

**BRUHAT DECOMPOSITION AND
NUMERICAL STABILITY**

O.H. ODEH, D.D. OLESKY & P. VAN DEN DRIESSCHE

DMS-751-IR

December 1996

BRUHAT DECOMPOSITION AND NUMERICAL STABILITY

O.H. ODEH

*Department of Computer Science
University of Victoria
Victoria, British Columbia
Canada V8W 3P6
oodeh@gulf.uvic.ca*

D.D. OLESKY¹

*Department of Computer Science
University of Victoria
Victoria, British Columbia
Canada V8W 3P6
dolesky@csr.uvic.ca*

P. VAN DEN DRIESSCHE ¹

*Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia
Canada V8W 3P4
pvdd@smart.math.uvic.ca*

December 11, 1996

¹ The research of this author was partially supported by the Natural Science and Engineering Research Council of Canada and the University of Victoria Committee on Faculty Research and Travel.

Abstract

For a real nonsingular n -by- n matrix A , there exists a decomposition $A = V\Pi U$, where Π is a permutation matrix and V, U are upper triangular matrices. When $\Pi^T V \Pi$ is lower triangular and U is normalized, such a decomposition is called the left Bruhat decomposition of A . An algorithm for computing the left Bruhat decomposition is given. For classes of matrices introduced by Wilkinson and recently (from a practical application) by Foster that have an exponential growth factor when Gaussian elimination with partial pivoting (GEPP) is applied, left Bruhat decomposition has at most linear growth. A partial pivoting strategy for Bruhat decomposition is also developed, and an explicit equivalence between GEPP and Bruhat decomposition with partial pivoting (BDPP) is derived. This equivalence implies that the growth factor for GEPP on A equals the growth factor for BDPP on ρA^T , where ρ is the permutation matrix that reverses the rows of A^T . BDPP is shown to give a growth factor of at most 2 when applied to any matrix for which GEPP gives the maximal growth factor of 2^{n-1} .

Keywords: Bruhat decomposition, Gaussian elimination, growth factor, numerical stability, partial pivoting.

AMS subject classifications. 65F05, 15A23

1. Introduction

Matrix factorization techniques are frequently used for solving nonsingular systems of linear equations. The most common factorization is LU decomposition, and Gaussian elimination with partial pivoting (GEPP) is the most common practical algorithm for computing an LU decomposition. However, other decompositions, such as LPR decomposition (see, e.g., Elsner [2], Gohberg and Goldberg [4]) and Bruhat decomposition, can also be used to solve linear systems.

Bruhat decomposition, known from the theory of linear algebraic groups ([5], [9]), was considered by Kolotilina and Yeremin [9] as an alternative to LU decomposition for solving sparse systems of linear equations. Kolotilina and Yeremin also gave relations between Bruhat decomposition and the other two decompositions given above, and sparsity of the Bruhat decomposition factors was considered in [8].

In the following sections we describe the left Bruhat decomposition, and give an algorithm for its computation (Algorithm 2.1), which is the analogue of an algorithm given in [9, Section 2] for the right Bruhat decomposition. In contrast with GEPP, Bruhat decomposition is numerically stable for the classes of matrices given by Wilkinson and Foster (Section 3). We also introduce a pivoting strategy for Bruhat decomposition (Algorithm 4.1), and derive explicit relationships (Corollary 4.5) between the factors that are determined by applying GEPP to A and Bruhat decomposition with partial pivoting (BDPP) to ρA^T , where ρ is the permutation matrix that reverses the order of the rows of A^T . We show that BDPP gives a growth factor of at most 2 when applied to matrices that give maximal growth when GEPP is applied (Section 5). BDPP is a practical algorithm for solving systems of linear equations, and is an alternative to consider when GEPP may be unstable.

