**ON THE MELLIN TRANSFORM OF A PRODUCT
OF HYPERGEOMETRIC FUNCTIONS**

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Abstract

We obtain representations for the Mellin transform of the product of generalized hypergeometric functions ${}_0F_1[-a^2x^2]{}_1F_2[-b^2x^2]$ for $a, b > 0$. The later transform is a generalization of the discontinuous integral of Weber and Schafheitlin; in addition to reducing to other known integrals (*e.g.*, integrals involving products of powers, Bessel and Lommel functions), it contains numerous integrals of interest that are not readily available in the mathematical literature. As a by-product of the present investigation, we deduce the second fundamental relation for ${}_3F_2[1]$. Furthermore, we give the sine and cosine transforms of ${}_1F_2[-b^2x^2]$.

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1. Introduction

Although definite integrals of products of two generalized hypergeometric functions have numerous applications in pure and applied mathematics (see, *e.g.*, [3]), not all such integrals have been collected in tables or are readily available in the mathematical literature. In what follows, we shall consider for $a > 0$ and $b > 0$ one such rather general improper integral, namely the Mellin transform:

$$F(s) := \int_0^\infty x^{s-1} {}_0F_1 \left[\overline{}; -a^2 x^2 \right] {}_1F_2 \left[\begin{matrix} \alpha \\ \beta, 1 + \nu \end{matrix}; -b^2 x^2 \right] dx. \quad (1.1)$$

Convergence of this integral will be discussed in Section 2.

If $\alpha = \beta$, the above Mellin transform reduces to the hypergeometric formulation of the discontinuous integral of Weber and Schafheitlin (*cf.* [11, p. 398] and [6, Equations (2.3) to (2.5)]) which, since we shall need it later, we record below:

$$\begin{aligned} & \int_0^\infty x^{s-1} {}_0F_1 \left[\overline{}; -a^2 x^2 \right] {}_0F_1 \left[\overline{}; -b^2 x^2 \right] dx \\ &= \frac{1}{2} b^{-s} \frac{\Gamma(1+\nu)\Gamma(\frac{s}{2})}{\Gamma(1+\nu-\frac{s}{2})} {}_2F_1 \left[\begin{matrix} \frac{s}{2}, \frac{s}{2} - \nu \\ 1 + \mu \end{matrix}; \frac{a^2}{b^2} \right] \end{aligned} \quad (1.2a)$$

$$(0 < a < b; 0 < \Re(s) < \Re(2 + \mu + \nu))$$

$$= \frac{1}{2} a^{-s} \frac{\Gamma(\frac{s}{2})\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(1+\mu+\nu-s)}{\Gamma(1+\mu-\frac{s}{2})\Gamma(1+\nu-\frac{s}{2})\Gamma(1+\mu+\nu-\frac{s}{2})} \quad (1.2b)$$

$$(b = a; 0 < \Re(s) < \Re(1 + \mu + \nu))$$

$$= \frac{1}{2} a^{-s} \frac{\Gamma(1+\mu)\Gamma(\frac{s}{2})}{\Gamma(1+\mu-\frac{s}{2})} {}_2F_1 \left[\begin{matrix} \frac{s}{2}, \frac{s}{2} - \mu \\ 1 + \nu \end{matrix}; \frac{b^2}{a^2} \right] \quad (1.2c)$$

$$(0 < b < a; 0 < \Re(s) < \Re(2 + \mu + \nu)).$$

The above integral is, in fact, continuous at $a = b$. It is called “discontinuous” because the representations on the right-hand sides of Equations (1.2a) and (1.2c) are not analytic continuations of each other. Thus “discontinuous” refers to a discontinuity in representation across $a = b$. It should be noted also that when $a = b$, if $\mu - \nu$ is an odd positive

integer, then Equation (1.2b) actually holds true for $0 < \Re(s) < \Re(2 + \mu + \nu)$ (see Watson's treatise [11, p. 403] for further details).

In addition, we record for later use the well-known Mellin transform (see, *e.g.*, [7, Section 8.4.47.(1), p. 717])

$$\int_0^\infty x^{s-1} {}_0F_1 \left[\begin{matrix} \text{---} \\ 1 + \mu \end{matrix}; -a^2 x^2 \right] dx = \frac{1}{2} a^{-s} \frac{\Gamma(1 + \mu) \Gamma(\frac{s}{2})}{\Gamma(1 + \mu - \frac{s}{2})}, \quad (1.3)$$

where $a > 0$ and $0 < \Re(s) < \Re(\frac{3}{2} + \mu)$.

Since we may write Bessel, Struve and Lommel functions, respectively, as (see, *e.g.*, [10, p. 44])

$$J_\mu(z) = \frac{(\frac{z}{2})^\mu}{\Gamma(1 + \mu)} {}_0F_1 \left[\begin{matrix} \text{---} \\ 1 + \mu \end{matrix}; -\frac{z^2}{4} \right], \quad (1.4)$$

$$H_\mu(z) = \frac{(\frac{z}{2})^{1+\mu}}{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2} + \mu)} {}_1F_2 \left[\begin{matrix} 1 \\ \frac{3}{2}, \frac{3}{2} + \mu \end{matrix}; -\frac{z^2}{4} \right],$$

and

$$s_{\mu,\nu}(z) = \frac{z^{1+\mu}}{(1 + \mu - \nu)(1 + \mu + \nu)} {}_1F_2 \left[\begin{matrix} 1 \\ \frac{1}{2}(3 + \mu - \nu), \frac{1}{2}(3 + \mu + \nu) \end{matrix}; -\frac{z^2}{4} \right],$$

once a representation in terms of generalized hypergeometric functions is deduced for the Mellin transform $F(s)$ (which will be done in Section 5), not only will we be able to obtain immediately all the known special cases (*e.g.*, integrals containing the product of two Bessel functions or the integral of Weber and Schafheitlin [11, pp. 401–403], products of Bessel and Lommel functions [7, Section 2.9.5.(1), p. 110], and products of Bessel and Struve functions [7, Section 2.7.14.(1), p. 88]), but we shall also be able to specialize, for example, to products of Bessel functions and the Fresnel sine and cosine integrals:

$$S(z) = \frac{1}{3} \sqrt{\frac{2}{\pi}} z^{3/2} {}_1F_2 \left[\begin{matrix} \frac{3}{4} \\ \frac{3}{2}, \frac{7}{4} \end{matrix}; -\frac{z^2}{4} \right]$$

and

$$C(z) = \sqrt{\frac{2}{\pi}} z^{1/2} {}_1F_2 \left[\begin{matrix} \frac{1}{4} \\ \frac{1}{2}, \frac{5}{4} \end{matrix}; -\frac{z^2}{4} \right],$$

and every other special case of ${}_1F_2$ that is of interest.

