

PARTLY ZERO EIGENVECTORS

JOHN S. MAYBEE, D.D. OLESKY

AND

P. VAN DEN DRIESSCHE

DMS-522-IR

October 1989
(Revised February 8, 1990)

PARTLY ZERO EIGENVECTORS

JOHN S. MAYBEE¹

Department of Mathematics
University of Colorado
Boulder, Colorado 80309
U.S.A.

D.D. OLESKY²

Department of Computer Science

and

P. VAN DEN DRIESSCHE³

Department of Mathematics and Statistics

University of Victoria
Victoria, B.C. V8W 2Y2
Canada

¹The work of this author was supported by the U.S. Office of Naval Research under contract N0014-88-K-0087.

²The work of this author was partially supported by NSERC Grant A-8214 and the University of Victoria President's Committee on Faculty Research and Travel.

³The work of this author was partially supported by NSERC Grant A-8965.

ABSTRACT

This paper is concerned with the forced presence or absence of zero components in an eigenvector. Relative to a fixed matrix A with eigenvalue λ , we characterize the strictly nonzero part of a partly zero eigenvector associated with λ . We also give a sufficient condition for a fixed matrix to have a partly zero eigenvector, and discuss several examples in which a matrix has one or more partly zero eigenvectors. Our main results, however, are qualitative in nature. We associate a zero/nonzero pattern class of matrices with a digraph, and characterize the set of pattern classes which requires all eigenvectors to be strictly nonzero. A sufficient condition is also given that identifies the components of a partly zero eigenvector which may be nonzero.

1. INTRODUCTION

Let A be an $n \times n$, $n \geq 2$, nonzero matrix with entries $\in \mathcal{E}$, and consider the equation $Au = \lambda u$ where $\lambda \in \mathcal{E}$ is an eigenvalue. Let the term λ -eigenvector denote an eigenvector u of A corresponding to this eigenvalue; we call (λ, u) an eigenpair. If such an eigenvector $u = (u_1, \dots, u_n)^T$ has $u_i \neq 0$ for all $i = 1, \dots, n$, then u is called a *strictly nonzero eigenvector*. On the other hand, if $u_i = 0$ for at least one i , then u is called a *partly zero eigenvector*. If $u_{i_1}, u_{i_2}, \dots, u_{i_p}$ are all the nonzero components of a partly zero eigenvector u and $J = \{i_1, i_2, \dots, i_p\}$, $1 \leq i_1 < i_2 < \dots < i_p \leq n$, then $u[J]$ is called the *strictly nonzero part* of u .

Our goal is to discuss possible zero entries of an eigenvector of A . In particular we aim to characterize the set of matrices in terms of their digraph which allow a partly zero eigenvector, and to determine J . We are thus primarily considering qualitative properties of eigenvectors. In contrast to eigenvalue properties, such eigenvector properties have received little attention in the literature. There is the classical Perron Frobenius theory; in particular the result that if A is nonnegative and irreducible then its nonnegative eigenvector is strictly positive (see, *e.g.* [1], Th. 2.1.4). In the case that λ is a multiple eigenvalue, partly zero λ -eigenvectors are discussed in [2, 3, 9]. As a sample of results, Fiedler [2] Cor. (2, 6), shows that if a symmetric, acyclic, irreducible matrix has a multiple eigenvalue λ , then every λ -eigenvector is partly zero. More recently, for the special case of tournament matrices, Maybee and Pullman [8] find that a 0-eigenvector can have many zero components, but conjecture that for $\lambda \neq 0$, λ -eigenvectors are all strictly nonzero.

We proceed first with some preliminary quantitative matrix results. Then, in section 3, we introduce necessary graph theoretic concepts and work with classes of matrices which have the same digraph. We continue with qualitative results in section 4, where we introduce the concept of an admissible set. Section 5 contains a discussion of other related aspects.

2. MATRIX RESULTS

We denote the complement of J by J' , the submatrix of A in rows J' and columns J by $A[J', J]$, and abbreviate the principal submatrix $A[J, J]$ by $A[J]$. Let P be a permutation matrix, and recall that PAP^T has the same eigenvalues as A and has an eigenvector Pu if and only if u is an eigenvector of A . Note that Pu is strictly nonzero \langle partly zero \rangle if and only if u is strictly nonzero \langle partly zero \rangle .

