NILPOTENT FACTORIZATION OF MATRICES

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CANADA

ABSTRACT

We show that, with the exception of 2×2 nonzero nilpotent matrices, every singular square matrix over an arbitrary field is a product of two nilpotent matrices.

A square matrix N is called nilpotent if $N^m = 0$ for some positive integer m . In [7], Wu proved that, with the exception of 2×2 nilpotent matrices of rank one, every singular square matrix over the complex field is a product of two nilpotent matrices, which may be taken to have the same rank as A . The purpose of this article is to prove that the same conclusion holds for matrices over an arbitrary field. After a conversation with T.J. Laffey, the author found out that Laffey has, independently, discovered the same result. His proof in [3] uses the factorization theorem for invertible matrices in [5] as well as a special case of Wu's result. Our proof given here is self-contained.

It may be of interest to point out that products of quasi-nilpotent operators on infinite dimensional Hilbert space were considered by Fong and the author [1]. We gave necessary conditions and sufficient conditions for an operator to be factored as a product of two quasi-nilpotent operators; in particular we showed that every compact operator is a product of two compact quasi-nilpotent operators.

We now fix some notation. The set of all $n \times n$ matrices over a field F is denoted by $M_n(F)$. Vectors in $F^{(n)}$ are written as column vector and are identified with $n \times 1$ matrices. We make no distinction between a matrix A in $M_n(F)$ and the induced linear operator $x \mapsto Ax$ on $F^{(n)}$. The null space and range of A are denoted by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ respectively. The transpose of A is denoted by A^t . The set of all vectors orthogonal to every element in a set of vectors S is denoted by $S^\perp$.

We start by proving the impossibility of nilpotent factorization in the exceptional case. This impossibility was noted in [1]. A proof was also given in [7].

LEMMA 1. *If A is a nonzero 2×2 nilpotent matrix, then A is not a product of two nilpotent matrices.*

Proof. By applying a similarity transformation, we may assume, without loss of generality, that $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. If $A = BC$ with nilpotent B and C , then we have necessarily that $\text{rank } B = \text{rank } C = \text{rank } A = 1$. Furthermore $\mathcal{N}(C) \subseteq \mathcal{N}(A)$, and the equality of ranks implies that $\mathcal{N}(C) = \mathcal{N}(A) = \text{span}\{e_2\}$, where $e_2 = (0,1)^t$. Similarly $\mathcal{R}(B) = \mathcal{R}(A) = \text{span}\{e_2\}$. Therefore $B = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ for some b and c in F . But then $BC = 0$, a contradiction. ■

REMARK. A rank one 2×2 nilpotent N can be written as product of three nilpotent matrices; indeed if N is a 2×2 Jordan cell, then $N = NN^tN$.

Next we deal with certain 3×3 matrices.

LEMMA 2. Let A be a 3×3 matrix over a field F , and assume that the characteristic polynomial of A is $p(t) = t^2(t-\alpha)$ for some $\alpha \in F$. Then A is a product of two nilpotent matrices B and C with $\text{rank } B = \text{rank } C = \text{rank } A$.

Proof. The case $A = 0$ is trivial. Otherwise A is similar to one of four canonical forms which factor as follows.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha & -1 \\ \alpha & -\alpha & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/\alpha \\ -1 & 0 & 0 \\ \alpha & 0 & -1 \end{bmatrix}, \quad \alpha \neq 0$$

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha \neq 0.$$

We start with a proposition on partitioning matrices which may be of independent interest.

PROPOSITION. *Let $A \in M_n(F)$. Then A is similar to a matrix of the form*

$$\begin{bmatrix} \alpha & c^t \\ b & D \end{bmatrix}$$

with $\alpha \in F$, $D \in M_{n-1}(F)$, $\text{rank } D = \text{rank } A - 1$, $b \in \mathcal{R}(D)$ and $c \in \mathcal{R}(D^+)$ if and only if $A^2 \neq 0$.

Proof. First assume that $A^2 \neq 0$, and let e_0 be a vector in $F^{(n)}$ such that $A^2 e_0 \neq 0$. Let $e_1 = Ae_0$, and $\mathcal{M}_1 = \text{span}\{e_1\}$. Since $\mathcal{M}_1$ and $\mathcal{N}(A)$ are independent, there exists a complement $\mathcal{M}_2$ to $\mathcal{M}_1$ such that $\mathcal{M}_2 \supseteq \mathcal{N}(A)$. We will show that the matrix representation of A with respect to the decomposition $F^{(n)} = \mathcal{M}_1 \oplus \mathcal{M}_2$ satisfies the desired conclusion.

