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Coefficient Estimates for a Subclass of Analytic Functions Associated with a Certain Leaf-Like Domain

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Abstract: First, by making use of the concept of basic (or q-) calculus, as well as the principle of subordination between analytic functions, generalization $\mathcal{R}_q(h)$ of the class $\mathcal{R}(h)$ of analytic functions, which are associated with the leaf-like domain in the open unit disk $\mathbb{U}$, is given. Then, the coefficient estimates, the Fekete–Szegö problem, and the second-order Hankel determinant $H_2(1)$ for functions belonging to this class $\mathcal{R}_q(h)$ are investigated. Furthermore, similar results are examined and presented for the functions $\frac{z}{f(z)}$ and $f^{-1}(z)$. For the validity of our results, relevant connections with those in earlier works are also pointed out.

Keywords: analytic functions; univalent functions; bounded turning functions; q-derivative (or q-difference) operator; principle of subordination between analytic functions; leaf-like domain; coefficient estimates; Taylor–Maclaurin coefficients; Fekete–Szegö problem; Hankel determinant

MSC: Primary 05A30; 30C45; Secondary 11B65; 47B38

1. Introduction, Definitions, and Motivation

The class of analytic functions in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is the set of complex numbers, is denoted by $\mathcal{H}(\mathbb{U})$. Let $\mathcal{A}$ be the subclass consisting of functions $f \in \mathcal{H}(\mathbb{U})$. We represent the functions class with series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U}), \tag{1}$$

that is, the following normalization condition is also satisfied:

$$f(0) = f'(0) - 1 = 0.$$

Furthermore, the function class comprised of all univalent functions in open unit disk $\mathbb{U}$ is represented by $\mathcal{S}$, which is a subclass of $\mathcal{A}$.

In the furtherance of the area of geometric function theory of complex analysis, several researchers have devoted their studies to the class of analytic functions and its subclasses as well. A function $f \in \mathcal{A}$ is known as starlike and is denoted by $\mathcal{S}^*$, which satisfies the following conditions:

$$f \in \mathcal{S} \qquad \text{and} \qquad \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{U}). \tag{2}$$

For two analytic functions f and g in $\mathbb{U}$, the function f is subordinate to g and written as:

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z),$$

if there exists a Schwarz function $w \in \mathcal{B}$, where:

$$\mathcal{B} := \{w : w \in \mathcal{A},\ \ w(0) = 0 \ \text{and}\ \ |w(z)| < 1 \quad (\forall\, z \in \mathbb{U})\}, \tag{3}$$

such that:

$$f(z) = g\big(w(z)\big).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next, the class of normalized analytic functions p in $\mathbb{U}$ is denoted by $\mathcal{P}$, which is given by:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{4}$$

such that:

$$\Re\{p(z)\} > 0.$$

The class $\mathcal{P}$ plays a central role in the theory of analytic functions, because almost all of the important subclasses of analytic functions were defined by using this class of functions.

Definition 1. *(See [1].) Let $\mathcal{S}^*(\varrho)$ denote the class of analytic functions f in the unit disk $\mathbb{U}$ normalized by:*

$$f(0) = f'(0) - 1 = 0$$

and satisfying the following condition:

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z^2} + z =: \varrho(z) \quad (\forall\, z \in \mathbb{U}), \tag{5}$$

where the branch of the square root is chosen as $\varrho(0) = 1$.

The function class $\mathcal{S}^*(\varrho)$ was defined and studied by Raina and Sokól [1]. Clearly, one can see that $\mathcal{S}^*(\varrho)$ is a function class of starlike functions subordinate to a shell-shaped region. These earlier authors derived results related to coefficient inequalities for this function class [1]. Later on, Priya and Sharma [2], who were essentially motivated by the work of Raina and Sokól [1], introduced a new class $\mathcal{R}(h)$ of functions associated with the leaf-like domain as follows.

