Ergodic Theory of Multidimensional Random Dynamical Systems

by

Li-Yu Shelley Hsieh

B.Sc, National Chengchi University, 1998
M.Sc, National Taiwan University, 2001

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Supervisory Committee

Dr. Christopher Bose, Supervisor (Department of Mathematics and Statistics)

Dr. Roderick Edwards, Member (Department of Mathematics and Statistics)

Dr. Anthony Quas, Member (Department of Mathematics and Statistics)

Dr. Arthur Watton, Outside Member (Department of Physics)

Dr. Arno Berger, External Examiner (University of Alberta)
Abstract

Given a random dynamical system $T$ constructed from Jabłoński transformations, consider its Perron-Frobenius operator $P_T$. We prove a weak form of the Lasota-Yorke inequality for $P_T$ and thereby prove the existence of BV-invariant densities for $T$. Using the Spectral Decomposition Theorem we prove that the support of an invariant density is open a.e. and give conditions such that the invariant density for $T$ is unique. We study the asymptotic behavior of the Markov operator $P_T$, especially when $T$ has a unique absolutely continuous invariant measure (ACIM). Under the assumption of uniqueness, we obtain spectral stability in the sense of Keller. As an application, we can use Ulam’s method to approximate the invariant density of $P_T$. 

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Chapter 1

Introduction

Ergodic theory is the field of mathematics which studies dynamical systems from the point of view of statistical behavior of orbits under a transformation. The basic ingredients are a state space (a measurable space), a measurable transformation acting on points in the state space and an invariant measure on the state space. Asymptotics are investigated with respect to the transformation and the invariant measure. For example, one can ask what fraction of the orbit of a single point, under the action of the transformation, will lie in a given measurable set. A very powerful theory can be brought to bear on questions of this type through Ergodic theory.

In a natural way, the action of a measurable transformation induces an action on measures supported on the state space. For instance, invariant measures are fixed points for this action. More generally, we can study the evolution or orbits of measures under this action.

In the case of absolutely continuous measures on the state space (measures which can be obtained by integrating a density function with respect to some natural reference measure on the space) this evolution of measures is facilitated by examining the evolution of their densities under the action of a linear operator, the so-called Perron-Frobenius operator. (See, for example Lasota and Mackey [29].) As an illustration, absolutely continuous invariant measures (ACIM) correspond to fixed densities for the Perron-Frobenius operator. In general, the main advantage of this method is that systems may exhibit very chaotic orbits in the phase space but still may have stable behavior when one views the related evolution of densities on the phase space. For example, there may be a global attracting fixed point for this “functional dynamical system”, the fixed point representing the density of the stable statistics for the system.

Another advantage of this functional dynamical viewpoint is to give a precise formulation of the notion of a random dynamical system. This is a special type of Markov process which may also exhibit stable statistics when treated with the machinery of Ergodic theory.

In this thesis we review the framework for studying a random dynamical system $T$, constructed from a family of nonsingular transformations $\{\tau_k\}_{k=1,\ldots,q}$ on a phase space $X$ and a probability vector $p = (p_1, \cdots, p_q)$. We analyze its associated Perron-Frobenius operator $P_T$ acting on $L^1(X, \mathcal{B}_X, \nu)$. To keep things manageable, we focus on the specific case where the phase space is the unit cube in $\mathbb{R}^n$ with Lebesgue
measure \( m_n \), as the reference measure. The phase space is denoted as \((I^n, \mathcal{B}_{I^n}, m_n)\). Our transformations are also restricted to a special class of multidimensional maps called \textit{Jabłoński transformations} [21].

This setting has been studied extensively in papers such as Boyarsky, Góra and Lou [7] and Kamthan and Mackey [22]. In particular the latter reference contains most of the results to be found in this thesis. Unfortunately, a key inequality early in that paper (the Lasota-Yorke inequality, Equation (5) in Proposition 3.1) does not hold in the generality claimed by the authors. This defect has been noted in the literature (See Math. Rev. MR1345800 (96g:58112) ) but has not yet been addressed in published work. In this thesis, we prove instead a weak-Lasota-Yorke inequality which allows us to recover most of the results of [22]. The main technical difference is in our use of the Spectral Decomposition Theorem (Theorem 4.4) instead of the Ionescu-Tulcea and Marinescu Theorem (Theorem 5.10) to obtain the results.

The presentation is organized as follows.

In Chapter 2, we give the background required to establish our main results. We present and discuss \textit{Tonelli variation} in \( n \) dimensions (see Clarkson and Adams [13]). We indicate the difference between Tonelli variation and classical, one-dimensional variation and its relation to the more modern notion of \textit{Generalized Variation} (see Giusti [17]). We introduce the Perron-Frobenius and the Koopman operators related to a nonsingular transformation, and the class of piecewise \( C^2 \) Jabłoński transformations. While these maps may seem very specialized, they form a natural basis for investigation since, if we are given a piecewise \( C^2 \) transformation \( \tau \) on a rectangular partition of the \( n \)-dimensional cube, then \( \tau \) can be approximated by a sequence of piecewise \( C^2 \) Jabłoński transformations (see [7]). The approximation holds in the following sense. Let \( \tau \) be a piecewise \( C^2 \) transformation on a rectangular partition of \( I^n \). Assume \( \tau \) is expanding in every coordinate direction. Then there is a sequence \( \{\tau_\eta\}_{\eta=1}^\infty \) of \( C^2 \) expanding Jabłoński transformations which converges pointwise to \( \tau \). Moreover, if for each \( \eta \) we have invariant densities \( f_\eta (P_\tau f_\eta = f_\eta) \), and if \( f^* \) is any weak limit point of the \( \{f_\eta\}_{\eta=1}^\infty \), then \( P_\tau f^* = f^* \). In particular, if \( \tau \) has a unique ACIM, \( f^* d m_n \), then \( f_\eta \to f^* \) weakly.

In Chapter 3, we give an intuitive description of a random dynamical system \( T \) on the probability space \((I^n, \mathcal{B}_{I^n}, m_n)\). The ingredients are a finite set of transformations from \( I^n \) to \( I^n \), \( \mathfrak{T} = \{\tau_1, \tau_2, \ldots, \tau_q\} \) and a probability vector \( p = (p_1, p_2, \ldots, p_q) \), that is, each \( p_k \geq 0 \) and \( \sum_{k=1}^q p_k = 1 \). Define \( T \) at each point \( x \) in \( I^n \) by choosing \( \tau_k \) with probability \( p_k \) and sending \( x \) to \( \tau_k(x) \). In some studies the probability vector \( p \) is allowed to vary with the point \( x \), or is dependent on the iteration number of the transformation. We deal only with constant \( p \) in this thesis. In Section 3.2 we explain how to view \( T \) as a Markov process, and we study \( T \) by using its Perron-Frobenius operator \( P_T \). We obtain a \textit{weak Lasota-Yorke inequality}, Equation (3.8), under an \textit{expanding-on-average} condition. This inequality is sufficient to obtain invariant densities for \( T \): if a random dynamical system \( T \) satisfies Equation (3.8), then \( P_T \) has at least one invariant density (Theorem 3.12).

In Chapter 4, we give the definitions of \textit{Markov operator} and \textit{constrictive operator}. We introduce the \textit{Spectral Decomposition Theorem} (see [29], Komornik and Lasota...
[26] or Lasota, Li and Yorke [28]), which allows us to analyze the set of invariant densities for the operator $P$. We get some results from the Spectral Decomposition Theorem directly and reprove the uniqueness theorem from Boyarsky and Lou [9] by using this theorem instead of using Ionescu-Tulcea and Marinescu Theorem, (see our Theorem 4.46) as the latter would require the strong Lasota-Yorke inequality. A more detailed description about the uniqueness of the invariant density of $P$ can be found in Section 4.4. The main result of this chapter is shown in Theorem 4.50: if any one of the individual transformations of the random dynamical system $T$ has a unique invariant ACIM with $\tau^N$-ergodic for all positive integers $N$, then so does $T$. This theorem is modified from [22, Theorem 3.3]. In Section 4.5, we consider the specific random dynamical system $T$ constructed from piecewise linear Markov transformations with respect to the partition $\mathcal{P}$. In this case, a piecewise linear Markov transformation preserves a finite-dimensional subspace of $L^1(X, \mathcal{B}_X, \nu)$ so that we have the following property. The Perron-Frobenius operator $P_T$ is represented by a Markov matrix $M_T$ with respect to $T$. Therefore, we can apply another useful tool, the (matrix) Perron-Frobenius Theorem, Theorem A.6 in Appendix, to indicate whether $P_T$ has a unique invariant density. The main theorem in this section, Theorem 4.56 is from [22, Theorem 3.4]. However, here we deal with the expanding-on-average condition instead of the more restrictive individually expanding condition required in [22].

In conclusion, we provide a self-contained and rigorous treatment of the existence of invariant densities for random transformations $T$ constructed from piecewise $C^2$ Jabłoński transformations. We study the set of invariant densities and give conditions under which this set contains a unique element, leading to a unique ACIM. We
study the asymptotic properties of the related Perron-Frobenius operator $P_T$ and it’s stability in the sense of Keller. Finally, as an application we show convergence of Ulam’s method of approximation for this class of transformations.
Chapter 2

Preliminaries

Let $I = [0,1]$ denote the unit interval in $\mathbb{R}$ and $I^n = [0,1]^n$ the unit cube in $\mathbb{R}^n$. Let $\mathcal{B}_{I^n}$ denote the Borel $\sigma$-algebra$^1$ on $I^n$, and $\mu$ be a measure on $(I^n, \mathcal{B}_{I^n})$. By a transformation $\tau : I^n \rightarrow I^n$, we mean a function

$$\tau(x) = (\varphi_1(x), \cdots, \varphi_n(x)),$$

where $x \equiv (x_1, \cdots, x_n) \in I^n$ and $\varphi_i(x) \in [0,1]$ for $i = 1,2,\cdots,n$. Such $\varphi_i$’s are called the components of $\tau$.

**Definition 2.1.** (finite partition)

$\mathcal{P} = \{D_1, \cdots, D_m\}$ is a finite partition of $I^n$ if $m$ is finite, $\bigcup_{j=1}^m D_j = I^n$ with measure zero boundaries of each $D_j$, and $D_j$’s are measurable and pairwise measurably disjoint. That is, for each pair $D_i$ and $D_j$, the measure of $D_i \cap D_j$ is equal to zero.

We say $\tau$ is piecewise $C^2$ if

1. there exists a finite partition $\mathcal{P} = \{D_j\}_{j=1}^m$ of $I^n$;
2. for each $D_j \in \mathcal{P}$, $\tau_j \equiv \tau|_{D_j}$ maps $D_j^0$, the interior set, to its image bijectively, and all $\tau_j$, $\tau_j'$ and $\tau_j''$ can be continuously extended to $D_j$.

We denote by $\mathfrak{m}_n$ the Lebesgue measure on $\mathbb{R}^n$ and $L^1(I^n) = L^1(I^n, \mathcal{B}_{I^n}, \mathfrak{m}_n)$ the collection of all integrable functions on $I^n$. On the measure space $(I^n, \mathcal{B}_{I^n}, \mu)$, we say $\tau$ is measurable if $\tau^{-1}(\mathcal{B}_{I^n}) \subseteq \mathcal{B}_{I^n}$. It means for any measurable set $A \in \mathcal{B}_{I^n}$, each component $\varphi_i$ satisfies $\varphi_i^{-1}(A) \in \mathcal{B}_{I^n}$.

Here are some general definitions in ergodic theory (see Walters [38]). We list them below for convenience.

**Definition 2.2.** For a measure space $(I^n, \mathcal{B}_{I^n}, \mu)$ and a measurable transformation $\tau : I^n \rightarrow I^n$, define that

1. $\mu$ is $\tau$-invariant if for all $A \in \mathcal{B}_{I^n}$, $\mu(\tau^{-1}A) = \mu(A)$. Equivalently, we say $\tau$ is measure preserving with respect to $\mu$. Moreover, define the measure $\mu \circ \tau^{-1}$ as

$$\mu \circ \tau^{-1}(A) = \mu(\tau^{-1}A).$$

$^1$\(\mathcal{B}_{I^n}\) is the smallest $\sigma$-algebra containing all open subsets in $I^n$. 
(2) $\mu$ is $\tau$-ergodic if for all $A \in \mathcal{B}_I^n$, $\tau^{-1}A = A$ implies $\mu(A) = 0$, or $\mu(I^n \setminus A) = 0$.

(3) $\mu$ is $\tau$-exact if for all $A \in \mathcal{B}_I^n$ with $\mu(A) > 0$, $\lim_{\tau \to \infty} \mu(\tau^rA) = 1$.

(4) $\tau$ is nonsingular with respect to $\mu$ if $\mu \circ \tau^{-1}$ is absolutely continuous with respect to $\mu$ and denoted as $\mu \circ \tau^{-1} << \mu$. That is, for every $A \in \mathcal{B}_I^n$, $\mu(A) = 0$ implies $\mu \circ \tau^{-1}(A) = 0$.

Here are some simple examples of the above definition. In our examples, take $\mu$ to be the Lebesgue measure $m$ in $\mathbb{R}$ and the transformation $\tau$ to be defined on the unit interval $I$ into itself.

**Example 2.3.**

1. For all $x \in I$, let $\tau(x) = 2x \mod 1$.

2. For an irrational number $\alpha$ and for all $x \in I$, define $\tau$ by $\tau(x) = x + \alpha \mod 1$.

3. For a real number $a > 1$ and for all $x \in I$, let $\tau(x) = ax \mod 1$.

4. For all $x \in I$, if $\tau(x) = x^2$, then $\tau$ is nonsingular. On the other hand, any constant or simple function is not a nonsingular transformation. For example, for all $x \in I$, let $\tilde{\tau}(x) = \frac{1}{3}$. If $A = \{\frac{1}{3}\}$, then $m(A) = 0$; but

$$m \circ \tilde{\tau}^{-1}(A) = m(\tilde{\tau}^{-1}A) = m([0, 1]) = 1.$$  

Such $\tilde{\tau}$ is not nonsingular. This means that $\mu \circ \tilde{\tau}^{-1}$ has a singular part with respect to the Lebesgue measure.

### 2.1 Koopman and Perron-Frobenius Operators

In this section, introduce two special operators related to $\tau$, the **Koopman operator** acting on $L^\infty(X, \mathcal{B}_X, \mu)$ and the **Perron-Frobenius operator** acting on $L^1(X, \mathcal{B}_X, \mu)$. The Koopman operator was first defined by Koopman in 1931 [27]. In this article, we deal with the special case on the unit cube in $n$ dimensions. Denote $L^\infty(I^n) \equiv L^\infty(I^n, \mathcal{B}_I^n, m_n)$ and $L^1(I^n) \equiv L^1(I^n, \mathcal{B}_I^n, m_n)$. The Koopman operator is the dual of the Perron-Frobenius operator which is presented in Proposition 2.9.

**Definition 2.4.** (Koopman operator) 

Let $\tau : I^n \to I^n$ be a nonsingular transformation with respect to the Lebesgue measure $m_n$. The Koopman operator, $K_\tau : L^\infty(I^n) \to L^\infty(I^n)$, with respect to the transformation $\tau$ is defined as follows. For all $g \in L^\infty(I^n)$,

$$K_\tau g = g \circ \tau.$$ 

In the definition of the Koopman operator, the condition of a nonsingular transformation is essential; otherwise, $K_\tau$ is not well-defined. For instance, take $\tau(x) = \frac{3}{4}$, for all $x \in I = [0, 1]$. Define $g$ and $\tilde{g} \in L^\infty(I)$ by

$$g(x) = 1 \text{ for all } x \in I \quad \text{and} \quad \tilde{g}(x) = \begin{cases} g(x), & x \in [0, 1] \setminus \{\frac{3}{4}\} \\ 0, & x = \frac{3}{4}. \end{cases}$$
Then $\tilde{g} = g$ almost everywhere (we abbreviate as a.e.). However, for all $x \in I$,

$$K_\tau g(x) = g \circ \tau(x) = g\left(\frac{3}{4}\right) = 1,$$

$$K_\tau \tilde{g}(x) = \tilde{g} \circ \tau(x) = \tilde{g}\left(\frac{3}{4}\right) = 0.$$  

It implies $K_\tau g \neq K_\tau \tilde{g}$. Thus, the operator $K_\tau$ is not well-defined.

**Definition 2.5. (density and its support)**

1. For every $f \in L^1(I^n)$, $f$ is called a density if $f \geq 0$ and $\|f\|_1 = 1$, where $\| \cdot \|_1$ is $L^1$-norm.

2. For every $f \in L^1(I^n)$, $\text{supp } f = \{x \in I^n : f(x) > 0\}$.

Note that the support of a given $L^1$ function is not a closed set. More precisely it is an element of the measure algebra of classes of sets under the equivalence relation of equality up to sets of measure zero.

The following construction gives us the idea of what the Perron-Frobenius operator is. Let $\tau : I^n \to I^n$ be nonsingular. For each $f \in L^1(I^n)$ and any set $A \in \mathcal{B}_{I^n}$, define $d\mu = f dm_n \circ \tau^{-1}$ as follows

$$\mu(A) = \int f dm_n \circ \tau^{-1}(A) \equiv \int_{\tau^{-1}(A)} f dm_n. \tag{2.1}$$

Since $\tau$ is nonsingular with respect to $m_n$,

$$m_n(A) = 0 \text{ implies } m_n(\tau^{-1}(A)) = 0.$$

By Equation (2.1),

$$m_n(\tau^{-1}(A)) = 0 \text{ implies } \mu(A) = 0.$$

Therefore, we get $\mu \ll m_n$. By the Radon-Nikodym Theorem (see Theorem A.4 in Appendix), there exists a unique $g \in L^1(I^n)$ such that

$$d\mu = f dm_n \circ \tau^{-1} = g dm_n.$$

Define an operator $P_\tau : L^1(I^n) \to L^1(I^n)$ such that $P_\tau f = g$. Hence, for any $A \in \mathcal{B}_{I^n}$,

$$\int_A P_\tau f dm_n = \int_A g dm_n = \int_A d\mu = \mu(A) = \int_{\tau^{-1}(A)} f dm_n.$$

Hence, for a nonsingular transformation $\tau$, such operator $P_\tau$ defined as above satisfies

$$\int_A P_\tau f dm_n = \int_{\tau^{-1}(A)} f dm_n. \tag{2.2}$$

From Equation (2.2), we know that $P_\tau$ is positive, preserves integrals, bounded and linear. The details are shown as below.
(a) For every set \( A \in \mathcal{B}_I \), any positive \( L^1 \) function \( f \) implies
\[
\int_A P_\tau f \, dm_n = \int_{\tau^{-1}(A)} f \, dm_n \geq 0.
\]
Hence \( P_\tau f \geq 0 \).

(b) By Equation (2.2) and \( \tau^{-1}(I^n) = I^n \), we have the fact that
\[
\int_{I^n} P_\tau f \, dm_n = \int_{\tau^{-1}(I^n)} f \, dm_n = \int_{I^n} f \, dm_n.
\]
Therefore, \( P_\tau \) preserves integrals. Moreover, if \( f \geq 0 \), then
\[
\|P_\tau f\|_1 = \int_{I^n} |P_\tau f| \, dm_n = \int_{I^n} P_\tau f = \int_{I^n} f = \int_{I^n} |f| = \|f\|_1.
\]

(c) For \(-|f| \leq f \leq |f|\),
\[
-P_\tau |f| \leq P_\tau f \leq P_\tau |f| \quad \text{implies} \quad |P_\tau f| \leq P_\tau |f|.
\]
Thus,
\[
\|P_\tau f\|_1 = \int_{I^n} |P_\tau f| \, dm_n \leq \int_{I^n} P_\tau |f| \, dm_n = \int_{I^n} |f| \, dm_n = \|f\|_1.
\]
Therefore, \( P_\tau \) is bounded.

(d) Given \( f_1 \) and \( f_2 \) in \( L^1(I^n) \), let \( P_\tau f_1 = g_1 \) and \( P_\tau f_2 = g_2 \) and for a fixed \( A \in \mathcal{B}_I \),
\[
\int_A P_\tau (f_1 + f_2) \, dm_n = \int_{\tau^{-1}(A)} (f_1 + f_2) \, dm_n = \int_{\tau^{-1}(A)} f_1 \, dm_n + \int_{\tau^{-1}(A)} f_2 \, dm_n \\
= \int_A P_\tau f_1 \, dm_n + \int_A P_\tau f_2 \, dm_n = \int_A (P_\tau f_1 + P_\tau f_2) \, dm_n.
\]
Since this equation is true for all \( A \in \mathcal{B}_I \),
\[
P_\tau (f_1 + f_2) = P_\tau f_1 + P_\tau f_2.
\]
Hence, \( P_\tau \) is linear.

**Lemma 2.6.** \( P_\tau \) defined as above is unique.

**Proof.** Assume both operators \( P_\tau \) and \( \tilde{P}_\tau \) satisfy Equation (2.2). For all \( A \in \mathcal{B}_I \) and for all \( f \) in \( L^1(I^n) \),
\[
\int_A P_\tau f \, dm_n = \int_{\tau^{-1}(A)} f \, dm_n = \int_A \tilde{P}_\tau f \, dm_n.
\]
Since \( \int_A (P_\tau f - \tilde{P}_\tau f) \, d\mathfrak{m}_n = 0 \) is true for all \( A \in \mathfrak{B}_I^n \) and for all \( f \in L^1(I^n) \),
\[
P_\tau f - \tilde{P}_\tau f = 0.
\]

Hence, \( P_\tau = \tilde{P}_\tau \).

From the above construction we have the following definition of the Perron-Frobenius operator.

**Definition 2.7. (Perron-Frobenius operator)**

Let \( \tau : I^n \to I^n \) be a nonsingular transformation. Suppose \( P_\tau : L^1(I^n) \to L^1(I^n) \) is a bounded linear operator with the property that for every measurable subset \( A \) and for every \( f \in L^1(I^n) \) we have
\[
\int_A P_\tau f \, d\mathfrak{m}_n = \int_{\tau^{-1}(A)} f \, d\mathfrak{m}_n.
\]

Then we say that \( P_\tau \) is the Perron-Frobenius operator associated to \( \tau \).

The operator \( P_\tau \) has the following properties from Pelikan [36] or [29].

**Proposition 2.8.** Let \( \tau : I^n \to I^n \) be a nonsingular transformation with respect to the Lebesgue measure \( \mathfrak{m}_n \) on \( \mathbb{R}^n \) and \( P_\tau : L^1(I^n) \to L^1(I^n) \) be its Perron-Frobenius operator, then

(a) For every \( N \in \mathbb{N} \), \( P_\tau P_N = P_N^N \).

(b) For every density \( f \), \( P_\tau f = f \) a.e. if and only if \( d\mu = f \, d\mathfrak{m}_n \) is \( \tau \)-invariant.

**Proof.** (a) For every \( N \in \mathbb{N} \),
\[
\int_A P_N^N f \, d\mathfrak{m}_n = \int_{\tau^{-1}(A)} P_{\tau^{-1}(A)}^N \, d\mathfrak{m}_n = \int_{\tau^{-2}(A)} P_{\tau^{-2}(A)}^{N-1} \, d\mathfrak{m}_n = \cdots = \int_{\tau^{-N}(A)} f \, d\mathfrak{m}_n = \int_{(\tau^N)^{-1}(A)} f \, d\mathfrak{m}_n = \int_A P_\tau f \, d\mathfrak{m}_n.
\]

The equation is true for all \( f \in L^1(I^n) \) and for all \( A \in \mathfrak{B}_I^n \), then
\[
P_{\tau N} = P_N^N.
\]

(b) Claim 1: \( P_\tau f = f \) a.e. implies \( d\mu = f \, d\mathfrak{m}_n \) is \( \tau \)-invariant.

Fix a set \( A \in \mathfrak{B}_I^n \), let \( \mu(A) = \int_A f \, d\mathfrak{m}_n \). For \( P_\tau f = f \),
\[
\mu(A) = \int_A f \, d\mathfrak{m}_n = \int_A P_\tau f \, d\mathfrak{m}_n = \int_{\tau^{-1}(A)} P_\tau f \, d\mathfrak{m}_n = \mu \circ \tau^{-1}(A).
\]

Since \( A \) is arbitrary, \( \mu \) is \( \tau \)-invariant.
Claim 2: \( d\mu = fd\mathbf{m}_n \) is \( \tau \)-invariant which implies \( P_{\tau}f = f \) a.e..

For all \( A \in \mathcal{B}_{I^n} \),

\[
\int_A P_{\tau}f\,d\mathbf{m}_n = \int_{\tau^{-1}(A)} f\,d\mathbf{m}_n = \int_{\tau^{-1}(A)} d\mu = \mu \circ \tau^{-1}(A)
\]

\[
= \mu(A) = \int_A d\mu = \int_A f\,d\mathbf{m}_n.
\]

Thus, \( P_{\tau}f = f \) a.e..

\( \square \)

By the definition of the Koopman operator \( K_{\tau} \) and the Perron-Frobenius operator \( P_{\tau} \), we establish some relations between \( K_{\tau} \) and \( P_{\tau} \).

**Proposition 2.9.** Let Koopman operator \( K_{\tau} \) and Perron-Frobenius operator \( P_{\tau} \) be associated with nonsingular transformation \( \tau \). Then

(a) For all \( f \in L^1(I^n) \) and for all \( g \in L^\infty(I^n) \), \( \int (P_{\tau}f)gd\mathbf{m}_n = \int f(K_{\tau}g)d\mathbf{m}_n \).

(b) \( K_{\tau} \) is the dual of \( P_{\tau} \).

**Proof.** (a) To prove \( \int (P_{\tau}f)gd\mathbf{m}_n = \int f(K_{\tau}g)d\mathbf{m}_n \) is equivalent to prove

\[
\int (P_{\tau}f)gd\mathbf{m}_n = \int f(g \circ \tau)d\mathbf{m}_n.
\] (2.3)

First, we show that Equation (2.3) holds when \( g \) is any characteristic function. Assume \( g = \chi_A \), for a measurable set \( A \). Then

\[
\int (P_{\tau}f)gd\mathbf{m}_n = \int_A P_{\tau}f\,d\mathbf{m}_n = \int_{\tau^{-1}(A)} f\,d\mathbf{m}_n,
\]

and

\[
\int f(g \circ \tau)d\mathbf{m}_n = \int f(\chi_A \circ \tau)d\mathbf{m}_n = \int_{\tau^{-1}(A)} f\,d\mathbf{m}_n.
\]

Hence, Equation (2.3) holds for every simple function. Now we only have to show the simple functions are dense in \( L^\infty(I^n) \). Let \( g \in L^\infty(I^n) \) with a finite number \( M > 0 \) such that \( \|g\|_\infty \leq M \). For a given \( \epsilon > 0 \), take a partition

\[
\{-M = a_0 < a_1 < \cdots < a_m = M\}
\]

such that \( a_j - a_{j-1} < \epsilon \), for all \( j = 1, \cdots, m \). Denote \( A_j = g^{-1}([a_{j-1}, a_j]) \). Let \( h = \sum_{j=1}^m \alpha_j \chi_{A_j} \) be a simple function for some constants \( \alpha_1, \cdots, \alpha_m \). Then

\[
\|g - h\|_\infty \leq \epsilon.
\]
Thus, simple functions are dense in $L^\infty(I^n)$. Hence, for all $g \in L^\infty(I^n)$,

$$\int (P_\tau f) gd\mathcal{m}_n = \int f (g \circ \tau) d\mathcal{m}_n.$$ 

Since $K_\tau g = g \circ \tau$, we get the desired result.

(b) Denote the $L^1 - L^\infty$ duality, $\langle \cdot, \cdot \rangle$, as below:

$$\langle f, g \rangle = \int fg d\mathcal{m}_n.$$ 

By the definition of an adjoint operator, for all $g \in L^\infty(I^n)$ and for all $f \in L^1(I^n)$,

$$\langle f, P_\tau^* g \rangle = \langle P_\tau f, g \rangle = \int (P_\tau f) gd\mathcal{m}_n = \int f (K_\tau g) d\mathcal{m}_n = \langle f, K_\tau g \rangle.$$ 

Therefore, $K_\tau$ is the dual of $P_\tau$ (i.e. $P_\tau$ is the predual of $K_\tau$).

In some special cases, if $A \subset I^n$ is an $n$-dimensional rectangle, then we get a specific form of $P_\tau f$ in Equation (2.2) (see Lasota and Mackey [29]). We describe this as follows.

**Lemma 2.10.** Assume $\tau : I^n \to I^n$ is piecewise $C^2$ on a partition $\mathcal{P} = \{D_j\}_{j=1}^m$ and $\tau(x) = (\varphi_1(x), \ldots, \varphi_n(x))$. Let $|J_\tau|(x) = \det \left( \frac{\partial (\varphi_1(x), \ldots, \varphi_n(x))}{\partial (x_1, \ldots, x_n)} \right)$ be the Jacobian determinant. For each $x = (x_1, \ldots, x_n) \in I^n$, define $A(x) = \prod_{i=1}^n [0, x_i]$. Then

(a) For $x \in I^n$ a.e.,

$$P_\tau f(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\tau^{-1}(A(x))} f(y) d\mathcal{m}_n(y).$$

(b) For $x \in I^n$ a.e.,

$$P_\tau f(x) = \sum_{\tau(y) = x} \frac{f(y)}{|J_\tau|(y)}. \quad (2.4)$$

**Proof.** (a) By Fubini’s theorem and the definition of the Perron-Frobenius operator,

$$\int_0^{x_n} \cdots \int_0^{x_1} P_\tau f(y) dm_n(y) = \int_{A(x)} P_\tau f(y) dm_n(y) = \int_{\tau^{-1}(A(x))} f(y) dm_n(y).$$
Take the derivative on both sides in order
\[ \partial x_n, \partial x_{n-1}, \ldots, \partial x_1 \]
and apply Lebesgue’s theorem, for \( x \in I^n \) a.e.,
\[ P_\tau f(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{\tau^{-1}(A(x))} f(y) dm_n(y). \]

(b) Since \( \tau \) is piecewise \( C^2 \) on \( P \), let \( \tau^{-1}(A(x)) = \bigcup_{j=1}^m \tau_j^{-1}(A(x)) \cap D_j^\phi \) be a disjoint union. Then,
\[ \int_{\tau^{-1}(A(x))} f(y) dm_n(y) = \sum_{j=1}^m \int_{\tau_j^{-1}(A(x)) \cap D_j^\phi} f(y) dm_n(y) \]
\[ = \sum_{\tau_j^{-1}(A(x)) \cap D_j^\phi \neq \emptyset, j \in \{1,2,\ldots,m\}} \int_{A(x) \cap \tau_j(D_j^\phi)} \frac{f(\tau_j^{-1}z)}{|J_\tau(\tau_j^{-1}z)|} dm_n(z) \]
by standard change of variables. Note that for \( x' \sim x \) sufficiently close, the indices in the sum do not depend on \( x' \). Hence we can differentiate both sides \( \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \) to obtain
\[ P_\tau f(x) = \sum_{\tau_j^{-1}(A(x)) \cap D_j^\phi \neq \emptyset, j \in \{1,2,\ldots,m\}} \frac{f(\tau_j^{-1}x)}{|J_\tau(\tau_j^{-1}x)|} \]
where we have again used Lebesgue’s theorem. Since the boundaries of \( D_j \) have measure zero, we can, up to measure zero, simplify the right hand side to
\[ \sum_{\tau_j^{-1}(A(x)) \cap D_j^\phi \neq \emptyset, j \in \{1,2,\ldots,m\}} \frac{f(\tau_j^{-1}x)}{|J_\tau(\tau_j^{-1}x)|} = \sum_{y: \tau(y) = x} \frac{f(y)}{|J_\tau(y)|} \]
as required. \( \square \)

### 2.2 Tonelli Variation on \( \mathbb{R}^n \)

In one-dimensional space, it is not difficult to define the total variation of a function. However, there is a small problem to make the same definition in a higher-dimensional space. Here we describe Tonelli Variation informally as below.

