

COMBINATORIAL EIGENVALUES OF MATRICES

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Abstract

Let S be a subset of diagonal entries of an $n \times n$ matrix A . When the members of S have a common value which is equal to an eigenvalue of A , then S is a critical diagonal set of A . The existence of such a set is equivalent to the matrix \tilde{A} , obtained from A by setting its diagonal entries equal to zero, having an $s \times t$ zero submatrix with $s + t \geq n + 1$. The case of equality characterizes a minimal critical diagonal set. Moreover, we use such a zero submatrix of \tilde{A} to identify the elements in a critical diagonal set. We use a combinatorial approach to study the eigenspace of an eigenvalue associated with a critical diagonal set and make observations regarding reducibility, the Frobenius normal form of A and the cardinality of critical diagonal sets.

1 Introduction

Let $A = [a_{ij}]$ be an $n \times n$ matrix with entries from any field; it is convenient to let $d_i \equiv a_{ii}$, $i = 1, 2, \dots, n$. We are interested in discovering when it happens that at least one diagonal entry of A is equal to an eigenvalue of A . An obvious example of this occurrence is a triangular matrix. A less obvious example is the 3×3 tridiagonal matrix

$$A = \begin{bmatrix} d_1 & a_{12} & 0 \\ a_{21} & d_2 & a_{23} \\ 0 & a_{32} & d_3 \end{bmatrix}. \quad (1.1)$$

If $d_1 = d_3$, their common value is an eigenvalue of A regardless of how values are assigned to the remaining nonzero entries of the matrix. Thus, for both the triangular and for the 3×3 tridiagonal matrix we can conclude the existence of a set of diagonal entries equal to an eigenvalue of the matrix.

It is from a combinatorial point of view that we study this problem; specifically, for what given zero/nonzero off-diagonal patterns of a matrix A does there exist a set of diagonal entries of A whose common value is an eigenvalue of A ?

We let $N \equiv \{1, 2, \dots, n\}$, $\delta \subseteq N$ and $\delta^c = N \setminus \delta$, and denote the spectrum of a matrix by $\sigma(\cdot)$.

Definition 1.1 A set $\{d_i : i \in \delta\}$ is called a *critical diagonal set* of an $n \times n$ matrix A with a specified zero/nonzero off-diagonal pattern, if for any $\hat{\lambda} \in \mathbb{C}$, when $d_i = \hat{\lambda}$, $i \in \delta$, then $\hat{\lambda} \in \sigma(A)$, regardless of how the nonzero off-diagonal entries and the remaining diagonal entries of A are chosen. The (arbitrary) value $\hat{\lambda}$ is called a *combinatorial eigenvalue* of A .

There are certainly matrices which do not have a critical diagonal set based upon their zero/nonzero off-diagonal patterns, for example, the 2×2 matrix with both off-diagonal entries different from zero, the 4×4 tridiagonal matrix and the $n \times n$ (irreducible) weakly cyclic matrices of index n . Also, a matrix may have more than one critical diagonal set; for example, any subset of the diagonal entries of a triangular matrix is a critical diagonal set. In general, the union of critical diagonal sets is a critical diagonal set, but a proper subset of a critical diagonal set is not necessarily a critical

diagonal set. For example, $\{d_1, d_3\}$ and $\{d_1, d_2, d_3\}$ are critical diagonal sets of the matrix A in (1.1), but $\{d_1\}$, $\{d_2\}$, $\{d_3\}$, $\{d_1, d_2\}$ and $\{d_2, d_3\}$ are not.

We are also interested in the eigenvectors associated with an eigenvalue corresponding to a critical diagonal set. Moreover, we study the algebraic and geometric multiplicity of these eigenvalues. An important point needs to be made and we illustrate it with two examples. We let $A_\lambda \equiv A - \lambda I$. The tridiagonal matrix in (1.1) above has the characteristic polynomial

$$\det A_\lambda = -(d_1 - \lambda)a_{23}a_{32} - (d_3 - \lambda)a_{12}a_{21} + (d_1 - \lambda)(d_2 - \lambda)(d_3 - \lambda).$$

Obviously $d_1 = d_3 = \hat{\lambda}$ implies that $\hat{\lambda}$ is a simple eigenvalue for almost all values a_{ij} . But if, in addition, we have the quantitative condition $a_{23}a_{32} = -a_{12}a_{21}$, then $\hat{\lambda}$ becomes instead a double eigenvalue (algebraic multiplicity 2). Because of the required extra condition, we see that, for almost all choices of the off-diagonal entries of the matrix A the eigenvalue $\hat{\lambda}$ is simple. Our second example is the 3×3 lower triangular matrix

$$L = \begin{bmatrix} d_1 & 0 & 0 \\ a_{21} & d_2 & 0 \\ a_{31} & a_{32} & d_3 \end{bmatrix},$$

with $\det L_\lambda = (d_1 - \lambda)(d_2 - \lambda)(d_3 - \lambda)$. Suppose we choose $d_1 = d_3 = \hat{\lambda}$, but $d_2 \neq \hat{\lambda}$. Then $\hat{\lambda}$ is a double eigenvalue of L . If $u = (u_1, u_2, u_3)^T$ is an eigenvector belonging to $\hat{\lambda}$, then

$$a_{21}u_1 + (d_2 - \hat{\lambda})u_2 = 0 \quad \text{and} \quad a_{31}u_1 + a_{32}u_2 = 0.$$