Recently, M.A. Chaudhry [2] reconsidered $F(s)$ specialized to Bessel and Lommel functions $s_{\mu,\nu}(z)$ and discussed the importance of these integrals in many applications. Unfortunately, Chaudhry's result [2, Equation (4)] holds true for the Lommel functions $S_{\mu,\nu}(z)$ and not for $s_{\mu,\nu}(z)$ and the right side of [2, Equation (6)] is in error by a factor of $\frac{1}{2}$. Aside from numerous typographical and other oversights in [2], the latter two mentioned equations produced further erroneous results. Prudnikov *et al.* [7, Section 2.22.4.(1), p. 337] records the Mellin transform of products of Bessel and generalized hypergeometric functions ${}_pF_q[z]$, but the result is obviously not correct when $p = q - 1$, since the integral must be discontinuous.

2. Convergence of the Integral (1.1)

We shall need the following asymptotic results: For $|x| \rightarrow \infty$ and $|\arg(x)| < \frac{\pi}{2}$,

$${}_0F_1 \left[\begin{matrix} \text{---} \\ 1 + \mu; \end{matrix} -x^2 \right] = \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} x^{-\frac{1}{2} - \mu} \cos \left[2x - \frac{\pi}{2} \left(\frac{1}{2} + \mu \right) \right] \quad (2.1)$$

and

$$\begin{aligned} {}_1F_2 \left[\begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -x^2 \right] &= \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\frac{1}{2})\Gamma(\alpha)} x^{-\frac{1}{2} - \nu + \alpha - \beta} \cos \left[2x - \frac{\pi}{2} \left(\frac{1}{2} + \nu + \beta - \alpha \right) \right] \\ &\quad + \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\beta - \alpha)\Gamma(1 + \nu - \alpha)} x^{-2\alpha}. \end{aligned} \quad (2.2)$$

The above results are further simplifications of more precise asymptotic formulas which we shall record in Section 4 (see Equations (4.1) and (4.2)). Note that Equation (2.2) reduces to Equation (2.1) if we set $\alpha = \beta$.

Replacing x by bx ($b > 0$) in Equation (2.2) and using Equation (2.1), it is easy to see that we may also write for $|x| \rightarrow \infty$ and $|\arg(x)| < \frac{\pi}{2}$:

$$\begin{aligned} {}_1F_2 \left[\begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -b^2 x^2 \right] &= \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\alpha)\Gamma(1 + \nu + \beta - \alpha)} {}_0F_1 \left[\begin{matrix} \text{---} \\ 1 + \nu + \beta - \alpha; \end{matrix} -b^2 x^2 \right] \\ &\quad + \frac{\Gamma(\beta)\Gamma(1 + \nu)}{\Gamma(\beta - \alpha)\Gamma(1 + \nu - \alpha)} (bx)^{-2\alpha}, \end{aligned} \quad (2.3)$$

which now yields an identity when $\alpha = \beta$.

Next, by multiplying Equation (2.3) by

$$x^{s-1} {}_0F_1 \left[\text{---}; 1 + \mu; -a^2 x^2 \right],$$

and, for $z > 0$ sufficiently large, integrating the result over (z, ∞) , we have for $a, b > 0$ the asymptotic result:

$$\begin{aligned} & \int_z^\infty x^{s-1} {}_0F_1 \left[\text{---}; 1 + \mu; -a^2 x^2 \right] {}_1F_2 \left[\begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -b^2 x^2 \right] dx \\ &= \frac{\Gamma(\beta)\Gamma(1+\nu)}{\Gamma(\alpha)\Gamma(1+\nu+\beta-\alpha)} \int_z^\infty x^{s-1} {}_0F_1 \left[\text{---}; 1 + \mu; -a^2 x^2 \right] \\ & \quad \cdot {}_0F_1 \left[\text{---}; 1 + \nu + \beta - \alpha; -b^2 x^2 \right] dx \\ & \quad + \frac{\Gamma(\beta)\Gamma(1+\nu) b^{-2\alpha}}{\Gamma(\beta-\alpha)\Gamma(1+\nu-\alpha)} \int_z^\infty x^{s-2\alpha-1} {}_0F_1 \left[\text{---}; 1 + \mu; -a^2 x^2 \right] dx. \end{aligned} \tag{2.4}$$

It is obvious that, in order for the Mellin transform $F(s)$ defined by Equation (1.1) to converge at its lower limit of integration, we must have $\Re(s) > 0$. Further, from the latter asymptotic result and the convergence criteria of Equations (1.2) and (1.3), we see that, for $b \neq a$, convergence at the upper limit of integration is attained, provided that

$$\Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

If $b = a$, $F(s)$ converges, provided that

$$0 < \Re(s) < \Re(1 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right);$$

but if $b = a$ and $\mu - \nu + \alpha - \beta$ is an odd positive integer, then $F(s)$ converges, provided that

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

We note that if $\alpha = \beta$, the condition $\Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right)$ becomes superfluous, since the second term in Equation (2.4) is no longer present. This evidently completes the analysis of the convergence of $F(s)$.