2. Description of the Bruhat Decomposition

Let A be a given n -by- n real nonsingular matrix. Then there exists a decomposition

$$A = V\Pi U \tag{2.1}$$

where V and U are n -by- n upper triangular matrices and Π is an n -by- n permutation matrix. The permutation matrix Π in (2.1) is uniquely determined by A [9]. A decomposition of the form (2.1) is called a *Bruhat decomposition* of the matrix A , and Π is called the *Bruhat permutation* of A . The decomposition (2.1) is called the *reduced on the left* Bruhat decomposition if the matrix $\Pi^T V \Pi$ is lower triangular, and *reduced on the right* if the matrix $\Pi U \Pi^T$ is lower triangular [9]. For the remainder of this paper, we work with the reduced on the left Bruhat decomposition with U normalized to have all diagonal entries equal to 1, and we refer to this as the *left Bruhat decomposition*. With this normalization, the left Bruhat decomposition of a given nonsingular matrix is unique.

The decomposition (2.1) can be computed by post-multiplication of A by $n - 1$ nonsingular matrices $U^{(i)}$, whose entries are chosen so as to introduce zeros into the matrix product. Let

$$A^{(0)} = A \quad \text{and} \quad A^{(i)} = A^{(i-1)} U^{(i)}, \quad 1 \leq i \leq n - 1,$$

so that $A^{(i)} = A U^{(1)} U^{(2)} \dots U^{(i)}$.

Denoting $A^{(i)} = [a_{jk}^{(i)}]$, the matrices $U^{(i)}$ can be written compactly as

$$U^{(i)} = I - e^{(i)} \left(m^{(i)} \right)^T,$$

where

$$m_j^{(i)} = \begin{cases} \frac{a_{r_i,j}^{(i-1)}}{a_{r_i,i}^{(i-1)}}, & \text{for } i+1 \leq j \leq n \\ 0, & \text{otherwise,} \end{cases}$$

$$e_j^{(i)} = \begin{cases} 1, & i = j \\ 0, & \text{otherwise,} \end{cases}$$

and r_i is the maximum row index such that $a_{r_i,i}^{(i-1)} \neq 0$. Thus, at the i th step, $a_{r_i,i}^{(i-1)}$ is the pivot entry, and multiplication by $U^{(i)}$ zeros out all entries of $A^{(i-1)}$ in row r_i and columns $i+1, \dots, n$. After $n-1$ elimination steps, $A^{(n-1)} = AU^{(1)}U^{(2)} \dots U^{(n-1)}$. Let $a_{r_n,n}^{(n-1)}$ denote the sole nonzero entry in column n of $A^{(n-1)}$, and $\Pi = [\pi_{jk}]$ be the permutation matrix with $\pi_{r_k,k} = 1$, for $1 \leq k \leq n$. Then, letting

$$V = A^{(n-1)} \Pi^T \tag{2.2}$$

and $U^{-1} = U^{(1)}U^{(2)} \dots U^{(n-1)}$ gives $A = V\Pi U$.

The following algorithm determines the factors of this decomposition.

Algorithm 2.1: Left Bruhat Decomposition**Input:** Nonsingular n -by- n matrix A **Output:** The matrices V , Π , and U , where the left Bruhat decomposition is $A = V\Pi U$ **Initialization:** $U = I$

```

for  $i = 1$  to  $n$ 
     $j = \max\{p \mid a_{pi} \neq 0\}$ 
     $\pi_{ji} = 1, \quad \pi_{\ell i} = 0$  for  $\ell \neq j$ 
     $\nu_{tj} = a_{ti}$  for  $1 \leq t \leq n$ 
    for  $k = i + 1$  to  $n$ 
         $m = \frac{a_{jk}}{a_{ji}}$ 
         $u_{ik} = m$ 
        for  $\ell = 1$  to  $j - 1$ 
             $a_{\ell k} = a_{\ell k} - m a_{\ell i}$ 
         $a_{jk} = 0$ 