The main idea of the total variation of \( f \) on a closed rectangle, \( A = \prod_{i=1}^n [a_i, b_i] \), is shown as following:

1. Identify one coordinate \( i \) and consider the other variables as constants.
Take the usual total variation of $f$ on the $i^{th}$ coordinate, so that we get a total variation in one dimension and denote as $V_{[a_i,b_i]}f$ in Equation (2.6). This is a real valued function of the remaining $(n - 1)$ variables.

Integrate the rest $(n - 1)$ dimensions and take the infimum over all $g = f$ in $L^1(I^n)$. Therefore, we get a number which we denote by $V_{A,i}f$ in Equation (2.7).

Go through $i = 1, \cdots, n$ and take the maximum value of $V_{A,i}f$. Finally, we get a total Tonelli Variation and denote as $V_A f$ in Equation (2.8).

The following definition of Tonelli Variation is used in Clarkson and Adams [13].

**Definition 2.11. (Tonelli Variation)**

For each $i$, define $\pi_i$ to be a projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ by

$$\pi_i(x) \equiv \pi_i(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n).$$

Let $A = \prod_{i=1}^n [a_i, b_i]$ be an $n$-dimensional rectangle and consider the set of all partitions $S_i$ on the $i^{th}$-coordinate as

$$S_i = \{x_0, x_1, \cdots, x_r \mid a_i = x_0 < x_1 < \cdots < x_r = b_i, \ r \in \mathbb{N}\}. \quad (2.5)$$

For a given function $f : A \to \mathbb{R}$, define $V_{[a_i,b_i]}f : \pi_i(A) \to \mathbb{R}$ by

$$(V_{[a_i,b_i]}f)(\pi_i x) = \sup \sum_{c=1}^r \left| f(x_c) - f(x_{c-1}) \right|, \quad (2.6)$$

where $x_c \equiv (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \equiv (x_i, \pi_i x)$. Let $m_{n-1}$ be the Lebesgue measure in $\mathbb{R}^{n-1}$ and define

$$V_{A,i}f = \inf_{g \equiv f} \int_{\pi_i(A)} (V_{[a_i,b_i]}g)(\pi_i x)dm_{n-1}(\pi_i x). \quad (2.7)$$

Set

$$V_A f = \max_{i=1, \cdots, n} V_{A,i}f. \quad (2.8)$$

Then $V_A f$ is called Tonelli Variation of $f$ on $A$.

**Lemma 2.12.** The appearance of $\inf_g \ f$ in the definition ensures that Tonelli Variation is well-defined on elements of $L^1(I^n)$. If $f_1 = f_2$ a.e., then for any $A \in \mathbb{B}_{I^n}$,

$$V_A f_1 = V_A f_2. \quad (2.9)$$

**Proof.** Consider the $i^{th}$ coordinate. If $g = f_1$ a.e. and $f_1 = f_2$ a.e., then $g = f_2$ a.e.
By the definition of Tonelli Variation,

\[
V_{A,f_1} = \inf_{g \sim f_1} \int_{A \cap (I^n)} (V_{[0,1]} g) \, dm - \inf_{g \sim f_2} \int_{A \cap (I^n)} (V_{[0,1]} g) \, dm = V_{A,f_2}.
\]

Similarly, we have \( V_{A,f_1} \leq V_{A,f_2} \). Hence, \( V_{f_1} = V_{f_2} \) if \( f_1 = f_2 \) a.e.

**Example 2.13.** Consider \( A = \left[ \frac{1}{4}, \frac{3}{4} \right] \times \left[ \frac{1}{3}, \frac{2}{3} \right] \) and define a function \( f : A \to \mathbb{R} \) as

\[
f(x_1, x_2) = f(x) = \begin{cases} 
  x_2, & x \in \left[ \frac{1}{4}, \frac{3}{4} \right] \times \left[ \frac{1}{3}, \frac{2}{3} \right] \cap \mathbb{Q} \\
  \frac{1}{2} x_2, & x \in \left[ \frac{1}{4}, \frac{3}{4} \right] \times \left( \left[ \frac{1}{3}, \frac{2}{3} \right] \setminus \mathbb{Q} \right).
\end{cases}
\]

Then

\[
V_{A,f} = \frac{1}{12}.
\]

**Proof.** For the given function \( f \),

\[
(V_{[1/4,3/4]} f)(x_2) = 0 \\
(V_{[1/3,2/3]} f)(x_1) = \infty.
\]

Now define \( g \) as \( g(x_1, x_2) = \frac{1}{2} x_2 \) on \( A \), so \( g \) a.e. on \( A \). Moreover,

\[
(V_{[1/4,3/4]} g)(x_2) = 0 \\
(V_{[1/3,2/3]} g)(x_1) = \frac{1}{2} \left( \frac{2}{3} - \frac{1}{3} \right) = \frac{1}{6}.
\]
In this case, a continuous function has the minimum value of variations in the same equivalence class of $L^1(A)$. Hence,

$$V_{A,1}f = \inf_{\tilde{f} \leq f} \int_1^2 V_{[\frac{1}{3}, \frac{2}{3}]} \tilde{f} = \int_1^2 V_{[\frac{1}{3}, \frac{2}{3}]} g = 0$$

$$V_{A,2}f = \inf_{\tilde{f} \leq f} \int_1^2 V_{[\frac{3}{4}, \frac{4}{3}]} \tilde{f} = \int_1^2 V_{[\frac{3}{4}, \frac{4}{3}]} g = \frac{1}{12}.$$ 

Therefore, $V_A f = \max_{i=1,2} V_{A,i} f = \frac{1}{12}$.

**Definition 2.14. (bounded Tonelli Variation)**

For a closed, $n$-dimensional rectangle, $A \subset I^n$, if $f \in L^1(A)$ and $V_A f < \infty$, then $f$ is called a function of bounded Tonelli Variation on $A$. Denote $BV(A)$ as follows

$$BV(A) = \{V_A f < \infty \mid f \in L^1(A), \ f : A \to \mathbb{R} \}$$

We list some properties of the Tonelli Variation which are the same as in classical, 1-dimensional variation. Not all the properties are the same when we use Tonelli Variation. For example, a basic identity for the classical, one-dimensional variation is that for a function $f$ and $a < c < b$,

$$V_{[a,b]} f = V_{[a,c]} f + V_{[c,b]} f.$$ 

Note that this identity fails for the definition of Tonelli Variation because of the appearance of $\inf_{g \leq f}$. We give an example (in Example 2.15) in one-dimensional case to show $V_{A\cup B} f > V_A f + V_B f$.

![Figure 2.2: $V_{A\cup B} f > V_A f + V_B f$](image)

**Example 2.15.** Let $A = [0, \frac{1}{2})$ and $B = [\frac{1}{2}, 1]$. Define $f : I \to \mathbb{R}$ by

$$f(x) = \begin{cases} 
\frac{1}{3} x, & \forall x \in [0, \frac{1}{2}) \\
\frac{2}{3} x, & \forall x \in [\frac{1}{2}, 1].
\end{cases}$$
Then, $V_{A \cup B}f > V_Af + V_Bf$.

Proof. By the definition of Tonelli Variation, when $n = 1$,

$$V_Af = \inf_{g \leq f} V_Ag.$$ 

Define a function $g \in L^1(I)$ by

$$g(x) = \begin{cases} 
\frac{1}{4}x, & \forall x \in (0, \frac{1}{2}) \\
\frac{3}{4}x, & \forall x \in (\frac{1}{2}, 1).
\end{cases}$$ 

It is clear that $f = g$ on $[0, 1]$ a.e. Besides,

$$\begin{align*}
V_{A \cup B}f &= \frac{1}{8} + \frac{1}{4} + \frac{3}{8} = \frac{3}{4} \\
V_{A \cup B}g &= \frac{1}{8} + \frac{1}{4} + \frac{3}{8} = \frac{3}{4},
\end{align*}$$

which implies

$$V_{A \cup B}f = \inf_{g \leq f} V_{A \cup B}g = \frac{3}{4}.$$ 

In addition,

$$\begin{align*}
V_Af &= \left(\frac{1}{8} + \frac{1}{2}\right) = \frac{3}{8} \quad \text{implies} \quad V_Af = \inf_{g \leq f} V_Ag = \frac{1}{8}, \\
V_Ag &= \frac{1}{8} \\
V_Bf &= \frac{3}{8} \quad \text{implies} \quad V_Bf = \inf_{g \leq f} V_Bg = \frac{3}{8}, \\
V_Bg &= \frac{3}{8},
\end{align*}$$

Therefore,

$$V_{A \cup B}f = \inf_{g \leq f} V_{A \cup B}g = \frac{3}{4} > \frac{1}{8} + \frac{3}{8} = V_Af + V_Bf.$$ 

\[\square\]

**Proposition 2.16.** Let $f, g \in L^1(I^n)$ and $A \subset I^n$ be a closed subrectangle. Assume $A = \prod_{i=1}^n [a_i, b_i]$.

(a) If $f \in BV(I^n)$, then $f|_A \in BV(A)$ and $V_A(f|_A) \leq V_nf.$

(b) If $f, g \in BV(A)$, then $V_A(f+g) \leq V_Af + V_Ag.$

(c) If $f, g \in BV(A) \cap L^\infty(I^n)$, then $fg \in BV(A)$ and

$$V_A(fg) \leq \|g\|_\infty V_Af + \|f\|_\infty V_Ag.$$ 

(d) Let $f|_A \in BV(A)$ and $l = \min_{i=1,\ldots,n} (b_i - a_i)$. Assume $l = l(A) > 0$. Then we have $f\chi_A \in BV(I^n)$ and

$$V_l(f\chi_A) \leq 2V_A(f|_A) + \frac{2}{l} \int_A |f|dm_n.$$
(e) Assume \( f \in BV(A) \) and \( g \in C^1(A) \) such that for all \( i = 1, 2, \ldots, n \), \( \frac{\partial g(x)}{\partial x_i} \) and \( g \) have continuous extensions to \( \partial A \), the boundary of \( A \). Set constants

\[
\Gamma_i = \sup_{x \in A} \left| \frac{\partial g(x)}{\partial x_i} \right| \quad \text{and} \quad \Gamma = \max_{i=1, \ldots, n} \Gamma_i.
\]

Then

\[
V_A(fg) \leq \|g\|_{\infty} V_A f + \Gamma \int_A |f| dm_n.
\]

(f) Suppose \( B = \prod_{i=1}^n [a_i, b_i] \) is also a closed subrectangle in \( I^n \) such that \( A \cup B \subset I^n \) is a rectangle and \( m_n(A \cap B) = 0 \). If \( f \in BV(A \cup B) \), then for each \( i = 1, 2, \ldots, n \),

\[
V_{A \cup B,i} f \geq V_{A,i} (f|_A) + V_{B,i} (f|_B).
\]

Proof. (a) Fix the \( i^{th} \) coordinate. For a given \( \epsilon > 0 \), choose \( \tilde{f} \overset{a.e.}{=} f \) such that

\[
\int_{\pi_i(I^n)} V_{[0,1]} \tilde{f}(\pi_i x) dm_{n-1}(\pi_i x) \leq V_{I^n,i} f + \epsilon.
\]

Since \( \tilde{f}|_A = f|_A \ a.e. \), for all \( \pi_i(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \pi_i(I^n) \),

\[
V_{[a_i, b_i]} \tilde{f}|_A(\pi_i x) \leq V_{[0,1]} \tilde{f}(\pi_i x).
\]

Hence,

\[
\int_{\pi_i(A)} V_{[a_i, b_i]} \tilde{f}|_A dm_{n-1} \leq \int_{\pi_i(A)} V_{[0,1]} \tilde{f} dm_{n-1}
\]

\[
\leq \int_{\pi_i(I^n)} V_{[0,1]} \tilde{f} dm_{n-1} \leq V_{I^n,i} f + \epsilon.
\]

Therefore, \( V_{A,i} (f|_A) \leq V_{I^n,i} f + \epsilon. \) Take maximum over \( i = 1, 2, \ldots, n. \) For an arbitrary \( \epsilon \), we have

\[
V_A(f|_A) \leq V_{I^n} f.
\]

(b) Fix the \( i^{th} \) coordinate. For a given \( \epsilon > 0 \), choose \( \tilde{f} \) and \( \tilde{g} \) in \( L^1(A) \) such that \( \tilde{f} = f \ a.e., \ \tilde{g} = g \ a.e. \) and

\[
\int_{\pi_i(A)} V_{[a_i, b_i]} \tilde{f}(\pi_i x) dm_{n-1}(\pi_i x) \leq V_{A,i} f + \epsilon;
\]

\[
\int_{\pi_i(A)} V_{[a_i, b_i]} \tilde{g}(\pi_i x) dm_{n-1}(\pi_i x) \leq V_{A,i} g + \epsilon.
\]

Clearly, \( \tilde{f} + \tilde{g} \in L^1(A) \), \( \tilde{f} + \tilde{g} = f + g \ a.e. \) and use sublinearity of classical,
one-dimensional variation to obtain

\[
\int_{\pi_i(A)} V_{[a_i,b_i]}(\tilde{f} + \tilde{g}) dm_{n-1} \leq \int_{\pi_i(A)} \left( V_{[a_i,b_i]} \tilde{f} + V_{[a_i,b_i]} \tilde{g} \right) dm_{n-1} \\
= \int_{\pi_i(A)} V_{[a_i,b_i]} \tilde{f} dm_{n-1} + \int_{\pi_i(A)} V_{[a_i,b_i]} \tilde{g} dm_{n-1} \\
\leq (V_{A,i} \tilde{f} + \epsilon) + (V_{A,i} \tilde{g} + \epsilon).
\]

\[\implies V_{A,i}(f + g) \leq V_{A,i}f + V_{A,i}g + 2\epsilon.\]

Using the same argument as in part (a), we get

\[V_A(f + g) \leq V_Af + V_Ag.\]

(c) Fix the \(i^{th}\) coordinate. For a given \(\epsilon > 0\), choose \(\tilde{f}\) and \(\tilde{g}\) in \(L^1(A)\) such that \(\tilde{f}\) and \(\tilde{g}\) have the same setting as in part (b). If \(\hat{f} = \min \{f, \|f\|_\infty\}\), then

\[V_{[a_i,b_i]} \hat{f}(\pi_i x) \leq V_{[a_i,b_i]} \tilde{f}(\pi_i x).\]

Similarly, \(\hat{g} = \min \{\tilde{g}, \|g\|_\infty\}\) implies \(V_{[a_i,b_i]} \hat{g}(\pi_i x) \leq V_{[a_i,b_i]} \tilde{g}(\pi_i x)\). Observe that

\[\hat{f} = f \text{ a.e., } \hat{g} = g \text{ a.e. and } \hat{f} \hat{g} = fg \text{ a.e.}\]

Then

\[V_{A,i}(fg) \leq \int_{\pi_i(A)} V_{[a_i,b_i]}(\tilde{f} \tilde{g}) dm_{n-1} \]
\[\leq \|g\|_\infty \int_{\pi_i(A)} V_{[a_i,b_i]} \hat{f} dm_{n-1} + \|f\|_\infty \int_{\pi_i(A)} V_{[a_i,b_i]} \hat{g} dm_{n-1} \]
\[\leq \|g\|_\infty (V_{A,i} \hat{f} + \epsilon) + \|f\|_\infty (V_{A,i} \hat{g} + \epsilon),\]

where we have again used a standard result from classical, one-dimensional variation at the second inequality. Hence, use the same argument as in part (a),

\[V_A(fg) \leq \|g\|_\infty V_A f + \|f\|_\infty V_A g.\]

(d) Fix the \(i^{th}\) coordinate. For a given \(\epsilon > 0\), choose \(\tilde{f} = f|_A\) a.e. in \(L^1(A)\) such that

\[\int_{\pi_i(A)} V_{[a_i,b_i]}(\tilde{f}(\pi_i x)) dm_{n-1}(\pi_i x) \leq V_{A,i}(f|_A) + \epsilon,\]

where \(\pi_i x = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)\) and \(x \equiv (x_i, \pi_i x)\). Now recall the formula in the classical, one-dimensional variation: if \(g \in BV[0, 1]\) and \([a, b] \subset [0, 1],\)
then exists $c \in [a, b]$ such that $g(c) \leq \frac{1}{b-a} \int_a^b g(y)dy$ and

$$V_{[0,1]}(g \chi_{[a,b]}) \leq V_{[a,b]}(g|_{[a,b]}) + |g(a)| + |g(b)|$$

$$\leq V_{[a,b]}(g|_{[a,b]}) + \left( V_{[a,c]}(g|_{[a,b]}) + \frac{1}{b-a} \int_a^b |g| \right)$$

$$+ \left( V_{[c,b]}(g|_{[a,b]}) + \frac{1}{b-a} \int_a^b |g| \right)$$

$$\leq 2V_{[a,b]}(g|_{[a,b]}) + \frac{2}{b-a} \int_a^b |g|.$$

Set $\hat{f} \in L^1(I^n)$ with $\hat{f} = \tilde{f}$ on $A$, $\hat{f} = f$ on $I^n \setminus A$ and $l = \min_{i=1,\ldots,n} |a_i - b_i|$. Then $\hat{f} \chi_A = f \chi_A$ a.e. and

$$V_{I^n,i}(f \chi_A) \leq \int_{\pi_i(I^n)} V_{[0,1]}(\hat{f} \chi_A) dm_{n-1} = \int_{\pi_i(A)} V_{[0,1]}(\hat{f} \chi_A) dm_{n-1}$$

$$\leq 2 \int_{\pi_i(A)} (V_{[a,b]}(\hat{f}|_{A})) (\pi_i x) dm_{n}(\pi_i x)$$

$$+ \frac{2}{l} \int_{\pi_i(A)} \left( \int_{a_i}^{b_i} |\tilde{f}|(x_i, \pi_i x) dm(x_i) \right) dm_{n}(\pi_i x)$$

$$= 2 \int_{\pi_i(A)} V_{[a,b]}(\tilde{f}) dm_{n-1} + \frac{2}{l} \int_{A} |\tilde{f}| dm_{n}$$

$$\leq 2(V_{A,i}(f|_A) + \epsilon) + \frac{2}{l} \int_{A} |f| dm_{n}.$$

Use the same argument as in part (a) to get

$$V_{I^n}(f \chi_A) \leq 2V_{A}(f|_A) + \frac{2}{l} \int_{A} |f| dm_{n}.$$

(e) Fix the $i^{th}$ coordinate. For a given $\epsilon > 0$, choose $\tilde{f} = f$ a.e. as in part (b). We use the well-known fact that in the classical, one-dimensional variation: a function $f \in BV[a,b]$ is Riemann integrable on $[a,b]$. Now, for all $\pi_i x \in \pi_i(A)$, we have $V_{[a,b]}(\tilde{f}(\pi_i x) < \infty$ since the function is $m_{n-1}$-integrable by choice. For $g$ is $C^1$,

$$|g(x)| \leq ||g||_{\infty} < \infty, \forall x \in I^n.$$

Following the notation on Equation (2.5) and (2.6), for the same $\epsilon$, there exists a partition such that

$$\sum_{c=1}^r |\tilde{f}(x_{i,c-1})| |x_{i,c-1} - x_{i,c}| \leq \int_{a_i}^{b_i} |\tilde{f}(x_i, \pi_i x)| dm(x_i) + \epsilon$$
and
\[ \sum_{c=1}^{r} |\tilde{f}(x_{ic}) - \tilde{f}(x_{ic-1})| \leq \sum_{c=1}^{r} |\tilde{f}(x_{i}) - \tilde{f}(x_{i-1})|g(x_{ic}) | \]
\[ + \sum_{c=1}^{r} |\tilde{f}(x_{ic-1})||g(x_{ic}) - g(x_{ic-1})| \]
\[ \leq \|g\|_{\infty} V_{[a_i, b_i]}(\tilde{f}(\pi; x)) + \Gamma \left( \int_{a_i}^{b_i} |\tilde{f}(x_i, \pi_i; x)|dm(x_i) + \epsilon \right). \]

Hence,
\[ V_{[a_i, b_i]}(\tilde{f}(\pi; x)) \leq \|g\|_{\infty} V_{[a_i, b_i]}(\tilde{f}(\pi; x)) + \Gamma \left( \int_{a_i}^{b_i} |\tilde{f}(x_i, \pi_i; x)|dm(x_i) + \epsilon \right). \]

Note that \( \tilde{f} = f \) a.e., so integration on \( \pi_i(A) \) yields
\[ V_{A,i}(f) \leq \int_{\pi_i(A)} V_{[a_i, b_i]}(\tilde{f}(\pi_i; x))dm_{n-1}(\pi_i; x) \]
\[ \leq \|g\|_{\infty} \int_{\pi_i(A)} V_{[a_i, b_i]}(\tilde{f}(\pi_i; x))dm_{n-1}(\pi_i; x) \]
\[ + \Gamma \int_{\pi_i(A)} \left( \int_{a_i}^{b_i} |\tilde{f}(x_i, \pi_i; x)|dm(x_i) + \epsilon \right)dm_{n-1}(\pi_i; x) \]
\[ \leq \|g\|_{\infty} V_{A,i}f + \epsilon + \Gamma \int_{A} |f|dm + \epsilon. \]

The same argument as above, we get
\[ V_{A}(fg) \leq \|g\|_{\infty} V_{A}f + \Gamma \int_{A} |f|dm. \]

(f) Since \( A \cup B \) is a closed rectangle, there exists a unique \( i_o \) such that
\[ [a_{i_o}, b_{i_o}] \cup [\bar{a}_{i_o}, \bar{b}_{i_o}] = [a_{i_o}, \bar{b}_{i_o}] \text{ with } \bar{a}_{i_o} = b_{i_o} \]
and
\[ [a_i, b_i] = [a_i, \bar{b}_i], \forall i \neq i_o. \]
For a given \( \epsilon > 0 \), choose \( \tilde{f} = f \) a.e. such that for each \( i = 1, 2, \cdots, n, \)
\[ \int_{\pi_i(A \cup B)} V_{[a_i, b_i]}\tilde{f}dm_{n-1} \leq V_{A \cup B,i}\tilde{f} + \epsilon. \]
Thus, for each coordinate \( i \neq i_o \), we have 
\[
\pi_i(A \cup B) = \pi_i(A) \cup \pi_i(B)
\]
and
\[
V_{A \cup B, i} f + \epsilon \geq \int_{\pi_i(A \cup B)} V_{[a_i, b_i]} \tilde{f} \, dm_{n-1}
\]
\[
= \int_{\pi_i(A)} V_{[a_i, b_i]} \tilde{f} \, dm_{n-1} + \int_{\pi_i(B)} V_{[a_i, b_i]} \tilde{f} \, dm_{n-1}
\]
\[
= \int_{\pi_i(A)} V_{[a_i, b_i]} (\tilde{f}|_A) \, dm_{n-1} + \int_{\pi_i(B)} V_{[a_i, b_i]} (\tilde{f}|_B) \, dm_{n-1}
\]
\[
\geq V_{A, i}(f|_A) + V_{B, i}(f|_B).
\]

Any arbitrary \( \epsilon > 0 \) yields
\[
V_{A \cup B, i} f \geq V_{A, i}(f|_A) + V_{B, i}(f|_B), \ \forall i \neq i_o.
\]

For the coordinate \( i_o \), we have 
\[
\pi_{i_o}(A \cup B) = \pi_{i_o}(A) = \pi_{i_o}(B)
\]
and
\[
V_{A \cup B, i_o} f + \epsilon \geq \int_{\pi_{i_o}(A \cup B)} V_{[a_{i_o}, b_{i_o}]} \tilde{f} \, dm_{n-1}
\]
\[
= \int_{\pi_{i_o}(A \cup B)} V_{[a_{i_o}, b_{i_o}]} (\tilde{f}|_{A \cup B}) \, dm_{n-1}
\]
\[
= \int_{\pi_{i_o}(A)} V_{[a_{i_o}, b_{i_o}]} (\tilde{f}|_A) \, dm_{n-1} + \int_{\pi_{i_o}(B)} V_{[a_{i_o}, b_{i_o}]} (\tilde{f}|_B) \, dm_{n-1}
\]
\[
\geq V_{A, i_o}(f|_A) + V_{B, i_o}(f|_B).
\]

Hence, we get the conclusion for each \( i = 1, 2, \ldots, n \)
\[
V_{A \cup B, i} f \geq V_{A, i}(f|_A) + V_{B, i}(f|_B).
\]

\[\square\]

2.3 Generalized Bounded Variation on \( \mathbb{R}^n \)

The notion of Tonelli Variation developed in the previous section is a classical approach to extending the notion of one-dimensional variation to multiple dimensions. There is another, more modern extension based on the well-known formula
\[
V_{[a, b]} f = \int_a^b |f'(t)| \, dt
\]
whenever \( f \in C^1([a, b]; \mathbb{R}) \) and the notion of (weak) generalized derivative.

**Definition 2.17.** Let \( A \subset \mathbb{R}^n \) be an open set, and let \( C_0^1(A; \mathbb{R}^n) \) denote the collection of compactly supported smooth vector fields on \( A \). Let \( \text{div} \) denote the divergence and
for \( f \in L^1(A) \) put

\[
GV_A(f) = \sup \left\{ \int_A f(x) \text{div} \omega(x) \, dm_n(x) : \omega \in C^1_0(A; \mathbb{R}^n), |\omega(x)|_2 \leq 1 \; \forall x \in A \right\},
\]

where \( |\omega|_2 = |(\omega_1, \ldots, \omega_n)|_2 = \left( \sum_{i=1}^n |\omega_i|^2 \right)^{1/2} \) denotes the ordinary vector 2-norm.

We use the term Generalized Variation of \( f \) over \( A \) for this quantity. If \( GV_A(f) \) is finite, then \( f \) is said to have bounded Generalized Variation over \( A \), and we write \( f \in BGV(A) \). The set of \( f \in BGV(A) \) equipped with the norm

\[
\|f\|_{BGV} = \|f\|_1 + GV_A(f)
\]

is a Banach space. More details can be found in E. Giusti [17]. Generalized Variation was developed for the theory of minimal surfaces.

The notion of Generalized Variation is extended to compact sets \( A \) with piecewise smooth boundaries by the following construction. First, assume \( f \) is \( C^2 \) on the interior of \( A \) \((A^o)\) and vanishing on the boundary \( \partial A \), then (by choosing \( \omega = -\nabla f/|\nabla f|_2 \) and using Stokes’ theorem)

\[
GV_A(f) := GV_{A^o}(f) = \int_A |\nabla f|_2 \, dm_n(x) = \int_A \left( \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i}(x) \right|^2 \right)^{1/2} \, dm_n(x) \tag{2.10}
\]

(provided that the integral is finite). For a general \( f \in L^1(A) \) we define

\[
GV_A(f) = \lim_N GV_A(f_N),
\]

where \( f_N \to f \) in \( L^1 \)-norm and \( f_N \) is a sequence of functions on \( A \) which are \( C^2 \) on interior of \( A \) and vanishing on \( \partial A \) as above. The technical details to justify this construction are also found in [17].

The connection between Tonelli Variation and Generalized Variation, when \( A \) is a rectangle, is the following.

**Lemma 2.18.** Let \( A \subseteq \mathbb{R}^n \) be a closed bounded rectangle and \( f \in L^1(A) \). Then

\[
GV_A(f) \leq nV_A(f).
\]

The proof of this is contained in Lemma 4.34 and Corollary 4.35 later in this document.

### 2.4 Piecewise \( C^2 \) Jabłoński Transformation

We introduce special transformations in a higher dimensional space as defined by Jabłoński [21]. We first define a rectangular partition and define a Jabłoński transformation on the rectangular partition.

**Definition 2.19.** (rectangular partition and join partition)
(1) A partition $\mathcal{P} = \{D_j\}_{j=1}^m$ (in the sense of Definition 2.1) is called a rectangular partition where each partition element $D_j$ is an $n$-dimensional rectangle.

(2) Given two partitions $\mathcal{P}_1 = \{D_1, \cdots, D_m\}$, $\mathcal{P}_2 = \{E_1, \cdots, E_l\}$, we define the join partition of $\mathcal{P}_1$ and $\mathcal{P}_2$ by

$$\mathcal{P} \equiv \mathcal{P}_1 \vee \mathcal{P}_2 = \{D_i \cap E_j | D_i \in \mathcal{P}_1, E_j \in \mathcal{P}_2 \text{ where } i = 1, \cdots, m; j = 1, \cdots, l\}.$$ 

**Definition 2.20. (Jabłoński transformation)**

Let $\mathcal{P} = \{D_1, \cdots, D_m\}$ be a rectangular partition. For $D_j \in \mathcal{P}$, let $D_j = \prod_{i=1}^n [a_{ij}, b_{ij}]$ and $\tau_j = \tau|_{D_j}$. For all $x \equiv (x_1, \cdots, x_n) \in D_j^\circ \subset I^n$ and for $i = 1, \cdots, n$, define $\varphi_{ij} : (a_{ij}, b_{ij}) \rightarrow [0, 1]$. If each $\tau_j$ has the following representation

$$\tau_j(x) = (\varphi_{1j}(x_1), \varphi_{2j}(x_2), \cdots, \varphi_{nj}(x_n)),$$  \hspace{1cm} (2.11)

then the transformation $\tau : I^n \rightarrow I^n$ is called a Jabłoński transformation with respect to the rectangular partition $\mathcal{P}$.

**Definition 2.21. Using the same notation as in Definition 2.20,**

(1) $\tau$ is called piecewise $C^2$ with respect to the partition $\mathcal{P}$ if for $i = 1, \cdots, n$ and $j = 1, \cdots, m$, each $\varphi_{ij}$ is $C^2$ bijective on $(a_{ij}, b_{ij})$, and $\varphi_{ij}$, $\varphi'_{ij}$ and $\varphi''_{ij}$ have continuous extensions to $[a_{ij}, b_{ij}]$.

(2) $\tau$ is expanding if for each $i = 1, \cdots, n$,

$$\inf_{\substack{x_i \in (a_{ij}, b_{ij}) \atop j=1, \cdots, m}} |\varphi'_{ij}(x_i)| > 1.$$ 

**Note:** If a Jabłoński transformation satisfies condition (1) in Definition 2.21, we call the transformation a piecewise $C^2$ Jabłoński transformation.

**Figure 2.3:** Rectangular Partition
Example 2.22. Let \( \tau : I^2 \to I^2 \) be defined on a rectangular partition \( \mathcal{P} \) of \( I^2 \) as follows: let \( \mathcal{P} = \{D_1, D_2, D_3, D_4\} \) and for \( j = 1, \cdots, 4 \), let \( \tau_j = \tau|_{D_j} \).

\[
\begin{align*}
D_1 &= [0, \frac{3}{5}] \times [0, \frac{1}{2}] ; \quad \forall \mathbf{x} \in D_1^0, \; \tau_1(\mathbf{x}) &= \left( \frac{4}{5}x_1 + \frac{1}{10}, \; 2x_2 \right), \\
D_2 &= [0, \frac{3}{5}] \times [\frac{1}{2}, 1] ; \quad \forall \mathbf{x} \in D_2^0, \; \tau_2(\mathbf{x}) &= \left( \frac{4}{5}x_1 + \frac{1}{10}, \; 2 - 2x_2 \right), \\
D_3 &= [\frac{3}{5}, 1] \times [0, \frac{2}{5}] ; \quad \forall \mathbf{x} \in D_3^0, \; \tau_3(\mathbf{x}) &= \left( x_1^2, \; \frac{3}{5}x_2 \right), \\
D_4 &= [\frac{3}{5}, 1] \times [\frac{2}{5}, 1] ; \quad \forall \mathbf{x} \in D_4^0, \; \tau_4(\mathbf{x}) &= \left( x_1^2, \; 2x_2 - 1 \right).
\end{align*}
\]

Then \( \tau \) is a piecewise \( C^2 \) expanding Jabłoński transformation.