3. The Incomplete Mellin Transform

If the upper limit of integration in Equation (1.1) is replaced by z such that $|z| < \infty$, the integral is said to be incomplete and we define the Euler-type integral:

$$F(s; z) := \int_0^z x^{s-1} {}_0F_1 \left[\begin{matrix} \text{---} \\ 1 + \mu \end{matrix}; -a^2 x^2 \right] {}_1F_2 \left[\begin{matrix} \alpha \\ \beta, 1 + \nu \end{matrix}; -b^2 x^2 \right] dx, \quad (3.1)$$

where, for convergence at the lower limit of integration, $\Re(s) > 0$. But now the parameters a and b may be arbitrary complex numbers.

The incomplete Mellin transform $F(s; z)$ is easily evaluated by expressing the functions ${}_0F_1$ and ${}_1F_2$ as hypergeometric sums, interchanging the resulting double sum and integral and then performing the required term-by-term integrations. Or the integral in Equation (3.1) may be evaluated by using a tabulated result in Exton's handbook [3, Equation (A.1.2.5), p. 172]. Either way we obtain $F(s; z)$ in terms of a single Kampé de Fériet function which is everywhere convergent in its two independent variables:

$$F(s; z) = \frac{z^s}{s} F_{1:1;2}^{1:0;1} \left[\begin{matrix} \frac{s}{2} : \text{---}; & \alpha; \\ 1 + \frac{s}{2} : 1 + \mu; & \beta, 1 + \nu; \end{matrix} -a^2 z^2, -b^2 z^2 \right], \quad (3.2)$$

where $\Re(s) > 0$. Regions of convergence for Kampé de Fériet and other generalized hypergeometric functions in two variables may be determined by using Horn's theorem for double series (see Srivastava and Karlsson [9, p. 57]).

Further, it is easy to show that the incomplete transform $F(s; z)$ in Equation (3.2) may be written in the following two different ways for $\Re(s) > 0$:

$$F(s; z) = \frac{z^s}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)_n (\alpha)_n}{\left(1 + \frac{s}{2}\right)_n (\beta)_n (1 + \nu)_n} \frac{(-b^2 z^2)^n}{n!} {}_1F_2 \left[\begin{matrix} \frac{s}{2} + n; \\ 1 + \frac{s}{2} + n, 1 + \mu; \end{matrix} -a^2 z^2 \right] \quad (3.3)$$

and

$$F(s; z) = \frac{z^s}{s} \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2}\right)_n}{\left(1 + \frac{s}{2}\right)_n (1 + \mu)_n} \frac{(-a^2 z^2)^n}{n!} {}_2F_3 \left[\begin{matrix} \frac{s}{2} + n, \alpha; \\ 1 + \frac{s}{2} + n, \beta, 1 + \nu; \end{matrix} -b^2 z^2 \right]. \quad (3.4)$$

4. Asymptotic Formulas for ${}_pF_{p+1}[-z^2]$

We shall want to evaluate $F(s; z)$ given by Equations (3.3) and (3.4) as $z \rightarrow \infty$, thereby obtaining the Mellin transform $F(s)$ defined by Equation (1.1). To this end, we record below somewhat simplified asymptotic formulas for the generalized hypergeometric function

$${}_pF_{p+1}[(a_p); (b_{p+1}); -z^2], \quad p = 0, 1, 2.$$

These results are special cases of a general formula due to C.S. Meijer (*circa* 1946) given by Luke [4, p. 203, Eq. (4)] for $|z| \rightarrow \infty$ and $|\arg(z)| < \frac{\pi}{2}$:

$${}_0F_1 \left[\begin{matrix} -; \\ a; \end{matrix} -z^2 \right] = \frac{\Gamma(a)}{\Gamma(\frac{1}{2})} \left(\frac{1}{z^2} \right)^\eta \left[1 + O \left(\frac{1}{z^2} \right) \right] \cos \xi(z), \quad (4.1)$$

where

$$\eta = \frac{1}{2} \left(a - \frac{1}{2} \right) \quad \text{and} \quad \xi(z) = 2z - \pi\eta + O \left(\frac{1}{z} \right);$$

$${}_1F_2 \left[\begin{matrix} a; \\ b, c; \end{matrix} -z^2 \right] = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b-a)\Gamma(c-a)} \left(\frac{1}{z^2} \right)^a \quad (4.2)$$

$$\cdot {}_3F_0 \left[a, 1+a-b, 1+a-c; ---; -\frac{1}{z^2} \right] + \frac{\Gamma(b)\Gamma(c)}{\Gamma(\frac{1}{2})\Gamma(a)} \left(\frac{1}{z^2} \right)^\eta \left[1 + O \left(\frac{1}{z^2} \right) \right] \cos \xi(z),$$

where

$$\eta = \frac{1}{2} \left(b + c - a - \frac{1}{2} \right) \quad \text{and} \quad \xi(z) = 2z - \pi\eta + O \left(\frac{1}{z} \right);$$

and

$$\begin{aligned} {}_2F_3 \left[\begin{matrix} a, b; \\ c, d, e; \end{matrix} -z^2 \right] &= \frac{\Gamma(b-a)}{\Gamma(b)} \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(c-a)\Gamma(d-a)\Gamma(e-a)} \left(\frac{1}{z^2} \right)^a \\ &\quad \cdot {}_4F_1 \left[\begin{matrix} a, 1+a-c, 1+a-d, 1+a-e; \\ 1+a-b; \end{matrix} -\frac{1}{z^2} \right] \\ &\quad + \frac{\Gamma(a-b)}{\Gamma(a)} \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(c-b)\Gamma(d-b)\Gamma(e-b)} \left(\frac{1}{z^2} \right)^b \\ &\quad \cdot {}_4F_1 \left[\begin{matrix} b, 1+b-c, 1+b-d, 1+b-e; \\ 1+b-a; \end{matrix} -\frac{1}{z^2} \right] \\ &\quad + \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \left(\frac{1}{z^2} \right)^\eta \left[1 + O \left(\frac{1}{z^2} \right) \right] \cos \xi(z), \end{aligned} \quad (4.3)$$

where

$$\eta = \frac{1}{2} (c + d + e - a - b - \frac{1}{2}) \quad \text{and} \quad \xi(z) = 2z - \pi\eta + O\left(\frac{1}{z}\right).$$