Definition 2. *(See [2].) A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(h)$, if it satisfies the following condition:*

$$f'(z) \prec z + \sqrt[3]{1+z^2}. \tag{6}$$

For convenience, now, we recall some firm footing concept details and definitions of the q-difference calculus, which will play a vital role in our presentation. Throughout the article, it should be understood that unless otherwise notified, we presume $0 < q < 1$ and that:

$$\mathbb{N} = \{1, 2, 3, \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, 3, \cdots\}).$$

Definition 3. *For $q \in (0,1)$, we define the q-number $[\lambda]_q$ by:*

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 4. *(See [3,4].) The q-derivative (or the q-difference) operator D_q is defined for a function f in a given subset of $\mathbb{C}$ by:*

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0). \end{cases} \tag{7}$$

We note from Definition 4 that the q-difference $D_q f(z)$ converges to the ordinary derivative $f'(z)$ as follows:

$$\lim_{q \longrightarrow 1-} \left(D_q f\right)(z) = \lim_{q \longrightarrow 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. Moreover, it is readily deduced from Equations (1) and (7) that:

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{8}$$

Up to date, the study of q-calculus has intensely fascinated researchers. This great concentration is due to its advantages in several fields of mathematics and physics. The significance of the operator D_q is quite obvious by its applications in the study of the several subclasses of analytic functions. For example, initially, in 1990, Ismail et al. [5] gave the idea of the q-extension of the class of starlike functions in $\mathbb{U}$. Historically speaking, a foothold usage of the q-calculus in the context of geometric functions theory was effectively invoked by Srivastava (see, for details, [6], p. 347 et seq.). Subsequently, remarkable research work has been done by many authors, which has played an important role in the development of geometric function theory. In particular, Srivastava et al. [7] studied the class of q-starlike functions in the conic region, while the upper bound of the third Hankel determinant for the class of q-starlike functions was investigated in [8]. Moreover, several authors (see, for example, [9–12]) published a set of articles in which they concentrated on the classes of q-starlike functions related to the Janowski or other functions from several different aspects. Additionally, a recently-published survey-cum-expository review article by Srivastava [13] is potentially useful for researchers and scholars working on these topics. In this survey-cum-expository review article [13], the mathematical explanation and applications of the fractional q-calculus and the fractional q-derivative operators in geometric function theory were systematically investigated. For some more recent investigations about the recent usages of the q-calculus in geometric function theory, we may refer the interested readers to [14–27].

Definition 5. *(See [5].) A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if it satisfies the following conditions:*

$$f(0) = f'(0) - 1 = 0 \tag{9}$$

and:

$$\left| \frac{z\left(D_q f\right)(z)}{f(z)} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}. \tag{10}$$

Then, on account of the last inequality, it is obvious that, in the limiting case $q \to 1-$:

$$\left| w - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}$$

the above closed disk is merely the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions turns into the prominent class $\mathcal{S}^*$. Analogously, on behalf of principle of subordination, one may express the relations in (9) and (10) as follows (see [28]):

$$\frac{z\left(D_q f\right)(z)}{f(z)} \prec \widehat{p}(z) \qquad \left(\widehat{p}(z) = \frac{1+z}{1-qz}\right).$$

Now, in order to define the new class $\mathcal{R}_q(h)$ of analytic functions that are associated with a certain leaf-like domain, we make use of the above-mentioned q-calculus and the principle of subordination between analytic functions and define the following.

Definition 6. *A function $f \in \mathcal{S}$ is said to be in the functions class $\mathcal{R}_q(h)$ if it satisfies the condition given by:*

$$z\left(D_q f\right)(z) \prec \phi(z) \qquad (z \in \mathbb{U}), \tag{11}$$

where:

$$\phi(z) = \frac{(1+q)z}{2+(1-q)z} + \sqrt[3]{1 + \left(\frac{1+(1+q)z}{2+(1-q)z}\right)^3}. \tag{12}$$

Remark 1. *It is easy to see that:*

$$\lim_{q \to 1-} \mathcal{R}_q(h) =: \mathcal{R}(h)$$

where $\mathcal{R}(h)$ is a function class introduced and studied by Priya and Sharma [2].

Definition 7. *(See [29].) The jth Hankel determinant is given, for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$, by:*

$$H_j(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+j-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+j-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(j-1)} \end{vmatrix}.$$

The determinant $H_j(n)$ has also been considered by several authors in the literature on the subject (see, for example, [8,30,31]). In particular, Noor [32] determined the rate of growth of $H_j(n)$ as $n \to 0$ for functions f given by Equation (1) with bounded boundary. Ehrenborg [33] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [34].

