mathcal{C}(T_{i(\gamma)}, \sigma)$ coincides with the ideal $\mathcal{C}(\pi, U)$ generated by $\{T_a T_b - T_{ab} \mid a, b \in C^*(\Gamma, \sigma)\}$.

Proof. Let $W_\gamma = \pi(i(\gamma))$, so that W is a unitary σ -representation of Γ satisfying (1.1). The covariance of (π, U) says that $U_s W_\gamma = \exp(2\pi i s \theta(\gamma)) W_\gamma U_s$, and hence for $f \in C_c(\mathbf{R})$ we have

$$W_\gamma \rho(\tilde{f}) = \int f(s) W_\gamma U_s ds = \left(\int f(s) e^{-2\pi i s \theta(\gamma)} U_s ds \right) W_\gamma.$$

Since the inverse Fourier transform of $s \rightarrow f(s) e^{-2\pi i s t}$ is $\tau_t(\tilde{f})$, it follows that (ρ, W) is a covariant representation of the twisted system $(C_0(\mathbf{R}), \Gamma, \tau \circ \theta, \sigma)$. Now if $\theta(\gamma) \geq 0$ we have

$$\begin{aligned} T_{i(\gamma)} &= P W_\gamma P = \rho(1_{[0,\infty)}) W_\gamma \rho(1_{[0,\infty)}) \\ &= \rho(1_{[0,\infty)}) \rho(\tau_{\theta(\gamma)}(1_{[0,\infty)})) W_\gamma \\ &= \rho(1_{[0,\infty)}) \rho(1_{[\theta(\gamma), \infty)}) W_\gamma \\ &= \rho(1_{[\theta(\gamma), \infty)}) W_\gamma, \end{aligned}$$

which, by reversing the argument, is just $W_\gamma P$. It follows immediately that $\gamma \rightarrow T_{i(\gamma)}$ is an isometric σ -representation of Γ^+ .

If $\theta(\gamma) < 0$, then the equation $W_{\gamma^{-1}} = \sigma(\gamma, \gamma^{-1})W_\gamma^*$ implies that

$$T_{i(\gamma)} = \sigma(\gamma, \gamma^{-1})T_{i(\gamma^{-1})}^*, \quad (1.3)$$

and hence lies in $C^*(T_{i(\gamma)}, \sigma)$. Since for every f in the dense subalgebra $C_c(\Gamma)$ of $C^*(\Gamma, \sigma)$ we have $T_f = \sum_\gamma f(\gamma)T_{i(\gamma)}$, we deduce that $C^*(T_{i(\gamma)}, \sigma)$ is all of $\mathcal{T}(\pi, U)$. Finally, we note that (1.3) implies that $\mathcal{C}(\pi, U)$ is actually generated by

$$\{T_{i(\gamma)}T_{i(\chi)} - \sigma(\gamma, \chi)T_{i(\gamma\chi)} \mid \gamma, \chi \in \Gamma\} = \{T_{i(\gamma)}T_{i(\chi)} - T_{i(\gamma)i(\chi)} \mid \gamma, \chi \in \Gamma\}.$$

Thus $\mathcal{C}(T_{i(\gamma)}, \sigma) \subset \mathcal{C}(\pi, U)$, and the reverse inclusion holds because the range of i spans a dense subspace of $C^*(\Gamma, \sigma)$.

We shall see later that, when $\theta(\Gamma)$ is dense in $\mathbf{R}$, every isometric σ -representation V arises this way: there is a covariant representation (π, U) of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$ such that $V_\gamma = T_{i(\gamma)}$ and $C^*(V_\gamma, \sigma) = \mathcal{T}(\pi, U)$. To prove this, we shall produce a covariant representation (ρ, W) of the twisted system $(C_0(\mathbf{R}), \Gamma, \tau \circ \theta, \sigma)$ with $V_\gamma = \rho(1_{[0, \infty)})W_\gamma\rho(1_{[0, \infty)})$, and reverse the process in the proof above to obtain a covariant representation of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$. (See Corollary 3.4 below.)

2. A dilation theorem.

Theorem 2.1. *Let Γ be a discrete group, $\sigma \in Z^2(\Gamma, \mathbf{T})$ and $\theta : \Gamma \rightarrow \mathbf{R}$ a homomorphism. Let $\Gamma^+ = \theta^{-1}([0, \infty))$, and suppose $V : \Gamma^+ \rightarrow B(H)$ is an isometric σ -representation. Then there is a unitary σ -representation $W : \Gamma \rightarrow U(K)$ and an isometry Y of H onto a subspace of K invariant under $\{W_\gamma : \gamma \in \Gamma^+\}$, such that $V_\gamma = Y^*W_\gamma Y$ for $\gamma \in \Gamma^+$.*

Proof. We follow [8]. We shall write $\gamma \geq \chi$ iff $\theta(\gamma) \geq \theta(\chi)$, but we stress that this is not a partial order in the usual sense if $\ker \theta \neq 0$. We take K to be the completion of the space

$$H_W = \left\{ f : \Gamma^+ \rightarrow H \mid \begin{array}{l} f(\gamma\chi) = \sigma(\gamma\chi\gamma^{-1}, \gamma)^{-1}V_{\gamma\chi\gamma^{-1}}(f(\gamma)) \\ \text{for } \gamma \geq \text{some } \gamma_f \text{ and all } \chi \in \Gamma^+ \end{array} \right\}$$

in the inner product

$$(f|g) = (f(\gamma)|g(\gamma)) \quad \text{for } \gamma \geq \max(\gamma_f, \gamma_g).$$

To justify this definition, we notice, first, that $\gamma \geq \chi \geq \max(\gamma_f, \gamma_g)$ implies $\chi^{-1}\gamma \in \Gamma^+$, and then writing $\gamma = \chi(\chi^{-1}\gamma)$ gives

$$\begin{aligned} (f(\gamma)|g(\gamma)) &= (V_{\chi(\chi^{-1}\gamma)\chi^{-1}}(f(\chi))|V_{\chi(\chi^{-1}\gamma)\chi^{-1}}(g(\chi))) \\ &= (f(\chi)|g(\chi)), \end{aligned}$$

so the inner product is well-defined; and second, that we are implicitly modding out the subspace

$$\{f \in H_W \mid \text{there exists } \gamma_f \text{ such that } f(\gamma) = 0 \text{ for } \gamma \geq \gamma_f\} \quad (2.1)$$

before completing. The isometry $Y : H \rightarrow K$ is defined by $Y(x)(\gamma) = V_\gamma(x)$: a direct computation shows $Y(x) \in H_W$, with $\gamma_{Y(x)} = e$, and $\|Y(x)\| = \|x\|$.

For $\gamma \in \Gamma^+$, we try to define an operator U_γ on K by

$$U_\gamma(f)(\chi) = \tilde{\sigma}(\chi\gamma\chi^{-1}, \chi)^{-1} V_{\chi\gamma\chi^{-1}}(f(\chi)) \text{ for } f \in H_W,$$

where $\tilde{\sigma}$ is the antisymmetrisation of σ defined by

$$\tilde{\sigma}(\chi, \eta) = \sigma(\chi, \eta)\sigma(\eta, \eta^{-1}\chi\eta)^{-1},$$

which is important because

$$V_\chi V_\eta = \tilde{\sigma}(\chi, \eta) V_\eta V_{\eta^{-1}\chi\eta}. \quad (2.2)$$

Once we have checked that $U_\gamma(f) \in H_W$, it should be obvious that U_γ is isometric and hence extends to an isometry on K . Well, for $\chi \geq \gamma_f$ we have

$$\begin{aligned} U_\gamma(f)(\chi\eta) &= \tilde{\sigma}(\chi\eta\gamma\eta^{-1}\chi^{-1}, \chi\eta)^{-1} V_{\chi\eta\gamma\eta^{-1}\chi^{-1}}(f(\chi\eta)) \\ &= \tilde{\sigma}(\chi\eta\gamma\eta^{-1}\chi^{-1}, \chi\eta)^{-1} V_{\chi\eta\gamma\eta^{-1}\chi^{-1}}(\sigma(\chi\eta\chi^{-1}, \chi)^{-1} V_{\chi\eta\chi^{-1}}(f(\chi))) \\ &= \left[\tilde{\sigma}(\chi\eta\gamma\eta^{-1}\chi^{-1}, \chi\eta\chi^{-1})^{-1} \tilde{\sigma}((\chi\eta\chi^{-1})(\chi\eta\gamma\eta^{-1}\chi^{-1})(\chi\eta\chi^{-1}), \chi)^{-1} \right] \\ &\quad \cdot \sigma(\chi\eta\chi^{-1}, \chi)^{-1} V_{\chi\eta\gamma\eta^{-1}\chi^{-1}} V_{\chi\eta\chi^{-1}}(f(\chi)) \quad \text{by [19, (1.1)]} \\ &= \sigma(\chi\eta\chi^{-1}, \chi)^{-1} \tilde{\sigma}(\chi\gamma\chi^{-1}, \chi)^{-1} [V_{\chi\eta\chi^{-1}} V_{\chi\gamma\chi^{-1}}(f(\chi))] \quad \text{by (2.2)} \\ &= \sigma(\chi\eta\chi^{-1}, \chi)^{-1} V_{\chi\eta\chi^{-1}}(U_\gamma(f)(\chi)), \end{aligned}$$

as required. Since U_γ is isometric, it will be unitary if it has dense range. However, if $f \in H_W$ and $\gamma \in \Gamma^+$, we set