Noting Equation (1.4), we see that Equation (4.1) is essentially the well-known asymptotic formula for Bessel functions. We remark also that the right members of Equations (4.1) to (4.3) may be written in further abbreviated, yet useful, approximate forms (*cf.* [5, p. 146]):

$${}_0F_1 \left[\begin{matrix} - \\ a; \end{matrix} -z^2 \right] \sim Az^{\frac{1}{2}-a} \cos(2z + B), \quad (4.4)$$

$${}_1F_2 \left[\begin{matrix} a; \\ b, c; \end{matrix} -z^2 \right] \sim Az^{-2a} + Bz^{\frac{1}{2}+a-b-c} \cos(2z + C), \quad (4.5)$$

and

$${}_2F_3 \left[\begin{matrix} a, b; \\ c, d, e; \end{matrix} -z^2 \right] \sim Az^{-2a} + Bz^{-2b} + Cz^{\frac{1}{2}+a+b-c-d-e} \cos(2z + D),$$

where $|z| \rightarrow \infty$, $|\arg(z)| < \frac{\pi}{2}$, and A, B, C, D are dependent on the parameters of the function ${}_pF_{p+1}[-z^2]$ ($p = 0, 1, 2$) and on $O\left(\frac{1}{z^2}\right)$ or $O\left(\frac{1}{z}\right)$.

5. Evaluation of the Mellin Transform (1.1)

Now that we have noted and developed some preliminaries, we are ready to derive our main result for the Mellin transform $F(s)$ defined in Equation (1.1) as

$$F(s) := \int_0^\infty x^{s-1} {}_0F_1 \left[\begin{matrix} - \\ 1 + \mu; \end{matrix} -a^2 x^2 \right] {}_1F_2 \left[\begin{matrix} \alpha; \\ \beta, 1 + \nu; \end{matrix} -b^2 x^2 \right] dx,$$

where $a > 0$ and $b > 0$. Thus we shall show that

$$F(s) = \frac{1}{2} a^{-s} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma(1 + \mu)}{\Gamma\left(1 + \mu - \frac{s}{2}\right)} {}_3F_2 \left[\begin{matrix} \alpha, \frac{s}{2}, \frac{s}{2} - \mu; \\ \beta, 1 + \nu; \end{matrix} \frac{b^2}{a^2} \right] \quad (b < a) \quad (5.1a)$$

and

$$\begin{aligned} F(s) = & \frac{1}{2} b^{-s} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\alpha - \frac{s}{2}\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma(\alpha) \Gamma\left(\beta - \frac{s}{2}\right) \Gamma\left(1 + \nu - \frac{s}{2}\right)} {}_3F_2 \left[\begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu; \\ 1 + \frac{s}{2} - \alpha, 1 + \mu; \end{matrix} \frac{a^2}{b^2} \right] \\ & + \frac{1}{2} a^{-s} \left(\frac{a^2}{b^2}\right)^\alpha \frac{\Gamma(1 + \mu) \Gamma(1 + \nu) \Gamma(\beta) \Gamma\left(\frac{s}{2} - \alpha\right)}{\Gamma(1 + \nu - \alpha) \Gamma(1 + \mu + \alpha - \frac{s}{2}) \Gamma(\beta - \alpha)} \end{aligned} \quad (5.1b)$$

$${}_3F_2 \left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2}; \end{matrix} \frac{a^2}{b^2} \right] \quad (a < b),$$

where, for convergence of the integral (see Section 2),

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha)$$

and

$$0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

If $\alpha = \beta$, the latter conditional inequality is superfluous and Equations (5.1a) and (5.1b) reduce, respectively, to Equations (1.2c) and (1.2a). The case $a = b$ will be discussed in Section 6. In Section 7 we shall use Equations (5.1) to derive the sine and cosine transforms of ${}_1F_2[-b^2x^2]$.

To derive Equation (5.1a), we approximate ${}_1F_2[-a^2z^2]$ in Equation (3.3) for large positive z by using Equation (4.2), thus obtaining

$$\begin{aligned} {}_1F_2\left[\begin{matrix} \frac{s}{2} + n; \\ 1 + \frac{s}{2} + n, 1 + \mu; \end{matrix} -a^2z^2\right] &= \frac{\Gamma(1 + \frac{s}{2} + n)\Gamma(1 + \mu)}{\Gamma(1 + \mu - \frac{s}{2} - n)} \left(\frac{1}{a^2z^2}\right)^{\frac{s}{2} + n} \\ &\quad \cdot {}_3F_0\left[\begin{matrix} \frac{s}{2} + n, 0, \frac{s}{2} - \mu + n; \\ - \end{matrix} -\frac{1}{a^2z^2}\right] \\ &\quad + \frac{\Gamma(1 + \frac{s}{2} + n)\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2} + n)} \left(\frac{1}{a^2z^2}\right)^{\frac{1}{2}(\frac{s}{2} + \mu)} \\ &\quad \cdot \left[1 + O\left(\frac{1}{z^2}\right)\right] \cos\left[2az - \frac{\pi}{2}\left(\frac{s}{2} + \mu\right) + O\left(\frac{1}{z}\right)\right]. \end{aligned}$$

Noting that ${}_3F_0[-\frac{1}{a^2z^2}] = 1$ for all integers $n \geq 0$ and using the identity

$$\Gamma(\alpha - n) = (-1)^n \frac{\Gamma(\alpha)}{(1 - \alpha)_n}, \quad (5.2)$$

we may then write

$$\begin{aligned} {}_1F_2\left[\begin{matrix} \frac{s}{2} + n; \\ 1 + \frac{s}{2} + n, 1 + \mu; \end{matrix} -a^2z^2\right] &= \frac{\Gamma(1 + \mu)\Gamma(1 + \frac{s}{2})}{\Gamma(1 + \mu - \frac{s}{2})} \left(1 + \frac{s}{2}\right)_n \left(\frac{s}{2} - \mu\right)_n \left(\frac{1}{az}\right)^s \left(-\frac{1}{a^2z^2}\right)^n \\ &\quad + \frac{s}{2} \left(\frac{1}{az}\right)^{\frac{s}{2} + \mu} \frac{\Gamma(1 + \mu)}{\Gamma(\frac{1}{2})} \frac{(1 + \frac{s}{2})_n}{(\frac{s}{2})_n} \left[1 + O\left(\frac{1}{z^2}\right)\right] \cos(2az + D), \end{aligned} \quad (5.3)$$

where D is obviously dependent on μ and $O\left(\frac{1}{z}\right)$.

