A CERTAIN FAMILY OF GENERATING FUNCTIONS FOR CLASSICAL ORTHOGONAL POLYNOMIALS

by

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The main object of this work is to present some important developments in the theory of generating functions for the classical orthogonal polynomials including, for example, the Jacobi polynomials (and, of course, the Gegenbauer or ultraspherical polynomials, the Legendre or spherical polynomials, and the Chebyshev polynomials of the first and second kinds), the Laguerre polynomials, and the Hermite polynomials, and the various generalizations and discrete analogues of these polynomials.

1. Introduction and Definitions

Consider an orthogonal system of functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ such that the inner product

$$(\varphi_m,\varphi_n) = \int_a^b \varphi_m(x)\varphi_n(x)\,d\mu(x) = \lambda_n\ \delta_{m,n} \tag{1.1}$$

$$(m,n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\quad \mathbb{N} = \{1,2,3,\cdots\}),$$

where $\delta_{m,n}$ is the Kronecker delta, (a,b) is a finite, one–sided infinite, or two–sided infinite interval on the real axis, and $d\mu(x)$ is a distribution along that interval. Here

$$\lambda_n = \|\varphi_n\|^2 = (\varphi_n,\varphi_n) \qquad (n \in \mathbb{N}_0) \tag{1.2}$$

and $\mu(x)$ is a non–decreasing function; if $\mu(x)$ is absolutely continuous, we may set

$$\mu'(x) = w(x), \tag{1.3}$$

and refer to $w(x)$ as the weight function of the orthogonal system $\{\varphi_n(x)\}_{n=0}^{\infty}$.

The family of the classical orthogonal polynomials forms a special type of the orthogonal system $\{\varphi_n(x)\}_{n=0}^{\infty}$ defined by (1.1). This family is led by the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which indeed are the most general of the three classes of orthogonal polynomials mentioned above. These polynomials can be defined by a Rodrigues type formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\ n!} \cdot D_x^n\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\} \qquad \left[D_x = \frac{d}{dx}\right], \tag{1.4}$$

which may be employed to derive a number of explicit hypergeometric representations for the Jacobi polynomials. For example, we have

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k}\binom{\beta+n}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \tag{1.5}$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n}\ {}_2F_1\left[\begin{matrix}-n,\ \alpha+\beta+n+1; \\ \alpha+1;\end{matrix}\ \frac{1-x}{2}\right]. \tag{1.6}$$

Throughout the present work, we shall make use of the binomial coefficient:

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} \quad (n \in \mathbb{N}); \quad \binom{\lambda}{0} = 1, \tag{1.7}$$

defined for an arbitrary (real or complex) parameter λ, so that

$$\binom{\lambda+n-1}{n} = (-1)^n\binom{-\lambda}{n} \quad (n \in \mathbb{N}_0) \tag{1.8}$$

and

$$\begin{bmatrix}\lambda\\ n\end{bmatrix} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}_0), \tag{1.9}$$

where $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if} \quad n \in \mathbb{N}, \end{cases} \tag{1.10}$$

in terms of the familiar Gamma function. Furthermore, ${}_2F_1$ occurring in the explicit representation (1.6) is the Gaussian hypergeometric function which corresponds to a special case

$$p - 1 = q = 1 \tag{1.11}$$

of the generalized hypergeometric function ${}_pF_q$ (with p numerator and q denominator parameters) defined by

$${}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z) = {}_pF_q\begin{bmatrix}\alpha_1,\cdots,\alpha_p; & \\ & z \\ \beta_1,\cdots,\beta_q; & \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n} \frac{z^n}{n!} \tag{1.12}$$

$$(p,q \in \mathbb{N}_0;\ p \leqq q + 1;\ \ p < q + 1 \ \text{ and } \ |z| < \infty;$$

$$p = q + 1 \ \text{ and } \ z \in \mathcal{U} = \{z : |z| < 1\};$$

$$p = q + 1, \ \ z \in \partial\mathcal{U}, \ \text{ and } \ \mathrm{Re}(\omega) > 0),$$

where, for convenience,

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j , \tag{1.13}$$

provided (of course) that no zeros appear in the denominator of (1.12). Clearly, since

$$(-N)_n = \begin{cases} \dfrac{(-1)^n \, N!}{(N-n)\,!} , & \text{if} \quad n = 0,1,\cdots,N, \\ \\ 0 , & \text{if} \quad n = N+1, N+2, N+3, \cdots, \end{cases} \tag{1.14}$$

the series in (1.12) would terminate when one (or more) of the numerator parameters

$$\alpha_1, \cdots, \alpha_p$$

is zero or a negative integer, and then the question of convergence of the series will not arise. Thus, if one of

$$\alpha_1, \cdots, \alpha_p$$

is a nonpositive integer $-N$, and there are no zeros in the denominator of (1.12), the function ${}_pF_q(z)$ would reduce to what may be called a *hypergeometric polynomial* of degree N in z. For such a hypergeometric polynomial, it is not difficult to show from the definitions (1.12) and (1.10) that

$$ {}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \cdots, \alpha_p ; \\ \\ \beta_1, \cdots, \beta_q ; \end{matrix} \; z \right] = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \, (-z)^n $$

$$ \cdot \; {}_{q+1}F_p \left[\begin{matrix} -n, 1-\beta_1-n, \cdots, 1-\beta_q-n ; \\ \\ 1-\alpha_1-n, \cdots, 1-\alpha_p-n ; \end{matrix} \; \frac{(-1)^{p+q}}{z} \right] \qquad (n \in \mathbb{N}_0), \tag{1.15}$$

which can be applied to rewrite the hypergeometric representation (1.6) in the form:

$$P_n^{(\alpha,\beta)}(x) = \begin{bmatrix}\alpha+\beta+2n\\ n\end{bmatrix} \left[\frac{x-1}{2}\right]^n \; {}_2F_1\left[\begin{matrix}-n,-\alpha-n;\\ \\ -\alpha-\beta-2n;\end{matrix}\;\frac{2}{1-x}\right]. \tag{1.16}$$