Remark 2. *By giving some particular values to j and n, the Hankel determinant $H_j(n)$ is reduced to the following form:*

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1a_3 - a_2^2.$$

We note that $H_2(1)$ is the well-known Fekete–Szegö functional (see, for instance, [35]). On the other hand, we have:

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_3 \end{vmatrix} = a_2a_4 - a_3^2,$$

where $H_2(2)$ is known as the second Hankel determinant.

Until now, very few researchers have studied the above determinants for the function class that is associated with a leaf-like domain. Therefore, in this paper, we are motivated to find estimates of the first few Taylor–Maclaurin coefficients of the functions f of the form (1) belonging to the class $\mathcal{R}_q(h)$, which is associated with a leaf-like domain. We also consider the estimates of the familiar functionals such as $|a_3 - \lambda a_2^2|$ and $|a_2a_4 - a_3^2|$. Finally, this work will be generalized and extended to hold true for the functions $\frac{z}{f(z)}$ and $f^{-1}(z)$.

2. Preliminary Results

Each of the following lemmas will be needed in our present investigation.

Lemma 1. *(See [36–38].) If:*

$$p(z) = 1 + p_1z + p_2z^2 + \cdots \in \mathcal{P},$$

then:

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right)$$

for some x $(|x| \leqq 1)$ and:

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1x - \left(4 - p_1^2\right)p_1x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some z $(|z| \leqq 1)$.

Lemma 2. *(See [39].) If $p(z) \in \mathcal{P}$, then, for any complex number μ,*

$$\left|p_2 - \mu p_1^2\right| \leqq 2\max\{1, |2\mu - 1|\}.$$

This result is sharp for the functions $p(z)$ given by:

$$p(z) = \frac{1+z^2}{1-z^2} \qquad \text{and} \qquad p(z) = \frac{1+z}{1-z}.$$

Lemma 3. *(See [40].) Let the function $p \in \mathcal{P}$ be given by (4). Then:*

$$|p_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

This inequality is sharp.

3. A Set of the Main Results

We begin this section by estimating the upper bound of the Taylor–Maclaurin coefficients for the functions belonging to the class $f \in \mathcal{R}_q(h)$.

Theorem 1. *If the function* $f \in \mathcal{R}_q(h)$ *has the form* (1), *then:*

$$|a_2| \leqq \frac{1+q}{2[2]_q}, \tag{13}$$

$$|a_3| \leqq \frac{1+q}{3(1+q+q^2)}, \tag{14}$$

and:

$$|a_4| \leqq \frac{q^2-3q+5}{2(1+q^2)}. \tag{15}$$

Proof. If we suppose that $f \in \mathcal{R}_q(h)$, then there exists a function $w(z) \in \mathcal{B}$ such that:

$$\left(D_q f\right)(z) = \phi\left(w(z)\right), \tag{16}$$

together with:

$$\phi\left(w(z)\right) = \frac{(1+q)w(z)}{2+(1-q)w(z)} + \left[1 + \left(\frac{(1+q)w(z)}{2+(1-q)w(z)}\right)^3\right]^{\frac{1}{3}}. \tag{17}$$

We now define a function $p(z)$ by:

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then, it is clear that $p \in \mathcal{P}$. The last relation can be restated in the following equivalent form:

$$w(z) = \frac{p(z)-1}{p(z)+1}. \tag{18}$$

Substitution of $w(z)$ from (18) into (17) yields:

$$\begin{aligned}\phi\left(w(z)\right) &= \frac{(1+q)(p(z)-1)}{1+3p(z)+(1-p(z))q} \\ &\quad + \left[1 + \left(\frac{(1+q)[p(z)-1]}{1+3p(z)+[1-p(z)]q}\right)^3\right]^{\frac{1}{3}} \\ &= 1 + \frac{(1+q)p_1}{4}z + \frac{(1+q)}{4}\left(p_2 - \frac{(3-q)}{4}p_1^2\right)z^2 + \cdots .\end{aligned} \tag{19}$$

From the right-hand side of (16), we find that:

$$\begin{aligned}\left(D_q f\right)(z) &= 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \\ &= 1 + [2]_q a_2 z + [3]_q a_3 z^2 + [4]_q a_4 z^3 + \cdots .\end{aligned} \tag{20}$$