Now, substituting the result in Equation (5.3) for ${}_1F_2[-a^2z^2]$ into Equation (3.3), we obtain

$$\begin{aligned} F(s; z) = & \frac{1}{2}a^{-s} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma(1+\mu)}{\Gamma\left(1+\mu-\frac{s}{2}\right)} {}_3F_2\left[\begin{matrix} \alpha, \frac{s}{2}, \frac{s}{2}-\mu; \\ \beta, 1+\nu; \end{matrix} \frac{b^2}{a^2}\right] \\ & + \frac{1}{2}a^{-\mu-\frac{3}{2}} \frac{\Gamma(1+\mu)}{\Gamma\left(\frac{1}{2}\right)} z^{s-\mu-\frac{3}{2}} {}_1F_2\left[\begin{matrix} \alpha; \\ \beta, 1+\nu; \end{matrix} -b^2z^2\right] \\ & \cdot \left[1 + O\left(\frac{1}{z^2}\right)\right] \cos(2az + D), \end{aligned} \quad (5.4)$$

where, for convergence of ${}_3F_2\left[\frac{b^2}{a^2}\right]$, we must have $b < a$. Next, denoting the second term in the above Equation (5.4) by $S_0(z)$ and using Equation (4.5) to approximate ${}_1F_2[-b^2z^2]$, we have

$$\begin{aligned} S_0(z) = & \left[\frac{1}{2}Aa^{-\mu-\frac{3}{2}} b^{-2\alpha} \frac{\Gamma(1+\mu)}{\Gamma\left(\frac{1}{2}\right)} z^{s-2\alpha-\mu-\frac{3}{2}} \right. \\ & \left. + \frac{1}{2}Ba^{-\mu-\frac{3}{2}} b^{\alpha-\beta-\nu-\frac{1}{2}} \frac{\Gamma(1+\mu)}{\Gamma\left(\frac{1}{2}\right)} z^{s-\mu-\nu+\alpha-\beta-2} \cos(2bz + C) \right] \\ & \cdot \left[1 + O\left(\frac{1}{z^2}\right)\right] \cos(2az + D). \end{aligned} \quad (5.5)$$

Finally, letting $z \rightarrow \infty$ we see that, since

$$\Re\left(s - 2\alpha - \mu - \frac{3}{2}\right) < 0$$

and

$$\Re(s - \mu - \nu + \alpha - \beta - 2) < 0$$

(both of which conditions secure the convergence of $F(s; z)$ to $F(s)$), $S_0(z)$ or the second term in Equation (5.4) vanishes. Thus Equation (5.4) yields Equation (5.1a) upon letting $z \rightarrow \infty$.

The derivation of Equation (5.1b) is similar to that of Equation (5.1a), but is somewhat more complex in its details, because the asymptotic formula for ${}_2F_3[-z^2]$ given by Equation (4.3) has three terms. Thus, for large z , by using the latter together with Equation (3.4)

and employing Equation (5.2), we obtain, for $a < b$,

$$F(s; z) = \frac{1}{2} b^{-s} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\alpha - \frac{s}{2}\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma(\alpha) \Gamma\left(\beta - \frac{s}{2}\right) \Gamma\left(1 + \nu - \frac{s}{2}\right)} {}_3F_2 \left[\begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu; \\ 1 + \frac{s}{2} - \alpha, 1 + \mu; \end{matrix} \frac{a^2}{b^2} \right] + S_1(z) + S_2(z), \quad (5.6)$$

where

$$S_1(z) = \frac{1}{2} b^{-2\alpha} z^{s-2\alpha} \frac{\Gamma\left(\frac{s}{2} - \alpha\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma\left(1 + \frac{s}{2} - \alpha\right) \Gamma(\beta - \alpha) \Gamma(1 + \nu - \alpha)} \cdot \sum_{n=0}^{\infty} \frac{\left(\frac{s}{2} - \alpha\right)_n}{\left(1 + \frac{s}{2} - \alpha\right)_n (1 + \mu)_n} \frac{(-a^2 z^2)^n}{n!} \cdot {}_4F_1 \left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu, \alpha - \frac{s}{2} - n; \\ 1 + \alpha - \frac{s}{2} - n; \end{matrix} -\frac{1}{b^2 z^2} \right], \quad (5.7)$$

and

$$S_2(z) = \frac{1}{2} z^{s+\alpha-\beta-\nu-\frac{3}{2}} \left[1 + O\left(\frac{1}{z^2}\right) \right] \cos \left[2bz - \frac{\pi}{2} \left(\beta - \alpha + \nu + \frac{3}{2} \right) + O\left(\frac{1}{z}\right) \right] \cdot {}_0F_1 \left[\begin{matrix} \text{---} \\ 1 + \mu; \end{matrix} -a^2 z^2 \right]. \quad (5.8)$$

