INTERCHANGING PARAMETERS OF THE HYPERGEOMETRIC DISTRIBUTION

by

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DMS–556–IR

September 1990
[Revised February 1991]

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Recently in a finite mathematics class, errors made by some of our students led us to rediscover an interesting property of the hypergeometric distribution. Our students accidentally demonstrated that it is possible to interchange certain parameters of the hypergeometric distribution and still obtain the correct probability values. The purpose of this note is to give a probabilistic rationale for this phenomenon.

Consider an experiment in which n balls are chosen at random without replacement from an urn that contains N balls of which K are black and $N - K$ are white, and suppose the random variable of interest is the number, X, of black balls chosen. Instead of using the correct hypergeometric formula

$$P(X=x) = \begin{bmatrix} K \\ x \end{bmatrix} \begin{bmatrix} N-K \\ n-x \end{bmatrix} \Big/ \begin{bmatrix} N \\ n \end{bmatrix},$$

some students wrongly interchanged K with n leading to the formula

$$P(X=x) = \begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} N-n \\ K-x \end{bmatrix} \Big/ \begin{bmatrix} N \\ K \end{bmatrix}.$$

The fact that this interchange does not change the probability values is verified immediately by expressing the hypergeometric formula in terms of factorials.

We offer the following probabilistic argument to demonstrate that the number of black balls drawn in both cases are identically distributed. We begin with an urn that contains N white balls. When Ms. Painter arrives, she picks K balls at random without replacement from the urn, then paints each of the drawn balls black with instant dry paint, and finally returns these K balls to the urn. When Mr. Carver arrives, he chooses n balls at random without replacement from the urn, then engraves each of the chosen balls with the letter C, and finally returns these n balls to the urn. Let the random variable X denote the number of black balls chosen (*i.e.* painted and engraved) when both Painter and Carver have finished their jobs. Since the tasks of Painter and Carver do not depend on which task is done first, the probability distribution of X is the same whether Painter does her job before or after Carver does his. If Painter goes first, then Carver chooses n balls from an urn that contains K black balls and $N - K$ white balls, so

$$P(X=x) = \binom{K}{x}\binom{N-K}{n-x}\Big/\binom{N}{n}$$

for x an integer satisfying $\max(0,n-(N-K)) \leq x \leq \min(n,K)$.

On the other hand, if Carver goes first, then Painter draws K balls from an urn that contains n balls engraved with C and $N - n$ balls not engraved, so

$$P(X=x) = \binom{n}{x}\binom{N-n}{K-x}\Big/\binom{N}{K}$$

for x an integer satisfying $\max(0,K-(N-n)) \leq x \leq \min(n,K)$.

Thus, changing the order of who goes first provides an urn–drawing explanation of why the probability distribution of X remains the same when K is interchanged with n.

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