Toward this end, let P be the projection on $\mathcal{M}_2$ along $\mathcal{M}_1$. Straightforward computation shows that

$$\mathcal{N}(PA) = \text{span}\{e_0\} + \mathcal{N}(A)$$

$$\mathcal{N}(AP) = \mathcal{M}_1 + \mathcal{N}(A)$$

$$\mathcal{N}(PAP) = \mathcal{M}_1 + \mathcal{N}(A).$$

Therefore

$$\text{rank PA} = \text{rank AP} = \text{rank PAP} = \text{rank A} - 1.$$

If the matrix of A relative to $\mathcal{M}_1 \oplus \mathcal{M}_2$ is

$$\begin{bmatrix} \alpha & c^t \\ b & D \end{bmatrix},$$

then

$$PA = \begin{bmatrix} 0 & 0 \\ b & D \end{bmatrix}, \quad AP = \begin{bmatrix} 0 & c^t \\ 0 & D \end{bmatrix}, \quad PAP = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

The fact that $\text{rank PA} = \text{rank PAP}$ implies that $b \in \mathcal{R}(D)$. Similarly $c \in \mathcal{R}(D^t)$ since $\text{rank AP} = \text{rank PAP}$. Finally $\text{rank A} = 1 + \text{rank PAP} = 1 + \text{rank D}$.

Conversely, if the matrix representation $\begin{bmatrix} \alpha & c^t \\ b & D \end{bmatrix}$ of A relative to a decomposition $F^{(n)} = \mathcal{V}_1 \oplus \mathcal{V}_2$ satisfies the conditions in the statement of the proposition, then we first show that $\mathcal{N}(A) \subseteq \mathcal{V}_2$. If that is not the case, then

$$\begin{bmatrix} \alpha & c^t \\ b & D \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = 0$$

for some $z \in \mathcal{V}_2$. Since $b = Db_o$, $c = D^t c_o$ for some $b_o, c_o \in \mathcal{V}_2$, then

$$\alpha + c_o^t Dz = 0 \quad \text{and} \quad b + Dz = 0.$$

Therefore $\alpha = c_o^t b$, which implies that

$$\begin{bmatrix} \alpha & c^t \\ b & D \end{bmatrix} = \begin{bmatrix} 1 & c_o^t \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b_o & I \end{bmatrix}.$$

It follows that $\text{rank } A = \text{rank } D$, a contradiction. This establishes that

$$\mathcal{N}(A) \subseteq \mathcal{V}_2.$$

If $A^2 = 0$, then $\mathcal{R}(A) \subseteq \mathcal{N}(A) \subseteq \mathcal{V}_2$, and so $\alpha = c = 0$. Again this implies that $\text{rank } A = \text{rank } D$, contradicting the assumption. This proves that $A^2 \neq 0$. ■

Next, we state and prove the main theorem.

THEOREM. *Let A be a singular $n \times n$ matrix over an arbitrary field. If A is not a nonzero 2×2 nilpotent, then A is a product of two nilpotent matrices each of which has rank equal to $\text{rank } A$.*

Proof. We start with the case $A^2 = 0$. The result is trivial if $A = 0$. If $\text{rank } A = 1$, then $n \geq 3$ and A is either a 3×3 nilpotent or similar to a direct sum of such a matrix with a zero matrix. In this case, the desired factorization is established by Lemma 1. If $\text{rank } A = k \geq 2$, then A is similar to a direct sum of a zero matrix of size $n - 2k$ and the $2k \times 2k$ matrix $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, and it suffices to establish the factorization for the latter matrix. Toward this end, let J be the upper triangular $k \times k$ Jordan cell and let E_{21} be the $k \times k$ matrix having 1 as the $(2,1)$ entry and zero for all other entries. It follows that

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} J & J^t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & E_{12} \\ 0 & J \end{bmatrix}$$

which gives a factorization into a product of two nilpotents each having rank k . This establishes the result if $A^2 = 0$.