$$\begin{aligned} g(\chi) &= \begin{cases} \tilde{\sigma}(\chi\gamma\chi^{-1}, \chi)\sigma(\chi\gamma\chi^{-1}, \chi\gamma^{-1})^{-1}f(\chi\gamma^{-1}) & \text{for } \chi \geq \gamma_f\gamma \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \sigma(\chi\gamma^{-1}, \gamma)^{-1}f(\chi\gamma^{-1}) & \text{for } \chi \geq \gamma_f\gamma \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $\chi \geq \gamma_f\gamma$, the second formula for g gives

$$\begin{aligned} g(\chi\eta) &= \sigma(\chi\eta\gamma^{-1}, \gamma)^{-1}f(\chi\gamma^{-1}(\gamma\eta\gamma^{-1})) \\ &= \sigma(\chi\eta\gamma^{-1}, \gamma)^{-1}\sigma(\chi\eta\chi^{-1}, \chi\gamma^{-1})^{-1}V_{\chi\eta\chi^{-1}}(f(\chi\gamma^{-1})) \\ &= \sigma(\chi\gamma^{-1}, \gamma)^{-1}\sigma(\chi\eta\chi^{-1}, \chi)^{-1}V_{\chi\eta\chi^{-1}}(f(\chi\gamma^{-1})) \\ &= \sigma(\chi\eta\chi^{-1}, \chi)^{-1}V_{\chi\eta\chi^{-1}}(g(\chi)), \end{aligned}$$

so that $g \in H_W$. On the other hand, the first formula for g shows that, again for $\chi \geq \gamma f \gamma$,

$$\begin{aligned} (U_\gamma g)(\chi) &= \tilde{\sigma}(\chi \gamma \chi^{-1}, \chi)^{-1} V_{\chi \gamma \chi^{-1}} (\tilde{\sigma}(\chi \gamma \chi^{-1}, \chi) \sigma(\chi \gamma \chi^{-1}, \chi \gamma^{-1})^{-1} f(\chi \gamma^{-1})) \\ &= \sigma((\chi \gamma^{-1}) \gamma (\chi \gamma^{-1})^{-1}, \chi \gamma^{-1})^{-1} V_{(\chi \gamma^{-1}) \gamma (\chi \gamma^{-1})^{-1}} (f(\chi \gamma^{-1})) \\ &= f((\chi \gamma^{-1}) \gamma) \\ &= f(\chi); \end{aligned}$$

thus $U_\gamma g$ differs from f by an element of the subspace (2.1), and $U_\gamma g = f$ in the completion K . Thus U_γ has dense range, and is unitary. We note that, from (2.2),

$$U_\gamma(Y(x))(\chi) = \tilde{\sigma}(\chi \gamma \chi^{-1}, \chi)^{-1} V_{\chi \gamma \chi^{-1}}(V_\chi(x)) = V_\chi(V_\gamma(x)) = Y(V_\gamma(x))(\chi),$$

so U_γ is a dilation of V_γ for each γ . From [19, (1.3)] we have

$$\begin{aligned} U_\gamma(U_\chi(f))(\eta) &= \tilde{\sigma}(\eta \gamma \eta^{-1}, \eta)^{-1} V_{\eta \gamma \eta^{-1}} (\tilde{\sigma}(\eta \gamma \eta^{-1}, \eta)^{-1} V_{\eta \chi \eta^{-1}}(f(\eta))) \\ &= \tilde{\sigma}(\eta \gamma \chi \eta^{-1}, \eta)^{-1} \sigma(\eta \gamma \eta^{-1}, \eta \chi \eta^{-1})^{-1} \sigma(\gamma, \chi) V_{\eta \gamma \eta^{-1}}(V_{\eta \chi \eta^{-1}}(f(\eta))) \\ &= \sigma(\gamma, \chi) \tilde{\sigma}(\eta \gamma \chi \eta^{-1}, \eta)^{-1} V_{\eta \gamma \chi \eta^{-1}}(f(\eta)) \\ &= \sigma(\gamma, \chi) U_{\gamma \chi}(f)(\eta), \end{aligned} \tag{2.3}$$

for any $\gamma, \chi \in \Gamma^+$, and it remains to extend U to a unitary representation of the whole group Γ .

The crucial observation here is that, for any $\gamma \in \Gamma$, we have either $\theta(\gamma) \geq 0$, in which case we define $W_\gamma = U_\gamma$, or $\theta(\gamma) < 0$, in which case we set $W_\gamma = \sigma(\gamma^{-1}, \gamma) U_{\gamma^{-1}}^*$. Since each U_γ is unitary, W is a well-defined map of Γ into $U(K)$, and we just have to prove it is a σ -representation. There seems no alternative to checking cases; recall in these calculations that $\sigma(e, \gamma) = \sigma(\gamma, e) = 1$.

Case (i): $\gamma \geq 0$ and $\chi \geq 0$. We did this in (2.3) above.

Case (ii): $\gamma < 0, \chi \geq 0$ and $\gamma \chi \geq 0$. Using (i), we have

$$\begin{aligned} W_\gamma W_\chi &= \sigma(\gamma^{-1}, \gamma) U_{\gamma^{-1}}^* U_\chi = \sigma(\gamma^{-1}, \gamma) U_{\gamma^{-1}}^* [\sigma(\gamma^{-1}, \gamma \chi)^{-1} U_{\gamma^{-1}} U_{\gamma \chi}] \\ &= \sigma(\gamma, \chi) (U_{\gamma^{-1}}^* U_{\gamma^{-1}}) U_{\gamma \chi} = \sigma(\gamma, \chi) W_{\gamma \chi}. \end{aligned}$$

Case (iii): $\gamma < 0, \chi \geq 0$ and $\gamma \chi < 0$. Then

$$\begin{aligned} W_\gamma W_\chi &= \sigma(\gamma^{-1}, \gamma) U_{\chi(\gamma \chi)^{-1}}^* U_\chi \\ &= \sigma(\gamma^{-1}, \gamma) [\sigma(\chi, \chi^{-1} \gamma^{-1})^{-1} U_\chi U_{(\gamma \chi)^{-1}}]^* U_\chi \\ &= \sigma(\gamma^{-1}, \gamma) [\sigma(\chi^{-1} \gamma^{-1}, \gamma \chi) \sigma(\chi, e) \sigma(\gamma^{-1}, \gamma \chi)^{-1}] U_{(\gamma \chi)^{-1}}^* \\ &= \sigma(e, \chi)^{-1} \sigma(\gamma, \chi) W_{\gamma \chi}. \end{aligned}$$

Case (iv): $\gamma \geq 0, \chi < 0$ and $\gamma\chi \geq 0$.

$$\begin{aligned}
W_\gamma W_\chi &= U_{(\gamma\chi)\chi^{-1}} \sigma(\chi^{-1}, \chi) U_{\chi^{-1}}^* \\
&= \sigma(\gamma\chi, \chi^{-1})^{-1} \sigma(\chi^{-1}, \chi) (U_{\gamma\chi} U_{\chi^{-1}}) U_{\chi^{-1}}^* \\
&= \sigma(\gamma, e)^{-1} \sigma(\gamma, \chi) U_{\gamma\chi} \quad \text{using } \sigma(\chi^{-1}, \chi) = \sigma(\chi, \chi^{-1}) \\
&= \sigma(\gamma, \chi) W_{\gamma\chi}.
\end{aligned}$$

Case (v): $\gamma \geq 0, \chi < 0$ and $\gamma\chi < 0$.

$$\begin{aligned}
W_\gamma W_\chi &= U_\gamma \sigma(\chi^{-1}, \chi) U_{(\gamma\chi)^{-1}}^* \\
&= U_\gamma \sigma((\chi^{-1} \gamma^{-1}) \gamma, \chi) [\sigma(\chi^{-1} \gamma^{-1}, \gamma) U_\gamma^* U_{(\gamma\chi)^{-1}}^*] \\
&= \sigma(\gamma, \chi) \sigma(\chi^{-1} \gamma^{-1}, \gamma\chi) U_{(\gamma\chi)^{-1}}^* \\
&= \sigma(\gamma, \chi) W_{\gamma\chi}.
\end{aligned}$$

Case (vi): $\gamma < 0$ and $\chi < 0$.

$$\begin{aligned}
W_\gamma W_\chi &= \sigma(\gamma^{-1}, \gamma) \sigma(\chi^{-1}, \chi) (U_{\chi^{-1}} U_{\gamma^{-1}})^* \\
&= \sigma(\gamma^{-1}, \gamma) \sigma((\chi^{-1} \gamma^{-1}) \gamma, \chi) [\sigma(\chi^{-1}, \gamma^{-1})^{-1} U_{(\gamma\chi)^{-1}}^*]^* \\
&= \sigma(\gamma^{-1}, \gamma) [\sigma(\chi^{-1} \gamma^{-1}, \gamma\chi) \sigma(\gamma, \chi) \sigma(\chi^{-1} \gamma^{-1}, \gamma)^{-1}] \sigma(\chi^{-1}, \gamma^{-1})^{-1} U_{(\gamma\chi)^{-1}}^* \\
&= \sigma(\gamma, \chi) [\sigma(\gamma^{-1}, \gamma) \sigma(\chi^{-1} \gamma^{-1}, \gamma)^{-1} \sigma(\chi^{-1}, \gamma^{-1})^{-1}] \sigma(\chi^{-1} \gamma^{-1}, \gamma\chi) U_{(\gamma\chi)^{-1}}^* \\
&= \sigma(\gamma, \chi) [\sigma(\chi^{-1}, e)^{-1}] W_{\gamma\chi}.
\end{aligned}$$

This completes the proof of the theorem.