LEMMA 2.1 *Matrix A has an eigenvalue λ and partly zero λ -eigenvector u with strictly nonzero part $u[J]$ if and only if*

- (i) *the proper principal submatrix $A[J]$ has an eigenvalue λ and a strictly nonzero λ -eigenvector,*

and (ii) $u[J]$ is orthogonal to the row space of $A[J', J]$, that is $A[J', J] u[J] = 0$.

Proof Assume the conditions on A and let P be a permutation matrix such that $Pu = u[J] \oplus 0_{n-p}$, so that the first $p \geq 1$ positions of Pu are nonzero, and the remaining $n - p$ positions are zero. Then

$$P(A - \lambda I)P^T \begin{bmatrix} u[J] \oplus 0_{n-p} \end{bmatrix} = 0.$$

This can be written in partitioned form as

$$\begin{bmatrix} A[J] & A[J, J'] \\ A[J', J] & A[J'] \end{bmatrix} \begin{bmatrix} u[J] \\ 0_{n-p} \end{bmatrix} = \lambda \begin{bmatrix} u[J] \\ 0_{n-p} \end{bmatrix}.$$

So

$$(A[J] - \lambda I_p) u[J] = 0 \tag{2.1}$$

and

$$A[J', J] u[J] = 0. \quad (2.2)$$

Equation (2.1) gives the submatrix condition (i) and (2.2) gives the orthogonality condition (ii). The converse is proved by reversing the steps. ■

Note, from Lemma 2.1 with $A[J', J] \equiv 0$, that a reducible matrix always has a partly zero eigenvector, whereas a diagonal matrix with distinct entries on the main diagonal has only partly zero eigenvectors. This latter statement is an instance of the more general quantitative result we now consider.

THEOREM 2.2 *Every eigenvector of A is partly zero if there exists a permutation matrix P such that*

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (2.3)$$

where A_{11} , A_{22} are square nonempty blocks with no eigenvalue common to both, and A_{12} is orthogonal to the eigenspace of A_{22} .

Proof Assume PAP^T has the form (2.3) and that $(\lambda_2, u^{(2)})$ is an eigenpair of A_{22} . As $A_{11} - \lambda_2 I$ is nonsingular, taking a λ_2 -eigenvector for PAP^T as $u^{(1)} \oplus u^{(2)}$, then $u^{(1)} = -(A_{11} - \lambda_2 I)^{-1} A_{12} u^{(2)} = 0$ (due to the orthogonality condition) which shows that every λ_2 -eigenvector of A is partly zero. Now assume that A_{11} has $(\lambda_1, u^{(1)})$ as an eigenpair. Then, as $A_{22} - \lambda_1 I$ is nonsingular, $(A_{22} - \lambda_1 I)u^{(2)} = 0$ implies that $u^{(2)} = 0$ and thus every λ_1 -eigenvector of A is partly zero. ■

As a special case of this theorem, suppose A is a block diagonal matrix. If every pair of diagonal blocks has no common eigenvalue, then clearly every eigenvector has to be

partly zero. The matrix $\text{diag}\{1,2,1\}$ shows that the converse of the latter statement is false, as its 1-eigenvector has a basis $\{(1,0,0)^T, (0,0,1)^T\}$ and its 2-eigenvector has a basis $\{(0,1,0)^T\}$. This example also illustrates the fact that if the geometric multiplicity of eigenvalue λ is at least 2, then this eigenspace has a basis of partly zero eigenvectors.

A nontrivial example for Theorem 2.2 is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

where the 2×2 block A_{22} has the eigenvalue 1 of algebraic multiplicity 2, but its eigenspace has dimension 1 (A_{22} is a defective matrix). Matrix A has a partly zero 1-eigenvector with associated $J' = \{1\}$, and a partly zero 2-eigenvector with associated $J' = \{2,3\}$. In fact, if every eigenvector of a matrix A in the form given by (2.3) is partly zero and $A_{12} \neq 0$, then A_{22} must be defective. This follows since, if A_{22} is nondefective, then the 0 matrix is the only matrix orthogonal to the eigenspace of A_{22} .