```

By construction, U is upper triangular. Thus, to prove that Algorithm 2.1 gives the left Bruhat decomposition of A , we show that V is upper triangular, and then show that $\Pi^T V \Pi$ is lower triangular. Let $\pi(j) = i$ if $\pi_{ji} = 1$; then $\pi^{-1}(i) = j$. From (2.2), for any fixed q , $\nu_{iq} = a_{i, \pi(q)}^{(n-1)}$. If $q = \max\{p \mid a_{p, \pi(q)}^{(n-1)} \neq 0\}$, then $a_{i, \pi(q)}^{(n-1)} = 0$ for $i > q$, hence V is upper triangular. Also by (2.2),

$$(\Pi^T V \Pi)_{rj} = (\Pi^T A^{(n-1)})_{rj} = a_{\pi^{-1}(r), j}^{(n-1)}.$$

But $a_{\pi^{-1}(r), r}^{(n-1)} \neq 0$ and $a_{\pi^{-1}(r), j}^{(n-1)} = 0$ for $j > r$ as these entries are eliminated in the r th step of the algorithm. Hence $\Pi^T V \Pi$ is lower triangular.

In general, Π cannot be determined from the zero-nonzero pattern of A ; it depends as well on the numerics. Even if matrix A does not have an LU decomposition, there exists a permutation matrix P such that PA has an LU decomposition. Such a permutation matrix is Π^T from the left Bruhat decomposition [9], because if $A = V\Pi U$, then $\Pi^T A = (\Pi^T V \Pi)U = LU$. This relationship between the left Bruhat decomposition of A and the LU decomposition (with U normalized) of $\Pi^T A$ shows that each of the triangular factors of the left Bruhat decomposition is uniquely determined.

3. Bruhat Decomposition of Matrices with Large γ for GEPP

For GEPP on a nonsingular matrix $A = [a_{jk}]$, the growth factor γ is defined as

$$\gamma = \max_{i,j,k} |a_{jk}^{(i)}| / \max_{j,k} |a_{jk}|,$$

where $A^{(i)} = [a_{jk}^{(i)}]$ is the derived matrix after the i th elimination step (see, *e.g.*, [7, p. 177] and [10, p. 151]). The computation of the solution x of a linear system $Ax = b$ may be unstable if the growth factor is very large [6]. Motivated by a backward error analysis for LU decomposition [7, p. 176] and its relationship to Bruhat decomposition, we define the *growth factor* for Bruhat decomposition as

$$\gamma_B = \max \left\{ \max_{i,j,k} |u_{jk}^{(i)}| / \max_{j,k} |a_{jk}|, \max_{i,j,k} |a_{jk}^{(i)}| / \max_{j,k} |a_{jk}| \right\}. \quad (3.1)$$

Wilkinson [11, p. 212] introduced an n -by- n matrix W_n that achieves the largest possible growth factor of 2^{n-1} when GEPP is applied. The Bruhat decomposition, on the other hand, gives $\gamma_B = 2$, as demonstrated in the following example.

Example 3.1

The left Bruhat decomposition of the 5-by-5 Wilkinson matrix is

$$\begin{aligned} W_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & -\frac{1}{2} & -\frac{1}{4} & 1 \\ 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

In general, application of Algorithm 2.1 to the n -by- n Wilkinson matrix W_n gives $\gamma_B = 2$.

Wilkinson also noted that matrices with large γ do not seem to arise in practical applications. However, recently, Foster [3] discussed a class of n -by- n matrices that arises in the numerical solution of Volterra integral equations and that for GEPP has growth

factor close to the maximal value of 2^{n-1} . In contrast, for Bruhat decomposition on an n -by- n matrix in Foster's class, the factors V and U can be explicitly determined, and γ_B is linear in n .

Bruhat decomposition is a good alternative to GEPP for the matrices above when the latter gives exponentially large growth factors. However, for some matrices both GEPP and Bruhat decomposition give exponential growth (for example, the block matrix given by Wright [12, equations (10) and (12)]). There are also examples of matrices for which Bruhat decomposition gives exponential growth, whereas GEPP gives constant growth; one such example is ρW_n , where ρ is the permutation matrix that reverses the rows of W_n .