3. C^* -algebras generated by isometries and twisted crossed products.

Let Λ be a subgroup of $\mathbf{R}$, viewed as a discrete group. We denote by $B_\Lambda(-\infty, \infty]$ the C^* -subalgebra of $L^\infty(\mathbf{R})$ generated by $\{1_{[\lambda, \infty)} : \lambda \in \Lambda\}$, and by $B_\Lambda(\mathbf{R})$ the C^* -subalgebra generated by $\{1_{[\lambda, \mu)} : \lambda, \mu \in \Lambda\}$. We write τ for the action of Λ by translation on either of these algebras: $\tau_\lambda(f)(\mu) = f(\mu - \lambda)$. (It is important that Λ be discrete: since $\|\tau_\lambda(1_{[\mu, \nu)}) - 1_{[\mu, \nu)}\|_\infty = 1$ for any $\lambda \neq 0$, τ will not be a continuous action otherwise.) We begin with some elementary observations about these algebras.

Proposition 3.1. *Let Λ be a discrete subgroup of $\mathbf{R}$.*

(1) *The algebras $B_\Lambda(\mathbf{R})$, $B_\Lambda(-\infty, \infty]$ are, respectively, the closures of the sets $\mathcal{A} = \text{sp}\{1_{[\lambda, \mu)} : \lambda, \mu \in \Lambda\}$ and $\text{sp}\{1_{[\lambda, \infty)}\}$, and $B_\Lambda(\mathbf{R})$ is contained in $B_\Lambda(-\infty, \infty]$.*

(2) *Cauchy sequences in $\mathcal{A} \subset L^\infty(\mathbf{R})$ converge uniformly; the limit is continuous at points of $\mathbf{R} \setminus \Lambda$, right continuous at points of Λ , and has left-hand limits at points of Λ . (Thus we can view elements of $B_\Lambda(-\infty, \infty]$ as functions with these properties.)*

(3) *There is a well-defined homomorphism $\epsilon_\infty : B_\Lambda(-\infty, \infty] \rightarrow \mathbf{C}$ such that $\epsilon_\infty(f) = \lim_{x \rightarrow \infty} f(x)$ for all f , and then we have a short exact sequence of C^* -algebras*

$$0 \longrightarrow B_\Lambda(\mathbf{R}) \longrightarrow B_\Lambda(-\infty, \infty] \xrightarrow{\epsilon_\infty} \mathbf{C} \longrightarrow 0.$$

(4) *If Λ is dense in $\mathbf{R}$, then $B_\Lambda(\mathbf{R})$ contains $C_0(\mathbf{R})$, and $B_\Lambda(-\infty, \infty]$ contains the algebra $C_0(-\infty, \infty]$ of continuous functions on $\mathbf{R}$ which vanish at $-\infty$ and have limits as $x \rightarrow \infty$.*

Proof. (1) is easy: one just has to check that $\mathcal{A}$ is itself a $*$ -algebra, and note that $1_{[\lambda, \mu)} = 1_{[\lambda, \infty)} - 1_{[\mu, \infty)}$. The point of (2) is that Cauchy sequences in B_Λ are *a priori* only Cauchy in the essential sup-norm; however, since all the non-empty sets of the form $[\lambda, \mu)$ have non-empty interior, this agrees with the sup-norm on elements of $\mathcal{A}$. It follows that Cauchy sequences converge uniformly, and the limit function has the properties described because the functions in $\mathcal{A}$ do. To see (3), just note that ϵ_∞ is well-defined on the dense subalgebra $\text{sp}\{1_{[\lambda, \infty)}\}$, is uniformly continuous there, and vanishes on $\mathcal{A}$. The last part requires a standard approximation argument.

Proposition 3.2. *Let Γ be a discrete group, $\theta : \Gamma \rightarrow \mathbf{R}$ a homomorphism, and $\sigma \in Z^2(\Gamma, \mathbf{T})$. Write $\Gamma^+ = \theta^{-1}([0, \infty))$, and suppose $V : \Gamma^+ \rightarrow B(H)$ is an isometric σ -representation.*

(1) *There is a covariant representation (ρ, W) of the twisted dynamical system $(B_{\theta(\Gamma)}(-\infty, \infty], \Gamma, \tau \circ \theta, \sigma)$ on a Hilbert space K containing H , such that H is the range of the projection $P = \rho(1_{[0, \infty)})$, and*

$$V_\gamma = PW_\gamma P = W_\gamma P \text{ for } \gamma \in \Gamma^+; \quad (3.1)$$

we can take W to be any dilation of V to a unitary σ -representation of Γ .

(2) The C^* -algebra $C^*(V_\gamma, \sigma)$ generated by $\{V_\gamma : \gamma \in \Gamma^+\}$ is the corner

$$P [\rho \times W (B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma)] P.$$

(3) If we identify $B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma$ with an ideal in $B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma$ in the usual way (cf. [23, 2.28], [18, 2.2]), then $P [\rho \times W (B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma)] P$ is the ideal $\mathcal{C}(V_\gamma, \sigma)$ in $C^*(V_\gamma, \sigma)$.

Proof. We let $W : \Gamma \rightarrow U(K)$ be a dilation of V — as given by Theorem 2.1, for example — and use the isometry Y to identify H with the closed W -invariant subspace $Y(H)$ of K ; thus if P is the projection onto H , equation (3.1) holds for all $\gamma \in \Gamma^+$. Since $\tau_{\theta(\gamma)}(1_{[0, \infty)}) = 1_{[\theta(\gamma), \infty)}$, if we are to have $P = \rho(1_{[0, \infty)})$ then the covariance of (ρ, W) forces

$$\rho(1_{[\theta(\gamma), \infty)}) = W_\gamma P W_\gamma^*. \quad (3.2)$$

Notice that if $\theta(\gamma) = \theta(\chi)$, then $V_{\chi^{-1}\gamma}$ is unitary on H ; thus the multiplicativity of $\text{Ad } W$ and (3.1) imply that

$$W_\gamma P W_\gamma = W_\chi W_{\chi^{-1}\gamma} P W_{\chi^{-1}\gamma}^* W_\chi^* = W_\chi V_{\chi^{-1}\gamma} V_{\chi^{-1}\gamma}^* W_\chi^* = W_\chi P W_\chi^*. \quad (3.3)$$

We can therefore use (3.2) to define $\rho(1_{[\theta(\gamma), \infty)})$. Naturally, we now want to define ρ by

$$\rho\left(\sum_{i=1}^n a_i 1_{[\theta(\gamma_i), \infty)}\right) = \sum_{i=1}^n a_i W_{\gamma_i} P W_{\gamma_i}^*;$$

this will be well-defined and extend by continuity to $B_{\theta(\Gamma)}(-\infty, \infty]$ provided

$$\left\| \sum_{i=1}^n a_i W_{\gamma_i} P W_{\gamma_i}^* \right\| \leq \left\| \sum_{i=1}^n a_i 1_{[\theta(\gamma_i), \infty)} \right\|_\infty. \quad (3.4)$$

In establishing (3.4), we may as well suppose $\theta(\gamma_1) \leq \theta(\gamma_2) \leq \dots \leq \theta(\gamma_n)$ (otherwise we can just reorder them). When $\theta(\gamma) \geq 0$, $V_\gamma V_\gamma^*$ is the projection onto $V_\gamma(H)$, and hence $V_\gamma V_\gamma^* \leq P$; from this, and a calculation like (3.3) above, we deduce that

$$W_\gamma P W_\gamma^* \leq W_\chi P W_\chi^* \quad \text{whenever } \theta(\gamma) \geq \theta(\chi).$$

Thus $\{W_{\gamma_i} P W_{\gamma_i}^*\}$ is a decreasing sequence of projections, and

$$\begin{aligned} & \left\| \sum_{i=1}^n a_i W_{\gamma_i} P W_{\gamma_i}^* \right\| \\ &= \left\| \left(\sum_{i=1}^n a_i \right) W_{\gamma_n} P W_{\gamma_n}^* + \sum_{k=2}^n \left(\sum_{i=1}^k a_i \right) (W_{\gamma_{k-1}} P W_{\gamma_{k-1}}^* - W_{\gamma_k} P W_{\gamma_k}^*) \right\| \\ &= \sup_{k \in S} \left| \sum_{i=1}^k a_i \right|, \end{aligned}$$

where S is the set of k for which $W_{\gamma_k} P W_{\gamma_k}^*$ is strictly smaller than $W_{\gamma_{k-1}} P W_{\gamma_{k-1}}^*$. Since by (3.3) $\theta(\gamma) = \theta(\chi)$ implies $W_\gamma P W_\gamma^* = W_\chi P W_\chi^*$,

$$\sup_{k \in S} \left| \sum_{i=1}^k a_i \right| = \sup_{\{k: \theta(\gamma_k) > \theta(\gamma_{k-1})\}} \left| \sum_{i=1}^k a_i \right| = \left\| \sum_{i=1}^n a_i 1_{[\theta(\gamma_i), \infty)} \right\|_\infty,$$

which gives (3.4) and hence shows ρ is well-defined. The covariance condition is certainly satisfied on $\text{sp}\{1_{[\theta(\gamma_i), \infty)}\}$, thus on all of $B_{\theta(\Gamma)}(-\infty, \infty]$ by continuity, and we have proved the first part.