Equating the coefficients of like powers of z, z^2, and z^3 from the relations (19) and (20), we get:

$$a_2 = \frac{1+q}{4[2]_q} p_1, \tag{21}$$

$$a_3 = \frac{(1+q)}{3(1+q+q^2)} \left(p_2 - \frac{(3-q)}{4}\ p_1^2 \right) \tag{22}$$

and:

$$a_4 = \frac{1}{2(1+q^2)} \left(\frac{(q^2-4q+7)}{12}\ p_1^3 + \frac{(q-3)}{2}\ p_2 p_1 + p_3 \right), \tag{23}$$

respectively. Thus, by applying Lemma 3 in (21), we obtain (13).

Next, Equation (22) can be reduced to the following form:

$$|a_3| = \frac{(1+q)}{3(1+q+q^2)} \left| p_2 - \eta p_1^2 \right|, \tag{24}$$

together with:

$$\eta = \frac{(3-q)}{4}.$$

Using (24) in conjunction with Lemma 2, we get (14).

Finally, we find from Equation (23) that:

$$|a_4| = \frac{1}{2(1+q^2)} \left| \frac{(q^2-4q+7)}{12}\ p_1^3 + \frac{(q-3)}{2}\ p_2 p_1 + p_3 \right|.$$

Substituting for the values of p_1 and p_2 from (21) and (22) and also by applying Lemma 3, one can obtain the result as in Equation (15). The proof of Theorem 1 is thus completed. □

Remark 3. *In the special case, if we let $q \to 1-$, Theorem 1 would coincide with the corresponding result of Priya and Sharma [2].*

Theorem 2. *If the function $f \in \mathcal{R}_q(h)$ has the form (1), then:*

$$|a_2 a_3 - a_4| \leqq \frac{q^4+4q^2+7}{24\Lambda(q)}, \tag{25}$$

together with:

$$\Lambda(q) = \frac{1}{(q^2+q+1)\,(q^2+1)}. \tag{26}$$

Proof. From (21)–(23), upon substituting for the values of a_2, a_3, and a_4, we have:

$$|a_2 a_3 - a_4| = \frac{1}{192\Lambda(q)} \left| \left(q^4 - 6q^3 + 22q^2 + 18q + 37\right) p_1{}^3 + 12\left(q^3 - 5q^2 - 5q - 7\right) p_1 p_2 + 48\left(q^2+q+1\right) p_3 \right|,$$

where $\Lambda(q)$ is given by (26). Substituting for p_2 and p_3 from Lemma 1, we obtain:

$$|a_2 a_3 - a_4| = \frac{1}{192\Lambda(q)} \left| \left(q^4 + 4q^2 + 7\right) p_1{}^3 + 6\left(q^3 - q^2 - q - 3\right)\left(4 - p_1{}^2\right) p_1 x - 12\left(q^2+q+1\right) \cdot \left(4 - p_1{}^2\right) p_1 x^2 + 24\left(q^2+q+1\right)\left(4-p_1{}^2\right)\left(1-|x|^2\right) z \right|.$$

We assume that:

$$|x| = t \in [0,1] \qquad \text{and} \qquad p_1 = p \in [0,2].$$

Then, using the triangle inequality, we deduce that:

$$|a_2a_3 - a_4| \leqq \frac{1}{192\Lambda(q)} \left\{ \left(q^4 + 4q^2 + 7\right) p^3 + 6 \left(3 + q + q^2 - q^3\right) \left(4 - p^2\right) pt + 12 \left(q^2 + q + 1\right) \cdot \left(4 - p^2\right) pt^2 + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) t^2 + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) \right\}.$$

We now define:

$$F_q(p,t) := \frac{1}{192\Lambda(q)} \left\{ \left(q^4 + 4q^2 + 7\right) p^3 + 6 \left(3 + q + q^2 - q^3\right) \left(4 - p^2\right) pt + 12 \left(q^2 + q + 1\right) \cdot \left(4 - p^2\right) pt^2 + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) t^2 + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) \right\}.$$