Now, by using Equation (4.4) we can rewrite Equation (5.8) as

$$S_2(z) = \frac{1}{2} A z^{s+\alpha-\beta-\mu-\nu-2} \left[1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + B) \cos(2bz + C), \quad (5.9)$$

where the definition of C is obvious. Further, by writing ${}_4F_1 \left[-\frac{1}{b^2 z^2} \right]$ in Equation (5.7) as a hypergeometric m -summation, interchanging it with its preceding n -summation, and then employing Equation (5.2), we arrive at

$$S_1(z) = \frac{1}{2} b^{-2\alpha} z^{s-2\alpha} \frac{\Gamma\left(\frac{s}{2} - \alpha\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma\left(1 + \frac{s}{2} - \alpha\right) \Gamma(\beta - \alpha) \Gamma(1 + \nu - \alpha)} \cdot \sum_{m=0}^{\infty} \frac{(\alpha)_m (1 + \alpha - \beta)_m (\alpha - \nu)_m \left(\alpha - \frac{s}{2}\right)_m}{\left(1 + \alpha - \frac{s}{2}\right)_m} \frac{\left(-\frac{1}{b^2 z^2}\right)^m}{m!} \cdot {}_1F_2 \left[\begin{matrix} \frac{s}{2} - \alpha - m; \\ 1 + \frac{s}{2} - \alpha - m, 1 + \mu; \end{matrix} -a^2 z^2 \right]. \quad (5.10)$$

We see from Equations (4.2) and (5.2) that ${}_1F_2[-a^2z^2]$ in Equation (5.10) may be written as

$$\begin{aligned}
{}_1F_2 \left[\begin{matrix} \frac{s}{2} - \alpha - m; \\ 1 + \frac{s}{2} - \alpha - m, 1 + \mu; \end{matrix} -a^2z^2 \right] &= \frac{z^{2\alpha-s}}{a^{s-2\alpha}} \frac{\Gamma(1+\mu)\Gamma(1+\frac{s}{2}-\alpha)}{\Gamma(1+\mu+\alpha-\frac{s}{2})} \\
&\cdot \frac{(-a^2z^2)^m}{(\alpha-\frac{s}{2})_m (1+\mu+\alpha-\frac{s}{2})_m} {}_3F_0 \left[\begin{matrix} \frac{s}{2} - \alpha - m, 0, \frac{s}{2} - \alpha - \mu - m; \\ \text{---}; \end{matrix} -\frac{1}{a^2z^2} \right] \\
&+ \frac{z^{-\frac{3}{2}-\mu}}{a^{\frac{3}{2}+\mu}} \frac{\Gamma(1+\mu)\Gamma(1+\frac{s}{2}-\alpha)}{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2}-\alpha)} \frac{(1+\alpha-\frac{s}{2})_m}{(\alpha-\frac{s}{2})_m} \left[1 + O\left(\frac{1}{z^2}\right) \right] \\
&\cdot \cos \left[2az - \frac{\pi}{2} \left(\frac{3}{2} + \mu \right) + O\left(\frac{1}{z}\right) \right].
\end{aligned} \tag{5.11}$$

Upon noting in Equation (5.11) that ${}_3F_0[-\frac{1}{a^2z^2}] = 1$ for all integers $m \geq 0$, we find from Equations (5.10) and (5.11) that

$$\begin{aligned}
S_1(z) &= \frac{1}{2}a^{-s} \left(\frac{a^2}{b^2} \right)^\alpha \frac{\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(\beta)\Gamma(\frac{s}{2}-\alpha)}{\Gamma(1+\nu-\alpha)\Gamma(1+\mu+\alpha-\frac{s}{2})\Gamma(\beta-\alpha)} \\
&\cdot \sum_{m=0}^{\infty} \frac{(\alpha)_m(1+\alpha-\beta)_m(\alpha-\nu)_m}{(1+\alpha-\frac{s}{2})_m(1+\mu+\alpha-\frac{s}{2})_m} \frac{\left(\frac{a^2}{b^2} \right)^m}{m!} \\
&+ \frac{1}{2} \frac{b^{-2\alpha}}{a^{\frac{3}{2}+\mu}} z^{s-\mu-2\alpha-\frac{3}{2}} \frac{\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(\beta)}{\Gamma(\frac{1}{2})\Gamma(\beta-\alpha)\Gamma(1+\nu-\alpha)} \left[1 + O\left(\frac{1}{z^2}\right) \right] \\
&\cdot \cos \left[2az - \frac{\pi}{2} \left(\frac{3}{2} + \mu \right) + O\left(\frac{1}{z}\right) \right] {}_3F_0 \left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ \text{---}; \end{matrix} -\frac{1}{b^2z^2} \right].
\end{aligned} \tag{5.12}$$

Equation (5.12) may be written more simply as

$$\begin{aligned}
S_1(z) &= \frac{1}{2}a^{-s} \left(\frac{a^2}{b^2} \right)^\alpha \frac{\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(\beta)\Gamma(\frac{s}{2}-\alpha)}{\Gamma(1+\nu-\alpha)\Gamma(1+\mu+\alpha-\frac{s}{2})\Gamma(\beta-\alpha)} \\
&\cdot {}_3F_2 \left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2}; \end{matrix} \frac{a^2}{b^2} \right] \\
&+ \frac{1}{2}A' \left[1 + O\left(\frac{1}{z^2}\right) \right] z^{s-\mu-2\alpha-\frac{3}{2}} \cos(2az + B') {}_3F_0 \left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ \text{---}; \end{matrix} -\frac{1}{b^2z^2} \right],
\end{aligned} \tag{5.13}$$

where $a < b$ and the definitions of A' and B' are obvious.