Except for the very special choice $a_{21}a_{32} = a_{31}(d_2 - \hat{\lambda})$, these equations imply that $u_1 = u_2 = 0$ so that the only eigenvector belonging to $\hat{\lambda}$ is a multiple of $(0, 0, 1)^T$. For almost all choices of the off-diagonal entries, the eigenvalue $\hat{\lambda}$ has geometric multiplicity one. From these two examples it is evident that there lurks the possibility that, because of numerical coincidence, the stated algebraic (or geometric) multiplicity of an eigenvalue may be different from its *combinatorial* multiplicity which we now define.

Definition 1.2 Let A be a matrix with a critical diagonal set $\{d_i : i \in \delta\}$. The combinatorial eigenvalue $\hat{\lambda} = d_i$, $i \in \delta$, has *combinatorial algebraic*

multiplicity r if the algebraic multiplicity of the eigenvalue $\hat{\lambda}$ is at least r , regardless of the choice of the nonzero off-diagonal entries and of d_j , $j \in \delta^c$, and equals r for some such choice of entries of A . The *combinatorial geometric multiplicity* of $\hat{\lambda}$ is similarly defined. We denote these by $cam(\hat{\lambda})$ and $cgm(\hat{\lambda})$ respectively.

These definitions incorporate the fact that off-diagonal nonzero entries can be arbitrarily chosen and therefore, as we state our results in terms of $cam(\hat{\lambda})$ and $cgm(\hat{\lambda})$, we do not have to add a caveat to this effect.

Note that $\hat{\lambda}$ is a combinatorial eigenvalue of A if and only if 0 is a combinatorial eigenvalue of $A_{\hat{\lambda}}$. The characterization of sign-pattern matrices requiring k eigenvalues equal to 0 is considered in [E, Section 4.3]. The relationship between our results and those of [E] is discussed in Section 3.

In Section 2 we present some fundamental combinatorial ideas and establish the notation necessary for the presentation of our results. Section 3 contains our main results on critical diagonal sets. Using the Frobenius-König theorem, we show that a necessary and sufficient condition for an $n \times n$ matrix to have a critical diagonal set is that the matrix \tilde{A} , obtained from A by setting the diagonal entries equal to zero, contains an $s \times t$ zero submatrix such that $s + t = n + p$, $p \geq 1$. Moreover, we show that every such submatrix of \tilde{A} contains at least p diagonal entries, which in turn yield a critical diagonal set of A . According to the definition, any superset of a critical diagonal set is also a critical diagonal set. As a consequence, in order to characterize critical diagonal sets in terms of the zero structure of A , one has to consider *minimal* critical diagonal sets (no subset of which is a critical diagonal set). We proceed by showing that the elements of a minimal critical diagonal set are precisely the diagonal entries corresponding to a zero $s \times t$ submatrix of \tilde{A} for which $s + t = n + 1$. In addition, the cardinality of a minimal critical diagonal set cannot exceed $\frac{n+1}{2}$. Next, we study the eigenspace of a combinatorial eigenvalue by relating its algebraic and geometric multiplicity to the size of the associated zero submatrix of \tilde{A} . In particular, we consider the case of a combinatorial eigenvalue corresponding to a minimal critical diagonal set, as well as the case of two overlapping $s_j \times t_j$ zero submatrices of \tilde{A} for which $s_j + t_j \geq n + 1$. In addition to the above results, we connect critical diagonal sets to reducibility of the pattern and also to the Frobenius normal form of a matrix, by showing

that the members of any one minimal critical diagonal set belong to the same irreducible component. Section 4 contains illustrative examples.

2 Some Fundamental Combinatorial Ideas

For an $n \times n$ matrix A , let $D = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix with diagonal entries equal to those of A and let $\tilde{A} \equiv A - D$; thus, with $\tilde{A} = [\tilde{a}_{ij}]$, $\tilde{a}_{ij} = a_{ij}$ for $i \neq j$ and $\tilde{a}_{ii} = 0$. Recall that $N = \{1, 2, \dots, n\}$. For $\alpha, \beta \subseteq N$ we let $A[\alpha \mid \beta]$ denote the submatrix in rows $\alpha \subseteq N$ and columns $\beta \subseteq N$. When $\alpha = \beta$, we denote the principal submatrix $A[\alpha \mid \alpha]$ of order $|\alpha|$ by $A[\alpha]$, where $|\alpha|$ denotes the cardinality of α . We also let $A(\alpha)$ denote the principal submatrix in the rows and columns α^c of A . If $A[\alpha \mid \beta] = 0$, we call this a *zero submatrix of size $|\alpha| + |\beta|$* . A zero submatrix is said to be *maximal* if it is not a proper submatrix of any other zero submatrix of A . We define the *principal rank* of a matrix A as the order of the largest nonsingular principal submatrix of A .