For part (2), note first that $V_\gamma = W_\gamma P = W_\gamma \rho(1_{[0, \infty)})$ is the image under $\rho \times W$ of the element $i_\Gamma(\gamma) i_{B_{\theta(\Gamma)}}(1_{[0, \infty)})$ of $B_{\theta(\Gamma)} \times \Gamma$, and since $V_\gamma = P W_\gamma P$, it lies in the corner. Conversely, for any $\gamma \in \Gamma^+$ and $\chi \in \Gamma^-$, $\rho(1_{[\theta(\gamma), \infty)}) = V_\gamma V_\gamma^*$ and

$$P W_\chi P = (P W_\chi^* P)^* = (P \sigma(\chi, \chi^{-1}) W_{\chi^{-1}} P)^* = (\sigma(\chi, \chi^{-1}) V_{\chi^{-1}})^*$$

both lie in $C^*(V_\gamma, \sigma)$, and since elements of the form $i_{B_{\theta(\Gamma)}}(1_{[\theta(\gamma), \infty)}) i_\Gamma(\chi)$ span the crossed product, (2) follows.

If $\theta(\chi) \geq 0$, then $V_\chi = P W_\chi P = W_\chi P$, so

$$V_\gamma V_\chi - \sigma(\gamma, \chi) V_{\gamma\chi} = P W_\gamma P W_\chi P - P \sigma(\gamma, \chi) W_{\gamma\chi} P = P (W_\gamma W_\chi - \sigma(\gamma, \chi) W_{\gamma\chi}) P = 0. \quad (3.5)$$

If $\theta(\chi) < 0$, then

$$\begin{aligned} V_\chi &= \sigma(\chi, \chi^{-1}) V_{\chi^{-1}}^* = P \sigma(\chi, \chi^{-1}) W_{\chi^{-1}}^* P = P W_\chi P \\ &= W_\chi (W_\chi^* \rho(1_{[0, \infty)}) W_\chi) P \\ &= W_\chi \rho(1_{[-\theta(\chi), \infty)}) P, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} V_\gamma V_\chi - \sigma(\gamma, \chi) V_{\gamma\chi} &= P W_\gamma W_\chi \rho(1_{[-\theta(\chi), \infty)}) P - P \sigma(\gamma, \chi) W_{\gamma\chi} P \\ &= -P \sigma(\gamma, \chi) W_{\gamma\chi} \rho(1_{[0, -\theta(\chi))}) P. \end{aligned} \quad (3.7)$$

Either way, $V_\gamma V_\chi - \sigma(\gamma, \chi) V_{\gamma\chi}$ belongs to $P[\rho \times W(B_{\theta(\Gamma)}(\mathbf{R}) \times \Gamma)]P$. On the other hand, because (3.7) holds for all γ , it implies that

$$P W_\eta \rho(1_{[\theta(\zeta), \theta(\xi))}) P = P W_\eta \rho(1_{[0, \theta(\xi))}) P - P W_\eta \rho(1_{[0, \theta(\zeta))}) P \quad (3.8)$$

lies in $\mathcal{C}(V_\gamma, \sigma)$ for all $\eta \in \Gamma$ and $\xi > \zeta > 0$ (take $\chi = -\xi$ and $-\zeta$ in (3.7)). Since the functions $1_{[\theta(\zeta), \theta(\xi))}$ span $B_\Gamma(\mathbf{R})$, elements of the form (3.8) span a dense subspace of $P[\rho \times W(B_{\theta(\Gamma)} \times \Gamma)]P$, the corner is contained in $\mathcal{C}(V_\gamma, \sigma)$, and the result follows.

Remark 3.3(1). Suppose (π, U) is a covariant representation of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$ as in Proposition 1.1, and (ρ, W) is the covariant representation of the twisted system $(\mathcal{B}(\mathbf{R}), \Gamma, \tau \circ \theta, \sigma)$ characterised by

$$\rho(\check{f}) = \int f(s) U_s ds, \quad W_\gamma = \pi(i(\gamma)). \quad (3.9)$$

(See the first part of the proof of Proposition 1.1.) Then the covariant representation in Proposition 3.2(1) is the restriction of (ρ, W) to $(B_{\theta(\Gamma)}, \Gamma, \tau \circ \theta, \sigma)$, and Propositions 1.1 and 3.2 together say that the Toeplitz algebra $\mathcal{T}(\pi, U)$ is the image of the corner

$$p [B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma] p = i_{B_{\theta(\Gamma)}}(1_{[0, \infty)}) [B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma] i_{B_{\theta(\Gamma)}}(1_{[0, \infty)}).$$

Thus we view this corner as a **universal Toeplitz algebra** for the system $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$.

(2). The ideal $\mathcal{C}(V_\gamma, \sigma)$ coincides with what Ji calls the twisted commutator ideal — the ideal $\mathcal{I}$ in $C^*(V_\gamma, \sigma)$ generated by the elements of the form

$$V_\gamma V_\chi - \tilde{\sigma}(\gamma, \chi) V_\chi V_{\chi^{-1}\gamma\chi} = V_\gamma V_\chi - \sigma(\gamma, \chi) \sigma(\chi, \chi^{-1}\gamma\chi)^{-1} V_\chi V_{\chi^{-1}\gamma\chi}$$

for $\gamma, \chi \in \Gamma$. Since each of these generators is the difference of two generators of $\mathcal{C}$, we only have to show that $V_\gamma V_\chi - \sigma(\gamma, \chi) V_{\gamma\chi}$ always lies in $\mathcal{I}$. If $\theta(\chi) \geq 0$, this is trivial by (3.5), so we suppose $\theta(\chi) < 0$. But then

$$V_\chi = P\sigma(\chi, \chi^{-1})W_{\chi^{-1}}^*P = \sigma(\chi, \chi^{-1})(PW_{\chi^{-1}}P)^* = \sigma(\chi, \chi^{-1})(W_{\chi^{-1}}P)^* = PW_\chi,$$

and

$$V_\chi V_{\chi^{-1}\gamma\chi} = PW_\chi PW_{\chi^{-1}\gamma\chi}P = PW_\chi W_{\chi^{-1}\gamma\chi}P = \sigma(\chi, \chi^{-1}\gamma\chi) V_{\gamma\chi},$$

so that

$$V_\gamma V_\chi - \sigma(\gamma, \chi) V_{\gamma\chi} = V_\gamma V_\chi - \sigma(\gamma, \chi) [\sigma(\chi, \chi^{-1}\gamma\chi)^{-1} V_\chi V_{\chi^{-1}\gamma\chi}]$$

belongs to $\mathcal{I}$.

Corollary 3.4. Let Γ be a discrete group, let $\sigma \in Z^2(\Gamma, \mathbf{T})$, and suppose $\theta : \Gamma \rightarrow \mathbf{R}$ is a homomorphism with dense range. Let α be the action of $\mathbf{R}$ on $C^*(\Gamma, \sigma)$ characterised by $\alpha_s(i(\gamma)) = \exp(2\pi i s \theta(\gamma)) i(\gamma)$. Then for every isometric σ -representation $V : \Gamma^+ \rightarrow B(H)$, there is a covariant representation (π, U) of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$ on a Hilbert space K containing H such that $V_\gamma = T_{i(\gamma)}$ and $C^*(V_\gamma, \sigma) = \mathcal{T}(\pi, U)$.

Proof. Since $\theta(\Gamma)$ is dense, $C_0(\mathbf{R})$ is contained in $B_{\theta(\Gamma)}(\mathbf{R})$, and the representation (ρ, W) constructed in Proposition 3.2(1) restricts to a covariant representation of $(C_0(\mathbf{R}), \Gamma, \tau \circ \theta, \sigma)$. If (π, U) is the covariant representation of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$ characterised by (3.9), then (3.1) says that $V_\gamma = T_{i(\gamma)}$, and the result follows from Proposition 1.1.

Theorem 3.5. Suppose Γ is a discrete group, $\sigma \in Z^2(\Gamma, \mathbf{T})$, and $\theta : \Gamma \rightarrow \mathbf{R}$ is a non-zero homomorphism such that $B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma$ is simple. Suppose $V : \Gamma^+ \rightarrow B(\mathcal{H})$ is an isometric σ -representation of $\Gamma^+ = \theta^{-1}([0, \infty))$ such that V_γ is unitary exactly when $\theta(\gamma) = 0$.

(1) The ideal $\mathcal{C}(V_\gamma, \sigma)$ is simple.

(2) The map $q : V_\gamma \rightarrow i(\gamma)$ extends to a homomorphism of $C^*(V_\gamma, \sigma)$ onto $C^*(\Gamma, \sigma)$, and there is a short exact sequence

$$0 \longrightarrow \mathcal{C}(V_\gamma, \sigma) \longrightarrow C^*(V_\gamma, \sigma) \xrightarrow{q} C^*(\Gamma, \sigma) \longrightarrow 0.$$

(3) If $Y : \Gamma^+ \rightarrow B(K)$ is another σ -representation of Γ^+ with the same properties, the map $V_\gamma \rightarrow Y_\gamma$ extends to a *-isomorphism of $C^*(V_\gamma, \sigma)$ onto $C^*(Y_\gamma, \sigma)$.