The form given in Theorem 2.2 is not necessary for a matrix A of size $n \geq 3$ to have all eigenvectors partly zero. An example is given by

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$$

with eigenvalues 0, 2 and 4 and corresponding J' sets equal to $\{1\}$, $\{3\}$ and $\{2\}$, so every eigenvector is partly zero.

3. QUALITATIVE RESULTS

Whereas the results of section 2 are concerned with a fixed matrix A , we now consider results concerning a class of matrices determined by an associated digraph. To make the ideas precise we need to introduce certain graph theoretic ideas; [4] and [10] are basic references for these. Digraph $D = (V, E)$ has vertex set $V = \{1, \dots, n\}$ and edge set $E = \{(i, j): \text{there is an edge from } i \text{ to } j, i \neq j\}$. We denote edge (i, j) by $i \rightarrow j$, and a directed simple cycle of length k by $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \rightarrow i_1$ where the vertices are distinct. With the $n \times n$ matrix $A = [a_{ij}]$ we associate the digraph $D(A) \equiv D$, where $(i, j) \in E$ precisely when $a_{ij} \neq 0$ and $i \neq j$. We make no restrictions on the a_{ii} entries or on the sign of the nonzero entries in the matrix; thus the edge set of $D(A)$ identifies the zero/nonzero pattern of *off-diagonal* entries of matrix A . Digraph $D(A)$ is strongly connected if and only if A is irreducible. Given a digraph $D = (V, E)$, the set of all matrices A such that $D(A) = D$ is called the *zero/nonzero pattern class of matrices* associated with D , and is denoted by \mathcal{P}_D . Following Johnson [7], we let \mathcal{P} denote a matrix property and say that \mathcal{P}_D *requires* property \mathcal{P} if every $A \in \mathcal{P}_D$ (with all choices of a_{ii}) has this property, and we say that \mathcal{P}_D *allows* property \mathcal{P} if some $A \in \mathcal{P}_D$ (with some choice of a_{ii}) has this property. We focus on the following matrix properties: "has all eigenvectors strictly nonzero" and "has a partly zero λ -eigenvector for some $\lambda \in \mathcal{C}$ ". The following result says that every zero/nonzero pattern class \mathcal{P}_D *allows* a strictly nonzero eigenvector; moreover, \mathcal{P}_D *allows* such a λ -eigenvector for all $\lambda \in \mathcal{C}$. The proof makes essential use of the fact that the diagonal entries of \mathcal{P}_D are unrestricted. We use \mathbf{e} to denote the vector which has each component equal to 1.

LEMMA 3.1 *Given any digraph D , \mathcal{P}_D allows a strictly nonzero eigenvector.*

Proof Choose a real matrix $B = [b_{ij}]$ with $D(B) = D$, $b_{ij} \geq 0$ for all i, j and B

row stochastic, that is $Be = e$. For any $\lambda \in \mathcal{C}$, let $A = B + (\lambda-1)I$. Then $D(A) = D(B) = D$ and $Ae = \lambda e$, so that e is a strictly nonzero λ -eigenvector of $A \in \mathcal{S}_D$. ■

We now introduce some notation for the above properties. Let

$$\begin{aligned} \text{ASN} &= \{ \mathcal{S}_D \text{ which allow a strictly nonzero eigenvector} \}, \\ \text{APZ} &= \{ \mathcal{S}_D \text{ which allow a partly zero eigenvector} \}, \\ \text{RSN} &= \{ \mathcal{S}_D \text{ which require all strictly nonzero eigenvectors} \}, \\ \text{RPZ} &= \{ \mathcal{S}_D \text{ which require all partly zero eigenvectors} \}. \end{aligned}$$

Note that APZ is the complement of RSN and ASN is the complement of RPZ .

Lemma 3.1 says that $\{\text{all } \mathcal{S}_D\} = \text{ASN}$, so $\text{RPZ} = \emptyset$ and $\text{ASN} = \text{RSN} \cup \text{APZ}$. The set RSN is characterized by the following result.

THEOREM 3.2 *The pattern class \mathcal{S}_D requires all eigenvectors to be strictly nonzero if and only if the digraph D is a directed simple cycle on all the vertices.*

Proof Assume without loss of generality that D is the directed simple cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$. If $Au = \lambda u$ for matrix $A \in \mathcal{S}_D$, then $u_i = 0$ implies $u_{i+1} = 0$ for all λ and for $i = 1, \dots, n$ (with $u_{n+1} = u_1$). Thus all eigenvectors of all $A \in \mathcal{S}_D$ (with D as above) are strictly nonzero, implying $\mathcal{S}_D \in \text{RSN}$.