To treat our problem combinatorially, the following concepts are introduced. Let $T = \{a_{i_1, j_1}, a_{i_2, j_2}, \dots, a_{i_k, j_k}\}$ be a set of nonzero entries of A . The set T is called a *transversal* of A if the sets $\{i_1, i_2, \dots, i_k\} \subseteq N$ and $\{j_1, j_2, \dots, j_k\} \subseteq N$ both consist of distinct integers. The length of the transversal T is k . A transversal is called *maximal* if its length is greater than or equal to the length of any other transversal of A . A transversal is called a *partial factor* of A if the row and column sets are equal.

We say that the matrix A with a given zero/nonzero pattern has *generic rank k* if the rank of A is less than or equal to k regardless of how the nonzero entries of A are chosen and is equal to k for some choice of these entries. Similarly we say A has *generic principal rank k_0* if the principal rank of A is less than or equal to k_0 regardless of how the nonzero entries of A are chosen and is equal to k_0 for some choice of these entries. Note that for a given $n \times n$ matrix :

- principal rank \leq generic principal rank \leq generic rank;
- generic rank $= n$ if and only if generic principal rank $= n$.

Also, from the definitions we have that :

- A has generic rank k if and only if the length of any maximal transversal of A is k ;
- A has generic principal rank k_0 if and only if the length of any maximal partial factor of A is k_0 .

Definition 2.1 An $n \times n$ matrix with generic principal rank less than n is called *combinatorially singular*.

Note that an $n \times n$ combinatorially singular matrix A cannot have a transversal of length n and that any matrix with the same zero/nonzero pattern as A must have zero determinant. The following equivalent condition for combinatorial singularity is classical; see [F] where it first appeared, or [MP] for an interesting discussion.

Theorem 2.2 [Frobenius–König] *An $n \times n$ matrix A is combinatorially singular if and only if there exists a zero submatrix $A[\alpha \mid \beta]$ of size $|\alpha| + |\beta| \geq n + 1$.*

In addition to the above combinatorial ideas, we use the following identity, recalling that $A_\lambda = A - \lambda I$.

$$\begin{aligned}
\det A_\lambda &= \det \tilde{A} + \sum_{i=1}^n (d_i - \lambda) \det \tilde{A}(i) + \sum_{1 \leq i_1 < i_2 \leq n} (d_{i_1} - \lambda)(d_{i_2} - \lambda) \det \tilde{A}(i_1, i_2) \\
&\quad + \dots + \sum_{1 \leq i_1 < \dots < i_{n-2} \leq n} (d_{i_1} - \lambda) \dots (d_{i_{n-2}} - \lambda) \det \tilde{A}(i_1, \dots, i_{n-2}) \\
&\quad + \prod_{i=1}^n (d_i - \lambda).
\end{aligned} \tag{2.1}$$

This formula is easy to verify (see, e.g., [PR]) and is particularly useful because of the way it relates an eigenvalue of A to the values of the entries of D . Note that there is no term involving the product of exactly $(n - 1)$ diagonal entries of A_λ .

Given a combinatorial eigenvalue $\hat{\lambda}$ of an $n \times n$ matrix A , we can use equation (2.1) in order to interpret $\text{cam}(\hat{\lambda})$ in terms of the generic principal rank of \tilde{A} . By definition of the combinatorial algebraic multiplicity, if $\text{cam}(\hat{\lambda}) = r$, then each term of the characteristic equation $\det A_\lambda = 0$ has at least r factors $(\hat{\lambda} - \lambda)$ and there is one such term with exactly r factors.

Observe now that, by (2.1), this can occur if and only if every principal minor of \tilde{A} of order greater than $(n - r)$ vanishes generically and at least one principal minor of order $(n - r)$ is generically nonzero.

3 Critical Diagonal Sets

The following two theorems connect the existence of a critical diagonal set with combinatorial singularity.

Theorem 3.1 *Let $\delta \subseteq N$ and $\hat{\lambda} \in \mathcal{C}$, and set $d_i = \hat{\lambda}$ for all $i \in \delta$. Then $S = \{d_i : i \in \delta\}$ is a critical diagonal set of A if and only if $A_{\hat{\lambda}}$ is combinatorially singular.*

Proof. This follows immediately from the definition of a critical diagonal set and combinatorial singularity. \square

Theorem 3.2 *For an $n \times n$ matrix A , the matrix \tilde{A} is combinatorially singular if and only if A has a critical diagonal set.*

Proof. If \tilde{A} is a combinatorially singular $n \times n$ matrix, then its generic principal rank is $n - q$, $q \geq 1$, and any maximal partial factor has length $n - q$. It follows that $\det \tilde{A}[\alpha] = 0$ if $|\alpha| > n - q$. Thus the first q terms on the right hand side of (2.1) vanish regardless of how the values of the nonzero entries of A are chosen. Since each of the remaining nonzero terms has at least one factor of the form $d_i - \lambda$, there exists a nonempty set of diagonal entries $\{d_{i_1}, d_{i_2}, \dots, d_{i_t}\}$ such that if $d_{i_k} = \hat{\lambda}$, $1 \leq k \leq t$, then $\det A_{\hat{\lambda}} = 0$. Thus A has a critical diagonal set. For the converse, suppose A has a critical diagonal set $\{d_{i_1}, d_{i_2}, \dots, d_{i_t}\}$. Then for $\hat{\lambda}$ chosen as above, by Theorem 3.1 $\det A_{\hat{\lambda}} = 0$ for all choices of the remaining nonzero entries of A . But, from (2.1), this implies that $\det \tilde{A} = 0$ for all choices of the nonzero entries of \tilde{A} , which implies that the generic principal rank of \tilde{A} is less than n and thus \tilde{A} is combinatorially singular. \square

In the following result, the existence of a zero submatrix in \tilde{A} is used to be more specific about its generic principal rank and the cardinality of a critical diagonal set of A .