Proof. Part (1) follows immediately from 3.2(3): a corner in a simple algebra is simple, because the corner is Morita equivalent to the whole algebra. For the other parts, return to the notation of Proposition 3.2. For $\theta(\gamma) > 0$, V_γ is not unitary, we have $V_\gamma V_\gamma^* \neq P$, and with ρ as in 3.2(2), we have $\rho \times W(i_{B_{\theta(\Gamma)}}(1_{[0, \theta(\gamma)]})) = \rho(1_{[0, \theta(\gamma)]}) = P - V_\gamma V_\gamma^* \neq 0$. Thus $\rho \times W$ is faithful on the simple algebra $B_{\theta(\Gamma)}(\mathbf{R}) \times \Gamma$, and because $B_{\theta(\Gamma)}(\mathbf{R}) \times \Gamma$ is an essential ideal in $B_{\theta(\Gamma)}(-\infty, \infty] \times \Gamma$, it is faithful there too. In particular, this implies by Proposition 3.2(2) that, if $p = i_{B_{\theta(\Gamma)}}(1_{[0, \infty)})$, then $\rho \times W$ is an isomorphism of $p[B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma]p$ onto $C^*(V_\gamma, \sigma)$. This gives (3).

To establish (2), we observe that the exact sequence of Proposition 3.2(3) induces a short exact sequence

$$0 \longrightarrow B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma \longrightarrow B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma \xrightarrow{\varepsilon_\infty \times \iota} C^*(\Gamma, \sigma) \longrightarrow 0$$

(see [23, 2.29; 18, 2.2]). Now given any exact sequence

$$0 \longrightarrow I \longrightarrow B \xrightarrow{q} C \longrightarrow 0,$$

and any projection $p \in M(B)$, compressing with p gives another exact sequence

$$0 \longrightarrow pIp \longrightarrow pBp \xrightarrow{q} q(p)Cq(p) \longrightarrow 0.$$

We apply this observation with $p = i_{B_{\theta(\Gamma)}}(1_{[0, \infty)}) \in M(B_{\theta(\Gamma)} \times \Gamma)$; since $\varepsilon_\infty(1_{[0, \infty)}) = 1$, $\varepsilon_\infty \times \iota(p)$ is the identity of $C^*(\Gamma, \sigma)$, and we obtain an exact sequence

$$0 \longrightarrow p[B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma]p \longrightarrow p[B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \sigma} \Gamma]p \xrightarrow{\varepsilon_\infty \times \iota} C^*(\Gamma, \sigma) \longrightarrow 0.$$

Replacing the first two terms by their isomorphic images under $\rho \times W$, and invoking Proposition 3.2, gives the result.

We intend to verify the hypothesis of simplicity on $B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma$ through applications of [23, 4.20], which says in particular that $C_0(\Delta) \times_{\tau, \sigma} \Gamma$ is simple when Γ acts freely and minimally on Δ . Although we shall not always be applying this result directly to the system $(B_{\theta(\Gamma)}(\mathbf{R}), \Gamma, \tau \circ \theta, \sigma)$, we shall always have to consider the action of Γ on $B_{\theta(\Gamma)}(\mathbf{R})^\wedge$, and we therefore give a description of this space now.

Lemma 3.6. *Suppose Λ is a dense subgroup of $\mathbf{R}$. Then every homomorphism of $B_\Lambda(\mathbf{R})$ into $\mathbf{C}$ is either evaluation ε_x at some point x of $\mathbf{R}$, or the map ε_λ^- which sends f to $\lim_{x \rightarrow \lambda^-} f(x)$ for some $\lambda \in \Lambda$. Whenever $x_n \rightarrow x^+$, we have $\varepsilon_{x_n} \rightarrow \varepsilon_x$, and whenever $x_n \rightarrow \lambda^-$ and $\lambda \in \Lambda$, we have $\varepsilon_{x_n} \rightarrow \varepsilon_\lambda^-$.*

Proof. It is clear from Proposition 2.1(2) that these are all well-defined homomorphisms with the stated continuity properties, so our problem is to show that these are the only homomorphisms on $B_\Lambda(\mathbf{R})$. The set $\{\varepsilon_x : x \in \mathbf{R}\}$ is dense in $B_\Lambda(\mathbf{R})^\wedge$, since $\{\varepsilon_x(f)\}$ determines f , and hence we only have to prove that $\{\varepsilon_x\} \cup \{\varepsilon_\lambda^-\}$ is closed in $B_\Lambda(\mathbf{R})^\wedge$. Suppose $\{\phi_n\}$ is a sequence in $\{\varepsilon_x\} \cup \{\varepsilon_\lambda^-\}$ and $\phi_n(f) \rightarrow \phi(f)$ for all $f \in B_\Lambda(\mathbf{R})$. The subalgebra $C_0(\mathbf{R})$ of $B_\Lambda(\mathbf{R})$ has maximal ideal space $\mathbf{R}$, so there are unique points $x_n, x \in \mathbf{R}$ such that $\phi_n(f) = f(x_n)$, $\phi(f) = f(x)$ for $f \in C_0(\mathbf{R})$, and then $\phi_n \rightarrow \phi$ implies $x_n \rightarrow x$. Therefore we have to prove that ϕ is either ε_x or ε_x^- . If $x \notin \Lambda$, every $f \in B_\Lambda$ is continuous at x , and $\phi_n(f)$, which is either $f(x_n)$ or $\lim_{y \rightarrow x_n^-} f(y)$, converges to $f(x)$; this forces $\phi(f) = f(x)$ and $\phi = \varepsilon_x$. If $x \in \Lambda$, there are several possibilities: (a) there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} > x$ for all k ; (b) $x_{n_k} < x$ for all sufficiently large n ; (c) there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = x$ for all k . In case (a), $\phi_{n_k}(f) = f(x_{n_k})$ or $f(x_{n_k}^-)$ converges to $f(x)$, and $\phi = \varepsilon_x$. In case (b), $\phi_n(f) = f(x_n)$ or $f(x_n^-)$ converges to $\varepsilon_x^-(f)$, and $\phi = \varepsilon_x^-$. Finally, if (c) happens, we can go to a further subsequence to ensure that ϕ_{n_k} is always one of ε_x or ε_x^- , and then ϕ is also. This completes the proof of the Lemma.

4. Examples and applications.

4(a). Dense subgroups of $\mathbf{R}$. We begin by deducing Douglas's and Ji's theorems from ours. Suppose Γ is a subgroup of $\mathbf{R}$, and $\sigma \in Z^2(\Gamma, \mathbf{T})$. To apply our theorem, we need to show that $B_\Gamma(\mathbf{R}) \times_{\tau, \sigma} \Gamma$ is simple, and this will follow from [23, 4.20] and the next lemma.

Lemma 4.1. *If Γ is a subgroup of $\mathbf{R}$ then Γ acts freely and minimally on $B_\Gamma(\mathbf{R})^\wedge$.*

Proof We first deal with the case where $\Gamma = \mathbf{Z}\alpha$ for some α : then $B_\Gamma = c_0(\mathbf{Z}\alpha)$, $\widehat{B}_\Gamma = \mathbf{Z}\alpha$, and Γ acts freely and transitively. So we may suppose Γ is dense in $\mathbf{R}$, and hence $\Delta = B_\Gamma(\mathbf{R})^\wedge$ is the union of $\{\varepsilon_x : x \in \mathbf{R}\}$ and $\{\varepsilon_\gamma^- : \gamma \in \Gamma\}$ (Lemma 3.6). Since we trivially have $\gamma \cdot \varepsilon_x = \varepsilon_{\gamma+x}$, $\gamma \cdot \varepsilon_\chi^- = \varepsilon_{\gamma+\chi}^-$, Γ certainly acts freely on Δ . Now let $\Gamma \cdot \phi$ be an orbit. If $\phi = \varepsilon_x$ for some x , and $y \in \mathbf{R}$, we can find sequences $\{\gamma_n\}$ and $\{\chi_n\}$ in Γ such that $\gamma_n + x \rightarrow y^+$ and $\chi_n + x \rightarrow y^-$. Then from Lemma 3.6 we have

$$\gamma_n \cdot \phi = \varepsilon_{\gamma_n+x} \rightarrow \varepsilon_y, \quad \chi_n \cdot \phi = \varepsilon_{\chi_n+x} \rightarrow \begin{cases} \varepsilon_y & \text{if } y \notin \Gamma \\ \varepsilon_y^- & \text{if } y \in \Gamma, \end{cases}$$

and hence $\overline{\Gamma \cdot \phi} = \Delta$. If $\phi = \varepsilon_\chi^-$, then $\Gamma \cdot \phi = \{\varepsilon_\gamma^- : \gamma \in \Gamma\}$; for $x \in \mathbf{R}$ we can find $\gamma_n \in \Gamma$ such that $\gamma_n \rightarrow x^+$, and then we have $\varepsilon_{\gamma_n}^- \rightarrow \varepsilon_x$, so that again $\Gamma \cdot \phi$ is dense.

Corollary 4.2. *Suppose Γ is a subgroup of $\mathbf{R}$ and $\sigma \in Z(\Gamma, \mathbf{T})$. Then the C^* -algebra $C^*(\Gamma, \sigma)$ generated by a non-unitary isometric σ -representation V of Γ is canonically unique, and there is a short exact sequence of C^* -algebras*

$$0 \longrightarrow \mathcal{C}(V_\gamma, \sigma) \longrightarrow C^*(V_\gamma, \sigma) \longrightarrow C^*(\Gamma, \sigma) \longrightarrow 0.$$

This is [11, Theorem 1 (i), (iii)], [9, Theorem 1 and its Corollary] when $\sigma = 1$, and Coburn's theorem [2] when $\Gamma = \mathbf{Z}$, $\sigma = 1$. Part (ii) of Ji's theorem also follows easily from the proof of Theorem 3.5: because the isomorphism class of the crossed product $B_\Gamma(-\infty, \infty] \times_{\tau, \sigma} \Gamma$ only depends on the class of σ in $H^2(\Gamma, \mathbf{T})$ [23, §2], the same is true of $C^*(V_\gamma, \sigma)$.