Differentiating $F_q(p,t)$ partially with respect to t, we have:

$$\frac{\partial F_q}{\partial t} := \frac{1}{192\Lambda(q)} \left\{ 6 \left(3 + q + q^2 - q^3\right) \left(4 - p^2\right) p + 24 \left(q^2 + q + 1\right) \cdot \left(4 - p^2\right) pt + 48 \left(q^2 + q + 1\right) \left(4 - p^2\right) t \right\}$$

which, after some elementary calculation, shows that:

$$\frac{\partial F_q(p,t)}{\partial t} > 0,$$

implying that $F_q(p,t)$ is an increasing function of t on the closed interval $[0,1]$. Thus, clearly, the maximum value of the function $F_q(p,t)$ is attained at $t = 1$, which is given by:

$$\max_{0 \leqq t \leqq 1} \{F_q(p,t)\} = F_q(p,1) = \frac{1}{192\Lambda(q)} \left\{ \left(q^4 + 4q^2 + 7\right) p^3 + 6 \left(3 + q + q^2 - q^3\right) \left(4 - p^2\right) p + 12 \left(q^2 + q + 1\right) \left(4 - p^2\right) p + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) \right\}.$$

Finally, we set:

$$G_q(p) = \frac{1}{192\Lambda(q)} \left\{ \left(q^4 + 4q^2 + 7\right) p^3 + 6 \left(3 + q + q^2 - q^3\right) \left(4 - p^2\right) p + 12 \left(q^2 + q + 1\right) \left(4 - p^2\right) p + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) + 24 \left(q^2 + q + 1\right) \left(4 - p^2\right) \right\}.$$

Then, since $p \in [0,2]$, it follows that:

$$G_q(2) \leqq \frac{q^4 + 4q^2 + 7}{24\Lambda(q)}.$$

This completes the proof of Theorem 2. □

If we let $q \to 1-$, Theorem 2 yields the following corollary.

Corollary 1. *(See [2].) Let the function f given by (1) be a member of the class $\mathcal{R}(h)$. Then:*

$$|a_2a_3 - a_4| \leqq \frac{1}{12}.$$

4. The Fekete–Szegö Problem for the Class $\mathcal{R}_q(h)$

We first prove the following result.

Theorem 3. *If the function $f \in \mathcal{R}_q(h)$ is of the form (1), then:*

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{(1+q)}{2(1+q+q^2)} \max\left\{1, \left|\frac{(1+q)\,(1-q\,(1-\mu))+\mu}{2(1+q)}\right|\right\}. \tag{27}$$

Proof. From Equations (21) and (22), we have:

$$a_3 - \mu a_2 = \left[\frac{(1+q)}{3(1+q+q^2)}\left\{p_2 - \frac{(3-q)}{4}{p_1}^2\right\} - \mu\left(\frac{1+q}{4[2]_q}p_1\right)^2\right].$$

After some suitable simplification, this last relation can be interpreted as follows:

$$\begin{aligned} |a_3 - \mu a_2| &= \left|\frac{(1+q)}{4(1+q+q^2)}\left(p_2 - \nu {p_1}^2\right)\right| \\ &= \frac{(1+q)}{4(1+q+q^2)}\left|\left(p_2 - \nu p_1^2\right)\right|, \end{aligned} \tag{28}$$

where:

$$\nu = \frac{(1+q)(3+q(\mu-1))+\mu}{4(1+q)}.$$

Now, taking into account (28) and Lemma 2, we obtain the assertion (27).

A closer examination of the proof shows that the equality in (27) is attained for:

$$|a_3 - \mu a_2| = \begin{cases} \frac{1+q}{2(1+q+q^2)} & \left(p(z) = \dfrac{1+z^2}{1-qz^2}\right) \\ \frac{1+q}{2(1+q+q^2)}\left|\frac{(1+q)(1-q(1-\mu))+\mu}{2(1+q)}\right| & \left(p(z) = \dfrac{1+z}{1-qz}\right). \end{cases}$$

The proof of Theorem 3 is thus completed □

Remark 4. *In the special case, if we let $q \to 1-$, Theorem 3 will yield the corresponding result that was already proven by Priya and Sharma (see [2]).*

5. Estimates of the Second Hankel Determinant

In this section, we prove the following result.