Theorem 3.3 *Let A be an $n \times n$ matrix and suppose $\tilde{A}[\alpha \mid \beta] = 0$ where $|\alpha| + |\beta| = n + p$, $p \geq 1$. Let $\gamma \equiv \alpha \cap \beta$. Then,*

- (i) *\tilde{A} has generic principal rank $\leq n - p$.*
- (ii) *$\{d_i : i \in \gamma\}$ is a critical diagonal set of A with $|\gamma| \geq p$.*
- (iii) *If $d_i = \hat{\lambda}$ for all $i \in \gamma$, then $\text{cam}(\hat{\lambda}) \geq p$.*

Proof.

(i) Observe that all of the nonzero entries of A are contained in at most $n - |\alpha|$ rows and $n - |\beta|$ columns of \tilde{A} . Thus the order of the largest principal submatrix of \tilde{A} containing a partial factor is less than or equal to $n - |\alpha| + n - |\beta|$. Since $|\alpha| + |\beta| = n + p$, this order must be $\leq n - p$.

(ii) It follows from Theorem 3.2 that A has a critical diagonal set. If $d_i = \hat{\lambda}$ for all $i \in \gamma$, then $\{d_i : i \in \gamma\}$ is a critical diagonal set of A , because $A_{\hat{\lambda}}$ contains the same zero submatrix as \tilde{A} and so, by Theorem 2.2, is combinatorially singular. Since $|\alpha \cup \beta| \leq n$ and $|\alpha| + |\beta| = n + p$, it follows that $|\gamma| \geq p$.

(iii) By (i), the first nonvanishing term on the righthand side of (2.1) has at least p factors of the form $(d_i - \lambda)$. Suppose $d_i = \hat{\lambda}$ when $i \in \gamma$. To prove (iii), according to the definition of $\text{cam}(\hat{\lambda})$, we must show that *all* nonzero terms on the righthand side of (2.1) have at least p factors of the form $(d_i - \lambda)$ for $i \in \gamma$. Equivalently, we must show that for each k , $0 \leq k \leq n - p - 2$, every principal submatrix $\tilde{A}[\theta]$ of order $n - p - k$ containing at least $|\gamma| - p + k + 1$ indices in γ must be combinatorially singular. Note that if $2k \geq n - |\gamma| - 1$, then $|\gamma| - p + k + 1 \geq n - p - k$ and all indices of θ are in γ , implying that $\tilde{A}[\theta] = 0$. To minimize the order of the zero submatrix in $\tilde{A}[\theta]$, first delete as many rows and columns from γ as possible, namely $\max\{0, p - k - 1\}$ rows and columns. Consider the case $p - k - 1 \geq 0$; then we must delete $2k + 1$ rows and columns not in γ (as $|\theta| = n - p - k$). Such a deletion can include at most one row or one column (not both) of a given zero submatrix. The size of the remaining zero submatrix in $\tilde{A}[\theta]$ is at least

$$\begin{aligned}
& (|\gamma| - p + k + 1) + (|\alpha| - |\gamma|) \\
& + (|\gamma| - p + k + 1) + (|\beta| - |\gamma|) - (2k + 1) \\
& = |\alpha| + |\beta| - 2p + 1
\end{aligned}$$

$$\geq n - p - k + 1, \text{ as } k \geq 0.$$

In the alternate case, namely when $p - k - 1 < 0$, we must delete $p + k$ rows and columns not in γ . The size of the remaining zero submatrix in $\tilde{A}[\theta]$ is $\geq |\alpha| + |\beta| - (p + k) = n - k \geq n - p - k + 1$, as $p \geq 1$. Thus, by Theorem 2.2, every required principal submatrix $\tilde{A}[\theta]$ is combinatorially singular. \square

As mentioned in the introduction, the above results can also be stated in terms of 0 being a combinatorial eigenvalue of $A_{\hat{\lambda}}$, or equivalently $A_{\hat{\lambda}}$ being combinatorially singular. This is the approach taken in [E, Section 4.3], where results equivalent to our Theorems 3.1, 3.2 and 3.3 (i) and (iii) are proved. In our proof of Theorem 3.3 (iii) and in subsequent results, the emphasis is placed on zero submatrices in \tilde{A} , leading to the identification of the elements of a critical diagonal set. In [E] the proofs are based on a continuity argument and the elements of a critical diagonal set are not specified.