4. A Pivoting Strategy for Bruhat Decomposition

We now present a pivoting strategy for Bruhat decomposition that, like the use of partial pivoting with Gaussian elimination, keeps the multipliers bounded by 1 and usually results in a stable computation. The decomposition is computed by post-multiplication of A by $n - 1$ pairs of nonsingular matrices $P^{(i)}U^{(i)}$ for $i = 1, 2, \dots, n - 1$, where $P^{(i)}$ is a permutation matrix and $U^{(i)}$ is chosen to introduce zeros into the matrix product. Let $A^{(0)} = A$ and $A^{(i)} = A^{(i-1)}P^{(i)}U^{(i)}$, so that

$$A^{(i)} = AP^{(1)}U^{(1)}P^{(2)}U^{(2)} \dots P^{(i)}U^{(i)}.$$

At the i th step of the decomposition, $P^{(i)}$ is chosen to interchange columns i and c of $A^{(i-1)}$, where c is such that

$$\max_{i \leq t \leq n} |a_{n-i+1,t}^{(i-1)}| = |a_{n-i+1,c}^{(i-1)}|.$$

Then $U^{(i)}$ is chosen so that $a_{n-i+1,r}^{(i)} = 0$, for $r = i + 1, i + 2, \dots, n$. That is, letting $A^{(i-1)}P^{(i)} = [\tilde{a}_{jk}^{(i-1)}]$, then $U^{(i)} = I - e^{(i)}(m^{(i)})^T$, where

$$m_j^{(i)} = \begin{cases} \frac{\tilde{a}_{n-i+1,j}^{(i-1)}}{\tilde{a}_{n-i+1,i}^{(i-1)}}, & \text{for } i + 1 \leq j \leq n \\ 0, & \text{otherwise.} \end{cases}$$

After $n - 1$ steps,

$$\begin{aligned}
A^{(n-1)} &= AP^{(1)}U^{(1)}P^{(2)}U^{(2)}\dots P^{(n-1)}U^{(n-1)} \\
&= \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(2)} & \dots & a_{1,n-2}^{(n-2)} & a_{1,n-1}^{(n-1)} & a_{1n}^{(n-1)} \\ a_{21}^{(1)} & a_{22}^{(2)} & \dots & a_{2,n-2}^{(n-2)} & a_{2,n-1}^{(n-1)} & \\ a_{31}^{(1)} & a_{32}^{(2)} & \dots & a_{3,n-2}^{(n-2)} & & \\ \vdots & \ddots & & & 0 & \\ a_{n1}^{(1)} & & & & & \end{bmatrix} \\
&= V\rho,
\end{aligned}$$

where V is an upper triangular matrix and the permutation matrix ρ reverses the columns of V .

The following algorithm essentially determines the factors of the above decomposition

$$A = V\rho \left(U^{(n-1)} \right)^{-1} P^{(n-1)} \left(U^{(n-2)} \right)^{-1} P^{(n-2)} \dots \left(U^{(1)} \right)^{-1} P^{(1)},$$

where we note that $(P^{(i)})^{-1} = P^{(i)}$ for $i = 1, 2, \dots, n - 1$. The one-dimensional array P has $P(i) = c$ if $P^{(i)}$ interchanges columns i and c of $A^{(i-1)}$. The i th row of the upper triangular matrix $(U^{(i)})^{-1}$ is stored in the i th row of an n -by- n matrix U . The reduced matrices $A^{(i)}$ overwrite A and the function $swap(i, c)$ is used to interchange columns i and c of A .

Algorithm 4.1: Bruhat Decomposition with Partial Pivoting (BDPP)

Input: Nonsingular n -by- n matrix A

Output: The essential components of the factors V , $(U^{(i)})^{-1}$ and $P^{(i)}$ of the Bruhat decomposition with partial pivoting of A .