Theorem 4. *If the function $f \in \mathcal{R}_q(h)$ has the form (1), then:*

$$\left|a_2 a_4 - a_3^2\right| \leqq \frac{q^6 + 4q^5 + 11q^4 + 4q^3 + 11q^2 + 4q + 1}{48(1+q^2)(1+q+q^2)^2}. \tag{29}$$

Proof. Let $f \in \mathcal{R}_q(h)$. Then, from Equations (21)–(23), we have:

$$\begin{aligned} \left|a_2 a_4 - a_3^2\right| = \Bigg| &\frac{\left(q^6 + 4q^5 + 11q^4 + 4q^3 + 11q^2 + 4q + 1\right) p_1^4}{768(1+q^2)(1+q+q^2)^2} \\ &+ \frac{(q-3)q^2 p_1^2 p_2}{32(1+q+q^2)^2(1+q^2)} + \frac{p_1 p_3}{16(1+q^2)} \\ &- \frac{(1+q^2)p_2^2}{16(1+q+q^2)^2}\Bigg|. \end{aligned}$$

Substituting for p_2 and p_3 and by using Lemma 1, we obtain

$$\left|a_2a_4 - a_3^2\right| = \left|\frac{\left(q^6+4q^5+11q^4+4q^3+11q^2+4q+1\right)p_1{}^4}{768(1+q^2)(1+q+q^2)^2}\right.$$
$$+\frac{(4-p_1^2)(1-|x|^2)zp_1}{32(1+q^2)} - \frac{q^2(1-q)(4-p_1^2)xp_1^2}{64(1+q^2)(1+q+q^2)^2}$$
$$\left.+\frac{(4-p_1^2)x^2p_1^2}{64(1+q^2)} + \frac{(1+q^2)(4-p_1^2)^2x^2}{64(1+q+q^2)^2}\right|. \tag{30}$$

We now set $p_1 = p$ and assume also, without restriction, that $p \in [0,2]$. Then, by applying the triangle inequality on (30), with $|x| = t \in [0,1]$, we find that:

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{\left(q^6+4q^5+11q^4+4q^3+11q^2+4q+1\right)p^4}{768(1+q^2)(1+q+q^2)^2}$$
$$+\frac{(4-p^2)p}{32(1+q^2)} + \frac{(4-p^2)t^2p}{32(1+q^2)}$$
$$+\frac{q^2(1-q)(4-p^2)tp^2}{64(1+q^2)(1+q+q^2)^2} + \frac{(4-p^2)t^2p^2}{64(1+q^2)} + \frac{(1+q^2)(4-p^2)^2t^2}{64(1+q+q^2)^2}.$$

By assuming further that:

$$F_q(p,t) = \frac{\left(q^6+4q^5+11q^4+4q^3+11q^2+4q+1\right)p^4}{768(1+q^2)(1+q+q^2)^2}$$
$$+\frac{(4-p^2)p}{32(1+q^2)} + \frac{(4-p^2)t^2p}{32(1+q^2)}$$
$$+\frac{q^2(1-q)(4-p^2)tp^2}{64(1+q^2)(1+q+q^2)^2} + \frac{(4-p^2)t^2p^2}{64(1+q^2)} + \frac{(1+q^2)(4-p^2)^2t^2}{64(1+q+q^2)^2}.$$

Differentiating $F_q(p,t)$ partially with respect to t, we have:

$$\frac{\partial F_q(p,t)}{\partial t} = \frac{(4-p^2)tp}{16(1+q^2)} + \frac{q^2(1-q)(4-p^2)p^2}{64(1+q^2)(1+q+q^2)^2} + \frac{(4-p^2)tp^2}{32(1+q^2)} + \frac{(1+q^2)(4-p^2)^2t}{32(1+q+q^2)^2} > 0,$$

which implies that, as a function of t, $F_q(p,t)$ increases on the closed interval $[0,1]$. This means that $F_q(p,t)$ has a maximum value at $t = 1$, which is given by:

$$\max_{0\leqq t\leqq 1}\{F_q(p,t)\} = F_q(p,1) = \frac{\left(q^6+4q^5+11q^4+4q^3+11q^2+4q+1\right)p^4}{768(1+q^2)(1+q+q^2)^2}$$
$$+\frac{(4-p^2)p}{32(1+q^2)} + \frac{(4-p^2)p}{32(1+q^2)}$$
$$+\frac{q^2(1-q)(4-p^2)p^2}{64(1+q^2)(1+q+q^2)^2} + \frac{(4-p^2)p^2}{64(1+q^2)} + \frac{(1+q^2)(4-p^2)^2}{64(1+q+q^2)^2}.$$