Now, combining Equations (5.6), (5.9), and (5.13), we find that

$$\begin{aligned}
 F(s; z) = & \frac{1}{2} b^{-s} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(\alpha - \frac{s}{2}\right) \Gamma(\beta) \Gamma(1 + \nu)}{\Gamma(\alpha) \Gamma\left(\beta - \frac{s}{2}\right) \Gamma\left(1 + \nu - \frac{s}{2}\right)} {}_3F_2 \left[\begin{matrix} \frac{s}{2}, 1 + \frac{s}{2} - \beta, \frac{s}{2} - \nu; \\ 1 + \frac{s}{2} - \alpha, 1 + \mu; \end{matrix} \frac{a^2}{b^2} \right] \\
 & + \frac{1}{2} a^{-s} \left(\frac{a^2}{b^2} \right)^\alpha \frac{\Gamma(1 + \mu) \Gamma(1 + \nu) \Gamma(\beta) \Gamma\left(\frac{s}{2} - \alpha\right)}{\Gamma(1 + \nu - \alpha) \Gamma\left(1 + \mu + \alpha - \frac{s}{2}\right) \Gamma(\beta - \alpha)} \\
 & \quad \cdot {}_3F_2 \left[\begin{matrix} \alpha, 1 + \alpha - \beta, \alpha - \nu; \\ 1 + \alpha - \frac{s}{2}, 1 + \mu + \alpha - \frac{s}{2}; \end{matrix} \frac{a^2}{b^2} \right] \\
 & + \frac{1}{2} A z^{s + \alpha - \beta - \mu - \nu - 2} \left[1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + B) \cos(2bz + C) \\
 & + \frac{1}{2} A' z^{s - \mu - 2\alpha - \frac{3}{2}} \left[1 + O\left(\frac{1}{z^2}\right) \right] \cos(2az + B') \\
 & \quad \cdot {}_3F_0 \left[\alpha, 1 + \alpha - \beta, \alpha - \nu; \text{---}; -\frac{1}{b^2 z^2} \right].
 \end{aligned} \tag{5.14}$$

Finally, recalling that

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \text{ and } 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right),$$

which secure the convergence of $F(s; z)$ to $F(s)$, upon letting $z \rightarrow \infty$, we see that the third and fourth terms in Equation (5.14) vanish, and we are left with Equation (5.1b). This evidently completes the derivation of Equations (5.1).

6. The Case $a = b$

The expressions on the right-hand sides of Equations (5.1a) and (5.1b) are not analytic continuations of each other. In particular, for $\alpha = \beta$ we noted this in Section 1 and mentioned also that the “discontinuous” nature of $F(s)$ refers to the discontinuity in its representation across $a = b$. However, $F(s)$ is continuous when $a = b$. To see this, since the first integral on the right-hand side of Equation (2.4) is continuous when $a = b$ (see [11, p. 403]), then so is the integral on the left-hand side. Thus also it is evident that $F(s)$ is continuous when a passes through b .

In addition, we showed in Section 2 that when $a = b$, a necessary condition that the integral $F(s)$ converges is that

$$0 < \Re(s) < \Re(1 + \mu + \nu + \beta - \alpha),$$

and *a fortiori* all three ${}_3F_2$ functions in Equations (5.1) converge absolutely when $a = b$, provided that the latter condition holds true. Thus, for $a > 0$,

$$0 < \Re(s) < \Re(1 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right),$$

we deduce, respectively, from Equations (5.1a) and (5.1b) that

$$\begin{aligned} \int_0^\infty x^{s-1} {}_0F_1 \left[\frac{\quad}{1+\mu}; -a^2x^2 \right] {}_1F_2 \left[\begin{matrix} \alpha; \\ \beta, 1+\nu; \end{matrix} -a^2x^2 \right] dx \\ = \frac{1}{2}a^{-s} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma(1+\mu)}{\Gamma\left(1+\mu-\frac{s}{2}\right)} {}_3F_2 \left[\begin{matrix} \alpha, \frac{s}{2}, \frac{s}{2}-\mu; \\ \beta, 1+\nu; \end{matrix} 1 \right] \end{aligned} \quad (6.1)$$

and

$$\begin{aligned} \int_0^\infty x^{s-1} {}_0F_1 \left[\frac{\quad}{1+\mu}; -a^2x^2 \right] {}_1F_2 \left[\begin{matrix} \alpha; \\ \beta, 1+\nu; \end{matrix} -a^2x^2 \right] dx \\ = \frac{1}{2}a^{-s} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\alpha-\frac{s}{2}\right)\Gamma(\beta)\Gamma(1+\nu)}{\Gamma(\alpha)\Gamma\left(\beta-\frac{s}{2}\right)\Gamma\left(1+\nu-\frac{s}{2}\right)} {}_3F_2 \left[\begin{matrix} \frac{s}{2}, 1+\frac{s}{2}-\beta, \frac{s}{2}-\nu; \\ 1+\frac{s}{2}-\alpha, 1+\mu; \end{matrix} 1 \right] \\ + \frac{1}{2}a^{-s} \frac{\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(\beta)\Gamma\left(\frac{s}{2}-\alpha\right)}{\Gamma(1+\nu-\alpha)\Gamma\left(1+\mu+\alpha-\frac{s}{2}\right)\Gamma(\beta-\alpha)} \\ \cdot {}_3F_2 \left[\begin{matrix} \alpha, 1+\alpha-\beta, \alpha-\nu; \\ 1+\alpha-\frac{s}{2}, 1+\mu+\alpha-\frac{s}{2}; \end{matrix} 1 \right]. \end{aligned} \quad (6.2)$$

When $\alpha = \beta$, the second right member in Equation (6.2) vanishes and it is easy to see that the right-hand sides of Equations (6.1) and (6.2) reduce, respectively, via Gauss's theorem to Equation (1.2b). However, the ${}_3F_2[1]$ functions in Equations (6.1) and (6.2) are not, in general, reducible, since their parameters are not interrelated [12].