Since any superset of a critical diagonal set is also a critical diagonal set, no converse of Theorem 3.3(ii) exists; i.e., given a critical diagonal set S of an $n \times n$ matrix A , there may not be a zero submatrix $\tilde{A}[\alpha | \beta]$ of size $|\alpha| + |\beta| = n + p$, $p \geq 1$, so that $S = \{d_i : i \in \alpha \cap \beta\}$. For instance, $\{d_1, d_2, d_3\}$ is a critical diagonal set of

$$A = \begin{bmatrix} d_1 & 0 & 0 & a_{14} \\ 0 & d_2 & 0 & a_{24} \\ a_{31} & 0 & d_3 & 0 \\ a_{41} & 0 & 0 & d_4 \end{bmatrix}, \quad (3.1)$$

but $\tilde{A}[\{1, 2, 3\}] \neq 0$. Observe now that, since the characteristic polynomial of A in (3.1) is

$$\det A_{\lambda} = \prod_{i=1}^3 (d_i - \lambda) - a_{41}a_{14}(d_2 - \lambda)(d_3 - \lambda),$$

in order for $\hat{\lambda}$ to be an eigenvalue of A either d_2 or d_3 *must* be set equal to $\hat{\lambda}$, while the values of d_1 and d_4 are actually immaterial. This observation suggests the following definition.

Definition 3.4 Let $S = \{d_i : i \in \delta\}$ be a critical diagonal set of a matrix A . We call S a *minimal critical diagonal set* of A if no proper subset of S is a critical diagonal set of A .

For example, $\{d_1, d_3\}$ is a minimal critical diagonal set of A in (1.1) but $\{d_1, d_2, d_3\}$ is not. Also $\{d_2\}$ and $\{d_3\}$ are the only minimal critical diagonal sets of A in (3.1). The next theorem provides us with a converse of Theorem 3.3(ii) when the critical diagonal set S is minimal.

Theorem 3.5 Let $\{d_i : i \in \delta\}$ be a minimal critical diagonal set of an $n \times n$ matrix A . Then there exists a zero submatrix $\tilde{A}[\alpha \mid \beta]$ of size $|\alpha| + |\beta| = n + 1$ such that $\delta = \alpha \cap \beta$.

Proof. Suppose that $S = \{d_i : i \in \delta\}$ is a minimal critical diagonal set of A and let $d_i = \hat{\lambda}$ for all $i \in \delta$. Then, by Theorem 3.1, the matrix $A_{\hat{\lambda}}$ is combinatorially singular and so, by Theorem 2.2, it must have a zero submatrix $A_{\hat{\lambda}}[\alpha \mid \beta]$ of size $|\alpha| + |\beta| \geq n + 1$. We then have that $|\alpha \cap \beta| \geq 1$ and hence $A_{\hat{\lambda}}$ must have a zero diagonal element. Furthermore, since we may in particular choose $d_j \neq \hat{\lambda}$ for all $j \in \delta^c$, we have that $\alpha \cap \beta \subseteq \delta$. But, by minimality of S , it follows that $\alpha \cap \beta = \delta$, or otherwise $S \supset \{d_i : i \in \alpha \cap \beta\}$, which by Theorem 3.3 is indeed a critical diagonal set of A . Next, observe that $\tilde{A}[\alpha \mid \beta] = 0$. In addition, if $|\alpha| + |\beta| > n + 1$, then $\tilde{A}[\alpha \mid \beta]$ contains a zero submatrix of size precisely $n + 1$, which therefore yields another critical diagonal set strictly contained in S ; this is a contradiction to the minimality of A which shows that $|\alpha| + |\beta| = n + 1$, completing the proof of the theorem. \square

Recall now the tridiagonal matrix in (1.1). In view of Theorem 3.5, d_2 cannot be a member of a minimal critical diagonal set of A , because every submatrix $\tilde{A}[\alpha \mid \beta]$ with $|\alpha| + |\beta| = 4$ and $\{2\} \in \alpha \cap \beta$ is nonzero. We can obtain, as a corollary to Theorem 3.5, the following bound on the cardinality of a minimal critical diagonal set.

Corollary 3.6 Let S be a minimal critical diagonal set of an $n \times n$ matrix A . Then,

$$|S| \leq \frac{n+1}{2}.$$

Proof. Let $S = \{d_i : i \in \delta\}$ be a minimal critical diagonal set. A particular consequence of Theorem 3.5 is that $\tilde{A}[\delta]$ is a zero submatrix of \tilde{A} of size

$2 \mid \delta \mid$. Suppose that $|\delta| > \frac{n+1}{2}$. Then, as $2 \mid \delta \mid > n+1$, $\tilde{A}[\delta]$ contains a zero submatrix $\tilde{A}[\alpha \mid \beta]$ such that $n+1 \leq |\alpha| + |\beta| < 2 \mid \delta \mid$ and therefore $\{d_i : i \in \alpha \cap \beta\} \subset S$ is a critical diagonal set of A . This contradicts the minimality of S , showing that $|\delta| \leq \frac{n+1}{2}$. \square

In the next theorem we connect the notion of a minimal critical diagonal set with the Frobenius normal form of a matrix. Recall that for any $n \times n$ matrix A , there exists a permutation matrix $P = [p_{ij}]$ such that PAP^T is a block upper triangular matrix, where each diagonal submatrix A_{kk} is either 1×1 or is irreducible (see [V 2.3]). This is called the Frobenius normal form of A . Since $\sigma(A) = \bigcup_k \sigma(A_{kk})$ and a permutation similarity simply permutes the diagonal entries of A , a critical diagonal set $\{d_i : i \in \delta\}$ of A corresponds to a critical diagonal set $\{d_{\pi(i)} : i \in \delta\}$ of PAP^T , where π is a permutation of N and $p_{i,\pi(i)} = 1$, $1 \leq i \leq n$.