Initialization: $U = I, P(j) = j$ for $j = 1, 2, \dots, n-1$

for $j = n$ to 2

$i = n - j + 1$

 find $c : \max_{i \leq t \leq n} |a_{jt}| = |a_{jc}|$

 if $c > i$ then

 swap (i, c)

$P(i) = c$

$\nu_{tj} = a_{ti}$ for $1 \leq t \leq j$

 for $k = i + 1$ to n

$m = \frac{a_{jk}}{a_{ji}}$

$u_{ik} = m$

 for $\ell = 1$ to $j - 1$

$a_{\ell k} = a_{\ell k} - m a_{\ell i}$

$a_{jk} = 0$

$\nu_{11} = a_{1n}$

The next theorem shows an equivalence between BDPP and GEPP.

Theorem 4.2. *Let A be an n -by- n nonsingular matrix. Suppose that*

$$L^{(n-1)} \bar{P}^{(n-1)} L^{(n-2)} \bar{P}^{(n-2)} \dots L^{(1)} \bar{P}^{(1)} A = \bar{U}$$

is the result of applying GEPP to A , where $L^{(i)}$ is the lower triangular matrix of multipliers and $\bar{P}^{(i)}$ is the permutation matrix associated with the i th step of GEPP. Suppose also that

$$\rho A^T P^{(1)} U^{(1)} P^{(2)} U^{(2)} \dots P^{(n-1)} U^{(n-1)} = V \rho$$

is the result of applying BDPP to $\rho A^T = B$. Then $P^{(i)} = \bar{P}^{(i)}$, $U^{(i)} = (L^{(i)})^T$, and $\rho(A^{(i)})^T = B^{(i)}$ for $1 \leq i \leq n-1$.

Proof. The proof is by induction. For $i = 1$, consider the first step of GEPP. Let

$$\max_{1 \leq j \leq n} |a_{j1}| = |a_{t1}|.$$

Thus the effect of $\bar{P}^{(1)}$ is to interchange rows 1 and t . Letting $\bar{P}^{(1)}A = [\tilde{a}_{jk}]$, then $L^{(1)} = I - m^{(1)}(e^{(1)})^T$, where $m_j^{(1)} = \frac{\tilde{a}_{j1}}{\tilde{a}_{11}}$ is the j th entry of the vector $m^{(1)}$ for $2 \leq j \leq n$ and $m_1^{(1)} = 0$. Thus $A^{(1)} = L^{(1)}P^{(1)}A$. Now consider the first step of BDPP applied to $B = \rho A^T$. Note that

$$b_{n-p+1,j} = \sum_{k=1}^n \rho_{n-p+1,k} a_{jk} = a_{jp} \quad \text{for } 1 \leq j, p \leq n.$$

Thus

$$\max_{1 \leq j \leq n} |b_{nj}| = \max_{1 \leq j \leq n} |a_{j1}| = |a_{t1}| = |b_{nt}|,$$

and the effect of $P^{(1)}$ is to interchange columns 1 and t , so that $P^{(1)} = (\bar{P}^{(1)})^T = \bar{P}^{(1)}$. Let $BP^{(1)} = [\tilde{b}_{jk}]$, and note that $BP^{(1)} = \rho(\bar{P}^{(1)}A)^T$. Now $U^{(1)} = I - e^{(1)}(x^{(1)})^T$, where

$$x_j^{(1)} = \frac{\tilde{b}_{nj}}{\tilde{b}_{n1}} = \frac{\tilde{a}_{j1}}{\tilde{a}_{11}} = m_j^{(1)} \quad \text{for } 2 \leq j \leq n \quad \text{and} \quad x_1^{(1)} = m_1^{(1)} = 0.$$

Thus $U^{(1)} = (L^{(1)})^T$, hence

$$B^{(1)} = BP^{(1)}U^{(1)} = \rho \left(L^{(1)} \bar{P}^{(1)} A \right)^T = \rho \left(A^{(1)} \right)^T.$$

Thus the statement is true for $i = 1$.