4(b). The integer Heisenberg group. Let Γ be the integer Heisenberg group, i.e.

$$\Gamma = \left\{ \begin{pmatrix} 1 & p & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, p \in \mathbf{Z} \right\};$$

we frequently write elements of Γ as triples $(m, n, p) \in \mathbf{Z}^3$, but the matrix realisation tells us how to multiply them:

$$(m_1, n_1, p_1)(m_2, n_2, p_2) = (m_1 + m_2 + p_1 n_2, n_1 + n_2, p_1 + p_2)$$

Note that the centre and commutator subgroup of Γ are both $Z = \{(m, 0, 0)\}$, which we identify with $\mathbf{Z}$ in the obvious way, and that $(m, n, p) \rightarrow (n, p)$ induces an isomorphism of Γ/Z onto $\mathbf{Z}^2$. The central extension

$$0 \longrightarrow Z \longrightarrow \Gamma \longrightarrow \mathbf{Z}^2 \longrightarrow 0$$

is not split (otherwise Γ would be isomorphic to $\mathbf{Z}^3$), and a corresponding cocycle $\omega \in \mathbf{Z}^2(\mathbf{Z}^2, \mathbf{Z})$ is given by

$$\omega((n_1, p_1), (n_2, p_2)) = p_1 n_2.$$

Any homomorphism $\theta : \Gamma \rightarrow \mathbf{R}$ necessarily factors through Γ/Z , and hence has the form $\theta(m, n, p) = \alpha n + \beta p$ for some $\alpha, \beta \in \mathbf{R}$; the image $\theta(\Gamma)$ is dense in $\mathbf{R}$ iff $\{\alpha, \beta\}$ is independent over $\mathbf{Q}$.

Although there are many non-trivial cocycles on Γ to which we can apply our results, we do not obtain any by lifting cocycles from $\Gamma/Z \cong \mathbf{Z}^2$: the inflation map $\text{Inf} : H^2(\Gamma/Z, \mathbf{T}) \rightarrow H^2(\Gamma, \mathbf{T})$ is the zero map. (For proofs of this, see [19, 1.4(1)] and [19, 1.4(4)] — although the second is done there for the real Heisenberg group, it carries over verbatim.) The non-trivial multipliers are given by pairs of real numbers: for $\theta, \psi \in \mathbf{R}$ we define a multiplier $\sigma = \sigma(\phi, \psi)$ of Γ by

$$\begin{aligned} & \sigma((m_1, n_1, p_1), (m_2, n_2, p_2)) \\ &= \exp 2\pi i (\phi(m_2 p_1 + p_1(p_1 - 1)n_2/2) + \psi(n_1(m_2 + p_1 n_2) + p_1 n_2(n_2 - 1)/2)), \end{aligned} \quad (4.1)$$

and the map $(\phi, \psi) \rightarrow \sigma(\phi, \psi)$ induces an isomorphism of $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ onto $H^2(\Gamma, \mathbf{T})$ (see [16, 1.1]). We want to prove that the twisted crossed product $B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma$ is typically simple, so that we can apply Theorem 3.5. The hypotheses of [23, 4.20] are not satisfied here, since the action $\tau \circ \theta$ factors through Γ/Z , and hence Γ does not act freely on $B_{\theta(\Gamma)}(\mathbf{R})^\wedge$. However, the cocycle $\sigma(\phi, \psi)$ does involve Z in a nontrivial way, and we shall still be able to prove simplicity by decomposing $B_{\theta(\Gamma)} \times \Gamma$ as an iterated crossed product $(B_{\theta(\Gamma)} \times Z) \times \Gamma/Z$, and applying [23, 4.20] to the outside product.

By [17, 4.1], then, there is a twisted action (β, v) such that

$$B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma \cong (B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} Z) \times_{\beta, v} \mathbf{Z}^2;$$

β is defined in terms of a section $c : \mathbf{Z}^2 \rightarrow \Gamma$ by

$$\beta_{n,p}(i_{B_{\theta(\Gamma)}}(a)) = i_{B_{\theta(\Gamma)}}(\tau_{\theta(c(n,p))}(a))$$

$$\beta_{n,p}(i_Z(m)) = \sigma(c(n,p), m) \sigma(c(n,p) m c(n,p)^{-1}, c(n,p))^{-1} i_Z(c(n,p) m c(n,p)^{-1})$$

We shall not need the precise formula for v given in [17, 4.1], but we do need to observe that it is centre-valued, since we shall want to use results from [23]. We choose the obvious section $c : (n, p) \rightarrow (0, n, p)$, observe that $c(n, p)m c(n, p)^{-1} = m$ because $Z = Z$ is central, and deduce that

$$\beta_{n,p}(i_Z(m)) = \exp 2\pi i(\phi p m + \psi n m) i_Z(m)$$

Since $Z \subset \ker \theta$, and σ is trivial on $Z \times Z$, we can identify $B_{\theta(\Gamma)} \times_{\tau \circ \theta, \sigma} Z$ with $B_{\theta(\Gamma)} \otimes C^*(Z)$, and $\beta_{n,p}$ is then $\tau_{\alpha n + \beta p} \otimes \widehat{\alpha}_{\exp 2\pi i(\phi p + \psi n)}$, where $\widehat{\alpha}$ is the dual action of $\mathbf{T} = \widehat{Z}$ characterised by $\widehat{\alpha}_z(i(m)) = z^m i(m)$. Under the Fourier transform of $C^*(Z)$ onto $C(\mathbf{T})$, $\widehat{\alpha}_z$ is carried into τ_z , and hence

$$B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \sigma} \Gamma \cong (B_{\theta(\Gamma)}(\mathbf{R}) \otimes C(\mathbf{T})) \times_{\beta, v} \mathbf{Z}^2,$$

where $\beta_{n,p} = \tau_{\alpha n + \beta p} \otimes \tau_{\exp 2\pi i(\phi p + \psi n)}$.

According to [23, 4.20], this last twisted crossed product will be simple if the action of $\mathbf{Z}^2$ on $(B_{\theta(\Gamma)}(\mathbf{R}) \otimes C(\mathbf{T}))^\wedge = \widehat{B}_{\theta(\Gamma)} \times \mathbf{T}$ is free and minimal. This action is given on the underlying copy of $\mathbf{R} \times \mathbf{T}$ by

$$(n, p) \cdot (x, e^{2\pi i y}) = (x + \alpha n + \beta p, e^{2\pi i(y + \psi n + \phi p)}),$$

and provided this action is minimal, we can use the argument of Lemma 4.1 above to deduce that $\mathbf{Z}^2$ acts minimally on $\widehat{B}_{\theta(\Gamma)} \times \mathbf{T}$; if it does act minimally, $\{\alpha, \beta\}$ must be independent and the action is automatically free. One would instinctively assume that this action on $\mathbf{R} \times \mathbf{T}$ is well-known to be generically minimal, but it does not seem to fit the standard models, and we have been forced to do a little work to find necessary and sufficient conditions for minimality. We thank Chris Bose for helping us do this.

Lemma 4.3. *Four real numbers $\alpha, \beta, \phi, \psi$ have the property that, for any $x, y \in \mathbf{R}$ and $\varepsilon > 0$, there exist $n, p \in \mathbf{Z}$ such that*

$$|n\alpha + p\beta - x| < \varepsilon \quad \text{and} \quad |e^{2\pi i(p\phi + n\psi)} - e^{2\pi i y}| < \varepsilon,$$

if and only if $\Delta = \alpha\phi - \beta\psi \neq 0$ and $\{\Delta, \alpha, \beta\}$ is independent over $\mathbf{Q}$.

Proof. We begin by observing that $\Delta \neq 0$ is certainly a necessary condition. For $\Delta = 0$ implies $(\psi, \phi) = \lambda(\alpha, \beta)$ for some $\lambda \in \mathbf{R}$, and then if $n\alpha + p\beta$ is near x , $n\psi + p\phi = \lambda(n\alpha + p\beta)$ cannot be near $y \bmod \mathbf{Z}$ unless $y + \mathbf{Z} \sim \lambda x + \mathbf{Z}$, so the property does not hold for *all* choices of x, y . We therefore suppose $\Delta \neq 0$. Then our property is equivalent to: for any $x, y \in \mathbf{R}, \varepsilon > 0$, there exist $n, p \in \mathbf{Z}$ and $k \in \mathbf{Z}$ such that

$$\left\| \begin{pmatrix} \alpha & \beta \\ \psi & \phi \end{pmatrix} \begin{pmatrix} n \\ p \end{pmatrix} - \begin{pmatrix} x \\ y + k \end{pmatrix} \right\| < \varepsilon,$$

or, equivalently,

$$\left\| \begin{pmatrix} n \\ p \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \psi & \phi \end{pmatrix}^{-1} \begin{pmatrix} x \\ y+k \end{pmatrix} \right\| < \varepsilon \left\| \begin{pmatrix} \alpha & \beta \\ \psi & \phi \end{pmatrix} \right\|^{-1}.$$

(It doesn't matter which norm we use here: for definiteness we choose the Euclidean one.)