If $A^2 \neq 0$, we proceed by induction on n . The result is obvious if $n = 1$. Next assume that the result is true for all square matrices with size less than n , and let $A \in M_n(F)$. By the proposition, we may assume that

$$A = \begin{bmatrix} \alpha & c^t \\ b & D \end{bmatrix}$$

with $b \in \mathcal{R}(D)$, $c \in \mathcal{R}(D^t)$ and $\text{rank } D = \text{rank } A - 1$. Therefore D is also singular. We will apply the induction hypothesis to D , but we start with the case where D is the exceptional 2×2 nonzero nilpotent. In this case A is similar to the matrix

$$\begin{bmatrix} \alpha & 0 & \beta \\ \gamma & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $\alpha, \beta, \gamma \in F$. The characteristic polynomial of this matrix is

$p(t) = t^2(t - \alpha)$, and we apply Lemma 2, to obtain the desired factorization for A .

We now consider the case that D is not a nonzero 2×2 nilpotent, so $D = N_1 N_2$ where N_1 and N_2 are nilpotents with

$$\text{rank } N_1 = \text{rank } N_2 = \text{rank } D.$$

Since $\mathcal{N}(D) \supseteq \mathcal{N}(N_2)$ and $\mathcal{R}(D) \subseteq \mathcal{R}(N_1)$, the rank condition implies that

$$\mathcal{N}(D) = \mathcal{N}(N_2)$$

and

$$\mathcal{R}(D) = \mathcal{R}(N_1).$$

Since $b \in \mathcal{R}(D)$, there exists $b_o \in F^{(n-1)}$ such that $b = N_1 b_o$. Similarly $c^t = c_o^t N_2$ for some $c_o \in F^{(n-1)}$.

We will establish the existence of vectors x and $y \in F^{(n-1)}$ such that

$$B = \begin{bmatrix} 0 & y^t \\ 0 & N_1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 \\ x & N_2 \end{bmatrix}$$

are the desired factors.

Toward this end let

$$x = b_o + \xi \quad \text{and} \quad y = c_o + \eta \tag{1}$$

where ξ and η are vectors in $F^{(n-1)}$ satisfying

$$\xi \in \mathcal{N}(N_1) \quad \text{and} \quad \eta \in \mathcal{N}(N_2^t) \tag{2}$$

and additional requirement to be determined presently. With x, y, B and C as above, we have $A = BC$ if and only if $\alpha = y^t x$, *i.e.*

$$c_o^t \xi + \eta^t b_o + \eta^t \xi = \alpha - c_o^t b_o. \tag{3}$$

Vectors ξ and η can be found so as to satisfy (2) and (3) unless

$$c_o \in \mathcal{N}(N_1)^\perp, \tag{4}$$

$$b_o \in \mathcal{N}(N_2^t)^\perp, \tag{5}$$

and

$$\mathcal{N}(N_1) \subseteq \mathcal{N}(N_2^t)^\perp. \tag{6}$$

However, $\mathcal{N}(N_2^t)^\perp = \mathcal{R}(N_2)$. This is a restatement of ([2], Theorem 22, p. 112). Now Equation (6) implies that $\mathcal{N}(N_1) \subseteq \mathcal{R}(N_2)$, and so $\text{rank } N_1 N_2 < \text{rank } N_2$, contradicting the choice of N_1 and N_2 . This shows that we can find ξ and η to satisfy (2) and (3) and hence $A = BC$, a product of two nilpotent matrices.

Furthermore

$$\text{rank } B \leq 1 + \text{rank } N_1 = 1 + \text{rank } D = \text{rank } A,$$

but the equation $A = BC$ implies that $\text{rank } B \geq \text{rank } A$, and hence $\text{rank } B = \text{rank } A$. Similarly $\text{rank } C = \text{rank } A$. This ends the proof. $\blacksquare$

We end by showing how factorization theorems such as the one presented here can be used to give alternate proofs of other factorization theorems. Several instances of this were presented in [5]. We illustrate by considering a theorem of Radjavi [4] which asserts that every complex square matrix A with real determinant is the product of four hermitian matrices. In the case A is singular, one may use the nilpotent factorization theorem to write A as a product of two nilpotents. Every nilpotent Jordan cell can easily be seen to be the product of two hermitian matrices, thus every nilpotent is a product of two hermitian matrices and hence every singular matrix is a product of four hermitian matrices.

REMARK. A more general factorization theorem on factoring a singular matrix as a product of matrices with prescribed eigenvalues is given in [6]. Applications similar to the one described in the preceding paragraph are presented in [6].

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