Proof. Clearly, \( \tau \) is piecewise \( C^2 \) on \( \mathcal{P} \) satisfying Equation (2.11). By the definition of the transformation \( \tau \), we have

\[
\begin{align*}
|\varphi_{11}^j(x_1)| &= \frac{4}{5}, & |\varphi_{21}^j(x_2)| &= 2, \\
|\varphi_{12}^j(x_1)| &= \frac{4}{5}, & |\varphi_{22}^j(x_2)| &= 2, \\
|\varphi_{13}^j(x_1)| &= 2x_1 \geq \frac{6}{5}, & |\varphi_{23}^j(x_2)| &= \frac{3}{2}, \\
|\varphi_{14}^j(x_1)| &= 2x_1 \geq \frac{6}{5}, & |\varphi_{24}^j(x_2)| &= 2.
\end{align*}
\]

\[
\implies \begin{cases} 
\inf_{x_1 \in (0,1)} |\varphi_{1j}^i(x_1)| = \frac{6}{5} > 1, \\
\inf_{x_2 \in (0,1)} |\varphi_{2j}^i(x_2)| = \frac{3}{2} > 1.
\end{cases}
\]

Therefore, \( \tau \) is a piecewise \( C^2 \) expanding Jabłoński transformation. \( \square \)

Remark 2.23. A Jabłoński transformation \( \tau \) maps topological rectangles (product of intervals) to measurable rectangles (product of measurable sets). Moreover, \( \tau^{-1} \) also has this property.

Following results are simple observations about combinations of Jabłoński transformations.

Lemma 2.24. (a) Let \( \tau_1 \) and \( \tau_2 \) be Jabłoński transformations defined on partitions \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) respectively. Then both \( \tau_1 \) and \( \tau_2 \) are also Jabłoński transformations on the join partition of \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

(b) Given a Jabłoński transformation \( \tau \) defined on the partition \( \mathcal{P} = \{D_1, \cdots, D_m\} \), for any integer \( N \in \mathbb{N} \), \( \tau^N \) is a Jabłoński transformation on the partition \( \mathcal{P}^N = \mathcal{P} \lor \tau^{-1}\mathcal{P} \lor \cdots \lor \tau^{-N+1}\mathcal{P} \). That is

\[
\mathcal{P}^N \equiv \{D_{j_1} \cap \cdots \cap \tau^{-N+1}(D_{j_N}) : D_{j_1}, \cdots, D_{j_N} \in \mathcal{P} \}. \quad (2.12)
\]

Proof. (a) Suppose \( \mathcal{P}_1 = \{D_i\}_{i=1}^{m_1} \) and \( \mathcal{P}_2 = \{E_j\}_{j=1}^{m_2} \), then

\[
\mathcal{P}_1 \lor \mathcal{P}_2 = \{D_i \cap E_j : i = 1, \cdots, m_1, \; j = 1, \cdots, m_2\}.
\]

For each \( D_i \cap E_j \in \mathcal{P}_1 \lor \mathcal{P}_2 \) and for all \( \mathbf{x} \in D_i \cap E_j \),
τ_1|_{D_i \cap E_j}(x) = τ_1|_{D_i}(x) \quad \text{and} \quad τ_2|_{D_i \cap E_j}(x) = τ_2|_{E_j}(x).

Since τ_1 is a Jabłoński transformation on P_1, τ_1 on partition P_1 ∨ P_2 satisfies the representation (2.11). Hence, τ_1 is also a Jabłoński transformation on P_1 ∨ P_2. Using the same argument for τ_2, then both τ_1 and τ_2 are Jabłoński transformations on the partition P_1 ∨ P_2.

(b) It is enough to prove that τ is a Jabłoński transformation on P then τ^2 is also a Jabłoński transformation on P^2. For any element D_i ∩ τ^(-1)(D_j) ∈ P^2, if for all x ∈ D_i ∩ τ^(-1)(D_j), then

τ(x) ∈ τ(D_i ∩ τ^(-1)(D_j)) ⊂ (τ(D_i) ∩ D_j) ⊂ D_j.

Thus, τ^2|_{D_i \cap τ^(-1)(D_j)}(x) = τ|_{D_j}(x). That is, τ^2 is a Jabłoński transformation on P^2. By induction, we get τ^N is a Jabłoński transformation on P^N.

We say that a sequence of functions \{f_N\} ⊂ L^1(I^n) is \textit{weakly convergent} to f if there exists an f ∈ L^1(I^n) such that for every h ∈ L^∞(I^n)

\lim_{N \to \infty} \int f_N h = \int f h.

\textbf{Definition 2.25.} [29] (weakly precompact)

For a subset \mathcal{F} in L^1(I^n), \mathcal{F} is called weakly precompact if every sequence of functions \{f_N\}_{N ∈ \mathbb{N}} in \mathcal{F} contains a weakly convergent subsequence \{f_{N_k}\}_{k ∈ \mathbb{N}} that converges to an \tilde{f} in L^1(I^n).


\mathcal{K} = \{f ≥ 0 \mid V_{I^n} f ≤ M, \|f\|_1 ≤ 1, \text{ where } f : I^n → \mathbb{R}\},

then \mathcal{K} is weakly precompact on L^1(I^n).

\textbf{Proposition 2.27.} Let f ∈ L^1(I^n) and τ : I^n → I^n be a piecewise C^2 Jabłoński transformation on a rectangular partition P. Suppose A ∈ P and A = \prod_{i=1}^{n}[a_i, b_i].

Define τ(x) = (ϕ_1(x_1), \cdots, ϕ_n(x_n)) on A. For each i = 1, \cdots, n,

λ_i = \inf_{x_i ∈ (a_i, b_i)} |ϕ'_i(x_i)| > 0 \quad \text{and} \quad Γ_i = \sup_{x_i ∈ (a_i, b_i)} \frac{|ϕ''_i(x_i)|}{|ϕ'_i(x_i)|^2}.

Let Γ = \max_{i=1,\cdots,n} Γ_i and \frac{1}{\lambda_i} = \|(ϕ'_i)^{-1}\|_∞. If f|_A ∈ BV(A), then

(a) τ(A) ⊂ I^n is a rectangle.

(b) For each i = 1, \cdots, n,

\[ V_{τ(A),i} ((f|_{J_i^{-1}}) \circ τ^{-1}) ≤ \|(ϕ'_i)^{-1}\|_∞ V_{A,i}(f|_A) + Γ \int_A |f| dm_A. \quad (2.13) \]
Proof. (a) Since $\tau$ is a $C^2$ Jabłoński transformation on $A$, $\tau(A)$ is a rectangle.

(b) For each $i = 1, 2, \cdots, n$ and for a given $\epsilon > 0$, choose $\tilde{f} \equiv f$ such that

$$\int_{\pi_i(A)} V_{[a_i, b_i]}(\tilde{f}|_A) dm_{n-1} \leq V_{A,i}(f|_A) + \epsilon.$$

By the definition of $\tau$, we have $|J_{\tau}(x)| = \prod_{i=1}^{n} |{\varphi}'_i(x_i)|$. From (a), we get a rectangle $\tau(A) = \prod_{i=1}^{n} I_i$. Hence,

$$V_i \equiv V_{\tau(A),i} (|(f|_{J_{\tau}}^{-1}) \circ \tau^{-1}) \leq \int_{\pi_i(\tau A)} V_{I_i} ((\tilde{f}|{J_{\tau}}^{-1}) \circ \tau^{-1}) dm_{n-1}$$

$$= \int_{\pi_i(\tau A)} \prod_{j=1, j \neq i}^{n} (({\varphi}'_j)^{-1} \circ \tau^{-1}) V_{I_i} ((\tilde{f}|({\varphi}'_i)^{-1}) \circ \tau^{-1}) dm_{n-1}$$

$$= \int_{\pi_i(A)} V_{[a_i, b_i]} (\tilde{f}|({\varphi}'_i)^{-1}) dm_{n-1}.$$

The last equality is followed by the standard change of variables. Furthermore, by Proposition 2.16 (e),

$$V_i \leq \int_{\pi_i(A)} (||({\varphi}'_i)^{-1}||_{\infty} V_{[a_i, b_i]}(\tilde{f}|_A) + \Gamma \int_{[a_i, b_i]} |\tilde{f}| dm) dm_{n-1}$$

$$= ||({\varphi}'_i)^{-1}||_{\infty} \int_{\pi_i(A)} V_{[a_i, b_i]}(\tilde{f}|_A) dm_{n-1} + \Gamma \int_{\pi_i(A)} \left( \int_{[a_i, b_i]} |f| dm \right) dm_{n-1}$$

$$\leq ||({\varphi}'_i)^{-1}||_{\infty} (V_{A,i}(f|_A) + \epsilon) + \Gamma \int_{A} |f| dm.$$

Since $\epsilon > 0$ is arbitrary, for each $i = 1, \cdots, n$, we get

$$V_{\tau(A),i} (|f|_{J_{\tau}}^{-1} \circ \tau^{-1}) \leq ||({\varphi}'_i)^{-1}||_{\infty} V_{A,i}(f|_A) + \Gamma \int_{A} |f| dm.$$

$\square$
Chapter 3

Random Dynamical System

A random dynamical system is constructed from finitely many transformations and a probability vector. An informal description of a random dynamical system $T$ is as follows: Given the collection of transformations $\mathcal{F} = \{\tau_k\}_{k=1}^q$ on $I^n$, a point $x \in I^n$ and the probability vector $p = (p_1, \cdots, p_q)$, we randomly select an image point $\tau_j(x)$ of $x$ with probability $p_j$. In other words, the image of $x$ under $T$ is a set of $q$ points in $I^n$, selected according to the probability law $p$. In this article, we only consider $p_k$'s that are constants. We use the notation $T = \{\tau_k; p_k\}_{k=1,\cdots,q}$ for a random dynamical system constructed from the given transformations in $\mathcal{F}$ and the probability vector $p$. For $\vec{k} \in \{1, \cdots, q\}^K$, we write

$$T_{\vec{k}}(x) = \tau_{k_K} \circ \tau_{k_{K-1}} \circ \cdots \circ \tau_{k_1}(x) \quad \text{and} \quad p_{\vec{k}} = p_{k_K} \cdot p_{k_{K-1}} \cdots p_{k_1}. \quad (3.1)$$

The following definition of a random dynamical system is used in Bahsoun, Bose and Quas [2], Bahsoun and Góra [3], Góra and Boyarsky [18] or Kifer [25].

**Definition 3.1. (random dynamical system on $I^n$ for position independent probability)**

For all $k = 1, \cdots, q$, let $\tau_k : I^n \rightarrow I^n$ be given transformations. Let $p = (p_1, \cdots, p_q)$ be a probability vector (i.e. $\sum_{k=1}^q p_k = 1$) with $p_k > 0$ for all $k$. A random dynamical system $T = \{\tau_k; p_k\}_{k=1,\cdots,q}$ is a Markov process with transition function

$$P(x, A) = \sum_{k=1}^q p_k \chi_A(\tau_k(x)), \quad (3.2)$$

where $\chi_A$ is the characteristic function of a measurable set $A$.

Note that, for $\vec{k} \in \{1, \cdots, q\}^K$,

$$P_{\vec{k}}(x, A) = \sum_{k_K, \cdots, k_1=1}^q p_{\vec{k}} \chi_A(T_{\vec{k}}(x)). \quad (3.3)$$

By Equation (3.2), we recognize the transition function as a type of Koopman operator, as in Definition 2.4. We define the Koopman operator with respect to the random dynamical system $T$ as below.
Definition 3.2. (Koopman operator w.r.t. T)

Let \( T = \{\tau_k; p_k\}_{k=1,\ldots,q} \) be a random dynamical system constructed from nonsingular transformations \( \{\tau_k\} \) on \( I^n \). For all \( g \in L^\infty(I^n) \), define \( K_T : L^\infty(I^n) \to L^\infty(I^n) \) by

\[
K_T g(x) = \sum_{k=1}^{q} p_k \cdot K_{\tau_k} g(x),
\]

where \( K_{\tau_k} g = g \circ \tau_k \).

From the previous chapter, for a nonsingular transformation \( \tau \), \( K_{\tau} \) is the dual of \( P_{\tau} \). To get the duality between the Koopman operator \( K_T \) and the Perron-Frobenius operator \( P_T \) with respect to \( T \), there is a natural way to define \( P_T \). By Proposition 2.9 (a),

\[
\int (P_{\tau} f) g \, dm_n = \int f(K_{\tau} g) \, dm_n.
\]

By the definition of \( K_T \), for all \( f \in L^1(I^n) \) and for all \( g \in L^\infty(I^n) \),

\[
\int f(K_T g) \, dm_n = \int \left( \sum_{k=1}^{q} p_k \cdot K_{\tau_k} g \right) \, dm_n = \sum_{k=1}^{q} p_k \int (K_{\tau_k} g) \, dm_n = \sum_{k=1}^{q} p_k \int (P_{\tau_k} f) g \, dm_n = \int \left( \sum_{k=1}^{q} p_k P_{\tau_k} f \right) g \, dm_n.
\]

Therefore, if we define \( P_T \equiv \sum_{k=1}^{q} p_k P_{\tau_k} \), we still have the dual relation between these two operators. That is, \( P_T^* = K_T \).

Definition 3.3. (Perron-Frobenius operator w.r.t. T)

Let \( T = \{\tau_k; p_k\}_{k=1,\ldots,q} \) be a random dynamical system constructed from nonsingular transformations \( \{\tau_k\}_{k=1}^{q} \) on \( I^n \). For all \( f \in L^1(I^n) \), define \( P_T : L^1(I^n) \to L^1(I^n) \) by

\[
P_T f = \sum_{k=1}^{q} p_k \cdot P_{\tau_k} f.
\]

(3.4)

Let \( \mu \) be a measure in the measure space \((I^n, \mathcal{B}_{I^n})\), and \( T = \{\tau_k; p_k\}_{k=1}^{q} \) be a random dynamical system defined as above. The measure \( \mu \) is called \( T \)-invariant, in Pelikan [36] or Morita [34], if for all \( A \in \mathcal{B}_{I^n} \),

\[
\mu(A) = \sum_{k=1}^{q} p_k \mu(\tau_k^{-1} A).
\]

Lemma 3.4. Let \( T = \{\tau_k; p_k\}_{k=1,\ldots,q} \) be a random dynamical system and \( P_T \) be its Perron-Frobenius operator. For every density \( f^* \), \( P_T f^* = f^* \) a.e. is the same as \( d\mu = f^* dm_n \) is \( T \)-invariant.
Proof. If $P_T f^* = f^*$ a.e. and $d\mu = f^* dm_n$, then for any measurable set $A$,

$$
\mu(A) = \int_A d\mu = \int_A f^* dm_n = \int_A P_T f^* dm_n = \int_A \sum_{k=1}^q p_k \int_{\tau_k^{-1}(A)} f^* dm_n = \int_A \sum_{k=1}^q p_k \mu(\tau_k^{-1}(A)).
$$

Since $A$ is an arbitrary measurable set, $\mu$ is $T$-invariant. On the other hand, for every measurable set $A$, if $d\mu = f^* dm_n$ is $T$-invariant, then

$$
\int_A P_T f^* dm_n = \sum_{k=1}^q p_k \int_{\tau_k^{-1}(A)} f^* dm_n = \int_{\tau_k^{-1}(A)} \sum_{k=1}^q p_k \mu(\tau_k^{-1}(A)) = \mu(A) = \int_A f^* dm_n.
$$

Thus, $P_T f^* = f^*$ a.e since $A$ is arbitrary. \hspace{1cm} \square

3.1 Weak Lasota-Yorke Inequality

We introduce the idea of expanding-on-average for the individual transformations in the random dynamical system $T$, as described in Boyarsky and Levesque [8], Boyarsky and Lou [10, 11] or Kamthan and Mackey [22]. This condition is weaker than the requirement that each transformation $\tau_k$ in $T$ be strictly expanding. We also present a weak form of the Lasota-Yorke inequality, in Proposition 3.6. This inequality helps us to control the variation term so that it goes to zero if we iterate $T$ many times.

**Definition 3.5.** (expanding-on-average)

Let $T = \{\tau_k; \ p_k\}_{k=1, \cdots, q}$ be a random dynamical system constructed from piecewise $C^2$ Jabłoński transformations (i.e. for $k = 1, \cdots, q$, each $\tau_k$ is defined on a rectangular partition $\mathcal{P}_k = \{D_{1,k}, \cdots, D_{m(k),k}\}$). Define $\mathcal{P} = \{D_j\}_{j=1}^m$ to be the join partition of $\mathcal{P}_1, \cdots, \mathcal{P}_q$. For each $D_j \in \mathcal{P}$, let $D_j \equiv \prod_{i=1}^n [a_{ij}, b_{ij}]$ and $\tau_{j,k} = \tau_k|_{D_j'}$. For $i = 1, \cdots, n$, let $\varphi_{ij,k}: (a_{ij}, b_{ij}) \rightarrow I$ be $C^2$ injective. Since $\tau_k$ is a Jabłoński transformation, for all $x \in D_j$, each $\tau_{j,k} \exists$ has the representation

$$
\tau_{j,k}(x) = (\varphi_{1j,k}(x_1), \cdots, \varphi_{nj,k}(x_n)).
$$

Define $\|\langle \varphi_{i,k}' \rangle^{-1}\|_{\infty} = \max_{j=1, \cdots, m} \left( \sup_{x_i \in (a_{ij}, b_{ij})} |\varphi_{ij,k}'(x_i)|^{-1} \right)$. If there exists a constant $\alpha$
satisfying $0 < \alpha < 1$ such that for each $i = 1, \cdots, n$,

$$\sum_{k=1}^{q} p_k \cdot \| (\varphi'_{i,k})^{-1} \|_\infty \leq \alpha,$$

(3.5)

then the random dynamical system $T$ is expanding-on-average.

Before introducing an important theorem in this chapter, we give a weak form of the Lasota-Yorke inequality described in the following proposition. The proof of Proposition 3.6 is similar as in Lasota and York [30], but here we deal with $n$ dimensions instead of one dimension.

**Key for the indices:**

$n =$ space dimension

$i =$ space coordinate, $i = 1, 2, \cdots, n$

$j =$ partition element, $D_j$, $j = 1, 2, \cdots, m$

$k =$ transformation numbers, $k = 1, 2, \cdots, q$

$k \in \{1, \cdots, q\}^K$; $K \in \mathbb{N}$, for different integer $K$ using different $\vec{k}$ for $T^K$

$p_{\vec{k}} = p_{k_1} \cdot p_{k_{(k-1)}} \cdots p_{k_2} \cdot p_k$ and $T_{\vec{k}}(x) = \tau_{k_k} \circ \tau_{k_{(k-1)}} \circ \cdots \circ \tau_{k_1}(x)$.

**Proposition 3.6.** Let $T = \{\tau_k; p_k\}_{k=1,\cdots,q}$ be a random dynamical system satisfying Equestion (3.5) which is constructed from piecewise $C^2$ Jabłoński transformations defined on the common partition $\mathcal{P}$. For the same $\alpha$ as in Equation (3.5), consider a constant $K \in \mathbb{N}$. For $\vec{k} \in \{1, \cdots, q\}^K$, let $T_{\vec{k}}$’s, as shown on Equation (3.1), be defined on the partition $\mathcal{P}^K = \{D_1, \cdots, D_m\}$, where $\mathcal{P}^K$ is the join of all $\{\mathcal{P}, T^{-1}\mathcal{P}, \cdots, T^{-K+1}\mathcal{P}\}$ (see Lemma 2.24 (b)). Assume $T_{ij,\vec{k}} = T_{\vec{k}}|_{D^o_j}$ has the representation

$$T_{ij,\vec{k}}(x) = (\varphi_{1j,\vec{k}}(x_1), \varphi_{2j,\vec{k}}(x_2), \cdots, \varphi_{nj,\vec{k}}(x_n)).$$

For all $x \in D_j$,

$$|J_{T_{ij,\vec{k}}}(x)| = \left| \frac{\partial (\varphi_{1j,\vec{k}}, \cdots, \varphi_{nj,\vec{k}})}{\partial (x_1, \cdots, x_n)} \right| = \left| \frac{\varphi'_{1j,\vec{k}}(x_1)}{\cdots} \frac{\varphi'_{nj,\vec{k}}(x_n)}{\cdots} \right|.$$

Let $g_{ij,\vec{k}}(x) = |J_{T_{ij,\vec{k}}^{-1}}(x)|$, so each $g_{ij,\vec{k}}$ is a $C^1$ real valued function of $n$ variables on $D_j$.

Define constants

$$\Gamma_{ij,\vec{k}} = \sup_{x \in D^o_j} \left| \frac{\partial g_{ij,\vec{k}}(x)}{\partial x_i} \right| \text{ and } \Gamma = \max_{i=1, \cdots, n; j=1, \cdots, m} \Gamma_{ij,\vec{k}}.$$

Given $D_j = \prod_{i=1}^{n} [a_{ij}, b_{ij}]$, define constants

$$l = \min_{i=1, \cdots, n; j=1, \cdots, m} |a_{ij} - b_{ij}| \text{ and } \gamma_K = 2\Gamma + \frac{2}{l}. $$
Then for every \( f \in L^1(I^n) \),
\[
V_{I^n} P^K_T f \leq 2\alpha^K V_{I^n} f + \gamma K \|f\|_1.
\]  \hfill (3.6)

**Note:** Equation (3.6) is a weak form of the Lasota-Yorke inequality. Usually the inequality
\[
V_{I^n} P_T f \leq \alpha V_{I^n} f + \gamma \|f\|_1
\]
is called Lasota-Yorke inequality, in [30].

**Proof.** Let \( \vec{k} \in \{1, \ldots, q\}^K \). By Equation (3.4) and (3.1),
\[
P_T f = \sum_{k_1=1}^{q} p_{k_1} P_{\tau_{k_1}} f;
\]
\[
P_T^2 f = \sum_{k_2=1}^{q} p_{k_2} P_{\tau_{k_2}} \left( \sum_{k_1=1}^{q} p_{k_1} P_{\tau_{k_1}} f \right) = \sum_{k_2=1}^{q} \sum_{k_1=1}^{q} p_{k_2} p_{k_1} P_{\tau_{k_2} \circ \tau_{k_1}} f
\]
\[
= \sum_{k_2, k_1=1}^{q} p_{k_2} p_{k_1} P_{\tau_{k_2} \circ \tau_{k_1}} f.
\]

Therefore,
\[
P_T^K f = \sum_{k_K, \ldots, k_1=1}^{q} p_{k_K} \cdots p_{k_1} P_{\tau_{k_K} \circ \cdots \circ \tau_{k_1}} f \equiv \sum_{\vec{k}} p_{\vec{k}} P_{T_{\vec{k}}} f. \]  \hfill (3.7)

We use some inequalities in Proposition 2.16 to prove this proposition. The symbol \( \leq^{(a)} \) is referred to the inequality in Proposition 2.16 (a) and so on. By Equation (3.7) and for each \( i = 1, \ldots, n \),
\[
V_{I^n, i} \left( P_T^K f(x) \right) = V_{I^n, i} \left( \sum_{\vec{k}} p_{\vec{k}} P_{T_{\vec{k}}} f(x) \right) \leq^{(b)} \sum_{\vec{k}} p_{\vec{k}} V_{I^n, i} \left( P_{T_{\vec{k}}} f(x) \right)
\]
\[
= \sum_{\vec{k}} p_{\vec{k}} V_{I^n, i} \left( \sum_{T_{\vec{k}}(y) = x} f(y) \right) \left| J_{T_{\vec{k}}}(y) \right|, \text{ by Lemma 2.10 (b)},
\]

\[
\leq^{(b)} \sum_{j=1}^{m} \sum_{\vec{k}} p_{\vec{k}} V_{I^n, i} \left( f(T_{j, \vec{k}}^{-1} x) \right) \left| J_{T_{j, \vec{k}}^{-1}}(T_{j, \vec{k}}^{-1} x) \right| \left( T_{j, \vec{k}}^{-1} x \right) \left| J_{T_{j, \vec{k}}^{-1}}(T_{j, \vec{k}}^{-1} x) \right| \chi_{T_{j, \vec{k}}^{-1}(D_j)}(x)
\]
\[
= \sum_{j=1}^{m} \sum_{\vec{k}} p_{\vec{k}} V_{I^n, i} \left( f(T_{j, \vec{k}}^{-1} x) \right) \left| J_{T_{j, \vec{k}}^{-1}}(T_{j, \vec{k}}^{-1} x) \right| \chi_{T_{j, \vec{k}}^{-1}(D_j)}(x)
\]
\[
= \sum_{j=1}^{m} \sum_{\vec{k}} p_{\vec{k}} V_{I^n, i} \left( f(T_{j, \vec{k}}^{-1} x) \right) \left| J_{T_{j, \vec{k}}^{-1}}(T_{j, \vec{k}}^{-1} x) \right| \chi_{T_{j, \vec{k}}^{-1}(D_j)}(x).
\]
\[ V_{J,\ell}^*(P_{T}^K f) \leq (d) 2 \sum_{j=1}^{m} \sum_{k} p_k V_{J,\ell}^*(D_j)i \left( (f|J_{T,j,k}^{-1}) \circ T_{j,k}^{-1} \right) \]
\[ + \frac{2}{l} \sum_{j=1}^{m} \sum_{k} p_k \int_{J_{T,j,k}(D_j)} \left| (f|J_{T,j,k}^{-1}) \circ T_{j,k}^{-1} \right| \, dm_n \]
\[ = A + \frac{2}{l} \sum_{k} p_k \left( \sum_{j=1}^{m} \int_{D_j} |f| \, dm_n \right) = A + \frac{2}{l} \|f\|_1. \]

Since
\[ |\varphi_{ij,k}(x_i)| = \left| (\varphi_{ij,k} \circ \varphi_{ij,k-1} \circ \cdots \circ \varphi_{ij,k_1})(x_i) \right| \]
\[ = |\varphi_{ij,k_1}(\varphi_{ij,k,K-1} \circ \cdots \circ \varphi_{ij,k_1}x_i)| \cdots |\varphi_{ij,k_1}(x_i)|, \]

\[ \|(|\varphi_{i,k,K}^i)^{-1}|_\infty = \max_{j=1,\ldots,m} \left( \sup_{x_i(a_{ij},b_{ij})} |\varphi_{ij,k}^i(x_i)|^{-1} \right) \]
\[ \leq \left( \max_{j=1,\ldots,m} \sup_{x_i(a_{ij},b_{ij})} |\varphi_{ij,k}^i(x_i)|^{-1} \right) \cdots \left( \max_{j=1,\ldots,m} \sup_{x_i(a_{ij},b_{ij})} |\varphi_{ij,k_1}^i(x_i)|^{-1} \right) \]
\[ = \|(|\varphi_{i,k,K}^i)^{-1}|_\infty \cdots \|(|\varphi_{i,k_1}^i)^{-1}|_\infty. \]

Thus, by Equation (3.1) and the above inequality,
\[ \sum_{k} p_k \|(|\varphi_{i,k,K}^i)^{-1}|_\infty = \sum_{k_1}^q p_{k,k_1-1,\ldots,k_1-1} \cdots p_{k_1} \|(|\varphi_{i,k_1}^i)^{-1}|_\infty \]
\[ \leq \sum_{k_1}^q p_{k_1} \cdots p_{k_1} \|(|\varphi_{i,k_1}^i)^{-1}|_\infty \cdots \|(|\varphi_{i,k_1}^i)^{-1}|_\infty \]
\[ = \left( \sum_{k_1}^q p_{k_1} \|(|\varphi_{i,k_1}^i)^{-1}|_\infty \right) \cdots \left( \sum_{k_1}^q p_{k_1} \|(|\varphi_{i,k_1}^i)^{-1}|_\infty \right). \]

By Equation (3.5), we get \( \sum_{k} p_k \|(|\varphi_{i,k,K}^i)^{-1}|_\infty \leq \alpha^K. \) Hence, by Proposition 2.27 (b)

\[ A = 2 \sum_{j=1}^{m} \sum_{k} p_k V_{J,\ell}^*(D_j) i \left( (f|J_{T,j,k}^{-1}) \circ T_{j,k}^{-1} \right) \]
\[ \leq 2 \sum_{j=1}^{m} \sum_{k} p_k \left( \|(|\varphi_{i,k_1}^i)^{-1}|_\infty \right) V_{D_j,i}(f|D_j) + 2\Gamma \sum_{j=1}^{m} \sum_{k} p_k \int_{D_j} |f| \, dm_n \]
\[ \leq 2\alpha^K \sum_{j=1}^{m} V_{D_j,i}(f|D_j) + 2\Gamma \|f\|_1 \leq (f) 2\alpha^K V_{J,\ell}^* f + 2\Gamma \|f\|_1. \]
Therefore,

\[ V_{I^n,T^K} f \leq A + \frac{2}{l} \|f\|_1 \leq 2\alpha^K V_{I^n,i} f + 2(\Gamma + \frac{1}{l})\|f\|_1 \]
\[ \equiv 2\alpha^K V_{I^n,i} f + \gamma_K \|f\|_1 \]
\[ \leq 2\alpha^K V_{I^n} f + \gamma_K \|f\|_1. \]

Thus,

\[ V_{I^n,T^K} f \leq 2\alpha^K V_{I^n} f + \gamma_K \|f\|_1. \]

\[ \square \]

**Remark 3.7.** In Proposition 3.6, constants \( \Gamma \) depends on \( T_k \), and \( l \) depends on the partition \( P^K \). Thus, \( \gamma_K = 2\Gamma + \frac{2}{l} \) is dependent on the fixed number \( K \). All of these constants are independent of any \( f \) in \( L^1(I^n) \).

By Equation (3.6), we have the following lemma immediately.

**Lemma 3.8.** Let \( K \in \mathbb{N} \) be a fixed constant such that \( 0 \leq 2\alpha^K < 1 \). For each positive integer \( N \), assume \( N = aK + b \) for \( a,b \in \mathbb{N} \) and \( 0 \leq b < K \). Define \( \beta = 2^{\frac{1}{\alpha}} \alpha < 1 \).

There exists a constant \( \gamma_0 = \gamma_0(K) \) which is independent of \( N \) such that we have the following inequality

\[ V_{I^n,P^N} f \leq 2\beta^N V_{I^n} f + \gamma_0 \|f\|_1. \] (3.8)

Note this inequality is useful when \( f \in BV \).