We now set:

$$G_q(p) = \frac{\left(q^6+4q^5+11q^4+4q^3+11q^2+4q+1\right)p^4}{768(1+q^2)(1+q+q^2)^2}$$
$$+\frac{(4-p^2)p}{32(1+q^2)} + \frac{(4-p^2)p}{32(1+q^2)}$$
$$+\frac{q^2(1-q)(4-p^2)p^2}{64(1+q^2)(1+q+q^2)^2} + \frac{(4-p^2)p^2}{64(1+q^2)} + \frac{(1+q^2)(4-p^2)^2}{64(1+q+q^2)^2}.$$

Then, since $p \in [0,2]$, it follows that:

$$G_q(2) \leqq \frac{(q^6 + 4q^5 + 11q^4 + 4q^3 + 11q^2 + 4q + 1)}{48(1+q^2)(1+q+q^2)^2},$$

which completes the proof of Theorem 4. □

Remark 5. *If, in Theorem 4, we let $q \to 1-$, we get the corresponding result due to Priya and Sharma [2].*

6. Coefficient Estimates for the Function $\frac{z}{f(z)}$

Let the function $G(z)$ be defined by:

$$G(z) := \frac{z}{f(z)} = z\left(\frac{1}{f(z)}\right) = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{31}$$

We now prove the following result.

Theorem 5. *Let the function $h(z)$ be defined by (12). Suppose also that:*

$$f \in \mathcal{R}_q(h) \qquad \text{and} \qquad G(z) = \frac{z}{f(z)}.$$

Then, for any $\sigma \in \mathbb{C}$, it is asserted that:

$$\left|b_2 - \sigma b_1^2\right| \leqq \frac{(1+q)}{2(1+q+q^2)} \max\left\{1, \left|\frac{2+q-\sigma(1+q+q^2)}{2(1+q)}\right|\right\}. \tag{32}$$

Proof. Since $f \in \mathcal{R}_q(h)$, we have:

$$z\left(\frac{1}{f(z)}\right) = 1 - a_2 z + (a_2 - a_3)\, z^2 + \left(a_2 a_3 - a_4 - \left({a_2}^2 - a_3\right) a_2\right) z^3 + \cdots. \tag{33}$$

Equating the coefficients of z and z^2 from (31) and (33), it can be deduced that:

$$b_1 = -a_2 \tag{34}$$

and:

$$b_2 = a_2 - a_3. \tag{35}$$

Thus, on account of (21), (22), (34), and (35), we get:

$$b_1 = -\frac{(1+q)}{4[2]_q} p_1 \tag{36}$$

and:

$$b_2 = -\frac{(1+q)}{4(1+q+q^2)}\left(p_2 - \frac{(3q+4)\, p_1^2}{4(1+q)}\right). \tag{37}$$

Now, for $\sigma \in \mathbb{C}$, we set:

$$b_2 - \sigma b_1^2 = -\frac{(1+q)}{4(1+q+q^2)}\left(p_2 - \xi {p_1}^2\right), \tag{38}$$

where:

$$\xi = \frac{4+3q-\sigma(1+q+q^2)}{4(1+q)}.$$

Thus, by applying Lemma 2 and after some suitable computation, Equation (38) is reduced to (32).

The sharpness of the estimate is given by:

$$\left|b_2 - \sigma {b_1}^2\right| = \begin{cases} \frac{(1+q)}{2(1+q+q^2)} & \left(p(z) = \dfrac{1+z^2}{1-qz^2}\right) \\ \frac{|2+q-\sigma(1+q+q^2)|}{4(1+q+q^2)} & \left(p(z) = \dfrac{1+z}{1-qz}\right). \end{cases}$$

Our demonstration of Theorem 5 is now complete. □

As a special case of Theorem 5, if we let $q \to 1-$, we get the following known result.