When $\min\{\mathrm{Re}(\alpha),\, \mathrm{Re}(\beta)\} > -1$, these polynomials are orthogonal with respect to the Beta distribution on $[-1,1]$:

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, P_m^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(x)\, dx$$

$$= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}\,\delta_{m,n} \tag{1.17}$$

$$(m,n \in \mathbb{N}_0;\;\; \min\{\mathrm{Re}(\alpha),\, \mathrm{Re}(\beta)\} > -1).$$

Various other members of the family, which are *special* cases of the Jacobi polynomials, include the Gegenbauer (or ultraspherical) polynomials $C_n^{\nu}(x)$, where

$$C_n^{\alpha+\frac{1}{2}}(x) = \begin{bmatrix}\alpha+n\\ n\end{bmatrix}^{-1} \begin{bmatrix}2\alpha+n\\ n\end{bmatrix} P_n^{(\alpha,\alpha)}(x)$$

$$= \sum_{k=0}^{n} \frac{(\alpha+\frac{1}{2})_k(\alpha+\frac{1}{2})_{n-k}}{k!\,(n-k)!}\, e^{i(n-2k)\theta} \qquad (x = \cos\,\theta), \tag{1.18}$$

the relatively more familiar Legendre (or spherical) polynomials:

$$P_n(x) = P_n^{(0,0)}(x) = C_n^{\frac{1}{2}}(x), \tag{1.19}$$

and the Chebyshev polynomials (of the first and second kinds):

$$\begin{cases} T_n(x) = \begin{bmatrix} n-\frac{1}{2} \\ n \end{bmatrix}^{-1} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = \frac{1}{2}\, n\, C_n^0(x), \\ \\ U_n(x) = \frac{1}{2}\begin{bmatrix} n+\frac{1}{2} \\ n+1 \end{bmatrix}^{-1} P_n^{(\frac{1}{2},\frac{1}{2})}(x) = C_n^1(x), \end{cases} \tag{1.20}$$

where, by definition,

$$C_n^0(x) = \lim_{\lambda \to 0} \left\{\lambda^{-1}\, C_n^\lambda(x)\right\}. \tag{1.21}$$

Two other important members of the family of the classical orthogonal polynomials are the Hermite polynomials:

$$\begin{aligned} H_n(x) &= \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!}{k!\,(n-2k)!}\,(2x)^{n-2k} \\ &= (2x)^n\, {}_2F_0\left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2}; \\ \text{———}; & \end{matrix}\; -\frac{1}{x^2}\right] \end{aligned} \tag{1.22}$$

and the Laguerre polynomials:

$$\begin{aligned} L_n^{(\alpha)}(x) &= \sum_{k=0}^{n} \begin{bmatrix} \alpha+n \\ n-k \end{bmatrix} \frac{(-x)^k}{k!} \\ &= \begin{bmatrix} \alpha+n \\ n \end{bmatrix}\, {}_1F_1\left[\begin{matrix} -n; \\ \alpha+1; \end{matrix}\; x\right]. \end{aligned} \tag{1.23}$$

Indeed, since

$$H_n(x) = (-1)^n\, 2^{n/2}\, n! \lim_{\alpha \to \infty} \left\{\alpha^{-n/2}\, L_n^{(\alpha)}(\alpha + x\sqrt{2\alpha})\right\} \tag{1.24}$$

and

$$L_n^{(\alpha)}(x) = \lim_{\beta \to \infty} \left\{P_n^{(\alpha,\beta)}\left[1 - \frac{2x}{\beta}\right]\right\}, \tag{1.25}$$

many of the properties of the Hermite and Laguerre polynomials can be deduced from those involving the classical Jacobi polynomials.

Another interesting class of orthogonal polynomials is provided by the generalized Bessel polynomials:

$$\begin{aligned} y_n(x,\alpha,\beta) &= \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+n+k-2}{k} k!\left(\frac{x}{\beta}\right)^k \\ &= {}_2F_0\left[\begin{matrix}-n,\ \alpha+n-1; \\ \underline{\qquad\quad};\end{matrix}\ -\frac{x}{\beta}\right], \end{aligned} \tag{1.26}$$

which were studied systematically by Krall and Frink [32] (and, more recently, by Grosswald [25]). In view of the relationships:

$$y_n(x,\alpha,\beta) = \lim_{\lambda \to \infty} \left\{\frac{n!}{(\lambda)_n} P_n^{(\lambda-1,\alpha-\lambda-1)}\left[1 + \frac{2\lambda x}{\beta}\right]\right\} \tag{1.27}$$

and

$$y_n(x,\alpha,\beta) = n!\left[-\frac{x}{\beta}\right]^n L_n^{(1-\alpha-2n)}\left[\frac{\beta}{x}\right], \tag{1.28}$$

the Bessel polynomials are also recoverable from the classical Jacobi and Laguerre polynomials.

The classical Jacobi, Laguerre, and Hermite polynomials, and many of their aforementioned relatives, are often characterized by one or the other of a number of properties which they have in common. Given a system of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$, the three most important ones of these characteristics of the classical Jacobi, Laguerre, and Hermite polynomials may be recalled as follows:

Property 1 (Sonine [45], Hahn [26], and Krall [31]). *The derivatives* $\{p_n'(x)\}_{n=1}^{\infty}$ *form a system of orthogonal polynomials.*

Property 2 (Bochner [2]). *The polynomials* $p_n(x)$ *satisfy a differential equation of the form:*

$$A(x)\,\frac{d^2y}{dx^2} + B(x)\,\frac{dy}{dx} + \lambda_n\,y = 0, \tag{1.29}$$

where $A(x)$ *and* $B(x)$ *are independent of* n, *and* λ_n *is independent of* x.