**Proof.** By Equation (3.6), for any constant \( N \in \mathbb{N} \),

\[ V_{I^n,P^N} f \leq 2\alpha^N V_{I^n} f + \gamma_N \|f\|_1. \]

Let \( \gamma_M = \max\{\gamma_1, \gamma_2, \cdots, \gamma_K\} \). Then for all \( b = 1, 2, \cdots, K - 1 \),

\[ V_{I^n,P^b} f \leq 2\alpha^b V_{I^n} f + \gamma_M \|f\|_1. \]

Therefore, for each \( N = aK + b \),

\[ V_{I^n,P^N} f = V_{I^n,P^{aK+b}} f \]
\[ \leq 2\alpha^K V_{I^n}(P^{(a-1)K+b} f) + \gamma_M \|f\|_1 \]
\[ \vdots \]
\[ \leq (2\alpha^K)^a V_{I^n}(P^b f) + ((2\alpha^K)^{a-1} + \cdots + 2\alpha^K + 1) \gamma_M \|f\|_1 \]
\[ \leq (2\alpha^K)^a (2\alpha^b V_{I^n} f + \gamma_M \|f\|_1) + ((2\alpha^K)^{a-1} + \cdots + 1) \gamma_M \|f\|_1 \]
\[ \leq (2\alpha^K)^a \cdot (2\alpha^b) V_{I^n} f + \frac{\gamma_M}{1 - 2\alpha^K} \|f\|_1. \]
Let \( \gamma_o = \frac{\gamma_M}{1 - 2\alpha K} \) and \( \beta = 2^b \alpha. \) Then, \( \beta^b = 2^b \alpha^b \geq \alpha^b \) and \( \beta^K = 2^a K. \) Therefore,

\[
V_I P_T^N f \leq (2^a K)^a \cdot (2^b) V_I f + \frac{\gamma_M}{1 - 2^a K} \| f \|_1
\]

\[
\leq (\beta^K)^a \cdot (2^b) V_I f + \gamma_o \| f \|_1
\]

\[
= 2^b V_I f + \gamma_o \| f \|_1.
\]

Thus, both \( \gamma_o \) and \( \beta \) are independent of \( N. \) For big enough \( N, \) \( 2^b N < 1 \) and

\[
V_I P_T^N f \leq 2^b V_I f + \gamma_o \| f \|_1.
\]

\[
3.2 \quad \text{Existence of Absolutely Continuous Invariant Measure}
\]

In this section, we present the main result on Theorem 3.12, which gives the existence of absolutely continuous invariant measure (ACIM). First, we state the mean ergodic theorem and a useful corollary in Dunford and Schwartz [14, VIII 5.1 and 5.3].

**Theorem 3.9.** [14] (mean ergodic theorem)

Suppose for every integer \( N \geq 1, A_N = \frac{1}{N} \sum_{s=0}^{N-1} P^s, \) where \( A_N \) is the average of the iterates of the bounded operator \( P \) on the Banach space \( X. \) If the sequence of operators \( \{A_N\} \) is uniformly bounded, then the set of those points \( x \) in \( X \) for which the sequence \( \{A_N x\} \) is weakly sequentially compact is a closed linear subspace consisting of all vectors \( x \) for which the set \( \{A_N x\} \) is weakly sequentially compact and \( \lim_{N \to \infty} \frac{1}{N} P^N x = 0. \)

Note that a set \( B \) is called **fundamental**, in [14], if the closure of the subspace spanned by \( B \) is equal to the whole space \( X. \)

**Corollary 3.10.** [14] Assume \( A_N \) is defined as above. If the sequence \( \{A_N\} \) is uniformly bounded, then it converges in the strong operator topology if and only if the sequence \( \frac{1}{N} P^N x \) converges to zero for every \( x \) in a fundamental set and the sequence \( \{A_N x\} \) is weakly sequentially compact for every \( x \) in a fundamental set.

We intend to apply the mean ergodic theorem and Corollary 3.10 to \( P = P_T \) in order to conclude convergence of the sequence \( \{\frac{1}{N} \sum_{s=0}^{N-1} P_T^s f\}. \)

**Proposition 3.11.** Let \( T = \{\tau_k; \ p_k\}_{k=1}^{P} \) be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise \( C^2 \) Jabłoński transformations. Let \( P_T \) be the Perron-Frobenius operator with respect to \( T. \) Then, for each \( f \in L^1(I^n), \)

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{s=0}^{N-1} P_T^s f
\]

exists in \( L^1(I^n). \)
Proof. By the weak Lasota-Yorke inequality, in Lemma 3.8, there exist constants $0 < \beta < 1$ and $\gamma_o < \infty$ such that for all $N \in \mathbb{N}$ and for all $f \in BV(I^n)$,

$$V_{I^n} P_T^N f \leq 2\beta^N V_{I^n} f + \gamma_o \|f\|_1.$$  

Let $A_N = \frac{1}{N} \sum_{s=0}^{N-1} P_{T_s}$ act on $L^1(I^n)$. Then for each $f \in BV(I^n)$,

$$\|A_N f\|_1 \leq \frac{1}{N} \sum_{s=0}^{N-1} \|P_{T_s} f\|_1 \leq \|f\|_1$$

and

$$V_{I^n} (A_N f) \leq \frac{1}{N} \sum_{s=0}^{N-1} (2\beta^s V_{I^n} f) + \gamma_o \|f\|_1 \leq \frac{2}{N(1-\beta)} V_{I^n} f + \gamma_o \|f\|_1.$$  

Therefore, there exist constants $L, M < \infty$ such that

$$\{A_N f\} \subset \{g \in L^1(I^n) : \|g\|_1 \leq \|f\|_1 = L, V_{I^n} g \leq M\}$$

which is weakly sequentially compact by Lemma 2.26. We also have

$$\left\| \frac{1}{N} P_T^N f \right\|_1 = \frac{1}{N} \|P_T^N f\|_1 \leq \frac{1}{N} \|f\|_1 \to 0.$$  

Since $BV \subset L^1$ is a dense linear subspace, $BV$ is a fundamental subset in $L^1$. Thus, applying Corollary 3.10 to $X = L^1(I^n)$ and $P = P_T$, we have that $A_N$ converges in the strong operator topology on $L^1(I^n)$, that is, for every $f$ in $L^1(I^n)$,

$$f^* = \lim_{N \to \infty} A_N f$$  

exists, where the limit is taken in $L^1$-norm. \hfill \Box

Now we present the important theorem, which guarantees that $P_T$ has at least one invariant density $f^*$. In other words, the result of the following theorem implies the existence of an absolutely continuous invariant measure, ACIM, (see Lasota and York [30]) for a random dynamical system $T$. Theorem 3.12 was established by Boyarsky and Lou [10], and it was proved under the condition of $\alpha < \frac{1}{3}$ in Equation (3.5), instead of $\alpha < 1$. In this article we prove Theorem 3.12 without assuming $\alpha < \frac{1}{3}$ and prove it by using the weak Lasota-Yorke inequality, described in Proposition 3.6.
Theorem 3.12. Let $T = \{\tau_k; p_k\}_{k=1,...,q}$ be a random dynamical system satisfying
Equation (3.5) which is constructed from piecewise $C^2$ Jabłoński transformations.
Then for each $L^1$ function $f$,

(a) there exists $f^* \in L^1$, $\lim_{N \to \infty} \frac{1}{N} \sum_{s=0}^{N-1} P_T^s f = f^*$.

(b) $P_T f^* = f^*$.

(c) If $f$ also satisfies $f \in BV(I^n)$, then $f^* \in BV(I^n)$.

Proof. (a) Follows from Proposition 3.11.

(b) Given $\epsilon > 0$, by (a), there exists $N_o \in \mathbb{N}$ such that for all $N \geq N_o$,

$$\left\| \frac{1}{N} \sum_{s=0}^{N-1} P_T^s f - f^* \right\|_1 < \epsilon.$$

That is, $P_T f^* = f^*$.

(c) By the calculation in the proof of Proposition 3.11, we know that if $f \in BV(I^n)$,
then there exist constants $L, M < \infty$ such that

$$\{A_N f\} \subset \{g \in L^1(I^n) : \|g\|_1 \leq \|f\|_1 = L, V_I g \leq M\} \equiv \mathcal{D}.$$

For $\mathcal{D}$ is a weakly sequentially compact subset of $L^1(I^n)$, and the $L^1$- limit of
$A_N f$ must be in $\mathcal{D}$. So does $f^*$ in $\mathcal{D} \subset BV(I^n)$.

$\Box$
Chapter 4

The Unique Invariant Density of a Markov Operator $P_T$

In Chapter 2, the Perron-Frobenius operator $P_\tau$ is the predual of the Koopman operator $K_\tau$. By Proposition 2.8 (b),

$$P_\tau f = f \text{ if and only if } d\mu = f dm_\tau \text{ is } \tau\text{-invariant}.$$  

This means, finding an ACIM (absolutely continuous invariant measure) is equivalent to finding an invariant density of $P_\tau$. In Chapter 3, for a random dynamical system $T = \{\tau_k; \ p_k\}_{k=1,\ldots,q}$, we have

$$P_T = \sum_{k=1}^{q} p_k P_{\tau_k}.$$  

In Theorem 3.12, it guarantees that there exists an eigenfunction corresponding to eigenvalue 1 for $P_T$, so there are three natural questions to ask:

1. What is the relation between an invariant density of a random dynamical system and its individual transformations? Theorem 4.50 and Theorem 5.7 would answer the question.

2. When can we get a unique invariant density? Theorem 4.46 shows how to get a unique ACIM under a single transformation, and Proposition 4.52 shows the way to get a unique ACIM of a random dynamical system.

3. Assuming the random dynamical system $T$ has a unique ACIM, how can we compute or approximate it? Theorem 5.28 discusses this point.

In this chapter, we focus on a unique solution of $P_T f = f$. In the last section, we show the uniqueness under a special case that consider the random dynamical system constructed from piecewise linear Markov transformations.

4.1 Constrictive Operator

We introduce the concept of being constrictive, in [28, 29], and consider the Spectral Decomposition Theorem for a constrictive Markov operator. The Spectral Decomposition Theorem plays an essential role to get uniqueness of the invariant density.
Definition 4.1. (Markov operator and constrictor)

(1) Let \( \mathcal{D} \) be the set of all densities in \( L^1 = L^1(I^n) \).
\[
\mathcal{D} = \{ f \in L^1 : f \geq 0, \|f\|_1 = 1 \}.
\]
A linear operator \( P : L^1 \to L^1 \) is called Markov if \( P(\mathcal{D}) \subset \mathcal{D} \).
Note: the Perron-Frobenius operator is a special case of a Markov operator.

(2) An operator \( P : L^1 \to L^1 \) is strongly constrictive if there exists a strongly compact (i.e. norm compact) set \( \mathcal{F} \subset L^1 \) such that for each \( f \in \mathcal{D} \),
\[
\lim_{N \to \infty} \left( \inf_{g \in \mathcal{F}} \|P^N f - g\|_1 \right) = 0.
\]

(3) An operator \( P : L^1 \to L^1 \) is weakly constrictive if there exists a weakly compact set \( \mathcal{F} \subset L^1 \) such that for each \( f \in \mathcal{D} \),
\[
\lim_{N \to \infty} \left( \inf_{g \in \mathcal{F}} \|P^N f - g\|_1 \right) = 0.
\]
In either case, the set \( \mathcal{F} \) is referred to as a constrictor.

The following lemma is from Boyarsky and Levesque [8, Theorem 1].

Lemma 4.2. [8] Let \( P : L^1 \to L^1 \) be a Markov operator. If \( P \) is weakly constrictive, then it is strongly constrictive.

The next proposition gives us a concrete idea about constrictive operators and piecewise \( C^2 \) Jabłoński transformations satisfying Equation (3.5), in [22, Theorem 4.2].

Proposition 4.3. Let \( T = \{\tau_k; p_k\}_{k=1,\ldots,q} \) be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise \( C^2 \) Jabłoński transformations, then the operator \( P_T \) is constrictive.

Proof. By Lemma 3.8, there exist \( \beta < 1, \gamma_o < \infty \) such that for all \( f \in BV(I^n) \) and for all \( N \in \mathbb{N} \),
\[
V_{I^n} P_T^N f \leq 2\beta^N V_{I^n} f + \gamma_o \|f\|_1.
\]
Let \( \mathcal{F} = \{ g \in \mathcal{D} : V_{I^n} g \leq 2\gamma_o \} \) be weakly compact by Lemma 2.26. Then, for all \( f \in BV(I^n) \cap \mathcal{D} \), we have
\[
\limsup_{N \to \infty} V_{I^n} P_T^N f \leq \gamma_o.
\]
Thus, for sufficiently large \( N_o = N_o(f) \) and for all \( N \geq N_o \),
\[
P_T^N f \in \mathcal{F}.
\]
Since $BV(I^n) \cap \mathcal{D}$ is dense in $\mathcal{D}$, for every $f \in \mathcal{D}$ and $\epsilon > 0$, we may choose $\tilde{f} \in BV(I^n) \cap \mathcal{D}$ and $N_o \in \mathbb{N}$ such that for all $N \geq N_o$ implies $P_T^N \tilde{f} \in \mathcal{F}$. Hence, for all $N \geq N_o$,
\[
\inf_{g \in \mathcal{F}} \|P_T^N f - g\|_1 \leq \|P_T^N f - P_T^N \tilde{f}\|_1 \leq \|f - \tilde{f}\|_1 < \epsilon.
\]
Therefore, $P_T : L^1 \to L^1$ is weakly (and hence strongly, by Lemma 4.2) constrictive.

Next, we consider the Spectral Decomposition Theorem, in Lasota and Mackey [29]. One can see the proof in Komornik and Lasota [26], Lasota, Li and Yorke [28] or Theorem A.24 in Appendix.

**Theorem 4.4. (Spectral Decomposition Theorem)**
Let $P : L^1 \to L^1$ be a constrictive Markov operator. There are two finite sets of nonnegative functions $\{g_i\}_{i=1}^r \subset \mathcal{D}$ and $\{h_i\}_{i=1}^r \subset L^\infty(I^n)$ for some $r \in \mathbb{N}$, and an operator $Q : L^1 \to L^1$ such that for all integers $N \geq 0$,
\[
P^N f = \sum_{i=1}^r \lambda_i(f) g_{\sigma^N(i)} + P^N Qf, \tag{4.1}
\]
where $\lambda_i(f) = \int_{I^n} fh_i dm_n$ with $\|h_i\|_\infty = 1$. The functions $g_i$’s and the operator $Q$ have the following properties:

(a) For all $i \neq j$, $i, j = 1, \cdots, r$, $g_i(x)g_j(x) = 0$ a.e. Hence, all $g_i$’s in Equation (4.1) have disjoint support up to measure.

(b) For each $i = 1, \cdots, r$, there exists a unique integer $\sigma(i)$ such that $Pg_i = g_{\sigma(i)}$, where $\sigma$ is a permutation on the numbers $\{1, 2, \cdots, r\}$.

(c) For each $f \in L^1$, $\|P^N Qf\|_1 \to 0$ as $N \to \infty$.

The results of the following lemma follow directly from the Spectral Decomposition Theorem 4.4.

**Lemma 4.5.** (a) For all $i, j = 1, \cdots, r$, $\lambda_i(g_j) = \begin{cases} 1, & i = j \\ 0, & \forall i \neq j \end{cases}$.

(b) For each $f \in \mathcal{D}$,
\[
\sum_{i=1}^r \lambda_i(f) = 1.
\]

(c) (1) If $f \in L^1$, then for each $i$, $|\lambda_i(f)| \leq \|f\|_1$.

(2) If $f \in L^1$ and $f \geq 0$, then $0 \leq \lambda_i(f) \leq \|f\|_1$.

(3) If $f \in L^1$ and $Pf = f$, then
\[
f = \sum_{i=1}^r \lambda_i(f) g_i.
\]
(d) From the representation on Equation (4.1), \( Q \) is automatically determined by

\[
Q f(x) = f(x) - \sum_{i=1}^{r} \lambda_i(f)g_i.
\]

Moreover, for each \( i = 1, \cdots, r \),

\[
Qg_i = 0.
\]

Proof. (a) Let \( l \) be the order of \( \sigma \), that is, \( \sigma^l = \text{id} \). By Theorem 4.4 (b), for all \( i = 1, \cdots, r \), \( P_i g_i = g_{\sigma(i)} \). Then \( P^l g_i = g_{\sigma^l(i)} = g_i, \ \forall i \). Let \( N \in \mathbb{N} \) be arbitrary and fix \( j \),

\[
g_j = P^N g_j = \sum_{i=1}^{r} \lambda_i(g_j)g_i + P^N Q(g_j).
\]

Thus,

\[
(1 - \lambda_j(g_j)) g_j = \sum_{i=1, i \neq j}^{r} \lambda_i(g_j)g_i + P^N Q(g_j).
\]

Let \( N \to \infty \) and apply Theorem 4.4 (c),

\[
(1 - \lambda_j(g_j)) g_j = \sum_{i=1, i \neq j}^{r} \lambda_i(g_j)g_i. \tag{4.2}
\]

By Theorem 4.4 (a), \( 1 - \lambda_j(g_j) = 0 \) and \( \lambda_i(g_j) = 0 \) for all \( i \neq j \).

(b) Assume \( \sigma^l = \text{id} \). By part (d), for all \( f \in \mathfrak{D} \) and any \( N \in \mathbb{N} \),

\[
P^N f = \sum_{i=1}^{r} \lambda_i(f)g_{\sigma^N(i)} + P^N Qf = \sum_{i=1}^{r} \lambda_i(f)g_i + P^N Qf.
\]

Integrate both sides on \( I^n \),

\[
1 = \int_{I^n} f dm_n = \int_{I^n} P^N f dm_n
\]

\[
= \sum_{i=1}^{r} \lambda_i(f) \int_{I^n} g_i dm_n + \int_{I^n} P^N Qf dm_n;
\]

\[
\implies 1 - \sum_{i=1}^{r} \lambda_i(f) = \int_{I^n} P^N Qf dm_n.
\]
Hence,
\[
\left| 1 - \sum_{i=1}^{r} \lambda_i(f) \right| = \left| \int_{I^n} P^N Q f \, d\mu \right| \leq \int_{I^n} |P^N Q f| \, d\mu \\
= \|P^N Q f\|_1 \longrightarrow 0.
\]

Hence, for every density \( f \), \( \sum_{i=1}^{r} \lambda_i(f) = 1 \).

(c) (1) By Hölder inequality,
\[
|\lambda_i(f)| \leq \int_{I^n} |fh_i| \leq \|f\|_1 \|h_i\|_{\infty} = \|f\|_1.
\]

(2) If \( f \geq 0 \), then by part (1),
\[
\|f\|_1 \geq \int_{I^n} |fh_i| \geq \int_{I^n} fh_i = \lambda_i(f) \geq 0.
\]

(3) Since \( P^N f = \sum_{i=1}^{r} \lambda_i(f)g_{\sigma^i} + P^N Q f \) and \( Pf = f \),
\[
\left| f - \sum_{i=1}^{r} \lambda_i(f)g_i \right| = \left| f - \sum_{i=1}^{r} \lambda_i(f)g_{\sigma^i} \right| \\
\leq \|P^N Q f\|_1 \longrightarrow 0.
\]

(d) For all \( f \in L^1 \), the equation \( Qf(x) = f(x) - \sum_{i=1}^{r} \lambda_i(f)g_i \) follows when \( N = 0 \). Moreover, for each \( g_i \), by part (a),
\[
Qg_i = g_i - \sum_{j=1}^{r} \lambda_j(g_i)g_j = g_i - g_i = 0.
\]

By the Spectral Decomposition Theorem, the following proposition implies that a constrictive Markov operator has an invariant density, in [8].

**Proposition 4.6.** Let \( P : L^1 \to L^1 \) be a constrictive Markov operator. Then \( P \) has an invariant density \( f^* \).

\[
f^* = \frac{1}{r} \sum_{i=1}^{r} g_i.
\]

**Proof.** Let \( f^* = \frac{1}{r} \sum_{i=1}^{r} g_i \neq 0 \). By Theorem 4.4 (b), for \( i = 1, \ldots, r \), we have
\( P g_i = g_{\sigma(i)} \). Therefore,

\[
P f^* = \frac{1}{r} \sum_{i=1}^{r} P g_i = \frac{1}{r} \sum_{i=1}^{r} g_{\sigma(i)} = \frac{1}{r} \sum_{i=1}^{r} g_i = f^*.
\]

The following theorem is from [29, Theorem 5.5.1]. We give the proof without assuming \( P 1 = 1 \).

**Theorem 4.7.** Let \( P : L^1 \rightarrow L^1 \) be a constrictive Markov operator. Then \( P \) has a unique invariant density, \( \frac{1}{r} \sum_{i=1}^{r} g_i \), if and only if the permutation \( \sigma \) is cyclic.

**Proof.** **Claim 1:** Uniqueness implies \( \sigma \) is cyclic.

If \( \sigma \) is not cyclic, then by the cycle decomposition theorem, \( \sigma \) is a product of disjoint cycles. i.e. \( \sigma = \sigma_1 \circ \sigma_2 \cdots \sigma_{\tilde{r}} \), where \( \sigma_1, \sigma_2, \cdots, \sigma_{\tilde{r}} \) are cyclic with order \( l_1, \cdots, l_{\tilde{r}} \) respectively and \( r \geq \tilde{r} \geq 2 \). Without lost of generality, suppose \( \tilde{r} = 2 \). For \( J_1 = \{ i : \sigma^l_1(i) = i \} \), \( J_2 = \{ j : \sigma^l_2(j) = j \} \). Define

\[
f_1 = \frac{1}{l_1} \sum_{i \in J_1} g_i \quad \text{and} \quad f_2 = \frac{1}{l_2} \sum_{j \in J_2} g_j.
\]

Then,

\[
P f_1 = \frac{1}{l_1} \sum_{i \in J_1} g_{\sigma(i)} = \frac{1}{l_1} \sum_{i \in J_1} g_{\sigma_1(i)} = \frac{1}{l_1} \sum_{i \in J_1} g_i = f_1,
\]

and

\[
P f_2 = \frac{1}{l_2} \sum_{j \in J_2} g_{\sigma(j)} = \frac{1}{l_2} \sum_{j \in J_2} g_{\sigma_2(j)} = \frac{1}{l_2} \sum_{j \in J_2} g_j = f_2.
\]

Therefore, \( P \) has more than one invariant density. It is a contradiction. Thus, \( \sigma \) should be cyclic.

**Claim 2:** If \( \sigma \) is cyclic, then \( P \) has a unique invariant density.

First, we prove that if \( \sigma \) is cyclic and \( f^* \) is an invariant density, then each

\[
\lambda_i(f^*) > 0, \ i = 1, \cdots, r.
\]

Since \( \sum_{i=1}^{r} \lambda_i(f^*) = 1 \), there exists some \( i_o \) such that \( \lambda_{i_o}(f^*) > 0 \). Fix \( j \in \{1, 2, \cdots, r\} \) and pick \( K \) such that \( \sigma^K(i_o) = j \). Then by Lemma 4.5; (c),

\[
\sum_{i=1}^{r} \lambda_i(f^*) g_{\sigma^K(i)} = P^K f^* = f^* = \sum_{i=1}^{r} \lambda_i(f^*) g_i.
\]

For the disjoint support of \( g_i \)’s, \( \lambda_j(f^*) g_j = \lambda_{i_o}(f^*) g_{\sigma^K(i_o)} = \lambda_{i_o}(f^*) g_j \). Hence \( \lambda_j(f^*) = \lambda_{i_o}(f^*) > 0 \). Since \( j \) is arbitrary, \( \lambda_j(f^*) > 0 \) for all \( j \). That is

\[
\text{supp } f^* = I^n \text{ mod zero}.
\]
Suppose there are two invariant densities, \(f^*\) and \(g^*\). Then both \(\text{supp} f^* = I^n\) and \(\text{supp} g^* = I^n \mod \text{zero}\). By Lemma 4.16, if \(f^*\) and \(g^*\) are linearly independent, then there exist two densities with disjoint support. It is a contradiction. Hence \(P\) has a unique invariant density.

**Corollary 4.8.** Use the same notation as in the Spectral Decomposition Theorem. Then the decomposition in Theorem 4.4 is unique up to the permutation of \(g_i\)’s.

**Proof.** Suppose we have two spectral decompositions for \(P\) with respect to functions \(g_i, \lambda_i\ i = 1, 2, \ldots r\) and \(G_j, \Lambda_j, j = 1, 2, \ldots R\). Let \(N\) and \(M\) be the orders of \(\sigma\) for \(g_i\) and \(G_j\) respectively, and let \(K = NM\). By considering powers \(tK\), for each \(j = 1, 2, \ldots R\) representing \(G_j\) in terms of the \(g_i\), we have

\[
G_j = \sum_{i=1}^{r} \lambda_i(G_j)g_i
\]  

(4.3)

For each such \(j\), let \(I_j = \{1 \leq i \leq r : \lambda_i(G_j) \neq 0\}\). By disjointness of the supports of \(G_j\), we have that \(\text{supp} \lambda_{j1} \cap \text{supp} \lambda_{j2} = \emptyset \mod \text{zero}\) if \(j_1 \neq j_2\). Therefore, the number of \(G_j\) cannot exceed the number of \(g_i\), in other words, \(R \leq r\). Reversing the argument (starting with \(g_i\)) we obtain \(r \leq R\). Hence \(R = r\). For each \(j\), \(I_j\) is a single index \(\{i_j\}\), that is

\[
G_j = \lambda_{i_j}(G_j)g_{i_j}
\]  

(4.4)

Keeping in mind that \(\sum_i \lambda_i(G_j) = 1\), we get

\[
G_j = g_{i_j}
\]  

(4.5)

Thus, the spectral decomposition is unique, up to a permutation of the \(g_i\). \(\square\)

### 4.2 Invariant Density and its Support

We introduce some basic properties of an invariant density and its support from Lemma 4.10 to Lemma 4.16. Here we extend the concept from one dimension into \(n\) dimensions. We say a set \(A\) is forward invariant under \(\tau\) if \(A \in \mathcal{B}_I^n\) and \(\tau(A) = A \mod \text{zero}\). i.e. \(m_n(\tau(A) \Delta A) = 0\), where \(\tau(A) \Delta A = (\tau(A) \setminus A) \cup (A \setminus \tau(A))\).

**Lemma 4.9.** Let \(\tau : I^n \to I^n\) be a piecewise 1–1 and \(C^1\) transformation. Then \(\tau\) maps nullsets into nullsets.

**Proof.** First, in Apostol [1, Section 11.32], we have transformation formula for \(n\) variables:

\[
\int_{\tau(A)} f dm_n = \int_A (f \circ \tau) |J_\tau| dm_n.
\]

In particular, if \(f = \chi_{I^n} \equiv 1\), then

\[
\int_{\tau(A)} 1 dm_n = \int_A |J_\tau| dm_n.
\]  

(4.6)
Since \( \tau \) is piecewise \( 1 - 1 \) and \( C^1 \), define \( \tau_j = \tau|_{D_j} \), where \( D_j \) is the element in the partition \( \mathcal{P} = \{ D_j \}_{j=1}^m \). Take a nullset \( N \) and let \( N_j = N \cap D_j \), so \( N = \bigcup_{j=1}^m N_j \). Each \( N_j \) is also a nullset. By Equation (4.6), for each \( j = 1, \ldots, m \),

\[
m_n(\tau_j N_j) = \int_{\tau_j(N_j)} 1(x) dm_n(x) = \int_{N_j} |J_{\tau_j}(y)| dm_n(y) = 0.
\]

Therefore, \( m_n(\tau N) = m_n(\bigcup_{j=1}^m \tau_j N_j) \leq \sum_{j=1}^m m_n(\tau_j N_j) = 0 \). Thus the transformation \( \tau \) maps nullsets into nullsets.

**Lemma 4.10.** [32] Let \( \tau : I^n \to I^n \) be a piecewise \( C^2 \) nonsingular transformation. For every density \( f \) with \( P_\tau f = f \), define \( S = \text{supp} f = \{ x \in I^n : f(x) > 0 \} \). Then \( S \) is a forward invariant set under \( \tau \).

**Proof.** For an invariant density \( f \), let \( S = \text{supp} f \) and denote \( S^c \) as the complement of \( S \), then \( f = 0 \) on \( S^c \). For \( P_\tau f = f \),

\[
1 = \int_S f dm_n = \int_S P_\tau f dm_n = \int_{\tau^{-1}(S)} f dm_n
= \int_{S \cap \tau^{-1}(S)} f dm_n + \int_{S \cap \tau^{-1}(S)} f dm_n = \int_{S \cap \tau^{-1}(S)} f dm_n + 0.
\]

Thus,

\[
\int_{S \setminus (S \cap \tau^{-1}(S))} f dm_n = \int_S f dm_n - \int_{S \cap \tau^{-1}(S)} f dm_n = 0.
\]

Since \( f > 0 \) on \( S \),

\[
\int_{S \setminus (S \cap \tau^{-1}(S))} f dm_n = 0 \implies m_n(S \setminus (S \cap \tau^{-1}(S))) = 0,
\]

so \( m_n(S) = m_n(S \cap \tau^{-1}(S)) \). That is, \( S \subset (\tau^{-1}(S) \cup N_1) \). By Lemma 4.9, \( \tau \) maps a nullset into a nullset, so we have \( \tau(S) \subset (\tau \circ \tau^{-1}(S) \cup \tau(N_1)) \subset (S \cup \tau(N_1)) \equiv S \cup N_2 \).

On the other hand,

\[
\int_{\tau(S)} f dm_n = \int_{\tau(S) \cap S} P_\tau f dm_n = \int_{\tau^{-1}(\tau(S) \cap S)} f dm_n
\geq \int_{S \cap \tau^{-1}(S)} f dm_n = \int_S f dm_n = 1.
\]

Thus, \( (\tau(S) \cup N_3) \supset S \). Therefore,

\[
\tau(S) \cup N_4 \equiv (\tau(S) \cup N_3) \cup N_2 \supset (S \cup N_2) \supset \tau(S).
\]

Hence, \( S = \tau(S) \) mod zero.
Lemma 4.11. Let $f$ be an invariant density of $P_\tau$ with $S = \text{supp } f$.
For $d\mu = f dm_n$,
\[ \mu(\tau^{-1}(S) \Delta S) = 0. \]

Proof. Since $P_\tau f = f$ and $S = \text{supp } f$,
\[ \int_{\tau^{-1}(S)} f dm_n = \int_S P_\tau f dm_n = \int_S f dm_n = 1. \]
Hence, $\tau^{-1}(S) \supset S$ mod zero. Since $\mu$ is $\tau$-invariant from Proposition 2.8 (b),
\[ \mu(\tau^{-1}(S) \Delta S) = 0. \]

Lemma 4.12. If $f$ is an invariant function of $P_\tau$, then $|f|$ is also invariant.

Proof. Since $P_\tau f = f$, $|f| = |P_\tau f| \leq P_\tau |f|$. By Proposition 2.9 (a), for $1 = \chi_{I^n}$,
\[ \int |f| \cdot 1 dm_n \leq \int (P_\tau |f|) \cdot 1 dm_n = \int |f| \cdot (K_\tau 1) dm_n \]
\[ = \int |f| \cdot 1 dm_n. \]
Thus, $\int (P_\tau |f| - |f|) = 0$, so $P_\tau |f| = |f|$ a.e. \hfill \Box

Lemma 4.13. [32] Let $f_1$ and $f_2$ be invariant densities of $P_\tau$, and $S_i$ be the support of $f_i$ denoted as $S_i = \text{supp } f_i$, $i = 1, 2$. Then

(a) $S_1$ and $S_2$ are forward invariant under $\tau$;
(b) $S_1 \cap S_2 = \text{supp } (\min\{f_1, f_2\})$.
(c) $S_1 \cup S_2 = \text{supp } (\max\{f_1, f_2\})$.
(d) $S_1 \cap S_2$ and $S_1 \cup S_2$ are forward invariant under $\tau$.

Proof. (a) Follows from Lemma 4.10.