Corollary 2. *(See [2].) Let the function $h(z)$ be defined by (12). If:*

$$f \in \mathcal{R}(h) \qquad \text{and} \qquad G(z) = \frac{z}{f(z)},$$

then, for any $\sigma \in \mathbb{C}$, it is asserted that:

$$\left|b_2 - \sigma b_1^2\right| \leqq \frac{1}{3} \max\left\{1, \left|\frac{3-3\sigma}{4}\right|\right\}.$$

7. Coefficient Estimates for the Function $f^{-1}(z)$

Here, in this section, we prove the following result.

Theorem 6. *If $f \in \mathcal{R}_q(h)$ and:*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w_n$$

is the inverse function of f with $|w| < r_0$, where r_0 is greater than the radius of the Koebe domain of the class $f \in \mathcal{R}_q(h)$, then, for arbitrary $\mu \in \mathbb{C}$, it is asserted that:

$$\left|d_3 - \mu d_2^2\right| \leqq \frac{(1+q)}{2(1+q+q^2)} \max\left\{1, \left|\frac{q^2+2q-\mu(1+q+q^2)+2}{2(1+q)}\right|\right\}. \tag{39}$$

The above-asserted estimate is sharp.

Proof. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by:

$$f^{-1}\big(f(z)\big) = f\big(f^{-1}(z)\big) = z \qquad (z \in \mathbb{U}).$$

By means of the above relation and (1), we find that:

$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z_n) = z. \tag{40}$$

It is also known that:

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w_n.$$

Making use of (33) and (40), it can be seen that:

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2d_2 + d_3)z^3 + \cdots = z. \tag{41}$$

Now, by equating the coefficients of z and z^2, we obtain:

$$d_2 = -a_2 \tag{42}$$

and:

$$d_3 = 2a_2^2 - a_3. \tag{43}$$

From (21), (22), (42), and (43), we can see that:

$$d_2 = -\frac{(1+q)}{4[2]_q}p_1$$

and:

$$d_3 = -\frac{(1+q)}{4(1+q+q^2)}\left(p_2 - \frac{\left(q^2+4q+5\right)p_1{}^2}{4(1+q)}\right).$$

For any $\sigma \in \mathbb{C}$, we set:

$$d_3 - \sigma d_2^2 = -\frac{(1+q)}{4(1+q+q^2)}\left(p_2 - \xi_1 p_1^2\right) \tag{44}$$

and:

$$\xi_1 = \frac{q^2+4q+5-\sigma(1+q+q^2)}{4(1+q)}.$$

Then, by applying Lemma 2, it is easy to observe that the inequality (44) reduces to (39).

The sharpness of the estimate is given by:

$$\left|d_3 - \mu d_2^2\right| = \begin{cases} \dfrac{(1+q)}{2(1+q+q^2)} & \left(p(z) = \dfrac{1+z^2}{1-qz^2}\right) \\[2ex] \dfrac{\left|q^2+2q-\mu(1+q+q^2)+2\right|}{4(1+q+q^2)} & \left(p(z) = \dfrac{1+z}{1-qz}\right). \end{cases}$$

This completes our proof of Theorem 5. □

As a special case of Theorem 5, if we let $q \to 1-$, we are led to the following known result.

Corollary 3. *(See [2].) If $f \in \mathcal{R}(h)$ and if:*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w_n$$

is the inverse function of f, then, for an arbitrary $\mu \in \mathbb{C}$, it is asserted that:

$$\left|b_2 - \sigma b_1^2\right| \leqq \frac{1}{3}\max\left\{1, \left|\frac{2-\mu}{4}\right|\right\}.$$

8. Conclusions

Here, in our present investigation, we first defined a new subclass $\mathcal{R}_q(h)$ of normalized analytic functions in the open unit disk $\mathbb{U}$, which is associated with a leaf-like domain and which involves the basic (or q-) calculus.

We then successfully investigated many properties and characteristics such as the estimates on the first few Taylor–Maclaurin coefficients, the Fekete–Szegö problem, and the second-order Hankel determinant $H_2(2)$. We also obtained several results for the functions $\frac{z}{f(z)}$ and $f^{-1}(z)$ associated with this newly generalized domain. Finally, we highlighted a number of known corollaries and consequences that are already available in the literature on the subject.

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