Property 3 (Tricomi [58]). *The polynomials* $p_n(x)$ *are given by a generalized Rodrigues formula* [*cf.* Equation (1.4)]:

$$p_n(x) = \frac{1}{K_n\,w(x)}\,D_x^n\Big\{w(x)[X(x)]^n\Big\}, \tag{1.30}$$

where K_n *is a constant,* $X(x)$ *is a polynomial in* x *whose coefficients are independent of* n, *and* $w(x)$ *is independent of* n.

There are a number of important instances of orthogonal polynomials of a *discrete* variable, that is, orthogonal polynomials for which the distribution function $\mu(x)$ in (1.1) is a *jump function.* A study of such orthogonal polynomials was initiated by Chebyshev ([14], [15], and [16]) who also introduced a general system of orthogonal polynomials of a discrete variable. These polynomials (popularly known as Hahn polynomials) may be defined by (*cf.* Chebyshev [16] and Hahn [27]; see also Karlin and McGregor [29])

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left[\begin{matrix} -n,\ \alpha+\beta+n+1,\ -x; \\ \\ \alpha+1,\ -N; \end{matrix}\ 1\right] \tag{1.31}$$

and satisfy the (discrete) orthogonality relation:

$$\sum_{x=0}^{N} \begin{bmatrix} \alpha+x \\ x \end{bmatrix} \begin{bmatrix} \beta+N-x \\ N-x \end{bmatrix} Q_m(x;\alpha,\beta,N)\, Q_n(x;\alpha,\beta,N) = 0 \tag{1.32}$$

$$(m \neq n \in \{0,1,\cdots,N\}).$$

The following limiting cases of the Hahn polynomials are worthy of note.

Krawtchouk polynomials:

$$K_n(x;p,N) = {}_2F_1\left[\begin{matrix} -n, & -x; \\ & -N; \end{matrix} \quad \frac{1}{p}\right]$$

$$= \lim_{\alpha \to \infty} \left\{ Q_n\left[x;\ \alpha,\ \frac{(1-p)\alpha}{p},\ N\right]\right\}; \tag{1.33}$$

Meixner polynomials:

$$M_n(x;\beta,c) = (\beta)_n\ {}_2F_1\left[\begin{matrix} -n, & -x; \\ & -\beta; \end{matrix} \quad 1 - \frac{1}{c}\right]$$

$$= (\beta)_n \lim_{N \to \infty} \left\{ Q_n\left[x;\ \beta-1,\ \frac{(1-c)N}{c},\ N\right]\right\}; \tag{1.34}$$

Poisson–Charlier polynomials:

$$c_n(x;\alpha) = {}_2F_0\left[\begin{matrix} -n, & -x; \\ \underline{\qquad}; & \end{matrix} \quad -\frac{1}{\alpha}\right]$$

$$= \lim_{N \to \infty} \left\{ Q_n(x;\ N-1,\ N^2\alpha^{-1},\ N)\right\}. \tag{1.35}$$

All these orthogonal polynomials of a discrete variable are characterized by (for example) a finite–difference analogue of Property 3 above (see, *e.g.*, Hahn [27], and

Weber and Erdélyi [59]).

Since

$$P_n^{(\alpha,\beta)}(1-2x) = \binom{\alpha+n}{n} \lim_{N \to \infty} \left\{Q_n(xN;\ \alpha,\beta,N)\right\}, \tag{1.36}$$

some of the properties of the classical Jacobi polynomials can be deduced from those involving the Hahn polynomials defined by (1.31). We also note the relationships:

$$c_n(\alpha;\ x) = n!\left[-\frac{1}{x}\right]^n L_n^{(\alpha-n)}(x) \tag{1.37}$$

between Poisson–Charlier and Laguerre polynomials, and

$$M_n(x;\ \beta,c) = n!\ P_n^{(\beta-1,-\beta-n-x)}\left[\frac{2}{c} - 1\right] \tag{1.38}$$

between Meixner and Jacobi polynomials.

One further set of orthogonal polynomials of the type described above is the set of Meixner–Pollaczek polynomials defined by (*cf.* Meixner [35] and Pollaczek [40])

$$P_n^{(\lambda)}(x;\varphi) = \binom{2\lambda+n-1}{n} e^{in\varphi}\ {}_2F_1\left[\begin{matrix}-n,\ \lambda+ix; \\ 2\lambda;\end{matrix}\ 1-e^{-2i\varphi}\right] \tag{1.39}$$

$$(\lambda > 0;\quad 0 < \varphi < \pi),$$

which satisfy the orthogonality property (1.1) with

$$-a = b = \infty \quad \text{and} \quad w(x) = \mu'(x) = e^{(2\varphi-\pi)x}\ |\Gamma(\lambda+ix)|^2. \tag{1.40}$$

The subject of orthogonal polynomials is treated, in a lucid and systematic manner, by Szegö [57] who also considered some of the aforementioned discrete orthogonal polynomials. Other useful references on this subject include Erdélyi *et al.* [20, Chapter 10], Rainville [42], Luke [33, Chapter 11], Chihara [18], Brezinski *et al.* [3], Nikiforov *et al.* ([37], [38], and [39]), and Nevai [36]. In this work we aim at presenting some important developments in the theory of generating functions for many of the aforementioned polynomials and for certain other associated polynomials.