(b) Let $g \equiv \min\{f_1, f_2\}$. For all $x$ in $S_1 \cap S_2$, $f_1(x) > 0$ and $f_2(x) > 0$. Then we have $g(x) > 0$. That is, $S_1 \cap S_2 \subset \text{supp } g$. On the other hand, for all $x$ in $\text{supp } g$, both $f_1(x) > 0$ and $f_2(x) > 0$. Thus, $\text{supp } g \subset S_1 \cap S_2$. Therefore,
\[ S_1 \cap S_2 = \text{supp } (\min\{f_1, f_2\}). \]

(c) Let $h \equiv \max\{f_1, f_2\}$. Use the same argument as part (b) to get the result.
(d) Let \( g \equiv \min\{f_1, f_2\} = \frac{1}{2}(f_1 + f_2 - |f_1 - f_2|) \). For both \( f_1 \) and \( f_2 \) are invariant functions, so are \( f_1 + f_2 \) and \( f_1 - f_2 \). By Lemma 4.12, \( |f_1 - f_2| \) is also an invariant function. Hence,

\[
P g = \frac{1}{2} (P(f_1 + f_2) - P(|f_1 - f_2|)) = \frac{1}{2} (f_1 + f_2 - |f_1 - f_2|) = g.
\]

From part (b), \( S_1 \cap S_2 = \text{supp} \ g \). Let \( g^* = \frac{g}{\|g\|_1} \). Both \( f_1 \geq 0 \) and \( f_2 \geq 0 \) imply \( g \geq 0 \). Thus we get \( g^* \geq 0, \|g^*\|_1 = 1 \) and \( \text{supp} \ g^* = \text{supp} \ g \). Therefore, \( S_1 \cap S_2 \) is a forward invariant set by Lemma 4.10. Similarly, let \( h \equiv \max\{f_1, f_2\} = \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|) \). Then

\[
Ph = \frac{1}{2} (P(f_1 + f_2) + P(|f_1 - f_2|)) = \frac{1}{2} (f_1 + f_2 + |f_1 - f_2|) = h.
\]

Let \( h^* = \frac{h}{\|h\|_1} \). By part (c) and Lemma 4.10, \( S_1 \cup S_2 \) is also forward invariant.

Note that even though both \( S_1 \) and \( S_2 \) are forward invariant sets, the difference of two forward invariant sets is not necessary forward invariant. The following example shows the differences between two forward invariant sets may not be forward invariant.

![Figure 4.1: \( \tau : I_1 \cup I_2 \cup I_3 \to [0, 1] \)](image)

**Example 4.14.** Let \( \tau : [0, 1] \to [0, 1] \) be defined by

\[
\tau(x) = \begin{cases} 
2x, & \forall x \in [0, \frac{1}{4}) = I_1 \\
x, & \forall x \in [\frac{1}{4}, \frac{1}{2}) = I_2 \\
-\frac{3}{2}x + \frac{7}{4}, & \forall x \in [\frac{1}{2}, 1] = I_3 
\end{cases}
\]

Let \( S_1 = I_1 \cup I_2 \) and \( S_2 = I_2 \cup I_3 \). Then both \( S_1 \), \( S_2 \) are forward invariant, but neither \( S_1 \setminus S_2 \) nor \( S_2 \setminus S_1 \) are forward invariant.
Proof. Since \( \tau(I_1) = I_1 \cup I_2, \tau(I_2) = I_2 \) and \( \tau(I_3) = I_2 \cup I_3 \),

\[
\begin{align*}
\text{m}(\tau(S_1) \Delta S_1) &= \text{m}(\tau(S_2) \Delta S_2) = \text{m}((I_1 \cup I_2) \Delta S_1) = 0 \\
\text{m}(\tau(S_2) \Delta S_2) &= \text{m}((I_2 \cup I_3) \Delta S_2) = 0.
\end{align*}
\]

Similarly, \( \text{m}(\tau(S_2 \setminus S_1) \Delta (S_2 \setminus S_1)) = \frac{1}{4} > 0 \). Therefore, neither \( S_1 \setminus S_2 \) nor \( S_2 \setminus S_1 \) are forward invariant.

\[ \square \]

**Remark 4.15.** In this example, \( P_{\tau} \) has at least two invariant functions. For example,

\[ g(x) = \chi_{(1,1/2)}(x) \]

and

\[ h(x) = x \cdot \chi_{(1/2,1)}(x). \]

Hence, \( P_{\tau}g = g \) and \( P_{\tau}h = h \). Moreover, \( \text{supp } g = \text{supp } h = I_2 \) is a forward invariant set. This is consistent with Lemma 4.10.

**Lemma 4.16.** [32] Let \( f_1 \) and \( f_2 \) be invariant densities of \( P_{\tau} \) and \( \tau : I^n \to I^n \) be a piecewise \( C^2 \) nonsingular transformation. If \( f_1 \) and \( f_2 \) are linearly independent, then there exist invariant densities \( g_1^* \) and \( g_2^* \) with disjoint support.

**Proof.** Let \( g_1 = \max\{(f_1 - f_2), 0\} \) and \( g_2 = \max\{-(f_1 - f_2), 0\} \). Then

\[
\begin{align*}
g_1 &\geq 0, \quad g_1 &= \frac{1}{2}(|f_1 - f_2| + f_1 - f_2); \\
g_2 &\geq 0, \quad g_2 &= \frac{1}{2}(|f_2 - f_1| + f_2 - f_1).
\end{align*}
\]

Since both \( f_1, f_2 \) are invariant, \( f_1 - f_2 \) is invariant. By Lemma 4.12, \( |f_1 - f_2| \) is also invariant. Hence

\[
P g_1 = \frac{1}{2} P(|f_1 - f_2| + f_1 - f_2) = \frac{1}{2}(|f_1 - f_2| + f_1 - f_2) = g_1.
\]

The same as \( P g_2 = g_2 \). Since \( f_1 \) and \( f_2 \) are linearly independent and

\[
\begin{align*}
S_1 &\equiv \text{supp } g_1 = \{x \in I^n : f_1(x) - f_2(x) > 0\}, \\
S_2 &\equiv \text{supp } g_2 = \{x \in I^n : f_2(x) - f_1(x) > 0\};
\end{align*}
\]

clearly \( S_1 \) and \( S_2 \) are disjoint and nonempty. Set

\[
g_1^* = \begin{cases} \frac{g_1}{\|g_1\|_1}, & g_1 \neq 0 \\ 0, & g_1 = 0 \end{cases} \quad \text{and} \quad g_2^* = \begin{cases} \frac{g_2}{\|g_2\|_1}, & g_2 \neq 0 \\ 0, & g_2 = 0 \end{cases}.
\]

Then, \( g_1^* \) and \( g_2^* \) are invariant densities with disjoint support. \[ \square \]
We introduce the following notation to be used throughout the rest of this article. Let \( \tau : I^n \to I^n \) be nonsingular.

\[
\mathcal{D} = \{ f \in L^1 : f \geq 0, \| f \|_1 = 1 \}.
\]
\[
\mathcal{D}_1 = \{ f \in \mathcal{D} : P_\tau f = f \}.
\]
\[
\mathcal{M} = \{ \mu : \mu \text{ is } \tau - \text{invariant measure} \}.
\]
\[
\mathcal{M}_1 = \{ \mu \in \mathcal{M} : \mu \ll m_n, \mu(I^n) = 1 \}.
\]

**Lemma 4.17.** Sets \( \mathcal{D}_1 \) and \( \mathcal{M}_1 \) are in one-to-one correspondence.

**Proof.** For each \( f \in \mathcal{D}_1 \), let \( d\mu = fdm_n \). By Proposition 2.8 (b), \( \mu \) is \( \tau \)-invariant. Since \( \| f \|_1 = 1 \),
\[
\mu(I^n) = \int_{I^n} d\mu = \int_{I^n} fdm_n = \| f \|_1 = 1.
\]

Thus, \( \mu \in \mathcal{M}_1 \). On the other hand, for each \( \mu \in \mathcal{M}_1 \), \( \mu \ll m_n \). There exists \( f \in L^1 \) such that \( d\mu = fdm_n \). Since \( \mu \) is \( \tau \)-invariant, for any measurable set \( A \),
\[
\int_A fdm_n = \int_A d\mu = \mu(A) = \mu(\tau^{-1}A) = \int_{\tau^{-1}(A)} d\mu = \int_A P_\tau f dm_n.
\]

Thus, \( P_\tau f \overset{a.e.}{=} f \). Since \( \mu(I^n) = 1 \),
\[
\| f \|_1 = \int_{I^n} fdm_n = \int_{I^n} d\mu = \mu(I^n) = 1.
\]

Hence, \( f \in \mathcal{D}_1 \). \( \square \)

For an invariant density and its support, we have the following lemmas. Moreover, if we only consider **ergodic** ACIMs, then there are useful properties about ergodic ACIMs from Proposition 4.23 to Lemma 4.29.

**Lemma 4.18.** Let \( P : L^1 \to L^1 \) be a constrictive Markov operator. Let the Spectral Decomposition of \( P \) be given by Equation (4.1). Assume the order of \( \sigma \) is equal to \( l \). Then

\[
\begin{align*}
(a) & \quad f \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f) g_i \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f) g_{\sigma(i)} \overset{a.e.}{=} \cdots \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f) g_{\sigma^{l-1}(i)}. \\
(b) & \quad \text{For each } j, \text{ either } \text{supp } g_j \subset \text{supp } f \text{ mod zero or } \text{supp } g_j \subset (\text{supp } f)^c \text{ mod zero. Moreover, for } J = \{ j \in \{1, \ldots, r\} : \text{supp } g_j \subset \text{supp } f \text{ mod zero} \}, \text{ supp } f = \bigcup_{j \in J} \text{supp } g_j \text{ mod zero}. \\
(c) & \quad \text{If } \text{supp } g_j \subset \text{supp } f \text{ mod zero, then } \text{supp } g_{\sigma(j)} \subset \text{supp } f \text{ mod zero.}
\end{align*}
\]
Proof. (a) By Lemma 4.5; (c), \( f = \sum_{i=1}^{r} \lambda_i(f)g_i \). Hence

\[
f \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f)g_i = \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f)g_{\sigma(i)} = \cdots = \sum_{i=1}^{r} \lambda_i(f)g_{\sigma^{i-1}(i)}.
\]

(b) For each \( j = 1, \cdots, r \), let \( S_j = \text{supp } g_j \). For almost every \( x \in S_j \),

\[
g_j(x) > 0 \text{ and } g_i(x) = 0, \forall i \neq j.
\]

By Lemma 4.5; (c), \( f \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f)g_i \) and \( \lambda_i(f) \geq 0 \). Hence,

1. if \( \lambda_j(f) > 0 \), then \( f(x) \overset{a.e.}{=} \lambda_j(f)g_j(x) > 0 \). Thus, \( S_j \subset \text{supp } f \) mod zero.
2. if \( \lambda_j(f) = 0 \), then \( f(x) \overset{a.e.}{=} \lambda_j(f)g_j(x) = 0 \). Thus, \( S_j \subset (\text{supp } f)^c \) mod zero.

Going through each \( j = 1, 2, \cdots, r \), we get \( \text{supp } f = \bigcup_{j \in J} \text{supp } g_j \) mod zero.

(c) Let \( x \in \text{supp } g_j = S_j \). Since \( S_j \subset \text{supp } f \), \( g_j(x) > 0 \) and \( f(x) > 0 \). By (a),

\[
f(x) \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f)g_i(x) = \lambda_j(f)g_j(x) > 0
\]

implies \( \lambda_j(f) > 0 \). By (a) again, \( f \overset{a.e.}{=} \sum_{i=1}^{r} \lambda_i(f)g_{\sigma(i)} \). Using the same argument as in part (b), if \( \lambda_j(f) > 0 \), then \( S_{\sigma(j)} \subset \text{supp } f \) mod zero.

\[\text{Lemma 4.19.} \text{ Let } P_\tau : L^1 \rightarrow L^1 \text{ be the Perron-Frobenius operator. For any measurable set } U, \text{ assume } S = \text{supp } P_\tau \chi_U \text{. Then } m_n(\tau(U) \Delta S) = 0.\]

Proof. For \( U \in \mathcal{B}_{I^n} \) and \( S = \text{supp } P_\tau \chi_U \),

\[
m_n(U) = \int_{I^n} \chi_U dm_n = \int_{I^n} P_\tau \chi_U dm_n = \int_S P_\tau \chi_U dm_n = \int_{\tau^{-1}(S) \cap U} \chi_U dm_n = m_n(\tau^{-1}(S) \cap U).
\]

Hence, \( \tau^{-1}(S) \cap U \mod zero \) which implies \( S \supset \tau \circ \tau^{-1}(S) \subset \tau(U) \mod zero \).

\[
\int_{\tau(U)} P_\tau \chi_U dm_n = \int_{\tau^{-1}(\tau(U))} \chi_U dm_n = \int_{U} \chi_U dm_n = m_n(U) = \int_{I^n} P_\tau \chi_U dm_n = \int_S P_\tau \chi_U dm_n.
\]

Thus, \( \tau(U) \supset S \mod zero \). Therefore, \( m_n(\tau(U) \Delta S) = 0. \)
Lemma 4.20. Let $P : L^1 \to L^1$ with $P \mathbf{1} = \mathbf{1}$ be a constrictive Markov operator. Let the Spectral Decomposition of $P$ be given by Equation (4.1). For each $i = 1, 2, \cdots, r$, let $A_i = \text{supp } g_i$, then

$$m_n \left( \tau(A_i) \Delta A_{\sigma(i)} \right) = 0. \tag{4.8}$$

Moreover, $m_n \left( \tau(A_i^c) \Delta A_{\sigma(i)}^c \right) = 0$ and $m_n \left( \tau^{-1}(A_i) \Delta A_{\sigma^{-1}(i)} \right) = 0$.

Proof. For each $i = 1, 2, \cdots, r$, since $P \tau g_i = g_{\sigma(i)}$,

$$\int_{A_i} g_i = 1 = \int_{A_{\sigma(i)}} g_{\sigma(i)} = \int_{A_{\sigma(i)}} P \tau g_i = \int_{\tau^{-1}(A_{\sigma(i)})} g_i.$$

Then, $\tau^{-1}(A_{\sigma(i)}) \supset A_i \mod \text{zero which implies}$

$$A_{\sigma(i)} \supset \tau \circ \tau^{-1}(A_{\sigma(i)}) \supset \tau(A_i) \mod \text{zero.}$$

On the other hand,

$$1 \geq \int_{\tau(A_i)} g_{\sigma(i)} = \int_{\tau(A_i)} P \tau g_i = \int_{\tau^{-1}(A_{\sigma(i)})} g_i \geq \int_{A_i} g_i = \int_{A_{\sigma(i)}} g_{\sigma(i)} = 1.$$

Thus, $\tau(A_i) \supset A_{\sigma(i)} \mod \text{zero.}$ Therefore, for each $i$,

$$m_n \left( \tau(A_i) \Delta A_{\sigma(i)} \right) = 0.$$

Fix $i = 1$. Since $P \tau$ has an invariant density with its support equal to $I^n$ up to measure zero, $I^n = \bigcup_{i=1}^r A_i \mod \text{zero.}$ Then,

$$m_n \left( \tau(A_i^c) \Delta A_{\sigma(i)}^c \right) = m_n \left( \tau \left( \bigcup_{i=2}^r A_i \right) \Delta \left( \bigcup_{i=2}^r A_{\sigma(i)} \right) \right) \leq^{*} m_n \left( \bigcup_{i=2}^r \left( \tau(A_i) \Delta A_{\sigma(i)} \right) \right) \leq \sum_{i=2}^r m_n \left( \tau(A_i) \Delta A_{\sigma(i)} \right) = 0.$$

Note that $\leq^{*}$ because of $\left( \bigcup_i A_i \right) \Delta \left( \bigcup_i B_i \right) \subseteq \bigcup_i (A_i \Delta B_i)$. The above inequality is true for all $i = 1, 2, \cdots, r$, so

$$m_n \left( \tau(A_i^c) \Delta A_{\sigma(i)}^c \right) = 0.$$

Fix $j \in \{1, 2, \cdots, r\}$, $m_n \left( \tau(A_j^c) \setminus A_{\sigma(j)}^c \right) = m_n \left( \tau(A_j^c) \cap A_{\sigma(j)} \right) = 0$. Since $\tau$ is non-
satisfying not only \( S \), but also \( S \) is a permutation, \( j = \sigma^{-1}(i) \) and

\[
0 = m_n \left( A_j^c \cap \tau^{-1}(A_{\sigma(j)}) \right) = m_n \left( A_{\sigma^{-1}(i)}^c \cap \tau^{-1}(A_i) \right).
\]

From the argument in the beginning, \( \tau^{-1}(A_{\sigma(j)}) \supset A_j \mod \text{zero} \). Therefore,

\[
\begin{align*}
m_n \left( \tau^{-1}(A_{\sigma(j)}) \Delta A_j \right) &= m_n \left( \tau^{-1}(A_{\sigma(j)}) \setminus A_j \right) = m_n \left( \tau^{-1}(A_{\sigma(j)}) \cap A_j^c \right) = 0; \\
m_n \left( \tau^{-1}(A_i) \Delta A_{\sigma^{-1}(i)} \right) &= m_n \left( \tau^{-1}(A_i) \setminus A_{\sigma^{-1}(i)} \right) = m_n \left( \tau^{-1}(A_i) \cap A_{\sigma^{-1}(i)}^c \right) = 0.
\end{align*}
\]

By Lemma 4.10 and 4.20, we have a special invariant density \( h \) with its support \( S \) satisfying not only \( m_n(\tau(S)\Delta S) = 0 \) but also \( m_n(\tau^{-1}(S)\Delta S) = 0 \).

**Lemma 4.21.** Let \( P_r : L^1 \to L^1 \) with \( P_r 1 = 1 \) be a constrictive Markov operator satisfying Equation (4.1) in the Spectral Decomposition Theorem. Let \( j_o \in \{1, 2, \cdots, r\} \) with order \( l \) and \( h = \frac{1}{l} \sum_{N=0}^{l-1} \sigma^{N}(j_o) \) with support \( S \). Then

\[
m_n(\tau^{-1}(S)\Delta S) = 0.
\]

**Proof.** By Equation (4.1), for each \( i = 1, 2, \cdots, r \), set \( \text{supp } g_i = A_i \). Then

\[
S = \bigcup_{N=0}^{l-1} A_{\sigma^N(j_o)}.
\]

Since \( \sigma^l(j_o) = j_o, \ \bigcup_{N=0}^{l-1} A_{\sigma^N(j_o)} = \bigcup_{N=0}^{l-1} A_{\sigma^{N+1}(j_o)} \). By Lemma 4.20,

\[
\begin{align*}
m_n \left( \tau^{-1}(S)\Delta S \right) &= m_n \left( \tau^{-1} \left( \bigcup_{N=0}^{l-1} A_{\sigma^N(j_o)} \right) \Delta \left( \bigcup_{N=0}^{l-1} A_{\sigma^N(j_o)} \right) \right) \\
&= m_n \left( \tau^{-1} \left( \bigcup_{N=0}^{l-1} A_{\sigma^{N+1}(j_o)} \right) \Delta \left( \bigcup_{N=0}^{l-1} A_{\sigma^N(j_o)} \right) \right) \\
&\leq m_n \left( \bigcup_{N=0}^{l-1} \left( \tau^{-1}(A_{\sigma^{N+1}(j_o)}) \Delta A_{\sigma^N(j_o)} \right) \right) \\
&\leq \sum_{N=0}^{l-1} m_n \left( \tau^{-1}(A_{\sigma^{N+1}(j_o)}) \Delta A_{\sigma^N(j_o)} \right) = 0.
\end{align*}
\]
Now we introduce the concept of $\tau$-ergodic. If the given measure $\mu$ is ergodic, we have Proposition 4.23.

**Definition 4.22.** (τ-ergodic)
A measure $\mu$ is ergodic with respect to the transformation $\tau$ if for any measurable set $S$ satisfying $\mu(\tau^{-1}(S)\Delta S) = 0$, then $\mu(S) = 0$ or 1.

**Proposition 4.23.** Let $P_\tau : L^1 \to L^1$ with $P_\tau 1 = 1$ be a constrictive Markov operator. Let the Spectral Decomposition of $P$ be given by Equation (4.1). If $d\mu = f d\mu_n$ is $\tau$-ergodic, and $j_o \in \{1, \ldots, r\}$ with the order of $j_o$ equal to $l$ and such that $\text{supp} \ g_{j_o} \subset \text{supp} \ f$, then

$$\text{supp} \ f = \bigcup_{N=0}^{l-1} \text{supp} \ g_{\sigma^N(j_o)} \mod \text{zero}.$$  

**Proof.** For $\text{supp} \ g_{j_o} \subset \text{supp} \ f \equiv S$, by Lemma 4.18 (c), it is clear that

$$A_o \equiv \bigcup_{N=0}^{l-1} \text{supp} \ g_{\sigma^N(j_o)} \subset S \mod \text{zero}.$$  

If $m_n(S \setminus A_o) \neq 0$, then there exists $j_1$ such that

$$\text{supp} \ g_{j_1} \subset (A_o \cap S) \mod \text{zero},$$

with the order of $j_1$ equal to $l_1$. Hence, by Lemma 4.18 (c) again,

$$A_1 \equiv \bigcup_{N=0}^{l_1-1} \text{supp} \ g_{\sigma^N(j_1)} \subset S \mod \text{zero}.$$  

Therefore, $(A_o \cup A_1) \subset S$ and $m_n(A_o \cap A_1) = 0$.

$$A_o = \text{supp} \left( \frac{1}{l} \sum_{N=0}^{l-1} g_{\sigma^N(j_o)} \right) \quad \text{and} \quad A_1 = \text{supp} \left( \frac{1}{l_1} \sum_{N=0}^{l_1-1} g_{\sigma^N(j_1)} \right).$$

Set $h_o = \frac{1}{l} \sum_{N=0}^{l-1} g_{\sigma^N(j_o)}$ and $h_1 = \frac{1}{l_1} \sum_{N=0}^{l_1-1} g_{\sigma^N(j_1)}$. Thus, $h_o$ and $h_1$ are invariant densities. By Lemma 4.21,

$$m_n(\tau^{-1}(A_o)\Delta A_o) = 0 \quad \text{and} \quad m_n(\tau^{-1}(A_1)\Delta A_1) = 0.$$  

Since $\mu \ll m_n$, $\mu(\tau^{-1}(A_o)\Delta A_o) = 0$ and $\mu(\tau^{-1}(A_1)\Delta A_1) = 0$. Moreover, by Lemma 4.11, $\mu(\tau^{-1}(S)\Delta S) = 0$. Thus, for the $\tau$-ergodic measure $\mu$, we get

$$\mu(A_o) = \mu(A_1) = \mu(S) = 1.$$  

However, $S \supset (A_o \cup A_1)$. It is a contradiction. Therefore, there is no such $j_1$ exists. Hence,

$$\text{supp} \ f = \bigcup_{N=0}^{l-1} \text{supp} \ g_{\sigma^N(j_o)} \mod \text{zero}. \quad \square$$

The following one lemma and two theorems are from Boyarsky and Góra [6].
Lemma 4.24. [6] If $\mu$ is a normalized $\tau$-ergodic measure, and $\nu << \mu$ is a normalized $\tau$-invariant measure, then $\nu = \mu$.

Theorem 4.25. [6] $\mu$ in $\mathcal{M}_1$ is $\tau$-ergodic if and only if $\mu$ is an extreme point of $\mathcal{M}_1$.

Proof. Suppose $\mu$ is $\tau$-ergodic, but there exist $\mu_1 \neq \mu_2$ in $\mathcal{M}_1$ and $t \in [0, 1]$ such that

$$\mu = t\mu_1 + (1-t)\mu_2.$$ 

Then $\mu_1 << \mu$ and $\mu_2 << \mu$. By Lemma 4.24, $\mu = \mu_1$ and $\mu = \mu_2$. It is a contradiction. Hence $\mu$ must be an extreme point of $\mathcal{M}_1$.

For the opposite direction, suppose $\mu$ in $\mathcal{M}_1$ is not ergodic, and $S \subset I^n$ with $\mu(\tau^{-1}S \Delta S) = 0$, $\mu(S) \neq 0, 1$. For any $A \in \mathcal{B}_{I^n}$, define two measures $\mu_1$ and $\mu_2$ by

$$\mu_1 = \mu|_{S} : \mu_1(A) = \frac{\mu(A \cap S)}{\mu(S)},$$

$$\mu_2 = \mu|_{S^c} : \mu_2(A) = \frac{\mu(A \cap S^c)}{\mu(S^c)}.$$ 

Set $t = \mu(S) > 0$. Then

$$\mu = t\mu_1 + (1-t)\mu_2.$$ 

Moreover, $\mu_1, \mu_2 \in \mathcal{M}_1$ as follows. From the above identity, $\mu_1 << \mu$ and $\mu_2 << \mu$. Since $\mu \in \mathcal{M}_1$, $\mu << m_n$ which implies $\mu_1 << m_n$ and $\mu_2 << m_n$. By the definition of $\mu_1, \mu_2$, $\mu_1(I^n) = \mu_2(I^n) = 1$ and $\mu_1, \mu_2$ are mutual singular, see Definition A.2 in Appendix. Hence $\mu_1 \neq \mu_2$. Besides, both $\mu_1$ and $\mu_2$ are $\tau$-invariant as below. Since $\mu(\tau^{-1}S \Delta S) = 0$, $\mu(\tau^{-1}S \cap S^c) = 0$ and $\mu(\tau^{-1}S^c \cap S) = 0$. Thus,

$$\mu \circ \tau^{-1}(A \cap S) = \mu(\tau^{-1}A \cap \tau^{-1}S)$$

$$= \mu(\tau^{-1}A \cap \tau^{-1}S \cap S) + \mu(\tau^{-1}A \cap \tau^{-1}S \cap S^c)$$

$$= \mu(\tau^{-1}A \cap S \cap \tau^{-1}S) + \mu(\tau^{-1}A \cap S \cap \tau^{-1}S^c)$$

$$= \mu(\tau^{-1}A \cap S).$$

So is $\mu \circ \tau^{-1}(A \cap S^c) = \mu(\tau^{-1}A \cap S^c)$. For $\mu$ is $\tau$-invariant,

$$\mu_1(A) = \frac{\mu(A \cap S)}{\mu(S)} = \frac{\mu \circ \tau^{-1}(A \cap S)}{\mu(S)} = \frac{\mu(\tau^{-1}A \cap S)}{\mu(S)} = \mu_1(\tau^{-1}A),$$

$$\mu_2(A) = \frac{\mu(A \cap S^c)}{\mu(S^c)} = \frac{\mu \circ \tau^{-1}(A \cap S^c)}{\mu(S^c)} = \frac{\mu(\tau^{-1}A \cap S^c)}{\mu(S^c)} = \mu_2(\tau^{-1}A).$$

Therefore, $\mu$ is not an extreme point in $\mathcal{M}_1$. 

Theorem 4.26. [6] Let $P_\tau : L^1 \to L^1$ be a constrictive Markov operator, and $\mathcal{M}_1$ be the collection of all normalized ACIM under $\tau$. If $\mu_1, \mu_2 \in \mathcal{M}_1$ are different normalized $\tau$-ergodic measures. Then $\mu_1$ and $\mu_2$ are mutually singular.

Hence, we conclude that there are finite many ergodic ACIMs in the probability space, which is described in Lemma 4.27.
Lemma 4.27. Let $P : L^1 \to L^1$ be a constrictive Markov operator satisfying Equation (4.1). If $d\mu = f dm$ is ergodic, then the number, $\tilde{n}$, of such ergodic ACIMs is less than the number, $r$, of the spectral decomposition. i.e.

$$\tilde{n} \leq r.$$  \hfill (4.9)

Moreover, $\tilde{n} = r$ if and only if the permutation $\sigma$ in the spectral decomposition is the identity permutation.

Proof. Suppose $f_1 \neq f_2$ are two ergodic invariant densities (meaning $\mu_i = f_i dm$ are ergodic ACIM). By Theorem 4.26, we know that $m_n(\text{supp}(f_1) \cap \text{supp}(f_2)) = 0$. On the other hand we have

$$f_1 = \sum_{i \in I_1} \lambda_i(f_1)g_i \text{ and } f_2 = \sum_{i \in I_2} \lambda_i(f_2)g_i,$$

where $I_1$ and $I_2$ are the sets of indices where $\lambda_i(f_1) > 0$ (respectively $\lambda_i(f_2) > 0$). Clearly $I_1 \cap I_2 = \emptyset$ and each $I_j$ is invariant under the permutation $\sigma$. It follows that there is a map from the set of ergodic densities to a disjoint collection of $\sigma$ invariant subsets of $\{1, 2, \ldots r\}$. Hence $\tilde{n} \leq r$. Furthermore, if $\tilde{n} = r$ then the disjoint subsets must be singletons, and by invariance, $\sigma = \text{id}$. \hfill $\square$

We give an example in one dimension to show that the number $\tilde{n}$ of ergodic ACIMs are strictly less than the number $r$ in the Spectral Decomposition Theorem.

![Figure 4.2: $\tau : I_1 \cup I_2 \to [0,1]$](image)

Example 4.28. Let $\tau : [0,1] \to [0,1]$ be defined on partition $\{D_1, D_2, D_3, D_4\}$ by

$$\tau(x) = \begin{cases} 
\tau_1(x) = 2x + \frac{1}{2}, & \forall x \in [0, \frac{1}{4}) = D_1 \\
\tau_2(x) = 2x, & \forall x \in [\frac{1}{4}, \frac{1}{2}) = D_2 \\
\tau_3(x) = 2x - 1, & \forall x \in [\frac{1}{2}, \frac{3}{4}) = D_3 \\
\tau_4(x) = 2x - \frac{3}{2}, & \forall x \in [\frac{3}{4}, 1] = D_4.
\end{cases}$$

Then we have $\tilde{n} = 1$ and $r = 2$.  

Proof. Using Proposition 4.3, in particular case $q = 1$, a piecewise $C^2$ expanding Jabłoński transformation $\tau$ implies $P_\tau$ is constrictive. Thus, there exists a decomposition by Theorem 4.4. Moreover, this decomposition is unique up to permutation by Corollary 4.8. Set $I_1 = [0, \frac{1}{2})$, $I_2 = [\frac{1}{2}, 1]$ and $g_i = 2\chi_{I_i}$; $i = 1, 2$ (i.e. $r = 2$). By Lemma 2.10 (b), or Equation (2.4),

$$
P_\tau g_1(x) = \begin{cases} 
\frac{1}{2}(g_1(y_3) + g_1(y_4)) = 0, & \forall x \in I_1 \\
\frac{1}{2}(g_1(y_1) + g_1(y_2)) = 2, & \forall x \in I_2.
\end{cases}
$$

Similarly, we have $P_\tau g_2(x) = g_1(x)$. Thus, the permutation $\sigma$ is cyclic. By Theorem 4.7, $\tau$ has a unique ACIM (i.e. $\tilde{n} = 1$).

**Lemma 4.29.** Let $P_\tau : L^1 \to L^1$ be a constrictive Markov operator with a unique invariant density $f$ such that $\mu = f dm$ is $\tau^N$-ergodic for every $N$. Then $r = 1$.

**Proof.** Let $g_i$ be the densities from the spectral decomposition and $A_i = \text{supp} g_i$. Set $A = \bigcup_{i=1}^r A_i \subseteq I^n$. By Theorem 4.7, since the invariant density for $P_\tau$ is unique, we have $f = \frac{1}{r} \sum_{i=1}^r g_i$ and $\sigma$ is cyclic. In particular, $P_\tau^r g_i = g_i$ for each density $g_i$ and $\text{supp} f = A$. Fix an $i$,

$$
\int_{\tau^{-r}(A_i)} g_i = \int_{A_i} P_\tau^r g_i = \int_{A_i} g_i = 1;
$$

$$
\int_{\tau^{-r}(A_i)} g_j = \int_{A_i} P_\tau^r g_j = \int_{A_i} g_j = 0 \ \forall j \neq i.
$$

Conclude $A_i \subseteq \tau^{-r}(A_i)$ and $m_n (\tau^{-r}(A_i) \cap A_j) = 0$. Thus, $\tau^{-r} A_i \Delta A_i$ is contained in $I^n \setminus A$. Besides,

$$
\mu(A) = \int_A f dm = 1.
$$

Now it follows that

$$
\mu ((\tau^r)^{-1} A_i \Delta A_i) = \mu(\tau^{-r} A_i \Delta A_i) \leq \mu(I^n \setminus A) = 0.
$$

Therefore, by ergodicity of $\tau^r$ with respect to $\mu$,

$$
\mu (A_i) = 0 \text{ or } 1.
$$

Hence $r = 1$. □

**4.3 Conditions of Uniqueness under a Single Transformation $\tau$**

In this section, we establish conditions sufficient for a piecewise $C^2$ Jabłoński transformation $\tau$ to have a unique ACIM. The main theorem is from Boyarsky and Lou [9]. We state this in Theorem 4.46 and give the proof by using the Spectral Decomposition Theorem instead of the Ionescu-Tulcea and Marinescu Theorem 5.10.
To prove Theorem 4.46, we need two main results. The support of invariant density is open mod zero, in Corollary 4.32, and there exists a point in $I^n$ with dense orbit under some conditions, in Lemma 4.45. We use some properties from Keller [23] to get open mod zero. Note that Keller’s paper has not been published. The properties are cited and translated by Góra [19]. Here are the notations.