Thus it is enough to apply the following Lemma with

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \psi & \phi \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -\beta/\Delta \\ \alpha/\Delta \end{pmatrix} = \text{last column in } \begin{pmatrix} \alpha & \beta \\ \psi & \phi \end{pmatrix}^{-1}.$$

Lemma 4.4. *Two real numbers (b, d) have the property that, for any $(w, z) \in \mathbf{R}^2$ and $\varepsilon > 0$, there exist $n, p \in \mathbf{Z}$ and $k \in \mathbf{Z}$ such that*

$$\left\| \begin{pmatrix} n \\ p \end{pmatrix} - \begin{pmatrix} w \\ z \end{pmatrix} - k \begin{pmatrix} b \\ d \end{pmatrix} \right\| < \varepsilon,$$

if and only if $\{1, b, d\}$ is independent over $\mathbf{Q}$.

Proof. Suppose first that $\{1, b, d\}$ is independent. Then the action

$$l \cdot (\zeta, \eta) = (e^{2\pi i l b} \zeta, e^{2\pi i l d} \eta)$$

of $\mathbf{Z}$ on $\mathbf{T}^2$ is minimal, and hence there exist l, K and $L \in \mathbf{Z}$ such that

$$\left\| l \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} K \\ L \end{pmatrix} \right\| < \varepsilon/2.$$

Now consider

$$\left\{ \begin{pmatrix} w \\ z \end{pmatrix} + k \left(l \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} K \\ L \end{pmatrix} \right) : k \in \mathbf{Z} \right\}.$$

Since the increment $\mathbf{v} = l(b, d) + (K, L)$ has length $< \varepsilon/2$, the union

$$\bigcup_k B \left(\begin{pmatrix} w \\ z \end{pmatrix} + \mathbf{v}; \varepsilon \right)$$

contains a strip of width $\sqrt{3}\varepsilon/2$ about the line $\{(w, z) + t\mathbf{v}\}$. The slope of this line is $(ld + L)/(lb + K)$, which is irrational by the hypothesis on (b, d) , and hence the image of the line is dense in $\mathbf{R}^2/\mathbf{Z}^2$. Thus any strip of positive width about this line contains $(0, 0) + \mathbf{Z}^2$, and there exist k, n, p such that $(n, p) \in B((w, z) + k\mathbf{v}; \varepsilon)$. But this says precisely that

$$\left\| \begin{pmatrix} n - kK \\ p - kL \end{pmatrix} - \begin{pmatrix} w \\ z \end{pmatrix} - kl \begin{pmatrix} b \\ d \end{pmatrix} \right\| < \varepsilon.$$

Conversely, suppose these are integers i, j, l such that $lb = i + jd$, z is arbitrary and p, k satisfy $|p - z - kd| < \varepsilon = 1/(4 \max\{j, l\})$. Then for any $n \in \mathbb{Z}$

$$\begin{aligned} |w + kb - n| &= \left| w - k\left(\frac{i + jd}{l}\right) - n \right| \\ &= \frac{1}{l} |lw - kjd - n'| \quad \text{for some } n' \in \mathbb{Z} \\ &\geq \frac{1}{l} \left| |lw - j(p - z) - n'| - |j(p - z) - jkd| \right|. \end{aligned}$$

Thus if we pick w such that $lw + jz = 1/2$, for any $n \in \mathbb{Z}$ we have

$$|w + kb - n| \geq \frac{1}{l} \left| \frac{1}{2} - j|(p - z) - kd| \right| \geq \frac{1}{l} \left| \frac{1}{2} - \frac{1}{4} \right| = \frac{1}{4l},$$

and we cannot find n, p and k as required for this choice of z, w and ε .

Remark. Notice that $\{n\alpha + p\beta : n, p \in \mathbb{Z}^2\}$ dense in $\mathbf{R}$ implies that $\{\alpha, \beta\}$ is independent over $\mathbf{Q}$, and $\{\exp 2\pi i(n\psi + p\phi)\}$ dense in $\mathbf{T}$ implies that at least one of ψ, ϕ is irrational. So it is reassuring that the independence of $\{\Delta, \alpha, \beta\}$ forces both of these.

We have now established simplicity of $B_{\theta(\Gamma)} \times_{\tau \circ \theta, \sigma} \Gamma$, and can apply Theorem 3.5. We obtain:

Corollary 4.5. *Let Γ be the integer Heisenberg group, define $\theta : \Gamma \rightarrow \mathbf{R}$ by $\theta(m, n, p) = \alpha n + \beta p$, and let $\sigma = \sigma(\phi, \psi)$ be the multiplier of Γ given by (4.1). Suppose that the set $\{\alpha\phi - \beta\psi, \alpha, \beta\}$ is independent over $\mathbf{Q}$. If V and W are isometric σ -representations of the semigroup $\Gamma^+ = \{\gamma \in \Gamma : \theta(\gamma) \geq 0\}$ such that V_γ and W_γ are unitary if and only if $\theta(\gamma) = 0$, then the map $V_\gamma \rightarrow W_\gamma$ extends to an isomorphism of $C^*(V_\gamma)$ onto $C^*(W_\gamma)$.*

4(c). The rotation algebra. We finish by looking at the case where Γ is again the integer Heisenberg group, but $\sigma = 1$. Our main theorem does not apply here, since $B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta} \Gamma$ is not simple, and indeed for any given homomorphism $\theta : \Gamma \rightarrow \mathbf{R}$ with dense range it is quite easy to write down isometric representations of Γ^+ which generate non-isomorphic C^* -algebras. To see this, we define a set of representatives $\{\omega_\phi : \phi \in [0, 1)\}$ for $H^2(\mathbb{Z}^2, \mathbf{T})$ by

$$\omega_\phi((n_1, p_1), (n_2, p_2)) = \exp(2\pi i \phi p_1 n_2),$$

and note that if $W : \mathbb{Z}^2 \rightarrow U(H)$ is a unitary ω_ϕ -representation, then

$$U(m, n, p) = e^{-2\pi i \phi m} W(n, p)$$

is an ordinary unitary representation of Γ . The homomorphism $\theta : \Gamma \rightarrow \mathbf{R}$ defined by $\theta(m, n, p) = \alpha n + \beta p$ gives us positive cones Γ^+ in Γ and $(\mathbb{Z}^2)^+$ in Γ/Z such that

$\Gamma^+/Z = (\mathbb{Z}^2)^+$; every isometric ω_ϕ -representation V of $(\mathbb{Z}^2)^+$ is obtained by compressing some ω_ϕ -representation W , and the corresponding compression of U gives an isometric representation V^1 of Γ^+ such that

$$V^1(m, n, p) = \epsilon^{-2\pi i \phi m} V(n, p) \quad \text{when} \quad \alpha n + \beta p \geq 0.$$

We then trivially have

$$C^*(V^1(m, n, p) : (m, n, p) \in \Gamma^+) = C^*(V(n, p) : (n, p) \in (\mathbb{Z}^2)^+).$$

However, we know from Ji's theorem (Corollary 4.2 above) that $C^*(V_\gamma^1)$ has $C^*(\mathbb{Z}^2, \omega_\phi)$ as its unique simple quotient, and since $C^*(\mathbb{Z}^2, \omega_\phi)$ is not isomorphic to $C^*(\mathbb{Z}^2, \omega_\tau)$ unless $\phi = \tau$ or $1 - \tau$, we deduce that there are many different C^* -algebras $C^*(V^1)$ generated by isometric representations V^1 of Γ^+ .

It is by now well-known that the C^* -algebra $C^*(\Gamma)$ of the integer Heisenberg group (the rotation algebra of [1]) provides a common framework for the rotation algebras, or, equivalently for the different twisted group algebras $C^*(\mathbb{Z}^2, \omega_\phi)$ of $\mathbb{Z}^2$: indeed, $C^*(\Gamma)$ can be realised as the algebra of sections of a C^* -bundle G over $\mathbb{T}$ with fibre $G_\phi = C^*(\mathbb{Z}^2, \omega_\phi)$ over $e^{2\pi i \phi} \in \mathbb{T}$ (see, for example, [1, Theorem 1.1], [19, 1.4(1)], and below). We shall show that the corner $p(B_{\theta(\Gamma)} \times_{\tau \circ \theta} \Gamma)p$ plays a similar role for the different C^* -algebras $C^*(V_\gamma, \omega_\phi) = \mathcal{T}_\gamma^{\omega_\phi}$ associated to projective isometric representations of $(\mathbb{Z}^2)^+ \cong \theta(\Gamma)^+$.