Theorem 3.7 *Let $\{d_i : i \in \delta\}$ be a minimal critical diagonal set of A . Then, all entries of the corresponding minimal critical diagonal set $\{d_{\pi(i)} : i \in \delta\}$ of the Frobenius normal form of A are in the same diagonal submatrix.*

Proof. Suppose $S = \{d_i : i \in \delta\}$ is a minimal critical diagonal set of A and let $d_i = \hat{\lambda}$ for all $i \in \delta$. Then $\hat{\lambda} \in \sigma(A)$. Consider now a diagonal submatrix A_{kk} of the Frobenius normal form of A , such that $d_{\pi(j)}$ is in A_{kk} , for some $j \in \delta$. We claim that $\hat{\lambda} \in \sigma(A_{kk})$ and all $d_{\pi(i)}$, $i \in \delta$ belong to A_{kk} . To prove these claims, observe that the spectrum of any one diagonal submatrix of the Frobenius normal form of A does not depend on the choice of entries of any other submatrix. As a consequence, if $\hat{\lambda} \notin \sigma(A_{kk})$, then $\hat{\lambda} \in \sigma(A)$ independently of the choice of $d_{\pi(j)}$, so that $S \setminus \{d_{\pi(j)}\}$ is a critical diagonal set of A , contradicting minimality of S . Similarly, since now $\hat{\lambda} \in \sigma(A_{kk})$, if for some $l \in \delta$ $d_{\pi(l)}$ is not in A_{kk} , then the diagonal entries of A_{kk} in δ form a critical diagonal set of A , a contradiction to the minimality of S , completing the proof of the theorem. \square

By Theorem 3.7, S is a minimal critical diagonal set of A if and only if S is a minimal critical diagonal set of some diagonal submatrix of the Frobenius normal form of A . As we will see in the examples in the next section, one such submatrix can contain more than one minimal critical diagonal set of A . The following result characterizes the case when an

$n \times n$ matrix has n minimal critical diagonal sets. An equivalent statement in terms of 0 eigenvalues is contained in [E], so we omit the proof.

Theorem 3.8 *Every diagonal element is a (minimal) critical diagonal set of a matrix A if and only if A is permutationally similar to a triangular matrix.*

The size of a zero submatrix of a combinatorially singular matrix is related to reducibility as follows.

Theorem 3.9 *Let A be an $n \times n$ matrix with $\tilde{A}[\alpha \mid \beta] = 0$ of size $|\alpha| + |\beta| = n + p$ and $|\alpha \cap \beta| = p$. Then A is reducible.*

Proof. Notice that, in order to show that A is reducible, it is enough to show that \tilde{A} , after an appropriate symmetric permutation, contains a zero block of size n whose entries do not intersect with the diagonal. Let us assume, without loss of generality, that the rows (and columns) of \tilde{A} are ordered as follows. First the rows in $\alpha \cap \beta$, next those in $\alpha \setminus (\alpha \cap \beta)$, then those in $N \setminus (\alpha \cup \beta)$, and finally the rows in $\beta \setminus (\alpha \cap \beta)$. Observe then that the zero block $\tilde{A}[\alpha \mid \beta \setminus (\alpha \cap \beta)]$ has size $|\alpha| + |\beta \setminus (\alpha \cap \beta)| = |\alpha| + |\beta| - p = n + p - p = n$, and $\alpha \cap (\beta \setminus (\alpha \cap \beta)) = \emptyset$, showing that A is reducible. \square

Also observe that if A is an $n \times n$ irreducible matrix, then there cannot be a zero submatrix $\tilde{A}[\alpha \mid \beta]$ with $|\alpha| + |\beta| = n + p$ and $p > n - 2$. However, the matrix A having off-diagonal entries a_{ij} nonzero if and only if $i = 1$ or $j = 1$ has a zero submatrix in \tilde{A} with $p = n - 2$.

We now return our attention to the structure of the eigenspace of a combinatorial eigenvalue $\hat{\lambda}$.

Theorem 3.10 *Let $\{d_i : i \in \delta\}$ be a minimal critical diagonal set of a matrix A . If $d_i = \hat{\lambda}$ for all $i \in \delta$, then $\text{cam}(\hat{\lambda}) = 1$.*

Proof. Let $S = \{d_i : i \in \delta\}$ be a minimal critical diagonal set of A . If $|\delta| = 1$, then the result is trivially true. If $|\delta| > 1$ and $\text{cam}(\hat{\lambda}) = m \geq 2$, then each term on the righthand side of (2.1) has a factor of the form $\prod_{j=1}^m (d_{i_j} - \lambda)$, where $d_{i_j} \in S$. In particular, for any fixed $l \in \delta$ each term on the righthand side of (2.1) contains a factor of the form $\prod_{k=1}^{m-1} (d_{i_k} - \lambda)$, where $d_{i_k} \in S \setminus \{d_l\}$. Consequently, $\{d_i : i \in \delta \setminus \{l\}\}$ is a critical diagonal set of A , contradicting the minimality of S . Thus $\text{cam}(\hat{\lambda}) = 1$. \square

In the next theorem, we investigate the lower bound for $\text{cam}(\hat{\lambda})$ which we have obtained in Theorem 3.3, when the critical diagonal set is the union of smaller critical diagonal sets of A .