The generating functions considered in this work are consequences of Lagrange's expansion (*cf.*, *e.g.*, [60, p. 133]):

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D_z^{n-1}\left\{ f'(z)[\varphi(z)]^n\right\}\Big|_{z=z_0}, \tag{1.41}$$

for an analytic function $f(z)$, holomorphic at $z = z_0$,

$$z = z_0 + w\ \varphi(z), \tag{1.42}$$

where $\varphi(z)$ is holomorphic at $z = z_0$, and

$$\varphi(z_0) \neq 0. \tag{1.43}$$

Upon differentiating both sides of (1.41) with respect to w, if we make use of the relationship (1.42), and replace $f'(z)\varphi(z)$ in the resulting equation by $f(z)$, we arrive at the alternative form (*cf.* Pólya and Szegö [41, p. 146, Problem 207]):

$$\frac{f(z)}{1-w\ \varphi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n\left\{f(z)[\varphi(z)]^n\right\}\Big|_{z=z_0}, \tag{1.44}$$

which we shall find to be more suitable to apply here than (1.41). Evidently, in their special case when $\varphi(z) = 1$, both (1.41) and (1.44) reduce immediately to Taylor's expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0). \tag{1.45}$$

2. Carltiz's Theorem and Its Multiparameter (and Multivariable) Extensions

For the classical Laguerre polynomials defined by (1.23), the generating functions

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\ t^n = (1-t)^{-\alpha-1} \exp\left[-\frac{xt}{1-t}\right] \qquad (|t| < 1) \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\ t^n = (1+t)^{\alpha} \exp(-xt) \qquad (|t| < 1) \tag{2.2}$$

are well–known in the literature (*cf.*, *e.g.*, Erdélyi *et al.* [20], Rainville [42], and McBride [34]). An interesting unification (and generalization) of a number of generating functions for the classical Leguerre polynomials, including (for example) (2.1) and (2.2), was given by Carlitz [8] in the form:

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x)\ t^n = \frac{(1+v)^{\alpha+1}}{1-\lambda v} \exp(-xv), \tag{2.3}$$

where α and λ are arbitrary (real or complex) numbers, and v is a function of t defined implicitly by

$$v = t(1+v)^{\lambda+1}, \qquad v(0) = 0. \tag{2.4}$$

Making use of the explicit hypergeometric representation given by (1.23), Carlitz's result (2.3) can be rewritten in the hypergeometric form:

$$\sum_{n=0}^{\infty} \binom{\alpha+(\lambda+1)n}{n}\ {}_1F_1\left[\begin{matrix} -n; \\ \alpha+\lambda n+1; \end{matrix}\ x\right] t^n$$

$$= \frac{(1+v)^{\alpha+1}}{1-\lambda v} \exp(-xv), \tag{2.5}$$

where v is given by (2.4). Since

$$\exp(z) = {}_0F_0\left[\begin{matrix} \text{——};\\ \\ \text{——}; \end{matrix}\; z\right], \tag{2.6}$$

in the generalized hypergeometric notation (1.12), Srivastava [47] applied the Laplace (and inverse Laplace) transform techniques in order to generalize (2.5), by the principle of multidimensional mathematical induction, to the form:

$$\sum_{n=0}^{\infty} \binom{\alpha+(\lambda+1)n}{n} \; {}_{p+1}F_{q+1}\left[\begin{matrix} -n,\ a_1,\cdots,a_p;\\ \\ \alpha+\lambda n+1,\ b_1,\cdots,b_q; \end{matrix}\; x\right] t^n$$

$$= \frac{(1+v)^{\alpha+1}}{1-\lambda v} \; {}_pF_q\left[\begin{matrix} a_1,\cdots,a_p;\\ \\ b_1,\cdots,b_q; \end{matrix}\; -xv\right], \tag{2.7}$$

where v is given (as before) by (2.4).

Two special cases of (2.7) when

$$p = 2 \quad \text{and} \quad q = 1 \tag{2.8}$$

are worthy of note. Indeed, in view of the hypergeometric representations of the types (1.6) and (1.16), we thus find the following generating functions for the Jacobi polynomials (*cf.* Srivastava [47]):

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-(\lambda+1)n)}(x)\; t^n$$

$$= (1+w)^{-\alpha-\beta} (1-\lambda w)^{-1} \left[1 + \frac{2w}{1-x}\right]^{\alpha}, \tag{2.9}$$

where w is a function of x and t defined by

$$w = \frac{1}{2}(1-x)t\,(1+w)^{\lambda+1}, \qquad w(x,0) = 0; \tag{2.10}$$

$$\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\beta-(\lambda+1)n)}(x)\; t^n$$

$$= (1+v)^{\alpha+1}\,(1-\lambda v)^{-1}\left[1-\frac{1}{2}(x-1)v\right]^{-\alpha-\beta-1}, \tag{2.11}$$

where v is given by (2.4).

Formulas (2.9) and (2.11) incorporate, as their particular cases, a large number of generating functions for the Jacobi polynomials considered, for instance, by Brown [4], Calvez and Génin [5], Feldheim [21], and others. It may be of interest to observe that Jacobi's generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\; t^n = 2^{\alpha+\beta}\, R^{-1}(1-t+R)^{-\alpha}\,(1+t+R)^{-\beta} \tag{2.12}$$

$$(R = (1-2xt+t^2)^{\frac{1}{2}})$$

is not derivable as a special case of (2.9) and (2.11). (For several interesting proofs of the classical result (2.12), the interested reader is referred (among other places) to Szegö [57, Section 4.4], Rainville [42, Section 140], Carlitz [7], Askey [1], Foata and Leroux [22], and Srivastava [50].) However, in view of the limit relationship (1.25), Carlitz's generating function (2.3) can readily be recovered from Srivastava's result (2.11) upon replacing x by $1 - 2x/\beta$ and letting $\beta \longrightarrow \infty$. Furthermore, since

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\, P_n^{(\beta,\alpha)}(-x), \tag{2.13}$$

which follows from Jacobi's generating function (2.12), Carlitz's result (2.3) can also be deduced as a limiting case of Srivastava's formula (2.9).