$\mathcal{P} = \{D_1, \ldots, D_n\}$, the rectangular partition for the Jabłoński transformation according to Definition 2.20.

$\mathcal{P}^s = \{D_{j_1} \cap \cdots \cap \tau^{-s+1}(D_{j_s}) : D_{j_1}, \ldots, D_{j_s} \in \mathcal{P}\}$, the join partition associated to $\tau$.

$\mathcal{U}_\infty(\tau) = \bigcap_{s \geq 0} \tau^{-s}(\mathcal{B}_{j_n})$, called asymptotic $\sigma$-algebra.

$\mathcal{B}_{s,k} = \{B \in \mathcal{P}^{s+k} : t = 1, \ldots, s, \tau^t(B) \in \mathcal{P}^{s+k-t}\} \subset \mathcal{P}^{s+k}$.

We say a set $S$ is open $\mathcal{m}_n$-a.e. if there exists an open set $U$ such that

$$\mathcal{m}_n(S \Delta U) = 0.$$  

**Lemma 4.30.** [19, 23] Let $\tau : I^n \to I^n$ be a piecewise $C^2$ Jabłoński transformation with $\|J_{\tau}^{-1}\|_\infty \leq \alpha < 1$ and $\mathcal{P}$ be the associated rectangular partition. If for $p > 1$, $f \in L^p$ is the density of the ACIM $\mu$, then

(a) There exist $0 < \beta < 1$ such that for any $D \in \mathcal{P}^s$, $\text{diam}(D) \leq \text{diam}(I^n)^p \beta^s$.

(b) For any given $\gamma > 0$, there exists a positive integer $K(\gamma)$ such that for every integer $k \geq K(\gamma)$ and for any $s \in \mathbb{N}$, we can find a collection of sets $\mathcal{B}_{s,k} \subset \mathcal{P}^{s+k}$ satisfying the following conditions.

1. For any $B \in \mathcal{B}_{s,k}$, $\tau^s(B) \in \mathcal{P}^k$.

2. $\mu(I^n \setminus \bigcup_{s,k} \mathcal{B}_{s,k}) \leq \gamma$.

3. For any $B \in \mathcal{B}_{s,k}$ and any measurable set $\tilde{B} \subset B$,

$$\left| \frac{\mathcal{m}_n(\tilde{B})}{\mathcal{m}_n(B)} - \frac{\mathcal{m}_n(\tau^s(\tilde{B}))}{\mathcal{m}_n(\tau^s(B))} \right| \leq \gamma \frac{\mathcal{m}_n(\tilde{B})}{\mathcal{m}_n(B)}.$$  

**Theorem 4.31.** [19, 23] Let $\tau$ satisfy the assumptions of Lemma 4.30 and let the asymptotic $\sigma$-algebra $\mathcal{U}_\infty(\tau)$ be finite $\mathcal{m}_n$-a.e.. Then

(a) For any $A \in \mathcal{U}_\infty(\tau)$, there exist $r \in \mathbb{N}$ and atoms $A_0, \cdots, A_{r-1} \in \mathcal{U}_\infty(\tau)$ pairwise disjoint such that for $i = 1, 2, \cdots, r-1$,

$$A = A_0, \quad \tau(A_{i-1}) \subset A_i \text{ and } \tau(A_{r-1}) \subset A \text{ mod zero}.$$  

(b) The atoms of $\mathcal{U}_\infty(\tau)$ are open sets up to measure zero.

**Corollary 4.32.** [19, 23] Let $\tau$ satisfy the assumptions of Lemma 4.30 and the asymptotic $\sigma$-algebra $\mathcal{U}_\infty(\tau)$ be finite. Assume $P_\tau 1 = 1$. For $p > 1$, if $f$ in $L^p$ is an invariant density, then the support of $f$ is an open set up to measure zero.
Proof. By Lemma 4.21, \( m_n(\tau^{-1}(S)\Delta S) = 0 \). Hence,

\[
S = \tau^{-1}S = \tau^{-2}S = \cdots \mod \text{zero}.
\]

Therefore, \( S \in \mathcal{U}_\infty(\tau) \). Now since the latter is a finite \( \sigma \)-algebra, every set in it, including \( S \) is a union of some of the finitely many atoms. By Theorem 4.31 (b), each atom in \( \mathcal{U}_\infty(\tau) \) is open \( \mod \text{zero} \). Hence, \( \text{supp } f \) is open \( \mod \text{zero} \).

In order to use Corollary 4.32 in our specific case on a piecewise \( C^2 \) Jabłoński transformation \( \tau \), we should check if an invariant density is in \( L^p \), where \( p > 1 \) (see Remark 4.36), and if the asymptotic \( \sigma \)-algebra \( \mathcal{U}_\infty(\tau) \) is finite (see Proposition 4.39). Theorem 4.33 is from Giusti [17].

**Theorem 4.33.** [17](Sobolev inequality)

Let \( B_\rho \subset \mathbb{R}^n \) be a ball with radius \( \rho \). Assume \( f \in BGV \) and set

\[
f_\rho = \frac{1}{m_n(B_\rho)} \int_{B_\rho} f dm_n.
\]

Then for a constant \( \gamma \) depending only on \( n \),

\[
\left( \int_{B_\rho} |f - f_\rho|^\frac{n}{n-1} dm_n \right)^{\frac{n-1}{n}} \leq \gamma GV_{B_\rho}(f).
\]  \( (4.10) \)

The following lemma is from Federer [15, Theorem 4.5.9 (27) and 4.5.10].

**Lemma 4.34.** [15] Let \( A = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n \) and \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) in \( C^1_0 \). If \( f \) is a real valued \( m_n \)-measurable function with \( n > 1 \), and \( f \) satisfies

(1) \( \int_K |f| dm_n < \infty \) for every compact \( K \subset \mathbb{R}^n \),

(2) \( \inf_{f \neq \hat{f}} \int_{\pi(A)} V_{[a_i, b_i]} \hat{f} dm_{n-1} < \infty \).

Then

\[
-\int_A f \cdot \frac{\partial \psi}{\partial x_i} dm_n \leq M(\psi) \left( \inf_{f \neq \hat{f}} \int_{\pi(A)} V_{[a_i, b_i]} \hat{f} dm_{n-1} \right),
\]  \( (4.11) \)

where \( M(\psi) = \sup \{ |\psi(x_1, x_2, \cdots, x_n)| : |x_i| \leq 1, \forall i = 1, 2, \cdots, n \} \).

From Lemma 4.34, we get the following corollary immediately.

**Corollary 4.35.** Let \( f \in BV \). Then

\[
GV_A f \leq n V_A f.
\]
Proof. For $\omega \in C^1_0(\mathbb{R}^n; \mathbb{R}^n)$, then each $\omega_i \in C^1_0(\mathbb{R}^n; \mathbb{R})$. Since $f \in BV$, $f$ satisfies conditions in Lemma 4.34. Hence, by Equation (4.11),

$$-\int_A f \cdot \frac{\partial \omega_i}{\partial x_i} \, dm \leq M(\omega_i) \left( \inf_{f \in f_{\pi_1(A)}} \int_{[a_i,b_i]} \hat{f} \, dm_{n-1} \right) = M(\omega_i)V_{A,i}f \leq M(\omega_i)V_A f \leq |\omega|_2 V_A f.$$ 

Hence,

$$\sum_{i=1}^n \left( -\int_A f \cdot \frac{\partial \omega_i}{\partial x_i} \, dm \right) \leq n|\omega|_2 V_A f.$$ 

Therefore

$$\sup_{|\omega|_2 \leq 1} \sum_{i=1}^n \left( -\int_A f \cdot \frac{\partial \omega_i}{\partial x_i} \, dm \right) \leq \sup_{|\omega|_2 \leq 1} n|\omega|_2 V_A f,$$

implies

$$GV_A f \leq nV_A f.$$ 

\[ \square \]

Remark 4.36. By Corollary 4.35 and the Sobolev inequality, we can conclude that any $f$ in BV implies $f$ in $L^p$, where $p = \frac{n}{n-1} > 1$.

Before we prove $\mathcal{U}_\infty(\tau)$ is finite, in Proposition (4.39), we introduce nice sets and their properties.

Definition 4.37. (nice set and nice function)

Any measurable set $A$ is called a nice set if $P^N_{\tau} \chi_A$ is also a characteristic function for each positive integer $N$, and its characteristic function $\chi_A$ is called a nice function.

Lemma 4.38. Let $A$ be the collection of all nice sets. Then $A$ is a finite $\sigma$-algebra.

Proof. See Lemma A.15 (b) in Appendix. \[ \square \]

Proposition 4.39. Let $P_\tau$ be a Markov operator. Assuming $P_\tau 1 = 1$, then

$$\mathcal{U}_\infty(\tau)$$

is finite.

Proof. Since $P_\tau 1 = 1$, $\tau$ is onto. For any set $A \in \mathcal{U}_\infty(\tau)$, there exists a sequence $\{A_N\}_{N=1}^\infty \subset \mathcal{B}_I^n$ such that

$$A = \tau^{-1}(A_1) = \tau^{-2}(A_2) = \cdots = \tau^{-N}(A_N) = \cdots.$$ 

We also have $A^c = \tau^{-1}(A^c_1) = \tau^{-2}(A^c_2) = \cdots = \tau^{-N}(A^c_N) = \cdots$. Fix $N \in \mathbb{N}$. Then $A = \tau^{-N}(A_N)$ and $A^c = \tau^{-N}(A^c_N)$ which implies

$$\tau^N(A) = \tau^N \circ \tau^{-N}(A_N) = A_N \text{ and } \tau^N(A^c) = \tau^N \circ \tau^{-N}(A^c_N) = A^c_N.$$
By Lemma 4.19, $m_n(\tau(U) \Delta \text{supp } P_{\tau} \chi_U) = 0$. Since this equation holds for every measurable set $U$,

$$m_n(\tau^N(A) \Delta \text{supp } P_{\tau}^N \chi_A) = 0 \text{ and } m_n(\tau^N(A^c) \Delta \text{supp } P_{\tau}^N \chi_{A^c}) = 0.$$ 

Therefore,

$$\left(\text{supp } P_{\tau}^N \chi_A\right) \cap \left(\text{supp } P_{\tau}^N \chi_{A^c}\right) = A_N \cap A_N^c = \emptyset \text{ mod zero.}$$

Since $P_{\tau} 1 = 1$,

$$P_{\tau}^N \chi_A + P_{\tau}^N \chi_{A^c} = P_{\tau}^N (\chi_A + \chi_{A^c}) = P_{\tau}^N 1 = 1.$$

For all $N \in \mathbb{N}$, $m_n \left( \text{supp } P_{\tau}^N \chi_A \cap \text{supp } P_{\tau}^N \chi_{A^c} \right) = 0$, $A$ must be a nice set. Thus, $A \subseteq A \cap A^c = \emptyset$ mod zero. □

So far we get the first result. We try to prove the existence of the dense orbits. First, we introduce some conditions on the given Jabłoński transformation so that the related $P_{\tau}$ is constrictive. Recall: for any integer $N$, if $\tau$ is a Jabłoński transformation on a rectangular partition $\mathcal{P}$, in Lemma 2.24 (b), then $\tau^N$ is also a Jabłoński transformation on a rectangular partition $\mathcal{P}^N$, the join of $\{\mathcal{P}, \cdots, \tau^{-N}(\mathcal{P})\}$ (see Equation (2.12)).

**Lemma 4.40.** Let $\tau : I^n \rightarrow I^n$ be a Jabłoński transformation on a rectangular partition, $\mathcal{P} = \{D_1, \cdots, D_m\}$ satisfying the following conditions:

1. For each $D_j = \prod_{i=1}^n [a_{ij}, b_{ij}] \in \mathcal{P}$, let

$$\tau_j = \tau|_{D_j} \text{ and } \tau_j(\mathbf{x}) = (\phi_{1j}(x_1), \cdots, \phi_{nj}(x_n)).$$

Every $N \in \mathbb{N}$, suppose $D_j^N = \prod_{i=1}^n [a_{ij}(N), b_{ij}(N)]$ is an element in the partition $\mathcal{P}^N$. Fix $N_0 \in \mathbb{N}$ and define

$$\tau_j^{N_0} = \tau^{N_0}|_{D_j^{N_0}} \text{ and } \tau_j^{N_0}(\mathbf{x}) = (\phi_{1j}^{N_0}(x_1), \cdots, \phi_{nj}^{N_0}(x_n)).$$

For $i = 1, \cdots, n$, the Jabłoński transformation satisfies

$$\inf_{x_i \in (a_{ij}, b_{ij})} |\phi_{ij}'(x_i)| = c > 0 \text{ and } \inf_{x_i \in (a_{ij}(N_0), b_{ij}(N_0))} |(\phi_{ij}^{N_0}(x_i))'| > 1. \quad (4.12)$$

2. $\tau$ is piecewise $C^2$.

Let $P_{\tau}$ be the Perron-Frobenius operator. Then $P_{\tau}$ is a constrictive operator.
Proof. Assume $N_o \in \mathbb{N}$ such that $\tau$ satisfies Equation (4.12). Let $\bar{\tau} = \tau^{N_o}$ and

$$\inf_{x_i \in \left\{a_{ij}(N_o) \leq x_i \leq b_{ij}(N_o)\right\}} \left|\left(\phi_{ij}^{N_o}(x_i)\right)\right| \geq \frac{1}{\alpha} > 1.$$ 

By Equation (3.6), for each $\tilde{K} \in \mathbb{N}$, $V_{I^n} P_{\tilde{K}}^\tau f \leq 2\alpha \tilde{K} V_{I^n} f + \gamma \|f\|_1$. Hence,

$$V_{I^n} P_{\tilde{K}}^{KN_o} f \leq 2\alpha \tilde{K} V_{I^n} f + \gamma KN_o \|f\|_1.$$ 

Since for each $N \in \mathbb{N}$, there exist $a, b \in \mathbb{N}$, $0 \leq b < KN_o$ such that $N = aKN_o + b$. For the same argument as in Lemma 3.8, define constants

$$\beta = \alpha 2^{KN_o}, \gamma_M = \max\{\gamma_1, \cdots, \gamma_{KN_o}\} \text{ and } \gamma_o = \frac{2\alpha}{1 - 2\alpha KN_o}.$$ 

The same proof as in Proposition 4.3, $P_\tau$ is a constrictive operator. \qed

**Remark 4.41.** Such $\tau$ in Lemma 4.40 satisfying conditions (1) and (2) is piecewise monotone, which means on each rectangular, $\tau$ is monotone on each individual coordinate.

Now we define communication property.

**Definition 4.42.** (communication property)

Given a rectangular partition $\mathcal{P} = \{D_1, \cdots, D_m\}$ and a transformation $\tau : I^n \rightarrow I^n$, if for every pair of elements $D_j, D_l \in \mathcal{P}$, there exist $u, v \in \mathbb{N}$ such that

$$D_j \subset \tau^u(D_l) \text{ and } D_l \subset \tau^v(D_j).$$

Then the partition $\mathcal{P}$ has the communication property under $\tau$.

From this definition, we get the following lemma.

**Lemma 4.43.** Let $\tau : I^n \rightarrow I^n$ be a piecewise $C^2$ Jabłoński transformation on a rectangular partition $\mathcal{P} = \{D_j\}_{j=1}^m$ with communication property. Assume $P_\tau$ be its Perron-Frobenius operator. Then $\tau$ is onto.

**Proof.** Suppose $\tau$ is not onto, then there exist $D_l \in \mathcal{P}$ and $A \in \mathfrak{B}_{I^n}$ with $m_n(A) > 0$ such that $A \subset (\tau(I^n)^c \cap D_l)$. By the communication property, for any $D_j \neq D_l$, there exist $u, v \in \mathbb{N}$ such that

$$\tau^u(D_j) \supset D_l \text{ and } \tau^v(D_l) \supset D_j.$$ 

Since $I^n \supset \tau(I^n)$,

$$\tau(I^n) \supset \tau^u(D_j) \supset D_l \supset A.$$ 

This is a contradiction. Hence, $\tau$ is onto. \qed

The following two lemmas are simply extended the results from one dimension to higher dimensions, see Boyarinksy and Scarowsky [12].
Lemma 4.44. Let \( \tau : I^n \to I^n \) be a Jabłoński transformation on a rectangular partition \( \mathcal{P} = \{D_j\}_{j=1}^m \) satisfying conditions (1) and (2) in Lemma 4.40. For each point \( x \in I^n \). Set the code\(^1\) of the orbit \( x \) by a sequence

\[
\langle x \rangle = \gamma_1 \gamma_2 \gamma_3 \cdots, \text{ if } x \in D_{\gamma_1}, \quad \tau(x) \in D_{\gamma_2}, \quad \tau^2(x) \in D_{\gamma_3}, \cdots,
\]

where every \( \gamma_j \in \{1, 2, \cdots, m\} \). If \( \rho = \gamma_1 \gamma_2 \gamma_3 \cdots \) is a sequence with the property

\[
\tau(D_{\gamma_j}) \supset D_{\gamma_{j+1}}, \quad \forall \gamma_j \in \{1, 2, \cdots, m\},
\]

then there exists an \( x \in I^n \) such that \( \langle x \rangle = \rho \).

Proof. Let \( E_N = \{x \in I^n : x \in D_{\gamma_1}, \quad \tau(x) \in D_{\gamma_2}, \cdots, \tau^{N-1}(x) \in D_{\gamma_N}\} \). Each \( E_N \) is a nonempty closed rectangle since \( \tau \) is piecewise \( C^2 \) monotone Jabłoński transformation. Note that \( E_N \) is closed because of piecewise \( C^2 \). Condition (1) implies \( \tau \) is monotone on each coordinate on the suitable rectangle from Remark 4.41. Piecewise monotone Jabłoński transformation implies that \( E_N \) is a rectangle. Hence,

\[
\bigcap_{N=0}^{\infty} E_N \neq \emptyset.
\]

Lemma 4.45. The same setting as in Lemma 4.44. Assume the partition \( \mathcal{P} \) has the communication property under \( \tau \). Then there exists an \( x \in I^n \) such that \( \{\tau^N(x)\}_{N=0}^{\infty} \) is dense in \( I^n \).

Proof. For \( \gamma_j \in \{1, 2, \cdots, m\} \), let

\[
J = \{\gamma_1 \gamma_2 \cdots \gamma_r : \tau(D_{\gamma_j}) \supset D_{\gamma_{j+1}}, j = 1, \cdots, r - 1; \ r = 1, 2, \cdots\}
\]

be the collection of all possible finite sequences. Then \( J \) is countable. For \( A_1, A_2, \cdots \) in \( J \), consider the sequence

\[
\langle x \rangle = A_1 B_1 A_2 B_2 A_3 B_3 \cdots,
\]

where each \( B_j \) is a finite sequence joining the last symbol of \( A_j \) and the first symbol of \( A_{j+1} \). Such \( B_j \)'s exist because of the communication property. Hence, there exists an \( x \in I^n \) equal to the coding \( \langle x \rangle \) followed by Lemma 4.44.

Now prove \( \{\tau^N(x)\}_{N=0}^{\infty} \) is dense in \( I^n \). Given \( y \in I^n \) and \( \epsilon > 0 \), assume \( \langle y \rangle = \beta_1 \beta_2 \beta_3 \cdots \). Since \( J \) contains all possible finite sequences, for any \( l \in \mathbb{N} \), there exists \( A_j \in J \) with length \( N \) such that

\[
A_j = \beta_1 \beta_2 \cdots \beta_l N_0.
\]

Say \( A_j \) is the coding of \( \langle x \rangle \) from the \( (N + 1) \)th symbol to the \( (N + l N_0) \)th symbol. Thus, for each \( i = 1, 2, \cdots, l N_0 \), \( \tau^{N+i}(x) \) and \( \tau^i(y) \) are in the same rectangle. Since

\(^1\)If \( \tau^j(x) \) lies in two elements \( D_s \cap D_t \), then we may pick either \( s \) or \( t \) for \( \langle x \rangle_{j+1} \).
\[ \tau : I^n \to I^n, \sup_{x \in I^n} |\tau(x)|_2 \leq \sqrt{n}. \] By Equation (4.12),

\[
|\tau^N(x) - y|_2 \leq \alpha |\tau^{N+N_0}(x) - \tau^{N_0}(y)|_2 \\
\vdots \\
\leq \alpha^l |\tau^{N+1N_0}(x) - \tau^{N_0}(y)|_2 \\
\leq 2\sqrt{n}\alpha^l < \epsilon,
\]

for \( l \) sufficiently large. Therefore, the orbit of \( x \) is dense in \( I^n \).

Theorem 4.46 shows uniqueness, see [9]. We give a new proof by using the Spectral Decomposition Theorem.

**Theorem 4.46.** Let \( \tau : I^n \to I^n \) be a Jabłoński transformation on a rectangular partition, \( P = \{D_1, \cdots, D_m\} \) with communication property under \( \tau \). Consider invariant densities in \( BV \). If \( \tau \) satisfies conditions (1) and (2) in Lemma 4.40 and \( P_\tau 1 = 1 \), then \( \tau \) admits a unique invariant density, namely, \( f^* = 1 \).

**Proof.** By Lemma 4.40, \( P_\tau \) is constrictive. Hence, \( P_\tau \) has the spectral representation by Theorem 4.4, so there exists an invariant density. Suppose \( P_\tau \) has two invariant densities \( f_1 \) and \( f_2 \) in \( BV \) with disjoint support followed by Lemma 4.16. By Remark 4.36, \( f_1, f_2 \in L^p \), where \( p > 1 \). Hence we can apply Corollary 4.32, \( \text{supp } f_1 \) and \( \text{supp } f_2 \) are open mod zero. Thus, there exist \( \tilde{f}_1 = f_1 \text{ a.e.} \) and \( \tilde{f}_2 = f_2 \text{ a.e.} \) such that

\[ U_1 = \text{supp } \tilde{f}_1 \text{ and } U_2 = \text{supp } \tilde{f}_2, \]

where \( U_1 \) and \( U_2 \) are open and forward invariant sets. Hence, for \( y_k \in U_k, \ k = 1, 2, \) there exists \( \epsilon > 0 \) such that

\[ B_\epsilon(y_k) \equiv \{ z \in I^n : |z - y|_2 < \epsilon \} \subset U_k, \ k = 1, 2. \]

By Lemma 4.45, there exists a point with a dense orbit. Say \( x \in I^n \) such that \( \{\tau^N(x)\}_{N=1}^\infty \) is dense in \( I^n \). Then, there exist \( N_1, N_2 \in \mathbb{N} \) such that

\[ |\tau^{N_1}(x) - y_1|_2 < \frac{\epsilon}{2}, \text{ and } |\tau^{N_2+N_1}(x) - y_2|_2 < \frac{\epsilon}{2}. \]

Thus, \( \tau^{N_1}(x) \in U_1 \) and \( \tau^{N_2+N_1}(x) \in U_2 \). For the open set \( U_1 \), there exists \( r > 0 \) and

\[ B_r(\tau^{N_1}(x)) \subset U_1 \text{ with } m_n(B_r(\tau^{N_1}x)) > 0. \]

Since \( \tau \) is piecewise \( C^2 \), for \( \tau^{N_2+N_1}(x) \in U_2 \),

\[ \tau^{N_2}(B_r(\tau^{N_1}x)) \subset U_2. \]

It is a contradiction to the forward invariant sets, so \( P_\tau \) has only one invariant density.
4.4 Conditions of Uniqueness under a Random Dynamical System $T$

In this section, we show the uniqueness under two different situations. If any individual transformation of the random dynamical system $T$ has a unique ACIM, then $T$ has a unique ACIM. This is described in Theorem 4.50. The other situation, we focus on uniqueness under an individual transformation of $T$, see Proposition 4.52.

We need the following two lemmas to prove Theorem 4.49.

**Lemma 4.47.** [8] Let $P_1$ and $P_2$ be Markov operators. Fix some $0 < t < 1$, and set $P = tP_1 + (1-t)P_2$. Let $P$ and $P_1$ be constrictive Markov operators, and the spectral decomposition of $P$ and $P_1$ be given by

$$P^N f = \sum_{i=1}^{r} \lambda_i(f) g_{\sigma^N(i)} + P^N Q f, \forall N \geq 0;$$

$$P_1^N f = \sum_{j=1}^{r_1} \Lambda_j(f) G_{\delta^N(j)} + P_1^N Q_1 f, \forall N \geq 0.$$

Assume order of $\sigma$ and $\delta$ are $l$ and $l_1$ respectively. Then for each $i = 1, 2, \cdots, r$, there exists $j$ depending on $i$ such that

$$m_n(\text{supp } g_i \cap \text{supp } G_j) \neq 0.$$

**Proof.** Since $P = tP_1 + (1-t)P_2$,

$$P^N f = t^N P_1^N f + [\text{terms involving } P_1 f \& P_2 f]$$

$$= t^N \left( \sum_{j=1}^{r_1} \Lambda_j(f) G_{\delta^N(j)} + P_1^N Q_1 f \right)$$

$$+ [\text{terms involving } P_1 f \& P_2 f].$$

Now, since $\|P_i f\|_1 \leq \|f\|_1$, $i = 1, 2$, we have

$$\| [\text{terms involving } P_1 f \& P_2 f] \|_1 \leq \sum_{j=1}^{N} \binom{N}{j} t^{N-j}(1-t)^j \|f\|_1$$

$$= (t + (1-t))^N \|f\|_1 - t^N \|f\|_1$$

$$= (1-t^N) \|f\|_1.$$

Apply this to one of the densities $g_i$ and power $lN$,

$$\left\| g_i - t^{lN} \left( \sum_{j=1}^{r_1} \Lambda_j(g_i) G_j \right) \right\|_1 \leq l^N \| P_1^N Q_1 g_i \|_1 + 1 - t^{lN}.$$
For all $N$ such that $\|P_1^{tN}Q_1 g_i\|_1 < 1$. One obtains

$$\left\| g_i - t^{1N} \left( \sum_{j=1}^{r_1} \Lambda_j(g_i) G_j \right) \right\|_1 < 1.$$  

Since $\|g_i\|_1 = 1$, there exists some $j$ depending on $i$ such that

$$m_n(\text{supp } g_i \cap \text{supp } G_j) \neq 0.$$ 

Lemma 4.47 implies that if the number of densities $r > r_1$, then at least two $g_i$’s intersect the same $G_j$ with positive measure. Assume by re-indexing if necessary $g_1$, $g_2$ and $G_1$ satisfy

$$m_n(\text{supp } g_1 \cap \text{supp } G_1) > 0;$$

$$m_n(\text{supp } g_2 \cap \text{supp } G_1) > 0.$$  

Set $A_i \equiv \text{supp } g_i \cap \text{supp } G_1$, for $i = 1, 2$. Let $\tilde{g}_1 = g_1|_{A_1}$ and $\tilde{g}_2 = g_2|_{A_2}$.

**Lemma 4.48.** [8] Let $P_1$, $P_2$ and $P$ be Markov operators as in Lemma 4.47 with $r$ and $r_1$ densities in their spectral representation respectively. Let

$$P_1^N f = \sum_{j=1}^{r_1} \Lambda_j(f) G_\delta^{N(j)} + P_1^{N} Q_1 f, \forall N \geq 0.$$  

Assume order of $\delta$ is $l_1$. For $r > r_1$, let $g_1$, $g_2$ and $G_1$ be defined as above. Then there exists an $N_o \in \mathbb{N}$ such that for all $N_1 \geq N_o$,

$$\|P_1^{l_1 N_1} (\tilde{g}_1 - \tilde{g}_2)\|_1 < \|\tilde{g}_1 - \tilde{g}_2\|_1.$$  

**Proof.** Since $g_1$ and $g_2$ have disjoint support, $\text{supp } \tilde{g}_1$ and $\text{supp } \tilde{g}_2$ are also disjoint. Hence $\|\tilde{g}_1 - \tilde{g}_2\|_1 = \|\tilde{g}_1\|_1 + \|\tilde{g}_2\|_1$. Assume $\|\tilde{g}_1\|_1 \geq \|\tilde{g}_2\|_1 > 0$. Choose $N_o \in \mathbb{N}$, such that $\forall N_1 \geq N_o$, 

$$\|P_1^{l_1 N_1} Q_1 (\tilde{g}_1 - \tilde{g}_2)\|_1 < \|\tilde{g}_2\|_1.$$
Therefore, since $\Lambda_j(\tilde{g}_1 - \tilde{g}_2) = 0$ for all $j \neq 1$ and $0 \leq \Lambda_1(\tilde{g}_1) \leq \|\tilde{g}_1\|_1$, we have

$$\|P^{l_1N_1}_1(\tilde{g}_1 - \tilde{g}_2)\|_1 \leq \left\| \sum_{j=1}^{r_1} \Lambda_j(\tilde{g}_1 - \tilde{g}_2) G_j \right\|_1 + \|P^{l_1N_1}_1 Q_1(\tilde{g}_1 - \tilde{g}_2)\|_1$$

$$= \|\Lambda_1(\tilde{g}_1 - \tilde{g}_2) G_1\|_1 + \|P^{l_1N_1}_1 Q_1(\tilde{g}_1 - \tilde{g}_2)\|_1$$

$$< \|\Lambda_1(\tilde{g}_1 - \tilde{g}_2)\|_1 + \|\tilde{g}_2\|_1$$

$$= \|\Lambda_1(\tilde{g}_1) - \Lambda_1(\tilde{g}_2)\| + \|\tilde{g}_2\|_1$$

$$\leq \max\{\|\Lambda_1(\tilde{g}_1)\|, \|\Lambda_1(\tilde{g}_2)\|\} + \|\tilde{g}_2\|_1$$

$$\leq \max\{\|\tilde{g}_1\|_1, \|\tilde{g}_2\|_1\} + \|\tilde{g}_2\|_1$$

$$= \|\tilde{g}_1\|_1 + \|\tilde{g}_2\|_1.$$

Hence,

$$\|P^{l_1N_1}_1(\tilde{g}_1 - \tilde{g}_2)\|_1 < \|\tilde{g}_1\|_1 + \|\tilde{g}_2\|_1 = \|\tilde{g}_1 - \tilde{g}_2\|_1.$$

\[\Box\]

**Theorem 4.49.** [8] Let $P_1$ and $P_2$ be Markov operators. Fix some $0 < t < 1$, and set $P = tP_1 + (1 - t)P_2$. Assume $P$ and $P_1$ are constrictive with $r$ and $r_1$ densities respectively in their spectral decompositions. Then

$$r \leq r_1.$$

**Proof.** Since both $P$ and $P_1$ are constrictive, let

$$P^N f = \sum_{i=1}^{r} \lambda_i(f) g_{\sigma^N(i)} + P^N Q f, \quad \forall N \geq 0;$$

$$P^{l_1}_1 f = \sum_{j=1}^{r_1} \Lambda_j(f) G_{\delta^N(j)} + P^{l_1}_1 Q_1 f, \quad \forall N \geq 0.$$

Suppose $r > r_1$. Let $g_1$, $g_2$ and $\tilde{g}_1$, $\tilde{g}_2$ be defined as in Lemma 4.48. Then for

$$P = tP_1 + (1 - t)P_2,$$

$$g_1 - g_2 = \sum_{j=1}^{r_1} \Lambda_j f \left[ g_{\delta^N(j)} - \tilde{g}_{\delta^N(j)} \right]$$

$$+ \left[ \text{terms involving } P_1(g_1 - g_2) \& P_2(g_1 - g_2) \right].$$

For all $N \in \mathbb{N}$, such that $lN \geq l_1N_1$ ($N_1$ from Lemma 4.48) we have

$$\|\left[ \text{terms involving } P_1(g_1 - g_2) \& P_2(g_1 - g_2) \right]\|_1 \leq (1 - t^{lN})\|g_1 - g_2\|_1.$$
Using \(\|g_1 - g_2\|_1 = 2\),
\[
2 = \|g_1 - g_2\|_1 \leq t^{IN} \|P_{1}^{IN}(g_1 - g_2)\|_1 + (1 - t^{IN}) \|g_1 - g_2\|_1 \\
= t^{IN} \|P_{1}^{(IN-I_1,N_1)} \circ P_{1}^{I_1,N_1}(g_1 - g_2)\|_1 + 2(1 - t^{IN}) \\
\leq t^{IN} \|P_{1}^{I_1,N_1}((g_1 - \tilde{g}_1) + (\tilde{g}_2 - g_2) + (\tilde{g}_1 - \tilde{g}_2))\|_1 + 2(1 - t^{IN}) \\
\leq t^{IN} (\|g_1 - \tilde{g}_1\|_1 + \|\tilde{g}_2 - g_2\|_1 + \|P_{1}^{I_1,N_1}(\tilde{g}_1 - \tilde{g}_2)\|_1) + 2(1 - t^{IN}).
\]

By Equation (4.13), for sufficiently large \(N_1\),
\[
\|P_{1}^{I_1,N_1}(\tilde{g}_1 - \tilde{g}_2)\|_1 < \|\tilde{g}_1 - \tilde{g}_2\|_1.
\]
Therefore,
\[
2 \leq t^{IN} (\|g_1 - \tilde{g}_1\|_1 + \|\tilde{g}_2 - g_2\|_1 + \|P_{1}^{I_1,N_1}(\tilde{g}_1 - \tilde{g}_2)\|_1) + 2(1 - t^{IN}) \\
< t^{IN} (\|g_1 - \tilde{g}_1\|_1 + \|\tilde{g}_2 - g_2\|_1 + \|\tilde{g}_1 - \tilde{g}_2\|_1) + 2(1 - t^{IN}) \\
= t^{IN} (\|g_1 - \tilde{g}_1\|_1 + \|g_2 - g_2\|_1 + \|\tilde{g}_1 - \tilde{g}_2\|_1) + 2(1 - t^{IN}) \\
= t^{IN} (\|g_1\|_1 + \|g_2\|_1) + 2(1 - t^{IN}) \\
= 2t^{IN} + 2(1 - t^{IN}) = 2.
\]

This is a contradiction. Hence \(r \leq r_1\). \(\square\)

Considering the case of a random dynamical system constructed from piecewise \(C^2\) Jabłoński transformations, we have the following theorem to get uniqueness, which is modified from [22, Theorem 3.3].