For the converse, assume that D is not a simple cycle of length n . Then either

- (i) D is not strongly connected, or
- (ii) D is strongly connected and there exists an index i such that $a_{ij} \neq 0$ and $a_{ik} \neq 0$ for some set $\{i, j, k\}$ of distinct vertices.

In case (i), $\mathcal{S}_D \notin \text{RSN}$ since if $A \in \mathcal{S}_D$ then A is reducible and consequently has a

partly zero eigenvector (see Lemma 2.1). In case (ii) there exists at least one nonzero off diagonal entry in each row. Let $\lambda = 1$, pick an index i satisfying (ii), set $x_i = 0$ and $x_p = 1$ for all $p \neq i$. Assign a_{pi} arbitrarily, subject only to $A \in \mathcal{S}_D$, $1 \leq p \leq n$. For any p such that the only nonzero off diagonal entry in row p is a_{pi} , set $a_{pp} = 1$; and set all other $a_{pp} = 0$. Now complete the construction of $A \in \mathcal{S}_D$ by taking entries which satisfy $\sum_{i \neq q} a_{iq} = 0$ (always possible as by (ii) there exist at least 2 nonzero entries in the sum) and $\sum_{\substack{q \neq p \\ q \neq i}} a_{pq} = 1$ for all $p \neq i$ such that $a_{pp} = 0$. Then $Ax = \lambda x$, where x has a zero component, so $\mathcal{S}_D \notin \text{RSN}$ as \mathcal{S}_D allows a partly zero eigenvector. ■

We illustrate the relationships between the sets ASN , RSN and APZ with some examples. Taking $n = 3$, Theorem 3.2 says that for all choices of the diagonal entries and for all nonzero choices of a_{12} , a_{23} and a_{31} , the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

has all eigenvectors strictly nonzero. If D is not a directed simple cycle, then $\mathcal{S}_D \in \text{APZ}$ and, of course, is in ASN . For example, let D denote the straight chain digraph on 3 vertices. Then, for any $a \neq 0$,

$$\begin{bmatrix} 0 & a & 0 \\ a & a & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

illustrates that $\mathcal{S}_D \in \text{APZ}$, and

$$\begin{bmatrix} 0 & a & 0 \\ a & a & a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -a \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

illustrates that $\mathcal{P}_D \in \text{ASN}$.

We conclude this section by emphasizing the significance of the flexibility in the choice of diagonal entries in the problems we have considered. For example, if D is the digraph in Figure 1 with a loop on vertex 1, so that $A \in \mathcal{P}_D$ if and only if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix}$$

with the specified pattern for the diagonal, then \mathcal{P}_D requires all eigenvectors to be strictly nonzero.

Figure 1

However, this result is not in conflict with Theorem 3.2 because our allow and require problems have arbitrary diagonal entries. We have not considered the analogous allow and require problems with a specified diagonal (*i.e.*, relative to digraphs having loops). Allow problems of this nature for tree graphs are considered in [6].

4. ADMISSIBLE SETS

We now introduce some more terminology to address the problem of identifying which components of a partly zero eigenvector can be nonzero.

Given a digraph $D = (V, E)$ and $i \in V$, the *outdegree* of i , written $\text{od}(i)$, is the number of vertices j in V such that there is an edge (i, j) in D . A *subdigraph* of

$D = (V, E)$ is a digraph $D_1 = (V_1, E_1)$, where $V_1 \subseteq V$ and $E_1 \subseteq E$. For $K \subseteq V$, we denote by $\langle K \rangle$ the subdigraph of D *induced* (or *generated*) by the vertices of K ; *i.e.*, K is the vertex set of this subdigraph, and the edge set is exactly the edges of E joining vertices of K . The subdigraph $\langle J', J \rangle$ is obtained from D by deleting all edges incident to a vertex of J' and all edges incident from a vertex of J . Given D , a matrix $A \in \mathcal{P}_D$ and $J \subset V$ with $n-1 \geq |J| \geq 2$, then $A[J]$ is called an *admissible principal (proper) submatrix* of A if every vertex $j' \in J'$ has $\text{od}(j') \geq 2$ in $\langle J', J \rangle$; *i.e.*, every j' has at least 2 edges directed towards vertices in J . A set J with this property is called an *admissible set* of D . Every strongly connected digraph which is not a simple cycle (on all the vertices) has an admissible set; see the proof of Theorem 3.2 where $J' = \{i\}$.