The generating functions (2.9), (2.11), and (2.12) can be unified (and extended) by the following generalization of Carlitz's result (2.3) due to Srivastava and Singhal [55]:

$$\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)\ t^n$$

$$= (1+\xi)^{\alpha+1}(1+\eta)^{\beta+1}\{1 - \lambda\xi - \mu\eta - (1+\lambda+\mu)\xi\eta\}^{-1}, \tag{2.14}$$

where ξ and η satisfy

$$(x+1)^{-1}\xi = (x-1)^{-1}\eta = \tfrac{1}{2}t(1+\xi)^{\lambda+1}(1+\eta)^{\mu+1}. \tag{2.15}$$

Motivated by (2.3) and (2.14), Carlitz [9] derived generating functions for certain general one– and two–parameter coefficients, which we recall here as

Theorem 1. *Let $A(z)$ and $B(z)$ be arbitrary functions which are analytic in a neighborhood of the origin, and assume that*

$$A(0) = B(0) = 1. \tag{2.16}$$

Define the coefficients $\{c_n^{(\alpha)}\}$ by means of the generating function:

$$A(z)[B(z)]^{\alpha} = \sum_{n=0}^{\infty} c_n^{(\alpha)} \frac{z^n}{n!}, \tag{2.17}$$

where α is an arbitrary complex number independent of z.

Then, for an arbitrary parameter λ independent of z,

$$\sum_{n=0}^{\infty} c_n^{(\alpha+\lambda n)} \frac{t^n}{n!} = \frac{A(u)[B(u)]^{\alpha+1}}{B(u) - \lambda u B'(u)}, \tag{2.18}$$

where

$$u = t\,[B(u)]^{\lambda}. \tag{2.19}$$

More generally, if the functions $A(z)$, $B(z)$, *and* $C(z)$ *are analytic in a neighborhood of the origin such that*

$$A(0) = B(0) = C(0) = 1, \tag{2.20}$$

and if

$$A(z)[B(z)]^{\alpha}[\,C(z)]^{\beta} = \sum_{n=0}^{\infty} d_n^{(\alpha,\beta)} \frac{z^n}{n!}, \tag{2.21}$$

then, for arbitrary parameters α, β, λ, *and* μ *independent of* z,

$$\sum_{n=0}^{\infty} d_n^{(\alpha+\lambda n,\beta+\mu n)} \frac{t^n}{n!} = \frac{A(v)\,[B(v)]^{\alpha}[C(v)\,]^{\beta}}{1 - v\{\lambda[B'(v)/B(v)] + \mu[C'(v)/C(v)]\}}, \tag{2.22}$$

where

$$v = t\,[B(v)]^{\lambda}[\,C(v)]^{\mu}. \tag{2.23}$$

The proof of Theorem 1 makes use of Taylor's expansion (1.45) *and* Lagrange's expansion (1.44), *both* with $z_0 = 0$. Indeed, by Taylor's expansion (1.45) with $z_0 = 0$, (2.17) yields

$$c_n^{(\alpha)} = D_z^n\{A(z)[B(z)]^{\alpha}\}\Big|_{z=0} \qquad (n \in \mathbb{N}_0), \tag{2.24}$$

so that

$$c_n^{(\alpha+\lambda n)} = D_z^n\{\,f(z)[\varphi(z)]^n\}\Big|_{z=0} \qquad (n \in \mathbb{N}_0), \tag{2.25}$$

where, for convenience,

$$f(z) = A(z)[B(z)]^{\alpha} \quad \text{and} \quad \varphi(z) = [B(z)]^{\lambda}. \tag{2.26}$$

It follows from (2.25) that

$$\sum_{n=0}^{\infty} c_n^{(\alpha+\lambda n)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \; D_z^n\{ f(z)[\varphi(z)]^n\}\Big|_{z=0}, \tag{2.27}$$

where $f(z)$ and $\varphi(z)$ are given by (2.26).

We now apply Lagrange's expansion (1.44) with $z_0 = 0$ in the form (*cf.* Pólya and Szegö [41, p. 146, Problem 207]):

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \; D_z^n\{ f(z)[\varphi(z)]^n\}\Big|_{z=0} = \frac{f(\zeta)}{1 - t\,\varphi'(\zeta)}, \tag{2.28}$$

where the functions $f(z)$ and $\varphi(z)$ are analytic about the origin, and ζ is given implicitly by

$$\zeta = t\,\varphi(\zeta) \qquad (\varphi(0) \neq 0),$$

and the assertion (2.18) follows readily from (2.27) under the constraints (2.16) and (2.19), it being understood that the choice of 1 in the conditions (2.16) [and (2.20)] is merely a convenient one and, in fact, that any nonzero constant values can be assumed for $A(0)$, $B(0)$, and $C(0)$.

The generating functions (2.3) and (2.14), and many of their special cases including (2.1), (2.2), (2.9), (2.11), and (2.12), have been recorded systematically by Hansen [28], and (more recently) by Srivastava and Manocha [53]. Additionally, Srivastava and Manocha [53, Chapter 7] have included a systematic presentation of the aforementioned work of Carlitz [53], and indeed also of Srivastava [48] who gave a multiparameter and multivariable extension of Carlitz's theorem (Theorem 1) in the following form:

Theorem 2. *Let* $A(z)$, $B(z)$, *and* $z^{-1}\,C(z)$ *be arbitrary functions which are* *analytic in a neighborhood of the origin, and assume that*

$$A(0) = B(0) = C'(0) = 1. \tag{2.29}$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}$ *by means of the generating function:*

$$A(z)[B(z)]^{\alpha} \exp(x\,C(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{z^n}{n!}, \tag{2.30}$$

where α *and* x *are arbitrary complex numbers independent of* z.