As we did above in the case of non-trivial σ , we first decompose the crossed product using [17, 4.1]:

$$B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta} \Gamma \cong (B_{\theta(\Gamma)}(-\infty, \infty] \times_\iota \mathbb{Z}) \times_{\beta, v} \mathbb{Z}^2. \quad (4.2)$$

This time, when we identify $B_{\theta(\Gamma)}(-\infty, \infty] \times_\iota \mathbb{Z}$ with $B_{\theta(\Gamma)} \otimes C(\mathbb{T})$, β is carried into $\tau \circ \theta \otimes \iota$ and v is $1 \otimes \omega$, where we view the cocycle $\omega \in Z^2(\mathbb{Z}^2, \mathbb{Z})$ characterising the central extension Γ as an element of $Z^2(\mathbb{Z}^2, C(\mathbb{T}, \mathbb{T}))$:

$$\omega((n_1, p_1), (n_2, p_2))(e^{2\pi i \phi}) = \exp(2\pi i \phi p_1 n_2) = \omega_\phi((n_1, p_1), (n_2, p_2)).$$

Evaluation at $e^{2\pi i \phi} \in \mathbb{T}$ induces a surjection

$$\begin{aligned} \varepsilon_\phi \times \iota : B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta} \Gamma &\cong (B_{\theta(\Gamma)}(-\infty, \infty] \otimes C(\mathbb{T})) \times_{(\tau \circ \theta), 1 \otimes \omega} \mathbb{Z}^2 \\ &\longrightarrow B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \omega_\phi} \mathbb{Z}^2 \end{aligned}$$

which carries $p = i_{B_{\theta(\Gamma)}}(1_{[0, \infty)})$ into the projection $p_\phi = i_{B_{\theta(\Gamma)}}(1_{[0, \infty)})$ in $B_{\theta(\Gamma)} \times_{\tau \circ \theta, \omega} \mathbb{Z}^2$, and hence compresses to a surjection

$$\eta_\phi : p[B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta} \Gamma]p \rightarrow p_\phi[B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \omega_\phi} \mathbb{Z}^2]p_\phi \cong \mathcal{T}_{\theta(\Gamma)}^{\omega_\phi} \quad (4.3)$$

We can now state our result precisely:

Proposition 4.6. *Let Γ be the integer Heisenberg group, suppose $\alpha, \beta \in \mathbf{R}$ are independent over $\mathbf{Q}$, and define $\theta : \Gamma \rightarrow \mathbf{R}$ by $\theta(m, n, p) = \alpha n + \beta p$. Then there is a C^* -bundle F over $\mathbf{T}$ with fibre $\mathcal{T}_{\theta(\Gamma)}^{\omega_\phi}$ over $e^{2\pi i\phi}$ and an isomorphism*

$$\Phi : p(B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta} \Gamma)p \rightarrow \Gamma(F),$$

defined using the maps η_ϕ of (4.3) by $\Phi(pap)(e^{2\pi i\phi}) = \eta_\phi(pap)$.

Proof. We can view $B_{\theta(\Gamma)} \otimes C(\mathbf{T})$ as the algebra of sections of the trivial C^* -bundle $\mathbf{T} \times B_{\theta(\Gamma)}$, and then the action $(\tau \circ \theta) \otimes \iota$ of $\mathbf{Z}^2$ is given by bundle automorphisms. Thus it follows from [18, 2.1] (cf. also [20, 21]) that the evaluation maps induce an isomorphism of $(B_{\theta(\Gamma)} \otimes C(\mathbf{T})) \times_{\tau \circ \theta \otimes \iota, 1 \otimes \omega} \mathbf{Z}^2$ onto the algebra $\Gamma(E)$ of sections of a C^* -bundle E with fibre $E_\phi = B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta, \omega_\phi} \mathbf{Z}^2$ over $e^{2\pi i\phi} \in \mathbf{T}$. Composing this with the decomposition (4.2) gives an isomorphism Ψ of $B_{\theta(\Gamma)} \times_{\tau \circ \theta} \Gamma$ onto $\Gamma(E)$ defined by $\Psi(a)(e^{2\pi i\phi}) = \varepsilon_\phi \times \iota(a)$. But it is easy to check that, if q is a projection in $\Gamma(E)$ such that $q(e^{2\pi i\phi}) \neq 0$ for all ϕ , then $q\Gamma(E)q$ is the algebra of sections of a C^* -bundle with fibres $q(e^{2\pi i\phi})E_\phi q(e^{2\pi i\phi})$. Since η_ϕ is the compression of $\varepsilon_\phi \times \iota$, this gives the result.

Remark. The same constructions realise $C^*(\Gamma) \cong C(\mathbf{T}) \times_{\iota, \omega} \mathbf{Z}^2$ and $p[B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta} \Gamma]p$ as the section algebras of bundles G and C over $\mathbf{T}$ with fibres $G_\phi = C^*(\mathbf{Z}^2, \omega_\phi)$ and $C_\phi = p_\phi(B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta, \omega_\phi} \mathbf{Z}^2)p_\phi$ respectively over $e^{2\pi i\phi} \in \mathbf{T}$. Since Proposition 2.2 identifies C_ϕ with the twisted commutator ideal $\mathcal{C}_{\theta(\Gamma)}^{\omega_\phi}$, Corollary 3.2 gives us an exact sequence.

$$0 \longrightarrow C_\phi \longrightarrow F_\phi \longrightarrow G_\phi \longrightarrow 0$$

for each $e^{2\pi i\theta} \in \mathbf{T}$, and we can view the exact sequence

$$\begin{aligned} 0 \longrightarrow p(B_{\theta(\Gamma)}(\mathbf{R}) \times_{\tau \circ \theta} \Gamma)p &= \Gamma(C) \\ &\longrightarrow p(B_{\theta(\Gamma)}(-\infty, \infty] \times_{\tau \circ \theta} \Gamma)p = \Gamma(F) \longrightarrow C^*(\Gamma) = \Gamma(G) \longrightarrow 0 \end{aligned}$$

as an exact sequence of C^* -bundles. (We do not know if anyone has ever carefully formulated a suitable notion of exact sequence from the bundle-theoretic point of view [10], but it seems clear from the continuous-field approach [7, Chap. 10] that it should amount to exactness of the induced sequence of section algebras.)

4(d). Toeplitz operators. We saw in §1 that the Toeplitz algebras $\mathcal{T}(\pi, U)$ of [11, 14] are generated by isometric σ -representations. It therefore follows from our Theorem 3.5 that these Toeplitz algebras are often independent of the representation chosen. More precisely:

Corollary 4.7. *Let $\Gamma, \theta, \sigma, \alpha$ be as in Proposition 1.2, and suppose that $B_{\theta(\Gamma)} \times_{\tau \circ \theta, \sigma} \Gamma$ is simple. Then for any two covariant representations $(\pi, U), (\mu, V)$ of $(C^*(\Gamma, \sigma), \mathbf{R}, \alpha)$, the map $T_a^{\pi, U} \rightarrow T_a^{\mu, V}$ extends to an isomorphism of $\mathcal{T}(\pi, U)$ onto $\mathcal{T}(\mu, V)$.*

Proof. To apply Theorem 3.5, we need to check that $T_{i(\gamma)} = T_{i(\gamma)}^{\pi, U}$ and $T_{i(\gamma)}^{\mu, V}$ are unitary precisely when $\gamma \in \Gamma^+$ satisfies $\theta(\gamma) = 0$. If (ρ, W) is the covariant representation of $(C_0(\mathbf{R}), \Gamma, \tau \circ \theta, \sigma)$ corresponding to (π, U) , then $\rho \times W$ extends to a representation of the simple algebra $B_{\theta(\Gamma)} \times_{\tau \circ \theta, \sigma} \Gamma$, which must be faithful. Since $B_{\theta(\Gamma)}$ sits faithfully inside $B_{\theta(\Gamma)} \times_{\tau \circ \theta, \sigma} \Gamma$, $\rho = \rho \times W|_{B_{\theta(\Gamma)}(\mathbf{R})}$ is faithful. Thus

$$\begin{aligned} T_{i(\gamma)} T_{i(\gamma)}^* &= \rho(1_{[0, \infty)}) W_\gamma \rho(1_{[0, \infty)}) W_\gamma^* \rho(1_{[0, \infty)}) \\ &= \rho(1_{[0, \infty)}) \rho(1_{[\theta(\gamma), \infty)}) \rho(1_{[0, \infty)}) \end{aligned}$$

is the identity on $PH_\pi = \rho(1_{[0, \infty)})H_\pi$ exactly when $\theta(\gamma) = 0$. We therefore deduce from Theorem 3.5 that $T_{i(\gamma)}^{\pi, U} \rightarrow T_{i(\gamma)}^{\mu, V}$ extends to an isomorphism of $\mathcal{T}(\pi, U) = C^*(T_{i(\gamma)}^{\pi, U}, \sigma)$ onto $\mathcal{T}(\mu, V) = C^*(T_{i(\gamma)}^{\mu, V}, \sigma)$. Since $\{i(\gamma) \mid \gamma \in \Gamma\}$ spans a dense subspace of $C^*(\Gamma, \sigma)$, and $T_{i(\gamma^{-1})}$ is a multiple of $T_{i(\gamma)}^*$, this isomorphism carries $T_a^{\pi, U}$ onto $T_a^{\mu, V}$ for all $a \in C^*(\Gamma, \sigma)$.

It follows from the calculations in §4(a), (b) that this Corollary applies to almost all multipliers on dense subgroups of $\mathbf{R}$ and the integer Heisenberg group. Therefore, in these cases, the Toeplitz extension $\mathcal{T}(\pi, U)$ is independent of the representation chosen, as conjectured by Lesch [14, Remark 3.2], in greater generality.

This last observation is of some importance for the real-valued index theorems of [4, 11], where the Fredholm index on the C^* -algebra $C^*(V_\gamma)$ is only defined through a particular representation of $C^*(V_\gamma, \sigma)$ on $H^2(\mathbf{R}, l^2(\Gamma)) \cong L^2([0, \infty), l^2(\Gamma))$. In general, the index theorems of [11, 5, 14] only work for Toeplitz operators defined using the induced representation $\text{Ind } \pi_\tau = \tilde{\pi}_\tau \times \lambda$ of $A \times_\alpha \mathbf{R}$, where π_τ is the GNS-representation of A associated to an invariant trace τ .

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