To illustrate the above ideas, consider the strongly connected digraph D in Figure 2.

Figure 2

If $A \in \mathcal{P}_D$, then

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & a_{26} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 & 0 & 0 & a_{59} \\ 0 & 0 & 0 & 0 & 0 & a_{66} & a_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{76} & a_{77} & a_{78} & 0 \\ a_{81} & a_{82} & 0 & 0 & 0 & 0 & 0 & a_{88} & 0 \\ 0 & 0 & a_{93} & 0 & 0 & a_{96} & 0 & 0 & a_{99} \end{bmatrix}, \quad (4.1)$$

where for $i \neq j$, $a_{ij} \neq 0$ if and only if $(i, j) \in D$, and all a_{ii} are arbitrary. For $J = \{1, 2, \dots, 7\}$, $J' = \{8, 9\}$ and $A[J]$ is an admissible principal submatrix since in $\langle J', J \rangle$, $\text{od}(8) = \text{od}(9) = 2$. This set J is an admissible set of D .

Let $D = (V, E)$ and let $W \subset V$ be the set of the *interior vertices*, that is $W = \{v \in V: \text{od}(v) \geq 2\}$. We call $U \subseteq W$ an *out-independent set* if for each $u \in U$ there exist at least two vertices v_1 and v_2 such that (u, v_1) and $(u, v_2) \in E$ and

$\{v_1, v_2\} \cap U = \emptyset$. The maximum cardinality of any out-independent set is called the *out-independence number* of D . For example, a directed cycle has out-independence number equal to 0 (as it has no interior vertices), and the out-independence number of the smallest non-caterpillar digraph, as given in Figure 3, is equal to 3 (with $W = \{2, 3, 4, 6\}$ and $U = \{2, 4, 6\}$).

Figure 3

The concept of an out-independent set of vertices is similar in spirit to the concept of an independent set of vertices as used by Hefner, Lundgren and Maybee in [5] for strong, unipathic digraphs.

Clearly there is a complementary relation between an out-independent set and an admissible set; given D , the set $J' \subset V$ is an out-independent set if and only if J is an admissible set of D . It follows that the size of the smallest admissible principal submatrix of A is given by n minus the out-independence number of $D(A)$, provided the out-independence number is at least 1. For the digraph in Figure 2, the out-independence number is 3 and the maximal out-independent sets are $J' = \{2, 5, 7\}$, $\{2, 7, 9\}$, $\{3, 5, 7\}$ and $\{3, 5, 8\}$. For the digraph in Figure 3, with out-independence number 3, the maximal out-independent set is $J' = \{2, 4, 6\}$. In the special case that D is a tree, the out-independence number and maximal sets J' can be determined by partitioning W into two sets; the out-independence number is the cardinality of the largest possible subset of W which contains no nodes which are adjacent in D .

In Lemma 2.1, the strictly nonzero part of an eigenvector of a fixed matrix is characterized, partly in terms of a proper principal submatrix. In the next two results, we will give an analogous sufficient condition using the concept of an admissible set for identifying the strictly nonzero part of a partly zero eigenvector of any matrix in certain pattern classes.

LEMMA 4.1 *Given any $\lambda \in \mathbb{C}$ and any digraph $D = (V, E)$, let $A \in \mathcal{P}_D$ and let $J \subset V$ be an admissible set of D . Then the nonzero entries of A can be chosen so that*

- (i) λ is an eigenvalue of $A[J]$ with a strictly nonzero λ -eigenvector e ,
and (ii) $A[J', J]e = 0$.