Theorem 3.11 *Let A be an $n \times n$ matrix and suppose that $\tilde{A}[\alpha_j | \beta_j] = 0$, $|\alpha_j| + |\beta_j| = n + p_j$, with $p_j \geq 1$, $\gamma_j = \alpha_j \cap \beta_j$ for $j = 1, 2$. If $d_i = \hat{\lambda}$ for all $i \in \gamma_1 \cup \gamma_2$, then*

$$\text{cam}(\hat{\lambda}) \geq \begin{cases} p_1 + p_2 - |\gamma_1 \cap \gamma_2| & \text{if } |\gamma_1 \cap \gamma_2| < \min(p_1, p_2) \\ \max(p_1, p_2) & \text{otherwise.} \end{cases}$$

Proof. Clearly $\{d_i : i \in \gamma_1 \cup \gamma_2\}$ is a critical diagonal set of A . In addition, we can show, as in Theorem 3.3 (iii), that every term of the characteristic polynomial of A which does not vanish has at least p_1 factors $(\hat{\lambda} - \lambda)$ from the set γ_1 and at least p_2 factors $(\hat{\lambda} - \lambda)$ from the set γ_2 . Observe that since $|\gamma_i| \geq p_i$, $i = 1, 2$, we have that if

$$\mathcal{M} \equiv \max\{\max(p_1, p_2), p_1 + p_2 - |\gamma_1 \cap \gamma_2|\}, \quad (3.2)$$

then $\text{cam}(\hat{\lambda}) \geq \mathcal{M}$. Observe now that if $|\gamma_1 \cap \gamma_2| \leq \min(p_1, p_2)$, then $\mathcal{M} = p_1 + p_2 - |\gamma_1 \cap \gamma_2|$, and if $|\gamma_1 \cap \gamma_2| > \min(p_1, p_2)$, then $\mathcal{M} = \max(p_1, p_2)$. \square

Our next goal is to provide an upper bound for the combinatorial algebraic multiplicity.

Theorem 3.12 *Let $\{d_i : i \in \delta\}$ be a critical diagonal set of an $n \times n$ matrix A . If $d_i = \hat{\lambda}$ for all $i \in \delta$, then $\text{cam}(\hat{\lambda}) \leq |\delta|$.*

Proof. Suppose $\{d_i : i \in \delta\}$ is a critical diagonal set of A and let $d_i = \hat{\lambda}$ for all $i \in \delta$. According to its definition, $\text{cam}(\hat{\lambda})$ cannot exceed the algebraic multiplicity of $\hat{\lambda}$ for any particular choice of the remaining nonzero entries of A . Therefore, in order to show that $\text{cam}(\hat{\lambda}) \leq |\delta|$, we need only to display a choice of the remaining nonzero entries of A , for which the algebraic multiplicity of $\hat{\lambda}$ is at most $|\delta|$. Choose then the d_j with $j \in \delta^c$ to be distinct and different from $\hat{\lambda}$. It follows that for off-diagonal entries sufficiently small, there are $n - |\delta| + 1$ disjoint Gerschgorin disks and only one of them contains $\hat{\lambda}$. Consequently, by Gerschgorin's theorem (see, e.g., [HJ]) A has at least $(n - |\delta|)$ eigenvalues different than $\hat{\lambda}$. \square

In the following result we consider a combinatorial eigenvalue resulting from a maximal zero submatrix of \tilde{A} and link its geometric multiplicity to the size of this submatrix.

Theorem 3.13 *Let A be an $n \times n$ matrix with $\tilde{A}[\alpha \mid \beta]$ a maximal zero submatrix of \tilde{A} of size $|\alpha| + |\beta| = n + p$, $p \geq 1$. If $d_i = \hat{\lambda}$, $i \in \alpha \cap \beta$, then $\text{cgm}(\hat{\lambda}) \geq p$.*

Proof. Observe that, if $d_i = \hat{\lambda}$ for all $i \in \alpha \cap \beta$, then $A_{\hat{\lambda}}$ contains the maximal zero submatrix $A_{\hat{\lambda}}[\alpha \mid \beta]$. Consider now the eigenvector equation $A_{\hat{\lambda}}u = 0$, where u is partitioned into the direct sum $u = u[\beta] \oplus u[\beta^c]$. Then, since $A_{\hat{\lambda}}[\alpha \mid \beta] = 0$, we have that

$$A_{\hat{\lambda}}[\alpha \mid \beta^c]u[\beta^c] = 0. \quad (3.3)$$

As the system in (3.3) always admits the trivial solution, we will in particular consider an eigenvector u satisfying $u[\beta^c] = 0$. In such a case one obtains

$$A_{\hat{\lambda}}[\alpha^c \mid \beta]u[\beta] = 0. \quad (3.4)$$

By the maximality of $A_{\hat{\lambda}}[\alpha \mid \beta]$ as a zero submatrix, equation (3.4) represents a homogeneous system of precisely $|\alpha^c| = n - |\alpha| = |\beta| - p$ equations in $|\beta|$ unknowns. Consequently, there exist at least p linearly independent eigenvectors corresponding to $\hat{\lambda}$. \square