Then, for arbitrary parameters λ *and* y *independent of* z,

$$\sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny) \frac{t^n}{n!}$$

$$= \frac{A(\zeta)\,[\,B(\zeta)]^{\alpha} \exp(x\,C(\zeta))}{1 - \zeta\{\lambda[B'(\zeta)/B(\zeta)] + y\,C'(\zeta)\}}, \tag{2.31}$$

where

$$\zeta = t\,[B(\zeta)]^{\lambda} \exp(y\,C(\zeta)). \tag{2.32}$$

More generally, if $A(z)$, $B_i(z)$, *and* $z^{-1}\,C_j(z)$ *are analytic about the origin such that*

$$A(0) = B_i(0) = C_j'(0) = 1 \qquad (i = 1,\cdots,r; \quad j = 1,\cdots,s), \tag{2.33}$$

and if

$$A(z) \prod_{i=1}^{r} \left\{[B_i(z)]^{\alpha_i}\right\} \exp\left[\sum_{j=1}^{s} x_j\,C_j(z)\right]$$

$$= \sum_{n=0}^{\infty} g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_s)\,\frac{z^n}{n!}\,, \tag{2.34}$$

then, for arbitrary parameters

$$\alpha_i,\quad \lambda_i,\quad x_j,\quad \textit{and}\quad y_j \quad (i = 1,\cdots,r;\quad j = 1,\cdots,s)$$

independent of z,

$$\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\cdots,\alpha_r+\lambda_r n)}(x_1+ny_1,\cdots,x_s+ny_s)\,\frac{t^n}{n!}$$

$$= \frac{A(w)\prod_{i=1}^{r}\left\{[B_i(w)]^{\alpha_i}\right\}\exp\left[\sum_{j=1}^{s} x_j\, C_j(w)\right]}{1 - w\left\{\sum_{i=1}^{r} \lambda_i[B_i'(w)/B_i(w)] + \sum_{j=1}^{s} y_j\, C_j'(w)\right\}}\,, \tag{2.35}$$

where

$$w = t\prod_{i=1}^{r}\left\{[B_i(w)]^{\lambda_i}\right\}\exp\left[\sum_{j=1}^{s} y_j\, C_j(w)\right]. \tag{2.36}$$

Remark 1. The choice of 1 in the conditions (2.29) and (2.33) is merely a convenient one; in fact, as already observed above in connection with the constraints (2.16) and (2.20) of Theorem 1, any nonzero constant values may be assumed for

$$A(0),\ B(0),\ C'(0),\ B_i(0),\quad \text{and}\quad C_j'(0)$$

$$(i = 1,\cdots,r;\ j = 1,\cdots,s).$$

Remark 2. For $x = y = 0$, the generating function (2.31) reduces to the assertion (2.18) of Theorem 1.

Remark 3. The general result (2.35) with

$$r = 2 \quad \text{and} \quad x_j = y_j = 0 \quad (j = 1,\cdots,s)$$

is essentially the same as the assertion (2.22) of Theorem 1.

The first assertion (2.30) of Theorem 2 may be deduced from the two-parameter result (2.22) in Theorem 1 by making some suitable notational and parametric changes. A direct proof of (2.30), as also of its multiparameter and multivariable extension (2.35), would follow the lines of Carlitz's proof of Theorem 1; it can easily be detailed here by applying Taylor's expansion (1.45) *and* Lagrange's expansion (1.44), *both* with $z_0 = 0$, *mutatis mutandis.*

Carlitz [9] applied Theorem 1 involving general one- and two-parameter coefficients to prove the following class of *mixed* generating functions for the Laguerre polynomials:

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x+ny)\, t^n = \frac{(1-\zeta)^{-\alpha-1} \exp\left[-\frac{x\zeta}{1-\zeta}\right]}{1 - \zeta(1-\zeta)^{-1}\,[\lambda - y(1-\zeta)^{-1}]}, \tag{2.37}$$

where

$$\zeta = t\,(1-\zeta)^{-\lambda} \exp\left[-\frac{y\zeta}{1-\zeta}\right]. \tag{2.38}$$

Setting $\zeta = w/(1+w)$, (4.13) assumes the form:

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x+ny)\, t^n = \frac{(1+w)^{\alpha+1} \exp(-xw)}{1 - w[\lambda - y(1+w)]}, \tag{2.39}$$

where

$$w = t\,(1+w)^{\lambda+1} \exp(-yw). \tag{2.40}$$

For $y = 0$, both (2.37) and (2.39) reduce immediately to Carlitz's earlier result (2.3) above.

Cigler [19] showed that some of Carlitz's results in [9] can alternatively be derived by using the theory of Sheffer sets (*cf.* Rota *et al.* [43], [44]).

Srivastava [48] applied Theorem 2 to derive *mixed* generating functions for various classes of polynomials. In particular, he considered the Srivastava–Singhal generating function [54, p. 78, Equation (3.2)]:

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x^{1/r}, r, p, k)\ z^n = (1-kz)^{-\alpha/k} \cdot \exp(px[1 - (1-kz)^{-r/k}]), \tag{2.41}$$

where $G_n^{(\alpha)}(x,r,p,k)$ are the polynomials considered by Srivastava and Singhal [54] in an attempt to present a unified study of various known generalizations of the classical Laguerre and Hermite polynomials, the parameters α, p, k, and r being arbitrary (with, of course, $k,r \neq 0$). Thus it follows from the assertion (2.31) of Theorem 2 that