**Theorem 4.50.** Let \(T = \{\tau_k; p_k\}_{k=1,...,q}\) be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise \(C^2\) Jabłoński transformations. If \(P_{\tau_1}\) is constrictive with a unique invariant density \(f\), and \(d\mu = fdm_\alpha\) is \(\tau_1^N\)-ergodic for any integer \(N\), then the random dynamical system \(T\) has a unique ACIM.

**Proof.** By Proposition 4.3, \(P_T\) is constrictive and by Proposition 4.6, \(P_T\) has an invariant density. That is, \(\bar{n} \geq 1\). For \(P_T = \sum_{k=1}^{q} p_kP_{\tau_k}\), let \(\bar{P} = \frac{1}{1-p_1} \sum_{k=2}^{q} p_kP_{\tau_k}\). Then \(\bar{P}\) is a Markov operator and

\[
P_T = p_1P_{\tau_1} + (1-p_1)\bar{P}.
\]

By Lemma 4.27 we know \(\bar{n} \leq r\), and by Theorem 4.49 we know \(r \leq r_1\). In addition, \(r_1 = 1\) from Lemma 4.29. Since \(\bar{n} \geq 1\), we get
\[
1 \leq \bar{n} \leq r \leq r_1 = 1.
\]

Hence \(\bar{n} = 1\). \(\square\)

**Remark 4.51.** In general, the converse situation in the Theorem 4.50 is not always true. We can have a unique ACIM of the random dynamical system, but it does
not imply any one of the individual transformations should have a unique ACIM. A counterexample is shown in Example 4.58.

Now, we get a conclusion of this section that if any one of the individual transformations of $T$ satisfies conditions in Lemma 4.40, then the Perron-Frobenius operator $P_T$ with respect to a random dynamical system $T$ has a unique invariant density, which is modified from [22, Proposition 3.3]).

**Proposition 4.52.** Let $T = \{\tau_k; p_k\}_{k=1,\ldots,q}$ be a random dynamical system satisfying equation (3.5) which is constructed from piecewise $C^2$ Jabłoński transformations on the partition $\mathcal{P} = \{D_1, \cdots, D_m\}$. If $P_{\tau_1} f = f$, $d\mu = f dm_n$ is $\tau_1^N$-ergodic for every integer $N$, and $\tau_1$ satisfies the following conditions:

1. For each $D_j = \prod_{i=1}^{n} [a_{ij}, b_{ij}] \in \mathcal{P}$, let $\tau_{j,1} = \tau_1|_{D_j}$ and $\tau_{j,1}(x) = (\varphi_{1j,1}(x_1), \cdots, \varphi_{nj,1}(x_n))$.

   For $i = 1, \cdots, n$ and for some $N_o \in \mathbb{N}$,
   
   $$\inf_{x_i \in (a_{ij}, b_{ij})} \left| \varphi'_{ij,1}(x_i) \right| > 0 \quad \text{and} \quad \inf_{x_i \in (a_{i(j(N_o)), b_{ij}(N_o)})} \left| (\varphi_{ij,1}'(x_i)) \right| > 1.$$

2. The partition $\mathcal{P}$ has the communication property under $\tau_1$.

Then the random dynamical system $T$ has a unique ACIM.

**Proof.** It is directly followed by the Theorem 4.46 and Theorem 4.50. \qed

### 4.5 Random Dynamical System Constructed from Piecewise Linear Markov Transformations

In this section, we only consider a special case that a random dynamical system is constructed from piecewise linear Markov transformations defined as below.

**Definition 4.53.** (Markov transformation)

1. A transformation $\tau : \mathbb{I}^n \to \mathbb{I}^n$ is called a Markov transformation with respect to a partition $\mathcal{P} = \{D_1, D_2, \cdots, D_m\}$ if for each $j = 1, 2, \cdots, m$, there exist some $D_{j1}, \cdots, D_{jl} \in \mathcal{P}$, such that
   
   $$\tau(D_j) = \bigcup_{r=1}^{l} D_{jr}. \tag{4.14}$$

2. If a transformation $\tau$ is piecewise linear on $\mathcal{P}$. That is, $\tau = \sum_{j=1}^{m} f_j \chi_{D_j}$, for some linear functions $f_1, \cdots, f_m$. We are interested in piecewise linear and Markov transformations. Such transformation is called piecewise linear Markov.
We introduce the Markov matrix induced by \( \tau \).

**Proposition 4.54.** If \( \tau : I^n \to I^n \) is a piecewise linear Markov transformation with respect to the partition \( \mathcal{P} = \{D_1, D_2, \cdots, D_m\} \). For each \( D_j \in \mathcal{P} \), let \( \tau_j = \tau|_{D_j} = (\varphi_{1j}(x), \cdots, \varphi_{nj}(x)) \) and \( |J_{\tau_j}| \) be the Jacobian determinant

\[
|J_{\tau_j}|(x) = \det \left( \frac{\partial(\varphi_{1j}, \cdots, \varphi_{nj})}{\partial(x_1, \cdots, x_n)} \right) = \begin{vmatrix}
\frac{\partial \varphi_{1j}(x)}{\partial x_1} & \cdots & \frac{\partial \varphi_{nj}(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial \varphi_{nj}(x)}{\partial x_1} & \cdots & \frac{\partial \varphi_{nj}(x)}{\partial x_n}
\end{vmatrix}.
\]

Denote \( \mathcal{C} \) as a collection of all the piecewise constant functions on partition \( \mathcal{P} \) by

\[
\mathcal{C} = \left\{ f \in L^1 \left| f = \sum_{j=1}^{m} c_j g_j ; g_j = \frac{\chi_{D_j}}{m_n(D_j)} , \text{ } c_j \text{ constants} \right. \right\}. \tag{4.15}
\]

Then

(a) the subset \( \mathcal{C} \) is a finite-dimensional space and \( \mathcal{C} \cong \mathbb{R}^m \). Moreover, we can write \( \mathbf{f} \) as an element in \( \mathbb{R}^m \) by \( \mathbf{f} = (c_1, \cdots, c_m) \) to be the associated element \( f \in \mathcal{C} \) by \( f = \sum_{j=1}^{m} c_j g_j \). Define the correspondence \( \mathbf{f} \leftrightarrow f \). That is,

\[
\mathbf{f} = (c_1, \cdots, c_m) \text{ if and only if } f = \sum_{j=1}^{m} c_j g_j. \tag{4.16}
\]

(b) For each \( D_j \in \mathcal{P} \), the Jacobian \( J_{\tau_j} \) is constant on \( D_j \).

(c) \( P_{\tau} \) maps \( \mathcal{C} \) to itself. Moreover, there exists a stochastic matrix \( M_{\tau} \) which is induced by \( \tau \). \( M_{\tau} \) is defined from \( \mathbb{R}^m \) to itself. For all \( \mathbf{f} \in \mathbb{R}^m \) and \( \mathbf{f} \leftrightarrow f \in \mathcal{C} \),

\[
P_{\tau} f \leftrightarrow \mathbf{f} M_{\tau}. \tag{4.17}
\]

**Proof.** Part (a) and (b) are straight forward. For the part (c), let \( \{g_i = \frac{\chi_{D_i}}{m_n(D_i)}\}_{i=1}^{r} \) be a basis, and \( M_{\tau} = (M_{ij})_{m \times m} \) be an \( m \times m \) matrix. For each \( i, j = 1, 2, \cdots, m \), define

\[
M_{ij} = \int_{D_j} P_{\tau} g_i = \frac{m_n(D_i \cap \tau^{-1}(D_j))}{m_n(D_i)}. \tag{4.18}
\]

Thus, for each \( i^{th} \)-row,

\[
\sum_{j=1}^{m} M_{ij} = \sum_{j=1}^{m} \frac{m_n(D_i \cap \tau^{-1}(D_j))}{m_n(D_i)} = \frac{m_n(D_i \cap \tau^{-1}(I^n))}{m_n(D_i)} = 1. \tag{4.19}
\]

Since \( P_{\tau} : \mathcal{C} \to \mathcal{C} \) is linear, it is the same as a linear transformation maps from \( \mathbb{R}^m \).
to \( \mathbb{R}^m \). For the basis \{\( g_1, \ldots, g_m \)\} and for each \( i = 1, \ldots, m \), define

\[
P_\tau g_i = \sum_{j=1}^m M_{ij} g_j.
\]

Hence by Equation (4.16), \( \vec{g}_i \leftrightarrow \sum_{j=1}^m b_j g_j \), \( b_j = 0 \) \( \forall j \neq i \) & \( b_i = 1 \). For any \( f \in \mathcal{C} \), there are constants \( c_1, \ldots, c_m \) such that \( f = \sum_{i=1}^m c_i g_i \). Then, \( P_\tau f \leftrightarrow fM_\tau \) since

\[
P_\tau f = \sum_{i=1}^m c_i P_\tau g_i = \sum_{i=1}^m c_i \left( \sum_{j=1}^m M_{ij} g_j \right) = \sum_{j=1}^m \left( \sum_{i=1}^m c_i M_{ij} \right) g_j
\]

\[
\leftrightarrow \left( \sum_{i=1}^m c_i M_{i1}, \sum_{i=1}^m c_i M_{i2}, \ldots, \sum_{i=1}^m c_i M_{im} \right)
\]

\[
= (c_1, c_2, \ldots, c_m) (M_{ij})_{m \times m} = \vec{f}M_\tau.
\]

Note that \( M_\tau \) is a Markov\(^2\) matrix.

---

**Lemma 4.55.** Let \( \mathcal{P} = \{D_j\}_{j=1}^m \) be an equal partition (i.e. each \( D_j \) has the same measure equal to \( \frac{1}{m} \)). Then \( M_\tau \) induced by \( \tau \) is stochastic, and every nonzero entry on the \( i^{th} \) row is equal to \( |J_\tau_i|^{-1} \).

**Proof.** For each \( g_i = \frac{\chi_{D_i}}{m_n(D_i)} = m \chi_{D_i} \), by Equation (4.18),

\[
M_{ij} = \frac{m_n(D_i \cap \tau^{-1}(D_j))}{m_n(D_i)} = m \int_{D_i} \chi_{\tau^{-1}(D_j)}
\]

\[
= m \int_{D_i} \chi_{D_j}(\tau_i(x))d_mn(x) = m \int_{\tau_i(D_i)} \chi_{D_j}(y)d_mn(y)
\]

\[
= \frac{m}{|J_\tau_i|} m_n(\tau_i(D_i) \cap D_j).
\]

Since \( \tau \) is a piecewise linear Markov transformation,

\[
m_n(\tau_i(D_i) \cap D_j) = \frac{1}{m} \text{ or } m_n(\tau_i(D_i) \cap D_j) = 0.
\]

Hence we get the desired result. \( \square \)

Consider the Perron-Frobenius operator \( P_\tau \) with respect to the random dynamical system \( T = \{\tau_k; p_k\}_{k=1}^g \) constructed from linear Markov transformations \( \{\tau_k\}_{k=1}^q \). By Definition 3.3, \( P_\tau f = \sum_{k=1}^q p_k P_{\tau_k} f \). Thus, we have the following theorem (see [22, Theorem 3.4]). Here we use the condition of expanding-on-average instead of strictly expanding. We say that a set \( S \) is forward invariant under \( T \), in [36], if \( S \in \mathfrak{F} \) and

\(^2\)A nonnegative square matrix with the row sums equal to one.
\[ \bigcup_{k=1}^{q} \tau_k S = S \text{ mod zero.} \]

**Theorem 4.56.** Let \( T = \{ \tau_k; \ p_k \}_{k=1,...,q} \) be a random dynamical system. Assume that \( \{ \tau_k \}_{k=1,...,q} \) have common partition \( \mathcal{P} = \{ D_1, \ldots, D_m \} \) such that each \( \tau_k \) is a piecewise linear Markov transformation with respect to \( \mathcal{P} \). For each \( D_j \in \mathcal{P} \), let
\[
\tau_{j,k} = \tau_k|_{D_j} \quad \text{and} \quad |J_{\tau_{j,k}}(x)| = |J_{\tau_{j,k}}|,
\]
where the Jacobian determinant \( |J_{\tau_{j,k}}| \) is constant on \( D_j \). Set \( C_k = \max_{j=1,...,m} |J_{\tau_{j,k}}^{-1}| \). Assume \( T \) satisfies the following equation,
\[
\sum_{k=1}^{q} p_k C_k \leq \beta < 1. \tag{4.20}
\]

(a) If \( P_T f^* = f^* \) and \( f^* \in BV \), then \( f^* \) is piecewise constant with respect to the partition \( \mathcal{P} \).

(b) For \( k = 1, \ldots, q \), each \( M_{\tau_k} \) is the Markov matrix induced by \( \tau_k \), then there exists an \( m \times m \) matrix \( M_T \) induced by \( T \). Let \( \tilde{f} \leftrightarrow f \in \mathcal{C} \approx \mathbb{R}^m \) be defined as in Equation (4.16). Define \( M_T : \mathbb{R}^m \to \mathbb{R}^m \) by
\[
\tilde{f} M_T = \sum_{k=1}^{q} p_k \tilde{f} M_{\tau_k},
\]
where \( \tilde{f} M_{\tau_k} = P_{\tau_k} f \) (see Equation (4.17)). Moreover, \( \tilde{f}^* \) associated with \( f^* \) is a right-eigenvector of \( M_T \).

(c) \( M_T \) has 1 as the eigenvalue of maximum modulus. All other eigenvalues \( \lambda \) satisfy \( |\lambda| \leq 1 \).

(d) If \( M_T \) is irreducible, \( f^* \) is the unique invariant density of \( P_T \) (up to constant multiples). It is equivalent to say that \( \tilde{f}^* \) is the unique right-eigenvector of \( M_T \).

**Proof.** (a) Consider \( P_T f^* = f^* \). Fix \( D_l \in \mathcal{P} \) and \( x \in D_l \). Since \( \tau_k \) is piecewise linear Markov, for all \( y \in D_l \) we have \( \chi_{\tau_{l,k}(D_l)}(y) = \chi_{\tau_{l,k}(D_l)}(y) \) and
\[
|J_{\tau_{l,k}}(y)| = |J_{\tau_{l,k}}|, \quad \forall y \in D_l.
\]

For any \( K \in \mathbb{N} \), let \( \vec{k} \in \{1, \ldots, q\}^K \),
\[
p_{\vec{k}} = p_{k_{K}} \cdot p_{k_{(K-1)}} \cdots p_{k_2} \cdot p_{k_1} \quad \text{and} \quad T_{\vec{k}}(x) = \tau_{k_{K}} \circ \tau_{k_{K-1}} \circ \cdots \circ \tau_{k_1}(x).
\]
For each $D_j \in \mathcal{P}$, $T_{j,k} = T_k|D_j$,

$$\|J_{T,j,k}^{-1}\|_\infty = \sup_{x \in D_j} |J_{T,j,k}^{-1}(x)| = \sup_{x \in D_j} \left( |\varphi'_{ij,k}(x_1)| \cdots |\varphi'_{ij,k}(x_n)| \right)^{-1} = \sup_{x \in D_j} \left( \prod_{i=1}^n |\varphi'_{ij,k}(x_i)|^{-1} \right)$$

$$\leq \left( \sup_{x \in D_j} \prod_{i=1}^n |\varphi'_{ij,k}(x_i)|^{-1} \right) \cdots \left( \sup_{x \in D_j} \prod_{i=1}^n |\varphi'_{ij,k}(x_i)|^{-1} \right)$$

$$= \|J_{T,j,k}^{-1}\|_\infty \cdots \|J_{T,j,k}^{-1}\|_\infty$$

$$= |J_{T,j,k}^{-1}| \cdots |J_{T,j,k}^{-1}|$$

$$\leq C_{k,k} \cdots C_{k,1} \equiv C_{k}.$$ 

Therefore,

$$|f^*(x) - f^*(y)| = |P_K^T f^*(x) - P_K^T f^*(y)| \leq \sum_k p_k |P_k f^*(x) - P_k f^*(y)|$$

$$\leq \sum_{j=1}^m \sum_{k} p_k \left| \frac{f^*(T_{j,k}^{-1} x)\chi_{T_{j,k}(D_j)}(x)}{|J_{T,j,k}^{-1}(T_{j,k}^{-1} x)|} - \frac{f^*(T_{j,k}^{-1} y)\chi_{T_{j,k}(D_j)}(y)}{|J_{T,j,k}^{-1}(T_{j,k}^{-1} y)|} \right|$$

$$= \sum_{j=1}^m \sum_{k} p_k \left| f^*(T_{j,k}^{-1} x)\chi_{T_{j,k}(D_j)}(x) - f^*(T_{j,k}^{-1} y)\chi_{T_{j,k}(D_j)}(y) \right| |J_{T,j,k}^{-1}|$$

$$\leq \sum_{j=1}^m \sum_{k} p_k \|J_{T,j,k}^{-1}\|_\infty \left| f^*(T_{j,k}^{-1} x)\chi_{T_{j,k}(D_j)}(x) - f^*(T_{j,k}^{-1} y)\chi_{T_{j,k}(D_j)}(y) \right|$$

$$\leq \sum_{j=1}^m \sum_{k} p_k C_k \left( \sum_{j=1}^m \left| f^*(T_{j,k}^{-1} x)\chi_{T_{j,k}(D_j)}(x) - f^*(T_{j,k}^{-1} y)\chi_{T_{j,k}(D_j)}(y) \right| \right)$$

$$\leq \sum_{k} p_k C_k V_{T^n} f^* = \sum_{k=1}^q p_{k,k} \cdots p_{k,1} C_{k,k} \cdots C_{k,1} V_{T^n} f^*$$

$$= \left( \sum_{k=1}^q p_{k,k} C_{k,k} \right) \cdots \left( \sum_{k=1}^q p_{k,1} C_{k,1} \right) V_{T^n} f^* \leq \beta^K V_{T^n} f^*.$$

Since $\beta < 1$ and $K$ is any integer, for all $y \in D_l$,

$$\lim_{K \to \infty} \|f^*(x) - f^*(y)\| \leq \lim_{K \to \infty} \beta^K V_{T^n} f^* = 0.$$

Thus, $f^*$ is constant on $D_l$. This is true for all $D_l \in \mathcal{P}$, so $f^*$ is piecewise constant.

(b) $T$ is a random dynamical system on the common partition $\mathcal{P}$. By Definition 3.3, $P_T f = \sum_{k=1}^q p_k P_{\tau_k} f$. By Proposition 4.54 (c), for every $f \leftrightarrow f \in \mathcal{C}$ and for each
$k = 1, 2, \cdots, q$, there exists a matrix $M_{\tau_k}$ such that $P_{\tau_k}f = \vec{f}M_{\tau_k}$. Thus,

$$P_Tf = \sum_{k=1}^{q} p_kP_{\tau_k}f = \sum_{k=1}^{q} p_k\vec{f}M_{\tau_k}.$$ 

Since $T$ is defined on the common partition, and $\sum_{k=1}^{q} p_kM_{\tau_k}$ is still a Markov matrix, for all $\vec{f} \leftrightarrow f \in \mathcal{C} \cong \mathbb{R}^m$, define $M_T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$\vec{f}M_T = \sum_{k=1}^{q} p_k\vec{f}M_{\tau_k}.$$ 

Hence, $M_T$ is an $m \times m$ matrix induced by $T$ constructed by piecewise linear Markov transformations, and $\vec{f}M_T = P_Tf$. Moreover, if $P_Tf^* = f^*$,

$$\vec{f}^*M_T = P_Tf^* = f^* \leftrightarrow \vec{f}^*.$$ 

Therefore, $\vec{f}^*$ is a right-eigenvector of $M_T$.

(c) This is a consequence of the Perron-Frobenius theorem (see Theorem A.6 in Appendix) applied to the Markov matrix $M_T$.

(d) We prove this property by getting a contradiction. Let $f_1, f_2 \in \mathcal{C}$ be invariant densities on $\mathcal{P}$. By Lemma 4.16, we can assume $f_1$ and $f_2$ are linear independent with disjoint support. Since $f_1, f_2 \in \mathcal{C}$; for each $i = 1, 2$, $S_i$ is a union of disjoint rectangles which are elements in $\mathcal{P}$. Assume $j_1, j_2 \in \{1, 2, \cdots, m\}$ such that

$$S_1 = \bigcup_{j_1} D_{j_1} \text{ and } S_2 = \bigcup_{j_2} D_{j_2}.$$ 

Since $M_T$ is irreducible, $\mathcal{P}$ has the communication property. Hence, there are $D_{i_0}, D_{j_0} \in \mathcal{P}$ with $m_n(D_{i_0}) > 0$ and $m_n(D_{j_0}) > 0$ such that

$$D_{i_0} \subset S_1 \text{ and } D_{j_0} \subset S_2.$$ 

Then there exist $u, v \in \mathbb{N}$, $\bar{k} \in \{1, \cdots, q\}^u$ and $\bar{h} \in \{1, \cdots, q\}^v$ such that

$$\left\{ \begin{array}{l}
\bigcup_{\bar{k}} T_{\bar{k}}(D_{i_0}) \supset D_{j_0}, \\
\bigcup_{\bar{h}} T_{\bar{h}}(D_{j_0}) \supset D_{i_0}.
\end{array} \right.$$ 

Since $S_1$ is an invariant set, $S_1 = \bigcup_{k=1}^{q} \tau_k(S_1)$ mod zero. Thus,

$$S_1 = \bigcup_{\bar{k}} T_{\bar{k}}(S_1) \text{ mod zero.}$$
For \( D_{j0} \subset S_2 \),

\[
D_{j0} \subset \bigcup_k T_k(D_{i0}) \subset \bigcup_k T_k(S_1) = S_1 \text{ mode zero}.
\]

Hence, \( D_{j0} \) is a subset of both sets \( S_1 \) and \( S_2 \) mod zero. Since \( D_{j0} \) has positive measure and \( S_1, S_2 \) are disjoint, it is impossible. Therefore, \( f_1 \) and \( f_2 \) must be linear dependent. In the other words, \( f_1 \) is equal to \( f_2 \) up to constant multiples. Thus, \( P_T \) has a unique invariant density.

\[
\begin{array}{c}
  \text{Figure 4.3: } \tau : [0, 1] \to [0, 1].
\end{array}
\]

**Example 4.57.** An example for a transformation has more than one invariant density. Assume \( \tau = \tau_1 \chi_{I_1} + \tau_2 \chi_{I_2} \) on the partition \( \mathcal{P} = \{ I_1 = [0, \frac{1}{2}), I_2 = [\frac{1}{2}, 1]\} \) by

\[
\tau(x) = \begin{cases}
  \tau_1(x) = x + \frac{1}{2}, & \forall x \in I_1 \\
  \tau_2(x) = x - \frac{1}{2}, & \forall x \in I_2.
\end{cases}
\]

Then \( P_\tau \) has more than one invariant function.

**Proof.** By the definition of \( \tau \), \( M_\tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) implies \( M_\tau^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). Let

\[
f(x) = \begin{cases}
  1, & \forall x \in (0, 1) \setminus \{\frac{1}{2}\} \\
  0, & x = 0, \frac{1}{2}, 1
\end{cases}
\]

and \( g(x) = 2x \mod 1, \forall x \in [0, 1] \).

Then,

\[
P_\tau f(x) = \sum_{\tau(y) = x} \frac{f(y)}{|\tau|} = \sum_{\tau(y) = x} f(y) = \begin{cases}
  1, & \forall x \in (0, 1) \setminus \{\frac{1}{2}\} \\
  0, & x = 0, \frac{1}{2}, 1
\end{cases} = f(x)
\]

so that \( \tilde{f} M_\tau = P_\tau f = f \); and

\[
P_\tau g(x) = \sum_{\tau(y) = x} \frac{g(y)}{|\tau|} = \sum_{\tau(y) = x} g(y) = g \left( x \pm \frac{1}{2} \right) = g(x).
\]

Hence, both \( f \) and \( 2g \) are invariant densities of \( P_\tau \).
Note that it is not a contradiction to the Perron-Frobenius theorem. If we restrict the space on \( C \), then \( P_\tau \) has a unique invariant density since \( g \) is not piecewise constant. Nor does it contradict Theorem 4.56 since the map \( \tau \) is not expanding.

Here we give an example on the space \( X = [-2, 2] \times [-3, 3] \) in two dimensions. It shows the possible case that even through each individual transformation of the random dynamical system \( T \) has more than one invariant density, we still can get a unique ACIM of \( T \) as long as the induced Markov matrix \( M_T \) is irreducible.

![Figure 4.4: Two transformations \( \tau_1 \) and \( \tau_2 \).](image)

**Example 4.58.** Consider the case in one dimension. Let partition \( P = \{ D_j \}_{j=1}^4 \), where \( D_1 = [0, \frac{1}{4}), D_2 = \left[ \frac{1}{4}, \frac{1}{2} \right), D_3 = \left[ \frac{1}{2}, \frac{3}{4} \right), D_4 = \left[ \frac{3}{4}, 1 \right] \). Define \( \tau_1, \tau_2 : [0,1] \to [0,1] \)

\[
\tau_1 = \begin{cases} 
2x, & x \in D_1 \\
2x - \frac{1}{2}, & x \in D_2 \\
2x - \frac{1}{2}, & x \in D_3 \\
2x - 1, & x \in D_4
\end{cases}; \quad \text{and} \quad \tau_2 = \begin{cases} 
2x, & x \in [0, \frac{1}{5}) \\
2x + \frac{1}{5}, & x \in \left[ \frac{1}{5}, \frac{1}{2} \right) \\
2x - \frac{1}{5}, & x \in D_2 \\
2x - \frac{3}{2}, & x \in [\frac{3}{2}, \frac{7}{4}) \\
2x - \frac{3}{2}, & x \in [\frac{3}{4}, \frac{7}{8}) \\
2x - 1, & x \in \left( \frac{7}{8}, 1 \right]
\end{cases}.
\]

Then for a random dynamical system \( T \) with \( p_1 + p_2 = 1 \), \( P_T f = p_1 P_{\tau_1} f + p_2 P_{\tau_2} f \), \( T \) has a unique ACIM (absolutely continuous invariant measure).

**Proof.** By the definition of \( \tau_1 \) and \( \tau_2 \), for all \( x \in [0,1] \), we have

\[ |J_{\tau_1}|(x) = 2, \text{ and } |J_{\tau_2}|(x) = 2. \]

Since

\[
\begin{align*}
\tau_1|_{D_1}(D_1) &= D_1 \cup D_2 = \tau_1|_{D_2}(D_2) \\
\tau_1|_{D_3}(D_3) &= D_3 \cup D_4 = \tau_1|_{D_4}(D_4)
\end{align*}
\]
we define $f_1 \equiv 1$ on $X$, and $f_2(x) = \begin{cases} 1 & \forall x \in D_1 \cup D_2 \\ 2 & \forall x \in D_3 \cup D_4 \end{cases}$. By Equation (2.4),

$$P_{\tau_1} f_1(x) = \sum_{\tau_1(y) = x} \frac{f_1(y)}{|J_{\tau_1}|(y)}$$

$$= \frac{1}{2} (f_1(y_1) + f_1(y_2)), \text{ where } \tau_1(y_1) = \tau_1(y_2) = x$$

$$= \frac{1}{2} (1 + 1) = 1 = f_1(x),$$

$$P_{\tau_1} f_2(x) = \sum_{\tau_1(y) = x} \frac{f_2(y)}{|J_{\tau_1}|(y)}$$

$$= \begin{cases} \frac{1}{2} (1 + 1) = 1, & \forall x \in D_1 \cup D_2 \\ \frac{1}{2} (2 + 2) = 2, & \forall x \in D_3 \cup D_4 \end{cases} = f_2(x).$$

Therefore, $P_{\tau_1} f_1 = f_1$ and $P_{\tau_1} f_2 = f_2$. By Proposition 2.8 (b), $\tau_1$ has at least two ACIMs. Similarly, we have

$$\tau_2|_{D_1(D_1)} = D_1 \cup D_4 = \tau_2|_{D_4(D_4)}$$

and $\tau_2|_{D_4(D_3)} = D_2 \cup D_3 = \tau_2|_{D_2(D_2)}$.