Suppose that the theorem is true for all i such that $1 \leq i \leq s < n - 1$, and consider the $(s+1)$ st step of GEPP. Let

$$\max_{s+1 \leq j \leq n} |a_{j,s+1}^{(s)}| = |a_{r,s+1}^{(s)}|.$$

Thus the effect of $\bar{P}^{(s+1)}$ is to interchange rows $(s+1)$ and r . Letting

$$\bar{P}^{(s+1)} A^{(s)} = [\tilde{a}_{jk}^{(s)}],$$

then $L^{(s+1)} = I - m^{(s+1)}(e^{(s+1)})^T$, where

$$m_j^{(s+1)} = \frac{\tilde{a}_{j,s+1}^{(s)}}{\tilde{a}_{s+1,s+1}^{(s)}} \quad \text{for } s+2 \leq j \leq n$$

and

$$m_j^{(s+1)} = 0 \quad \text{for } 1 \leq j \leq s+1.$$

Thus $A^{(s+1)} = L^{(s+1)} \bar{P}^{(s+1)} A^{(s)}$. Now consider the $(s+1)$ st step of BDPP applied to $B = \rho A^T$. By the induction hypothesis, $B^{(s)} = \rho (A^{(s)})^T$, and consequently

$$\max_{s+1 \leq j \leq n} |b_{n-s,j}^{(s)}| = \max_{s+1 \leq j \leq n} |a_{j,s+1}^{(s)}| = |a_{r,s+1}^{(s)}| = |b_{n-s,r}^{(s)}|.$$

Thus the effect of $P^{(s+1)}$ is to interchange columns $(s+1)$ and r , so that

$$P^{(s+1)} = (\bar{P}^{(s+1)})^T = \bar{P}^{(s+1)}.$$

Let $B^{(s)} P^{(s+1)} = [\tilde{b}_{jk}^{(s)}]$, and note that $B^{(s)} P^{(s+1)} = \rho (\bar{P}^{(s+1)} A^{(s)})^T$. Now

$$U^{(s+1)} = I - e^{(s+1)} (x^{(s+1)})^T,$$

where

$$x_j^{(s+1)} = \frac{\tilde{b}_{n-s,j}^{(s)}}{\tilde{b}_{n-s,s+1}^{(s)}} = \frac{\tilde{a}_{j,s+1}^{(s)}}{\tilde{a}_{s+1,s+1}^{(s)}} = m_j^{(s+1)} \quad \text{for } s+2 \leq j \leq n$$

and

$$x_j^{(s+1)} = m_j^{(s+1)} = 0 \quad \text{for } 1 \leq j \leq s+1.$$

Thus $U^{(s+1)} = (L^{(s+1)})^T$, and

$$B^{(s+1)} = B P^{(s+1)} U^{(s+1)} = \rho \left(L^{(s+1)} \bar{P}^{(s+1)} A \right)^T = \rho \left(A^{(s+1)} \right)^T,$$

completing the proof. \blacksquare

Remark 4.3. Consider GEPP applied to $A^T \rho$. From Theorem 4.2, this is equivalent to application of BDPP to $\rho(A^T \rho)^T = A$.

An immediate consequence of Theorem 4.2 is the following, which shows that Bruhat decomposition with partial pivoting on A determines the left Bruhat decomposition of a column permutation of A . This result is analogous to a well known result for GEPP.

Corollary 4.4. Suppose A is an n -by- n nonsingular matrix and let

$$A P^{(1)} U^{(1)} P^{(2)} U^{(2)} \dots P^{(n-1)} U^{(n-1)} = V \rho$$

be the result of BDPP applied to A . Then there exist a permutation matrix P and an upper triangular matrix U such that $AP = V\rho U$.