If we equate the right members of Equations (6.1) and (6.2), divide the result by $\frac{1}{2}a^{-s}$, and then set

$$a = \frac{s}{2} - \mu, \quad b = \frac{s}{2}, \quad c = \alpha, \quad e = \beta, \quad \text{and} \quad f = 1 + \nu, \quad (6.3)$$

we deduce the second fundamental relation for ${}_3F_2[a, b, c; e, f; 1]$, which is

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, b, c; \\ e, f; \end{matrix} 1 \right] &= \frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(c-b)}{\Gamma(e-b)\Gamma(f-b)\Gamma(1+b-a)\Gamma(c)} \\ &\quad \cdot {}_3F_2 \left[\begin{matrix} b, b-e+1, b-f+1; \\ 1+b-c, 1+b-a; \end{matrix} 1 \right] \\ &\quad + \frac{\Gamma(1-a)\Gamma(e)\Gamma(f)\Gamma(b-c)}{\Gamma(e-c)\Gamma(f-c)\Gamma(1+c-a)\Gamma(b)} {}_3F_2 \left[\begin{matrix} c, c-e+1, c-f+1; \\ 1+c-b, 1+c-a; \end{matrix} 1 \right]. \end{aligned} \quad (6.4)$$

The conditional inequality $0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right)$ with the relevant substitutions of Equations (6.3) may now be waived by appealing to the principle of analytic continuation. That Equation (6.4) manifests itself as a corollary is both surprising and interesting, especially since Bailey [1] derives it by considering a certain contour integral, deforming its contour in two different ways, and then computing the integrals via residues [1, p. 15].

Finally, we recall that in Section 2 we showed, in the case $a = b$, that $F(s)$ converges also when

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right),$$

provided that $\mu - \nu + \alpha - \beta$ is an odd positive integer. That Equation (6.1) holds true in this case certainly seems plausible, since it is true for $\alpha = \beta$. It should be remarked also that, in an earlier work, Srivastava and Exton [8] considered a generalization of the Weber-Schafheitlin integral given in Equation (1.2) for the product of several Bessel (or ${}_0F_1$) functions.

7. The Sine and Cosine Transforms of ${}_1F_2[-b^2x^2]$

For brevity, we define

$$S(a, b) := \int_0^\infty \sin(2ax) {}_1F_2[\alpha; \beta, \gamma; -b^2x^2] dx$$

and

$$C(a, b) := \int_0^\infty \cos(2ax) {}_1F_2[\alpha; \beta, \gamma; -b^2x^2] dx,$$

which are, respectively, the sine and cosine transforms of ${}_1F_2[-b^2x^2]$.

Since

$$\sin z = z {}_0F_1 \left[\text{---}; \frac{3}{2}; -\frac{1}{4}z^2 \right]$$

and

$$\cos z = {}_0F_1 \left[\text{---}; \frac{1}{2}; -\frac{1}{4}z^2 \right],$$

it is easy to deduce from Equations (5.1) that

$$S(a, b) = \begin{cases} \frac{1}{2a} {}_3F_2 \left[\begin{matrix} \frac{1}{2}, 1, \alpha; \\ \beta, \gamma; \end{matrix} \frac{b^2}{a^2} \right] & (0 < b < a) \\ \frac{a}{b^2} \frac{(\beta-1)(\gamma-1)}{\alpha-1} {}_3F_2 \left[\begin{matrix} 1, 2-\beta, 2-\gamma; \\ \frac{3}{2}, 2-\alpha; \end{matrix} \frac{a^2}{b^2} \right] + \frac{\sqrt{\pi}}{2a} \left(\frac{a^2}{b^2} \right)^\alpha \\ \cdot \frac{\Gamma(\beta)\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)\Gamma(\frac{1}{2}+\alpha)} {}_2F_1 \left[\begin{matrix} 1+\alpha-\beta, 1+\alpha-\gamma; \\ \frac{1}{2}+\alpha; \end{matrix} \frac{a^2}{b^2} \right] & (0 < a < b) \end{cases}$$

and

$$C(a, b) = \begin{cases} 0 & (0 < b < a) \\ \frac{\sqrt{\pi}}{2b} \frac{\Gamma(\alpha-\frac{1}{2})\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta-\frac{1}{2})\Gamma(\gamma-\frac{1}{2})} {}_2F_1 \left[\begin{matrix} \frac{3}{2}-\beta, \frac{3}{2}-\gamma; \\ \frac{3}{2}-\alpha; \end{matrix} \frac{a^2}{b^2} \right] + \frac{\sqrt{\pi}}{2a} \left(\frac{a^2}{b^2} \right)^\alpha \\ \cdot \frac{\Gamma(\frac{1}{2}-\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)} {}_2F_1 \left[\begin{matrix} 1+\alpha-\beta, 1+\alpha-\gamma; \\ \frac{1}{2}+\alpha; \end{matrix} \frac{a^2}{b^2} \right] & (0 < a < b), \end{cases}$$

where

$$0 < \Re(\alpha) < \Re(\beta + \gamma - \frac{1}{2}).$$

8. Concluding Remarks

It should be mentioned that, by using a general result for the Mellin transform of a product of generalized hypergeometric functions in [7, Section 2.22, p. 333], $F(s)$ defined by Equation (1.1) may be represented by Meijer's G -function. Thus, for $a > 0$ and $b > 0$, we have

$$F(s) = \frac{1}{2} a^{-s} \frac{\Gamma(1+\mu)\Gamma(1+\nu)\Gamma(\beta)}{\Gamma(\alpha)} G_{3,3}^{1,2} \left(\frac{b^2}{a^2} \left| \begin{matrix} 1-\frac{s}{2}, 1-\alpha, 1+\mu-\frac{s}{2} \\ 0, -\nu, 1-\beta \end{matrix} \right. \right), \quad (8.1)$$

where

$$0 < \Re(s) < \Re(2 + \mu + \nu + \beta - \alpha) \quad \text{and} \quad 0 < \Re(s) < \Re\left(\frac{3}{2} + 2\alpha + \mu\right).$$

Furthermore, by using formulas for reducing the G -function to generalized hypergeometric functions (see, *e.g.*, [4, Section 6.5, p. 230]), we may obtain Equations (5.1) from Equation (8.1). Nonetheless, the derivation of Equations (5.1) presented herein is elementary in the sense that it does not require knowledge of the G -function and its properties. In addition, the results given by Equations (6.1), (6.2), and (6.4) require a detailed analysis of the continuity and convergence criteria for $F(s)$ when $a = b$, and, therefore, may not be deduced *directly* from Equation (8.1).

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