Proof It follows from Lemma 3.1 that the entries of $A[J]$ can be chosen so that $D(A[J]) = \langle J \rangle$ and $A[J]$ satisfies (i). Considering now (ii), let $k \in J'$ and note that for any $A \in \mathcal{P}_D$, every component of the vector $A[J', J]e$ has the form $\sum_{j \in J} a_{kj}$, where $a_{kj} \neq 0$ for at least two $j \in J$ (because $A[J]$ is an admissible principal submatrix). The a_{kj} can always be chosen different from zero so that $\sum_{j \in J} a_{kj} = 0$. Thus the entries of $A[J', J]$ can be chosen to satisfy (ii), proving the theorem. (Note that the entries of $A[J, J']$ and $A[J']$ are arbitrary.) ■

The above lemma enables us to prove the following, which says that a zero/nonzero pattern class of matrices \mathcal{P}_D with an admissible principal submatrix allows a partly zero λ -eigenvector u for all $\lambda \in \mathcal{C}$ and the strictly nonzero part of u is specified by the admissible set.

THEOREM 4.2 *Given any $\lambda \in \mathbb{C}$ and any digraph $D = (V, E)$, if J is an admissible set of D , then \mathcal{P}_D allows a partly zero λ -eigenvector u with strictly nonzero part $u[J]$.*

Proof Suppose that J is an admissible set of D and $A \in \mathcal{P}_D$. Then by Lemma 4.1, the entries of $A[J]$ can be chosen so that λ is an eigenvalue of $A[J]$ with a strictly nonzero λ -eigenvector $u[J] = e$, and the rows of $A[J', J]$ can be chosen so that $A[J', J]e = 0$. Therefore, by Lemma 2.1, for any choice of the remaining entries, the matrix A has an eigenvalue λ and a partly zero λ -eigenvector u with strictly nonzero part $u[J]$. ■

Note that J' is an out-independent set and $u[J'] = 0$. The values of $a_{kj'}$ for $j' \in J'$ remain as free parameters in the matrix A except as specified by the digraph. The converse of Theorem 4.2 is not true in general; for example, consider the digraph D with the edge set $E = \emptyset$. However, if D is strongly connected, then the converse of Theorem 4.2 holds (by Theorem 3.2). So APZ is the set of pattern classes of reducible matrices and those irreducible matrices which have an admissible principal submatrix.

We conclude this section by returning to the example (4.1) with digraph in Figure 2. Let $J = \{1, 2, \dots, 7\}$ and take $\lambda = 5 + i$. We proceed as in the proof of Lemma 3.1, and take $B[J]$ to be row stochastic, $A[J] = B[J] + (4+i)I_7$ and each component of $u[J]$ equal to 1. Now set $u_8 = u_9 = 0$. Equations from rows 8 and 9 give $a_{81} + a_{82} = 0$ and $a_{93} + a_{96} = 0$. Clearly these entries of A can be chosen to satisfy these equations, thus completing our matrix construction. For this same example we could alternatively choose as our admissible set $J = \{1, 2, \dots, 8\}$. A construction similar to that described above gives a $(5+i)$ -eigenvector u with $u[J]$ as the strictly nonzero part (with each component equal to 1), and $u_9 = 0$.

5. DISCUSSION

Our results have focussed on zero/nonzero pattern matrices. We have shown that all patterns allow a strictly nonzero eigenvector, and have characterized patterns which require all eigenvectors to be strictly nonzero and patterns which allow a partly zero eigenvector. We have identified the strictly nonzero components of a partly zero eigenvector for a fixed matrix and given an analogous sufficient condition for a pattern class.

We could also consider other classes, namely zero/nonzero patterns that *allow all* strictly nonzero \langle partly zero \rangle eigenvectors and those that *require a* strictly nonzero \langle partly

zero) eigenvector. We conjecture that the set which requires *a* strictly nonzero eigenvector in fact requires *all* strictly nonzero eigenvectors (that is, it is equal to RSN). It is also possible to discuss related problems for *signed* digraphs (see, *e.g.* [6]) where the signs of related matrix entries are important. As illustrated previously, specification of main diagonal entries of matrices (loops in related digraphs) also gives other classes of related problems.