Proof. Let

$$L^{(n-1)} \bar{P}^{(n-1)} L^{(n-2)} \bar{P}^{(n-2)} \dots L^{(1)} \bar{P}^{(1)} A^T \rho = L^{-1} \bar{P} A^T \rho = \bar{U}$$

be the result of applying GEPP to $A^T \rho$ (see, e.g., [1, p. 123] and [10, p. 125]). Thus

$$\begin{aligned} \bar{U}^T &= \rho A \bar{P}^T (L^{-1})^T \\ &= \rho A \bar{P}^{(1)} \left(L^{(1)} \right)^T \dots \bar{P}^{(n-1)} \left(L^{(n-1)} \right)^T \\ &= \rho A P^{(1)} U^{(1)} \dots P^{(n-1)} U^{(n-1)}, \end{aligned}$$

by Theorem 4.2 and Remark 4.3. Hence $A \bar{P}^T (L^{-1})^T = V\rho$, which implies that $A \bar{P}^T = V\rho L^T$, giving the required result with $P = \bar{P}^T$ and $U = L^T$. ■

We summarize the relationship between GEPP and BDPP in the following corollary.

Corollary 4.5. *Suppose A is an n -by- n nonsingular matrix. If the result of applying GEPP to A is $\bar{P}A = L\bar{U}$, and the result of applying BDPP to ρA^T is $\rho A^T P = V\rho U$, then*

$$\bar{P} = P^T, \quad L = U^T, \quad \text{and} \quad \bar{U} = \rho V^T \rho.$$

By virtue of the relations between the Bruhat decomposition and the LU decomposition, and between BDPP and GEPP, both Algorithms 2.1 and 4.1 require about $n^3/3$ flops (see, e.g., [1]).

5. Stability of BDPP

For BDPP the growth of entries in U is bounded by 1; thus, from (3.1), the growth factor for BDPP is

$$\gamma_{BP} = \max_{i,j,k} |a_{jk}^{(i)}| / \max_{j,k} |a_{jk}|.$$

For ρW_n , the row reversal of the Wilkinson matrix, it can be shown that $\gamma = 2$, $\gamma_B = 2^{n-1}$ and $\gamma_{BP} = 2$. The transpose of the Wilkinson matrix, W_n^T , is another matrix that has an exponential growth factor ($\gamma_B = 2^{n-1}$) when Algorithm 2.1 is applied, and a constant growth factor ($\gamma_{BP} = 4$) when Algorithm 4.1 is applied. Note that by the equivalence in

Theorem 4.2, γ for A equals γ_{BP} for ρA^T . Thus $\gamma_{BP} \leq 2^{n-1}$, and this upper bound is realized, for example, by ρW_n^T .

We now show that $\gamma_{BP} \leq 2$ for every n -by- n real matrix that has $\gamma = 2^{n-1}$ when GEPP is applied. The following theorem due to Higham and Higham characterizes this class of matrices, which includes W_n .

Theorem 5.1 [6, Theorem 2.2]. *All real n -by- n matrices for which $\gamma = 2^{n-1}$ are of the form*

$$A = DM \begin{bmatrix} T & \vdots & \theta d \\ 0 & \vdots & \end{bmatrix},$$

where $D = \text{diag}(\pm 1)$, M is unit lower triangular with $m_{ij} = -1$ for $i > j$, $T = [t_{ij}]$ is a nonsingular upper triangular matrix of order $n-1$, $d = [1 \ 2 \ 4 \ \dots \ 2^{n-1}]^T$, and θ is a scalar such that

$$\theta = |a_{1n}| = \max_{i,j} |a_{ij}|.$$

For example, the general form of a 5-by-5 matrix with $D = I$ having $\gamma = 2^4$ is

$$A = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & \theta \\ -t_{11} & t_{22} - t_{12} & t_{23} - t_{13} & t_{24} - t_{14} & \theta \\ -t_{11} & -(t_{22} + t_{12}) & t_{33} - (t_{23} + t_{13}) & t_{34} - (t_{24} + t_{14}) & \theta \\ -t_{11} & -(t_{22} + t_{12}) & -(t_{33} + t_{23} + t_{13}) & t_{44} - (t_{34} + t_{24} + t_{14}) & \theta \\ -t_{11} & -(t_{22} + t_{12}) & -(t_{33} + t_{23} + t_{13}) & -(t_{44} + t_{34} + t_{24} + t_{14}) & \theta \end{bmatrix}.$$