Notice that in Theorem 3.13, A is implicitly required to have a partly zero eigenvector. In [MOV] it is shown that an $n \times n$ irreducible weakly cyclic matrix of index n (i.e., one whose zero/nonzero off-diagonal pattern is a cycle of length n , see [V]) cannot have a partly zero eigenvector. This is in agreement with the fact that such a matrix cannot have a critical diagonal set, since \tilde{A} cannot in this case be combinatorially singular (see Theorem 3.2).

4 Examples

We present some examples in order to illustrate our results.

Example 4.1 Consider the matrix

$$A = \begin{bmatrix} d_1 & a_{12} & 0 & 0 \\ a_{21} & d_2 & a_{23} & a_{24} \\ a_{31} & a_{32} & d_3 & 0 \\ 0 & a_{42} & 0 & d_4 \end{bmatrix} \quad \text{with} \quad \tilde{A} = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{bmatrix}.$$

Here \tilde{A} is irreducible, has generic principal rank three and maximal zero submatrices $\tilde{A}[\{1,4\} \mid \{1,3,4\}]$ and $\tilde{A}[\{1,3,4\} \mid \{3,4\}]$. For both these submatrices $p = 1$. Observe that, by Theorem 3.5, d_2 cannot be contained in any minimal critical diagonal set of A . The sets $\gamma_1 = \{1,4\}$, $\gamma_2 = \{3,4\}$ and $\gamma_1 \cup \gamma_2 = \{1,3,4\}$ all yield critical diagonal sets of A . Only the first two are minimal. It is easy to verify that setting $d_1 = d_4 = \hat{\lambda}$, or $d_3 = d_4 = \hat{\lambda}$, or even $d_1 = d_3 = d_4 = \hat{\lambda}$ yields $\hat{\lambda}$ as a combinatorial eigenvalue with $\text{cam}(\hat{\lambda}) = \text{cgm}(\hat{\lambda}) = 1$.

Example 4.2 Let

$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & a_{14} & 0 & 0 & a_{17} \\ a_{21} & d_2 & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & a_{32} & d_3 & a_{34} & 0 & 0 & a_{37} \\ 0 & a_{42} & 0 & d_4 & 0 & 0 & 0 \\ 0 & a_{52} & a_{53} & a_{54} & d_5 & 0 & a_{57} \\ 0 & a_{62} & a_{63} & a_{64} & 0 & d_6 & a_{67} \\ 0 & a_{72} & 0 & 0 & 0 & 0 & d_7 \end{bmatrix}.$$

\tilde{A} is irreducible and has maximal zero submatrices $\tilde{A}[\{4,7\} \mid \{1,3,4,5,6,7\}]$ and $\tilde{A}[\{1,3,4,5,6,7\} \mid \{1,5,6\}]$ with $p_1 = 1$ and $p_2 = 2$ respectively. Here $\gamma_1 = \{4,7\}$ and $\gamma_2 = \{1,5,6\}$. Observe that γ_1 yields a minimal critical diagonal set, but γ_2 does not (since $\{d_1, d_5\}$, $\{d_1, d_6\}$ and $\{d_5, d_6\}$ are also minimal critical diagonal sets of A). It is easy to see that \tilde{A} has generic principal rank 4. If $d_4 = d_7 = \hat{\lambda}$, then $\hat{\lambda}$ is a combinatorial eigenvalue of A with $\text{cam}(\hat{\lambda}) = \text{cgm}(\hat{\lambda}) = 1$. If $d_1 = d_5 = d_6 = \hat{\lambda}$, then $\text{cam}(\hat{\lambda}) = \text{cgm}(\hat{\lambda}) = 2$. Finally, if $d_1 = d_4 = d_5 = d_6 = d_7 = \hat{\lambda}$, then $\text{cam}(\hat{\lambda}) = 3$ and $\text{cgm}(\hat{\lambda}) = 2$.

Example 4.3 Consider the matrix

$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & d_2 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & d_3 & a_{34} & 0 \\ 0 & 0 & a_{43} & d_4 & a_{45} \\ 0 & 0 & 0 & a_{54} & d_5 \end{bmatrix}.$$

\tilde{A} has the maximal zero submatrices $\tilde{A}[\{1, 2, 3, 4, 5\} \mid \{1\}]$, $\tilde{A}[\{2, 3, 4, 5\} \mid \{1, 2\}]$ and $\tilde{A}[\{3, 5\} \mid \{1, 2, 3, 5\}]$, each with $p = 1$. Observe that A is reducible. The minimal critical diagonal sets of A are $\{d_1\}$, $\{d_2\}$ and $\{d_3, d_5\}$, each one of which, (see Theorem 3.7), belongs to one diagonal block of the Frobenius normal form of A .

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