References

- [1] A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press (1979).
- [2] M. Fiedler, Eigenvectors of acyclic matrices, *Czech. Math. J.* **25**(1975), 607–618.
- [3] J. Genin and J.S. Maybee, Mechanical vibration trees, *J. Math. Anal. Appl.* **45**(1974), 746–763.
- [4] F. Harary, R.Z. Norman and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, Wiley (1965).
- [5] K. Hefner, R. Lundgren and J.S. Maybee, Biclique covers and partitions of unipathic digraphs. Preprint (1989).
- [6] C. Jeffries and P. van den Driessche, Eigenvalues of matrices with tree graphs, *Linear Algebra Appl.* **101**(1988), 109–120.
- [7] C.R. Johnson, Combinatorial matrix analysis, an overview, *Linear Algebra Appl.* **107**(1988), 3–15.
- [8] J.S. Maybee and N. Pullman, Tournament matrices and their generalizations, I, *Linear and Multilinear Alg.*, to appear.
- [9] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, *J. SIAM*, **8**(1960), 376–387.
- [10] F. Roberts, *Discrete Mathematical Models*, Prentice–Hall (1976).

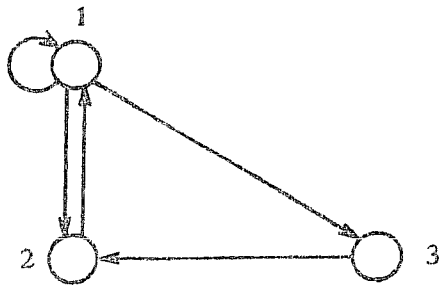


FIGURE 1

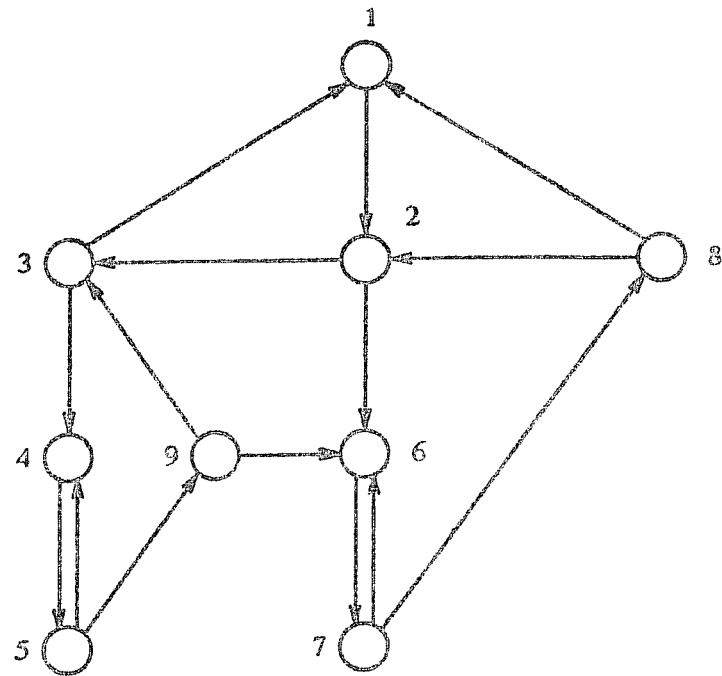


FIGURE 2

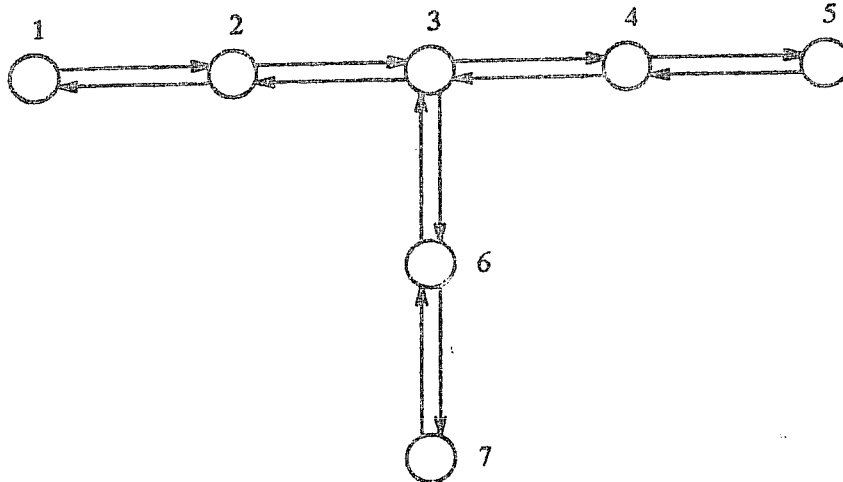


FIGURE 3