Theorem 5.2. *Let A be a real n -by- n matrix for which $\gamma = 2^{n-1}$ when GEPP is applied. Then application of BDPP to A gives $\gamma_{BP} \leq 2$.*

Proof. As A is assumed to have $\gamma = 2^{n-1}$, matrix A must be of the form given in Theorem 5.1. At the first step of BDPP on A , if

$$\theta = \max_{1 \leq q \leq n-1} |a_{nq}| = |a_{n1}| = |t_{11}|,$$

then no interchange is performed; however, if

$$\theta > \max_{1 \leq q \leq n-1} |a_{nq}| \quad \text{or} \quad \theta = \max_{2 \leq q \leq n-1} |a_{nq}| = |a_{nk}|$$

with $k \in [2, \dots, n-1]$, then $P^{(1)}$ interchanges columns 1 and n . (Note that this includes a tie-breaking strategy for BDPP.) After one step of Algorithm 4.1,

$$\max_{j,k} |a_{jk}^{(1)}| / \max_{j,k} |a_{jk}| \leq 2.$$

The resulting matrix $A^{(1)}$ can be partitioned as

$$A^{(1)} = \begin{bmatrix} z & \vdots & H \\ & \vdots & 0 \end{bmatrix},$$

where z is either column 1 or column n of A , and H is an $(n-1)$ -by- $(n-1)$ upper Hessenberg matrix with $h_{i,n-1} = 0$ for $i = 2, \dots, n-1$. Thus further steps require only column permutations (but no eliminations). Thus $\gamma_{BP} \leq 2$. ■

We conjecture that if an n -by- n nonsingular matrix A can be written as $A = R + xy^T$, where R is an upper triangular matrix, then $\gamma_{BP} \leq 2(n-1)$. The matrices of Theorem 5.1 and the matrices of Foster [3] are of this form.

Acknowledgment

We thank Jaroslaw Oleszczuk for a translation of reference [8], and the referees for constructive comments.

References

- [1] B. Datta, *Numerical Linear Algebra and Applications*, Brooks/Cole Publishing Company, Toronto, 1995.
- [2] L. Elsner, On some algebraic problems in connection with general eigenvalue algorithms, *Linear Algebra Appl.* 48(1982), 123–138.
- [3] L.V. Foster, Gaussian elimination with partial pivoting can fail in practice, *SIAM J. Matrix Anal. Appl.* 15(1994), 1354–1362.
- [4] I. Gohberg and S. Goldberg, Finite dimensional Weiner-Hopf equations and factorizations of matrices, *Linear Algebra Appl.* 48(1982), 219–236.
- [5] D. Yu. Grigor'ev, Additive complexity in direct computation, *Theoret. Comput. Sci.* 19(1982), 39–67.
- [6] N.J. Higham and D.J. Higham, Large growth factors in Gaussian elimination with pivoting, *SIAM J. Matrix Anal. Appl.* 10(1989), 155–164.
- [7] N.J. Higham, *Accuracy and Stability of Numerical Algorithms*, Society for Industrial and Applied Mathematics, Philadelphia, 1996.
- [8] L. Yu. Kolotilina, On the sparsity of factors of the Bruhat decomposition of nonsingular matrices, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 202(1992), 5–17 (In Russian). (MR 94m:15008).

- [9] L. Yu. Kolotilina and A. Yu. Yeremin, Bruhat decomposition and solution of sparse linear algebraic systems, *Sov. J. Numer. Anal. Math. Modelling* **2**(1987), 421–436.
- [10] G.W. Stewart, *Introduction to Matrix Computations*, Academic Press, New York, 1973.
- [11] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965.
- [12] S.J. Wright, A collection of problems for which Gaussian elimination with partial pivoting is unstable, *SIAM J. Sci. Comput.* **14**(1993), 231–238.