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\lambda n)}([x+ny]^{1/r}, r, p, k)\ t^n = \frac{(1-\alpha)^{-\alpha/k} \exp(px[1 - (1-\zeta)^{-r/k}])}{1 - k^{-1}\ \zeta(1-\zeta)^{-1}\ [\lambda - rpy(1-\zeta)^{-r/k}]}, \tag{2.42}$$

where

$$\zeta = kt(1-\zeta)^{-\lambda/k} \exp(py[1 - (1-\zeta)^{-r/k}]). \tag{2.43}$$

Putting $\zeta = w/(1+w)$ in (2.42), we have

$$\sum_{n=0}^{\infty} G_n^{(\alpha+\lambda n)}([x+ny]^{1/r},r,p,k)\, t^n$$

$$= \frac{(1+w)^{\alpha/k} \exp(px[1-(1+w)^{r/k}])}{1 - k^{-1}\, w[\lambda - rpy(1+w)^{r/k}]}\,, \tag{2.44}$$

where

$$w = kt(1+w)^{1+\lambda/k} \exp(py[1-(1+w)^{r/k}]). \tag{2.45}$$

By employing the known relationships [54]

$$G_n^{(\alpha+1)}(x,1,1,1) = L_n^{(\alpha)}(x) \tag{2.46}$$

and

$$G_n^{(0)}(x,2,1,-1) = \frac{(-x)^n}{n!} H_n(x) = G_n^{(1-n)}(x,2,1,1), \tag{2.47}$$

it is not difficult to deduce, from (2.42) and (2.44), the corresponding *mixed* generating functions for the classical Laguerre and Hermite polynomials. More importantly, since (*cf.*, *e.g.*, Srivastava and Manocha [53, p. 381, Equation 7.6(19)])

$$G_n^{(\alpha+1)}(x,1,1,k) = k^n\, Y_n^{\alpha}(x;k), \tag{2.48}$$

where $Y_n^{\alpha}(x;k)$ are one class of the biorthogonal polynomials considered by Konhauser [30] for $\alpha > -1$ and $k \in \mathbb{N}$, (2.42) readily yields the *mixed* generating function:

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n}(x+ny;k)\, t^n$$

$$= \frac{(1-\zeta)^{-(\alpha+1)/k} \exp(x[1 - y(1-\zeta)^{-1/k}])}{1 - k^{-1} \zeta(1-\zeta)^{-1} [\lambda - y(1-\zeta)^{-1/k}]}, \tag{2.49}$$

where

$$\zeta = t(1-\zeta)^{-\lambda/k} \exp(y[1 - (1-\zeta)^{-1/k}]), \tag{2.50}$$

while (2.44) reduces to the form:

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n}(x+ny;k)\, t^n$$

$$= \frac{(1+w)^{(\alpha+1)/k} \exp(x[1 - (1+w)^{1/k}])}{1 - k^{-1} w[\lambda - y(1+w)^{1/k}]}, \tag{2.51}$$

where

$$w = t(1+w)^{1+\lambda/k} \exp(y[1 - (1+w)^{1/k}]). \tag{2.52}$$

It should be noted in passing that

$$Y_n^{\alpha}(x;1) = L_n^{(\alpha)}(x) \qquad (\alpha > -1; \quad n \in \mathbb{N}_0), \tag{2.53}$$

and that the polynomials $Y_n^{\alpha}(x;2)$ were encountered earlier by Spencer and Fano [46] in certain analytical calculations involving the penetration of Gamma rays through matter (see also Srivastava [49]).

For $y = 0$, the generating functions (2.49) and (2.51) immediately yield the following *equivalent* forms of a result due to Calvez and Génin [6, p. A41, Equation (2)]:

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n}(x;k)\, t^n$$

$$= \frac{(1-\zeta)^{-(\alpha+1)/k} \exp(x[1-(1-\zeta)^{-1/k}])}{1-\lambda k^{-1}\, \zeta(1-\zeta)^{-1}}\,, \tag{2.54}$$

where

$$\zeta = t\,(1-\zeta)^{-\lambda/k}; \tag{2.55}$$

$$\sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n}(x;k)\, t^n$$

$$= \frac{(1+w)^{(\alpha+1)/k} \exp(x[1-(1+w)^{1/k}])}{1-\lambda k^{-1}\, w}\,, \tag{2.56}$$

where

$$w = t\,(1+w)^{1+\lambda/k}. \tag{2.57}$$

In view of the relationship (2.53), this last generating function (2.56) reduces, when $k = 1$, to Carlitz's result (2.3).

3. Further Applications and Analogues

Since

$$G_n^{(\alpha)}(x,r,\beta,-1) = \frac{(-x)^n}{n!} H_n^r(x,\alpha+n,\beta) \tag{3.1}$$

or, alternatively,

$$G_n^{(\alpha-n+1)}(x,r,\beta,1) = \frac{(-x)^n}{n!} H_n^r(x,\alpha,\beta), \tag{3.2}$$

the generating functions (2.42) and (2.44) would apply also to the generalized Hermite polynomials $H_n^r(x,\alpha,\beta)$ introduced by Gould and Hopper [24] by means of the Rodrigues formula:

$$H_n^r(x,\alpha,\beta) = (-1)^n x^{-\alpha} \exp(\beta x^r) D_x^n\{x^\alpha \exp(-\beta x^r)\}. \tag{3.3}$$

Clearly, we have [*cf.* Equation (2.47)]

$$H_n^2(x,0,1) = H_n(x), \tag{3.4}$$

in terms of the classical Hermite polynomials defined by (1.22).

For numerous further applications of Theorem 1 and Theorem 2, involving other sequences of special functions and polynomials, we refer the interested reader to the works by Carlitz [9], Srivastava [48], Srivastava and Manocha [53, Chapter 7], and Srivastava *et al.* [52]. By making use of some generating functions of Srivastava and Buschman [51], involving certain classes of generalized hypergeometric polynomials occurring in (1.15) and (2.7), Chandel and Yadava [12] have presented several additional applications of Theorem 2. Carlitz and Srivastava [10], on the other hand, have recently applied these general results with a view to obtaining several new generating functions for the classical Hermite polynomials.

We now turn to some binomial analogues of Theorem 2. Indeed, by virtue of the elementary Eulerian integral:

$$\int_0^\infty u^{\mu-1}\, e^{-su}\, du = \frac{\Gamma(\mu)}{s^{\mu}} \qquad (\mathrm{Re}(s) > 0;\quad \mathrm{Re}(\mu) > 0),$$

if we replace x in the definition (2.30) by xu, multiply both sides by $u^{\beta-1}\, e^{-u}$, and integrate the resulting equation with respect to u over the semi–infinite interval $(0,\infty)$, we obtain

$$A(z)[B(z)]^{\alpha}[1-x\,C(z)]^{-\beta} = \sum_{n=0}^{\infty} F_n^{(\alpha,\beta)}(x)\,\frac{z^n}{n!}\,, \tag{3.5}$$

where

$$F_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\beta)}\int_0^\infty u^{\beta-1}\, e^{-u} f_n^{(\alpha)}(xu)\, du. \tag{3.6}$$

Proceeding in this manner and applying Theorem 2, or (alternatively) by writing

$$[1 - x\,C(z)]^{-1} \quad \text{for} \quad C(z),$$

and

$$F_n^{(\alpha,\beta)}(x) \quad \text{for} \quad d_n^{(\alpha,\beta)},$$

in Theorem 1, it is not difficult to obtain the following binomial analogue of Theorem 2 (*cf.* Srivastava and Manocha [53, p. 398, Problem 30]; see also Chandel and Yadava [11], [13]):

Theorem 3. *Under the hypotheses surrounding the constraints* (2.29) *and* (2.23) *of Theorem* 2, *let the sequence of functions* $\{F_n^{(\alpha,\beta)}(x)\}$ *be given by* (3.5), *and define the multiparameter and multivariable sequence*

$$\left\{G_n^{(\alpha_1,\cdots,\alpha_r;\,\beta_1,\cdots,\beta_s)}(x_1,\cdots,x_s)\right\}$$

by means of the generating function:

$$A(z) \prod_{i=1}^{r} \left\{[B_i(z)]^{\alpha_i}\right\} \prod_{j=1}^{s} \left\{[1 - x_j C_j(z)]^{-\beta_j}\right\}$$

$$= \sum_{n=0}^{\infty} G_n^{(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_s)}(x_1,\cdots,x_r) \frac{z^n}{n!}, \tag{3.7}$$

the parameters α, β, *and* x, *and*

$$\alpha_i\ ,\ \beta_j\ ,\quad \text{and}\quad x_j \qquad (i = 1,\cdots,r;\ j = 1,\cdots,s),$$

being independent of z.

Then

$$\sum_{n=0}^{\infty} F_n^{(\alpha+\lambda n,\beta+\mu n)}(x) \frac{t^n}{n!}$$

$$= \frac{A(\zeta)[B(\zeta)]^{\alpha} [1 - x C(\zeta)]^{-\beta}}{1 - \zeta\{\lambda[B'(\zeta)/B(\zeta)] + \mu x\ C'(\zeta)/[1 - x C(\zeta)]\}}, \tag{3.8}$$

where

$$\zeta = t[B(\zeta)]^{\lambda} [1 - x C(\zeta)]^{-\mu}, \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} G_n^{(\alpha_1+\lambda_1 n,\cdots,\alpha_r+\lambda_r n;\beta_1+\mu_1 n,\cdots,\beta_s+\mu_s n)}(x_1,\cdots,x_r) \frac{t^n}{n!}$$

$$= \frac{A(w) \prod_{i=1}^{r} \left\{[B_i(w)]^{\alpha_i}\right\} \prod_{j=1}^{s} \left\{[1 - x_j\ C_j(w)]^{-\beta_j}\right\}}{1 - w\left\{\sum_{i=1}^{r} \lambda_i[B_i'(w)/B_i(w)] + \sum_{j=1}^{s} \mu_j x_j C_j'(w)/[1 - x_j C_j(w)]\right\}}, \tag{3.10}$$

where

$$w = t \prod_{i=1}^{r} \left\{[B_i(w)]^{\lambda_i}\right\} \prod_{j=1}^{s} \left\{[1 - x_j C_j(w)]^{-\mu_j}\right\} . \tag{3.11}$$

The multiparameter and multivariable generating function (2.35), and its binomial analogue (3.10), can be applied to a generalized Lauricella polynomial of the type

$$F_D^{(s)}(-n,\beta_1,\cdots,\beta_s;\ \alpha;\ x_1,\cdots,x_s);$$

see, for details, Srivastava [48], and Chandel and Yadava [13]. These latter authors [13] have also applied the assertion (3.8) of Theorem 3 to the generalized Humbert polynomials defined by Gould [23] by means of the generating function:

$$\sum_{n=0}^{\infty} P_n(m,x,y,\rho,\sigma)\, t^n = (\sigma - mxt + yt^m)^\rho, \tag{3.12}$$

where $m \in \mathbb{N}$ and the other parameters are unrestricted in general.

By employing such relationships as (1.37) and (1.38), and the hypergeometric representations in (1.31), (1.33), (1.34), (1.35), and (1.39), it is not difficult to specialize some of the above results in terms of various polynomials of a discrete variable.

Many of the generating functions considered or referred to above have tremendous potential for applications. For example, the generating function (2.14) was applied recently by Chen and Ismail [17] in order to determine the asymptotic behavior of the Jacobi polynomials

$$P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)$$

when $n \to \infty$ and α, β, λ, μ, and x remain fixed; in fact, these authors have similarly applied Carlitz's generating function (2.37) for the classical Laguerre polynomials. On the other hand, Strehl [56] has presented an interesting combinatorial proof of the Srivastava–Singhal generating function (2.14).

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