Define $g_1 \equiv 1$ on $[0, 1]$, and $g_2(x) = \begin{cases} 1 & \forall x \in D_1 \cup D_4 \\ 2 & \forall x \in D_2 \cup D_3 \end{cases}$. For the same argument as above, one can get that both $g_1$, $g_2$ are fixed points of $P_{\tau_2}$. That is, $\tau_2$ has at least two ACIMs. From this, we only have the conclusion that $T$ has at most two ACIMs. However, the Markov matrix induced by $T$ is irreducible. By Theorem 4.56 (d), $T$ has a unique ACIM. Moreover, the Markov matrices induced by $\tau_1$, $\tau_2$ and $T$ are shown as follows:

$$M_{\tau_1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad M_{\tau_2} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$M_T = p_1 M_{\tau_1} + p_2 M_{\tau_2} = \begin{bmatrix} \frac{p_1}{2} & \frac{p_2}{2} & 0 & \frac{p_2}{2} \\ \frac{p_2}{2} & \frac{p_1}{2} & \frac{p_2}{2} & 0 \\ \frac{p_2}{2} & 0 & \frac{p_1}{2} & \frac{p_1}{2} \\ \frac{p_2}{2} & 0 & \frac{p_1}{2} & \frac{p_1}{2} \end{bmatrix}.$$
Chapter 5

Asymptotic Behavior

In Chapter 4, we already know that every constrictive Markov operator has a spectral decomposition. From the Spectral Decomposition Theorem, it is easier for us to analyze the iteration of the given operator and its asymptotic behavior. In the first section, we focus on the situation of infinite iterations on one single operator. In Section 5.2, we are interested in a sequence of operators \( \{P_\eta\}_{\eta \in \mathbb{N}} \) and their approximation under \( \|\cdot\| \)-norm, as defined on Equation (5.2). For the last section, we use Ulam’s method to check the convergence of the sequence \( \{P_\eta\} \) and the approximation of an invariant \( f^* \) of the given operator \( P \).

5.1 Asymptotic Periodicity and Stability

In this section, we consider all the densities in \( L^1 \), denoted as

\[ \mathcal{D} = \{ f \in L^1 \mid f \geq 0, \|f\|_1 = 1 \}. \]

**Definition 5.1.** [29] *(asymptotic periodicity)*

If a Markov operator \( P : L^1 \to L^1 \) has a spectral representation as in Equation (4.1),

\[ Pf = \sum_{i=1}^{r} \lambda_i(f) g_{\sigma(i)} + PQf, \]

then we say the sequence \( \{P^N\} \) is asymptotically periodic.

By the definition of asymptotic periodicity and the Spectral Decomposition Theorem, we have the following theorem, see [22, Theorem 4.2].

**Theorem 5.2.** Let \( T = \{\tau_k; p_k\}_{k=1,\ldots,q} \) be a random dynamical system satisfying the Equation (3.5) which is constructed from piecewise \( C^2 \) Jabłoński transformations. Then the sequence \( \{P_T^N\}_{N \in \mathbb{N}} \) is asymptotically periodic.

**Proof.** By Proposition 4.3, \( P_T \) is constrictive. By the Spectral Decomposition Theorem 4.4, \( P_T \) has the spectral representation, as shown in Equation (4.1). Then \( \{P_T^N\}_{N \in \mathbb{N}} \) is asymptotically periodic. \( \square \)

By the Spectral Decomposition Theorem, an asymptotically periodic sequence \( \{P^N\} \) implies that for each density \( f \), the sequence of densities \( \{P^Nf\}_{N \in \mathbb{N}} \) is precompact.
That is, there exists a convergent subsequence. Hence, one may ask how to get the whole sequence to converge. Moreover, if all sequences \( \{ P^N f \}_{N \in \mathbb{N}} \) with different density \( f \) converges to the same invariant density \( f^* \) of \( P \), then \( \{ P^N \}_{N \in \mathbb{N}} \) is called \textit{asymptotically stable}. The definition is shown as below.

\textbf{Definition 5.3.} (asymptotic stability)

Let \( P : L^1 \rightarrow L^1 \) be a Markov operator. If

1. there exists a density \( f^* \) such that \( Pf^* = f^* \), and
2. for every density \( f \), \( \lim_{N \to \infty} \| P^N f - f^* \|_1 = 0 \).

Then the sequence \( \{ P^N \}_{N \in \mathbb{N}} \) is said to be asymptotically stable.

\textbf{Definition 5.4.} (exact)

Let \( \tau : I^n \rightarrow I^n \) be a nonsingular transformation, and \( P_\tau : L^1 \rightarrow L^1 \) be its Perron-Frobenius operator with \( P_\tau f^* = f^* \). If for every density \( f \), \( \{ P^N_\tau f \}_{N \in \mathbb{N}} \) strongly converges to \( f^* \) then the transformation \( \tau \) is called exact. We also call the operator \( P_\tau \) exact.

\textbf{Remark 5.5.} From above, we know that \( \tau \) (or \( P_\tau \)) is exact if and only if \( \{ P^N_\tau \}_{N \in \mathbb{N}} \) is asymptotically stable.

If a constrictive Markov operator \( P \) is asymptotically stable, then we get more information about the representation on Equation (4.1). \( P \) is asymptotically stable, which implies not only \( P \) has a unique invariant density but also equivalent to \( r = 1 \), the number of decompositions in the spectral representation. This idea is described as in Proposition 5.6 (see [29, Theorem 5.5.2 and 5.5.3], for the case \( f^* = 1 \)). Here we give the direct proof without using the condition \( P1 = 1 \).

\textbf{Proposition 5.6.} Let \( P \) be a constrictive Markov operator and \( r \) be the number of densities in its spectral representation. Then \( P \) is exact if and only if \( r = 1 \).

\textit{Proof.} First, we show that \( P \) is exact implies \( r = 1 \).

Suppose the order of \( \sigma \) is equal to \( l \) and \( r > 1 \). By the Spectral Decomposition Theorem,

\[ P^{NI} g_1 \to g_1 \] while \[ P^{NI} g_2 \to g_2. \]

Since \( g_1, g_2 \in \mathcal{D} \), and \( g_1 \neq g_2 \), we have a contradiction to the exact. Hence \( r = 1 \).

On the other hand, \( r = 1 \) means that there is only one density \( g \) in the spectral representation. Thus, for all \( N \in \mathbb{N} \) and for all \( f \in \mathcal{D} \),

\[ P^N f = \lambda(f)g + P^N Qf. \]

Hence, for all \( f \in \mathcal{D} \),

\[ 1 = \lim_{N \to \infty} \int P^N f = \int \lambda(f)g = \lambda(f) \int g = \lambda(f). \]
Therefore,
\[ \lim_{N \to \infty} \|P^N f - \lambda(f)g\|_1 = \lim_{N \to \infty} \|P^N f - g\|_1 = 0. \]

Hence, \(g\) is the unique invariant density of \(P\). \(\Box\)

The following theorem (see [22, Theorem 5.1]) is mainly dependent on Proposition 5.6. From this theorem, it allows us to analyze a random dynamical system \(T\) by studying its individual transformations.

**Theorem 5.7.** [22] Let \(T = \{\tau_k; p_k\}_{k=1}^q\) be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise \(C^2\) Jabłoński transformations. If any one of \(\{\tau_k\}_{k=1}^q\) is exact, then the sequence \(\{P_T^N\}_{N \in \mathbb{N}}\) is asymptotically stable.

**Proof.** By Proposition 4.3, \(P_T\) is constrictive. Since the random dynamical system satisfies (Equation 3.5), some \(\tau_k\) must satisfies condition (1) in Lemma 4.40. Without loss of generality, we assume \(\tau_1\) satisfies this condition and is exact. Hence, \(P_{\tau_1}\) is constrictive and \(r_1 = 1\) by Proposition 5.6. Moreover,

\[ P_T = \sum_{k=1}^q p_k P_{\tau_k} = p_1 P_{\tau_1} + (1 - p_1) \bar{P}, \]

where \(\bar{P} = \frac{1}{1 - p_1} \sum_{k=2}^q p_k P_{\tau_k}\) is a Markov operator. Assume \(T\) has \(\tilde{n}\) ergodic ACIMs, then by Lemma 4.27 and Theorem 4.49,

\[ 1 \leq \tilde{n} \leq r \leq r_1 = 1. \]

Therefore, \(P_T\) is exact, so \(\{P_T^N\}_{N \in \mathbb{N}}\) is asymptotically stable. \(\Box\)

**5.2 Stability of the Given Operator \(P\)**

In this section, we define a class of operators under some conditions so that we can approximate the given operator \(P\) by using a sequence of operators \(\{P_\eta\}_{\eta \in \mathbb{N}}\). The main result of this section is shown at Theorem 5.18.

**Definition 5.8.** Let \(P : L^1 \to L^1\) be an operator satisfying the following conditions

1. \(P \geq 0\) and for all \(f \in L^1\), \(\int Pfdm_n = \int f dm_n\) (implies \(\|P\|_1 = 1\)).
2. There exist constants \(0 < \beta < 1\) and \(C > 0\) such that for all \(f \in BV\),

\[ \|Pf\|_{BV} \leq \beta \|f\|_{BV} + C\|f\|_1. \]

(5.1)
3. The image of any bounded subset of \(BV\) under \(P\) is precompact in \(L^1\).

\(i.e. P : BV \to L^1\) is a compact operator.

Then define:
(a) \( \mathcal{S} \) to be the collection of all the operators satisfying conditions (1), (2) and (3).

(b) \( \mathcal{S}(\beta,C) \) to be a subclass of \( \mathcal{S} \) with fixed constants \( 0 < \beta < 1 \) and \( C > 0 \). For operator \( P : BV \to L^1 \) define \( \| \cdot \|_\text{-norm} \) as

\[
\| P \| \equiv \sup \{ \| Pf \|_1 : f \in BV, \| f \|_{BV} < 1 \}.
\] (5.2)

A sequence \( \{P_\eta\}_{\eta \in \mathbb{N}} \) of operators is called \( \mathcal{S} \)-bounded if there are constants \( C > 0 \) and \( 0 < \beta < 1 \) such that for all \( \eta \in \mathbb{N} \), \( P_\eta \in \mathcal{S}(\beta,C) \).

**Definition 5.9.** [33] (quasi-compact)
A linear operator \( P \) in a Banach space is quasi-compact if

(1) there is an \( M \), so \( \| P^N \| \leq M \) for all \( N \in \mathbb{N} \), and

(2) there is a compact operator \( Q \) and some power \( m > 0 \) such that \( \| P^m - Q \| < 1 \).

The following theorem is the basis for much modern analysis of the Perron-Frobenius operator in the literature. The original paper is [20]. A modern version in notation coincident with our setup may be found in Keller [24].

**Theorem 5.10.** (Ionescu-Tulcea and Marinescu Theorem)
Let \( P : L^1 \to L^1 \). If \( P \in \mathcal{S} \), then \( P \) is a quasi-compact operator on \((BV, \| \cdot \|_{BV})\). In particular,

(a) \( P \) has only finitely many eigenvalues \( \lambda_1, \ldots, \lambda_r \) of modulus 1. The corresponding eigenspaces \( E_i \) are finite-dimensional subspaces of \( BV \).

(b) There is a finite sequence of projections \( \{\Phi_i\}_{i=1}^r \) such that each \( \Phi_i \) projects \( L^1 \) onto the eigenspace \( E_i \). All \( \Phi_i \)'s satisfy \( \| \Phi_i \|_1 \leq 1 \), \( \Phi_i \circ \Phi_j = 0 \) for all \( i \neq j \), and an operator \( Q : L^1 \to L^1 \) to be linear with the following conditions

1. \( \sup_{N} \| Q^N \|_1 \leq r + 1 \) and \( Q(BV) \subseteq BV \).

2. For some bounded \( M > 0 \) and \( 0 < q < 1 \), for every \( N \in \mathbb{N} \),

\[
\| Q^N \|_{BV} \leq q^N M.
\]

3. For each \( i = 1, \ldots, r \), \( \Phi_i \circ Q = Q \circ \Phi_i = 0 \).

Then \( P \) has the following representation

\[
P = \sum_{i=1}^{r} \lambda_i \Phi_i + Q \tag{5.3}
\]

Moreover, for each \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and \( f \in L^1 \), the limit of the following exists in \( L^1 \).

\[
\Phi(\lambda, P)(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} (\lambda P)^j(f)
\] (5.4)
and
\[ \Phi(\lambda, P) = \begin{cases} \Phi_i & \text{if } \lambda = \lambda_i \\ 0 & \text{otherwise.} \end{cases} \]

From Theorem 5.10, we get the following properties.

**Lemma 5.11.** (a) For each \( i = 1, \cdots, r \), \( \Phi_i P = P \Phi_i = \lambda_i \Phi_i \), and \( QP = PQ = Q^2 \).

(b) For each \( N \in \mathbb{N} \), \( P^N = \sum_{i=1}^r \lambda_i^N \Phi_i + Q^N \).

**Proof.** (a) For each \( j = 1, \cdots, r \)
\[
P \Phi_j = \sum_{i=1}^r \lambda_i \Phi_i \Phi_j + Q \Phi_j = \lambda_j \Phi_j = \Phi_j P.
\]
\[
PQ = \sum_{i=1}^r \lambda_i \Phi_i Q + QQ = Q^2 = PQ.
\]

(b) By (a),
\[
P^N = P^{N-1} \left( \sum_{i=1}^r \lambda_i \Phi_i + Q \right) = \sum_{i=1}^r \lambda_i P^{N-1} \Phi_i + P^{N-1} Q
\]
\[
= \sum_{i=1}^r \lambda_i^2 P^{N-2} \Phi_i + P^{N-2} Q^2 = \cdots = \sum_{i=1}^r \lambda_i^N \Phi_i + Q^N.
\]

Now we describe the meaning of stable under a sequence of operators.

**Definition 5.12.** (stochastically prestable and stable)

(a) For each \( P \in \mathcal{T} \), \( P \) is called stochastically prestable if for each \( \mathcal{T} \)-bounded sequence \( \{P_\eta\}_{\eta \in \mathbb{N}} \), \( \lim_{\eta \to \infty} \|P_\eta - P\| = 0 \) implies
\[
\lim_{\eta \to \infty} \| (\Phi(1, P_\eta) - Id) \Phi(1, P) \|_1 = 0.
\]

(b) For each \( P \in \mathcal{T} \), \( P \) is called stochastically stable if for each \( \mathcal{T} \)-bounded sequence \( \{P_\eta\}_{\eta \in \mathbb{N}} \), \( \lim_{\eta \to \infty} \|P_\eta - P\| = 0 \) implies
\[
\lim_{\eta \to \infty} \| \Phi(1, P_\eta) - \Phi(1, P) \|_1 = 0.
\]

**Lemma 5.13.** By the definition, stochastically stable implies stochastically prestable.
Proof. Given $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $\eta \geq N_\epsilon$,

$$
\| (\Phi(1, P_\eta) - Id) \Phi(1, P) \|_1 = \| \Phi(1, P_\eta) \Phi(1, P) - \Phi(1, P) \|_1 \\
= \| \Phi(1, P_\eta) \Phi(1, P) - \Phi^2(1, P) \|_1 \\
\leq \| \Phi(1, P_\eta) - \Phi(1, P) \|_1 \| \Phi(1, P) \|_1 \\
< \epsilon \| \Phi(1, P) \|_1 \leq \epsilon.
$$

The main theorem to follow, Theorem 5.18 says $P \in \mathcal{F}$ is stable if and only if $P$ has a unique invariant density (up to constant multiples). Before getting into that point, we introduce two technical lemmas (see [24]) and give more details of the proofs.

**Lemma 5.14.** For $P, R \in \mathcal{F}$, assume $P = \sum_{i=1}^{r} \lambda_i \Phi_i + Q$. Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$
A_\lambda = \sum_{\lambda_i \neq \lambda \atop i=1}^{r} \frac{\bar{\lambda} \cdot \lambda_i}{\lambda - \bar{\lambda}_i} \Phi_i + (1 - \bar{\lambda}) \Phi(\lambda, P) + (\lambda Id - Q)^{-1}.
$$

Then we have the following properties:

(a) $A_\lambda$ is a bounded linear operator on $BV$.

(b) $A_\lambda = (\lambda Id - (P - \Phi(\lambda, P)))^{-1}$.

(c) $(\Phi(\lambda, P) - Id) \Phi(\lambda, R) = A_\lambda (P - R) \Phi(\lambda, R)$.

Proof. (a) From the definition of $A_\lambda$, it is enough to check if $(\lambda Id - Q)^{-1}$ is linearly bounded. First check $(\lambda Id - Q)$ is invertible. By Theorem 5.10 (b), (2), there exist $M > 0$ and $0 < q < 1$, for all $N \in \mathbb{N}$,

$$
\| Q^N \|_{BV} < q^N M.
$$

Let $S_N = (\lambda Id - Q) \left( \bar{\lambda} \sum_{j=0}^{N} (\bar{\lambda}Q)^j \right) - Id$. Then

$$
\| S_N \|_{BV} = \| (\lambda Id - Q) \left( \bar{\lambda} \sum_{j=0}^{N} (\bar{\lambda}Q)^j \right) - Id \|_{BV} \\
= \| \sum_{j=0}^{N} (\bar{\lambda}Q)^j - \bar{\lambda}Q \sum_{j=0}^{N} (\bar{\lambda}Q)^j - Id \|_{BV} \\
= \| \sum_{j=0}^{N} (\bar{\lambda}Q)^j - \sum_{j=0}^{N+1} (\bar{\lambda}Q)^j \|_{BV} \\
= \| (\bar{\lambda}Q)^{N+1} \|_{BV} \leq q^{N+1} M.
$$
Since $\|S_N\|_{BV} \to 0$ as $N \to \infty$,

$$(\lambda Id - Q) \left( \sum_{j=0}^{\infty} (\bar{\lambda}Q)^j \right) = Id \quad \text{and} \quad \left( \bar{\lambda} \sum_{j=0}^{\infty} (\bar{\lambda}Q)^j \right) (\lambda Id - Q) = Id.$$  

Hence, $\lambda Id - Q$ is invertible and $(\lambda Id - Q)^{-1} = \bar{\lambda} \sum_{j=0}^{\infty} (\bar{\lambda}Q)^j$. Therefore, $(\lambda Id - Q)^{-1}$ is linearly bounded so that $A_\lambda$ is also linearly bounded.

(b) By (a), we have $(\lambda Id - Q)^{-1} = \bar{\lambda} \sum_{j=0}^{\infty} (\bar{\lambda}Q)^j$. Define $S'_N$ as

$$S'_N \equiv \sum_{\lambda_j \neq \lambda} \frac{\bar{\lambda} \cdot \lambda_i}{\lambda - \lambda_i} \Phi_i + (1 - \bar{\lambda})\Phi(\lambda, P) + \bar{\lambda} \sum_{j=0}^{N} (\bar{\lambda}Q)^j.$$  

Then $S'_N \to A_\lambda$ as $N \to \infty$. Let $S'_N(\lambda Id - P + \Phi(\lambda, P)) \equiv C_1 + C_2 + C_N$.

$$C_1 \equiv \sum_{\lambda_j \neq \lambda} \frac{\bar{\lambda} \cdot \lambda_i}{\lambda - \lambda_i} \Phi_i(\lambda Id - P + \Phi(\lambda, P))$$

$$= \sum_{\lambda_j \neq \lambda} \frac{\lambda_i}{\lambda - \lambda_i} \Phi_i - \sum_{\lambda_j \neq \lambda} \frac{\bar{\lambda} \cdot \lambda_i}{\lambda - \lambda_i} \Phi_i P + \sum_{\lambda_j \neq \lambda} \frac{\bar{\lambda} \cdot \lambda_i}{\lambda - \lambda_i} \Phi_i\Phi(\lambda, P)$$

$$= \sum_{\lambda_j \neq \lambda} \frac{\lambda_i}{\lambda - \lambda_i} \Phi_i - \sum_{\lambda_j \neq \lambda} \frac{\bar{\lambda} \cdot \lambda_i^2}{\lambda - \lambda_i} \Phi_i + 0$$

$$= \sum_{\lambda_j \neq \lambda} \frac{\lambda_i}{\lambda - \lambda_i} (1 - \bar{\lambda} \cdot \lambda_i) \Phi_i = \sum_{\lambda_j \neq \lambda} \bar{\lambda} \cdot \lambda_i \Phi_i.$$

$$C_2 \equiv (1 - \bar{\lambda})\Phi(\lambda, P)(\lambda Id - P + \Phi(\lambda, P))$$

$$= \lambda(1 - \bar{\lambda})\Phi(\lambda, P) - (1 - \bar{\lambda})\Phi(\lambda, P)P + (1 - \bar{\lambda})\Phi(\lambda, P)\Phi(\lambda, P)$$

$$= \lambda(1 - \bar{\lambda})\Phi(\lambda, P) - \lambda(1 - \bar{\lambda})\Phi(\lambda, P) + (1 - \bar{\lambda})\Phi(\lambda, P)$$

$$= (1 - \bar{\lambda})\Phi(\lambda, P).$$
\( C_N \equiv \lambda^N \sum_{j=0}^{N} (\bar{\lambda} Q)^j (\lambda Id - P + \Phi(\lambda, P)) \)

\[
= \sum_{j=0}^{N} (\bar{\lambda} Q)^j - \lambda \sum_{j=0}^{N} (\bar{\lambda} Q)^j P + \lambda \sum_{j=0}^{N} (\bar{\lambda} Q)^j \Phi(\lambda, P) \\
= \sum_{j=0}^{N} (\bar{\lambda} Q)^j - \lambda P - \bar{\lambda} \sum_{j=0}^{N} (\bar{\lambda} Q)^j P + \bar{\lambda} \sum_{j=1}^{N} (\bar{\lambda} Q)^j \Phi(\lambda, P) \\
= \sum_{j=0}^{N} (\bar{\lambda} Q)^j - \lambda \left( \sum_{i=1}^{r} \lambda_i \Phi_i + Q \right) - \bar{\lambda} \sum_{j=1}^{N} \bar{\lambda} Q^{j+1} + \bar{\lambda} \Phi(\lambda, P) + 0 \\
= \sum_{j=0}^{N} (\bar{\lambda} Q)^j - \lambda \sum_{i=1}^{r} \lambda_i \Phi_i - \bar{\lambda} Q - \sum_{j=2}^{N+1} (\bar{\lambda} Q)^j + \bar{\lambda} \Phi(\lambda, P) \\
= \sum_{j=0}^{N} (\bar{\lambda} Q)^j - \lambda \sum_{i=1}^{r} \lambda_i \Phi_i - \sum_{j=1}^{N+1} (\bar{\lambda} Q)^j + \bar{\lambda} \Phi(\lambda, P) \\
= Id - (\bar{\lambda} Q)^{N+1} - \sum_{\lambda_i \neq \lambda}^{r} \bar{\lambda} \cdot \lambda_i \Phi_i - \Phi(\lambda, P) + \bar{\lambda} \Phi(\lambda, P) \\
= Id - (\bar{\lambda} Q)^{N+1} - \sum_{\lambda_i \neq \lambda}^{r} \bar{\lambda} \cdot \lambda_i \Phi_i - (1 - \bar{\lambda}) \Phi(\lambda, P). \\
\]

Sum of them,

\[
\sum C_1 + C_2 + C_N = \sum_{\lambda_i \neq \lambda}^{r} \bar{\lambda} \cdot \lambda_i \Phi_i + (1 - \bar{\lambda}) \Phi(\lambda, P) \\
\begin{align*}
&+ Id - (\bar{\lambda} Q)^{N+1} - \sum_{\lambda_i \neq \lambda}^{r} \bar{\lambda} \cdot \lambda_i \Phi_i - (\bar{\lambda} - 1) \Phi(\lambda, P) \\
= Id - (\bar{\lambda} Q)^{N+1}.
\end{align*}
\]

\[
\|S_N' \cdot (\lambda Id - P + \Phi(\lambda, P)) - Id\|_{BV} = \| - (\bar{\lambda} Q)^{N+1}\|_{BV} \\
= \|Q^{N+1}\|_{BV} \leq M \cdot q^{N+1} \rightarrow 0.
\]

Therefore, we have

\[
A_\lambda = (\lambda Id - (P - \Phi(\lambda, P)))^{-1}.
\]

(c) From (b), \( A_\lambda = (\lambda Id - (P - \Phi(\lambda, P)))^{-1} = \bar{\lambda} \sum_{j=0}^{\infty} \bar{\lambda}^j (P - \Phi(\lambda, P))^j. \)
Let \( S_N^r \equiv \bar{\lambda} \sum_{j=0}^N \bar{\lambda}^j (P - \Phi(\lambda, P))^j \). Then \( S_N^r \to A_\lambda \) and

\[
S_N^r (P - R)\Phi(\lambda, R) = S_N^r (P\Phi(\lambda, R) - R\Phi(\lambda, R))
= S_N^r (\lambda \Phi(\lambda, P)\Phi(\lambda, R) - \lambda \Phi(\lambda, R))
= \sum_{j=0}^N \bar{\lambda}^j (P - \Phi(\lambda, P))^j (\Phi(\lambda, P) - Id)\Phi(\lambda, R)
= (\Phi(\lambda, P) - Id)\Phi(\lambda, R) + \sum_{j=1}^N \bar{\lambda}^j (\Phi(\lambda, P) - Id)\Phi(\lambda, R)
\equiv (\Phi(\lambda, P) - Id)\Phi(\lambda, R) + \sum_{j=1}^N \bar{\lambda}^j C(j),
\]

where

\[
C(1) = (P - \Phi(\lambda, P)) (\Phi(\lambda, P) - Id)\Phi(\lambda, R)
= (P\Phi(\lambda, P) - P - \Phi^2(\lambda, P) + \Phi(\lambda, P))\Phi(\lambda, R)
= \left( \lambda \Phi(\lambda, P) - \sum_{i=1}^r \lambda_i \Phi_i - Q \right) \Phi(\lambda, R)
= \lambda \Phi(\lambda, P)\Phi(\lambda, R) - \lambda \Phi(\lambda, P)\Phi(\lambda, R) - Q\Phi(\lambda, R)
= -Q\Phi(\lambda, R) = 0.
\]

\[\implies C(j) = (P - \Phi(\lambda, P))^{j-1}C(1) = 0, \forall j \in \mathbb{N}.\]

Thus, \( S_N^r (P - R)\Phi(\lambda, R) = (\Phi(\lambda, P) - Id)\Phi(\lambda, R). \) Therefore,

\( A_\lambda (P - R)\Phi(\lambda, R) = (\Phi(\lambda, P) - Id)\Phi(\lambda, R). \)

\( \square \)

**Lemma 5.15.** [24] Let \( P \) in \( \mathfrak{T} \) and \( R \) in \( \mathfrak{T}(\beta, C) \). For a fixed \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \), assume \( P = \sum_{i=1}^r \lambda_i \Phi_i + Q \). Then there are three constants \( B_P \) (dependent on \( P \) only),

\[ B_{P,\lambda} \equiv \sum_{\lambda_i \neq \lambda} \frac{1}{|\lambda - \lambda_i|} + |1 - \bar{\lambda}| \cdot \|\Phi(\lambda, P)\|_1 \]

and \( \Gamma = \frac{C}{1-\beta} \) such that \( \|P - R\| \leq 1 \), which implies

\[ \|(\Phi(\lambda, P) - Id)\Phi(\lambda, R)\|_1 \leq (B_P + B_{P,\lambda})\Gamma \|P - R\| \left( 2 + \frac{\ln \|P - R\|}{\ln q} \right). \]
Proof. By Equation (5.4), for each $f \in BV$, since

$$\|\Phi(\lambda, R)f\|_{BV} \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \|R^j f\|_{BV}$$

$$\leq \limsup_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} \left( \beta^j \|f\|_{BV} + \frac{C}{1-\beta} \|f\|_1 \right)$$

$$\leq \limsup_{N \to \infty} \left( \frac{\|f\|_{BV}}{N(1-\beta)} + \frac{C}{1-\beta} \|f\|_1 \right) = \Gamma \|f\|_1.$$

(5.5)

Since $\|Q^N\|_{BV} \leq Mq^N$, we have

$$(\lambda Id - Q)^{-1} = \lambda \sum_{j=0}^{\infty} (\lambda Q)^j = \lambda \sum_{j=0}^{N-1} (\lambda Q)^j + (\lambda Q)^N (\lambda Id - Q)^{-1}.$$ \hspace{1cm} (5.6)

$$\| (\lambda Id - Q)^{-1} \|_{BV} \leq \left\| \lambda \sum_{j=0}^{N-1} (\lambda Q)^j \right\|_{BV} + \left\| (\lambda Q)^N (\lambda Id - Q)^{-1} \right\|_{BV}$$

$$\leq \sum_{j=0}^{N-1} \|Q^j\|_{BV} + \|(\lambda Id - Q)^{-1}\|_{BV} \|Q^N\|_{BV}$$

$$\leq \sum_{j=0}^{N-1} q^j M + (q^N M) \|(\lambda Id - Q)^{-1}\|_{BV}$$

$$\leq \frac{M}{1-q} + (q^N M) \|(\lambda Id - Q)^{-1}\|_{BV} \to M \frac{1}{1-q}.$$ 

Therefore, By Lemma 5.14 (c),

$$L \equiv \|(\Phi(\lambda, P) - Id)\Phi(\lambda, R)f\|_1 = \|A_{\lambda}(P-R)\Phi(\lambda, R)f\|_1$$

$$\leq \left\| \sum_{\lambda_i \neq \lambda} \frac{\tilde{\lambda} \cdot \lambda_i}{\lambda - \lambda_i} \Phi_i(P-R)\Phi(\lambda, R)f\right\|_1$$

$$+ \| (1 - \tilde{\lambda}) \Phi(\lambda, P)(P-R)\Phi(\lambda, R)f\|_1 + \| (\lambda \cdot Id - Q)^{-1}(P-R)\Phi(\lambda, R)f\|_1$$

$$\leq \left( \sum_{\lambda_i \neq \lambda} \frac{1}{|\lambda - \lambda_i|} \left\| \Phi_i\right\|_1 + |1 - \tilde{\lambda}| \cdot \|\Phi(\lambda, P)\|_1 \right) \|(P-R)\Phi(\lambda, R)f\|_1$$

$$+ \| (\lambda Id - Q)^{-1}(P-R)\Phi(\lambda, R)f\|_1 \leq B_{P,\lambda} \|(P-R)\Phi(\lambda, R)f\|_1 + S,$$
where $S \equiv \| (\lambda Id - Q)^{-1}(P - R)\Phi(\lambda, R)f \|_1$. By Equation (5.6),

$$S \equiv \left\| \left( \frac{\lambda}{1} \sum_{j=0}^{N-1} (\lambda Q)^j + (\lambda Q)^N (\lambda Id - Q)^{-1} \right) (P - R)\Phi(\lambda, R)f \right\|_1$$

$$\leq \sum_{j=0}^{N-1} \| (\lambda Q)^j \|_1 \| (P - R)\Phi(\lambda, R)f \|_1 + \| (\lambda Q)^N (\lambda Id - Q)^{-1}(P - R)\Phi(\lambda, R)f \|_1$$

$$\leq N(r + 1)\| (P - R)\Phi(\lambda, R)f \|_1 \quad \text{(by Theorem 5.10 (b), (1))}$$

$$+ \| P - R \| \cdot \| (\lambda Q)^N (\lambda Id - Q)^{-1}\Phi(\lambda, R)f \|_{BV}$$

$$\leq N(r + 1)\| (P - R)\Phi(\lambda, R)f \|_1 + (q^N M) \left( \frac{M}{1 - q} \right) \| \Phi(\lambda, R)f \|_{BV}.$$  

By Equation (5.5),

$$L \leq (B_{P,\lambda} + N(r + 1))\| (P - R)\Phi(\lambda, R)f \|_1 + \frac{q^N M^2}{1 - q} \| \Phi(\lambda, R)f \|_{BV}$$

$$\leq (B_{P,\lambda} + N(r + 1))\| P - R \| \cdot \| \Phi(\lambda, R)f \|_{BV} + \frac{q^N M^2}{1 - q} \| \Phi(\lambda, R)f \|_{BV}$$

$$\leq \Gamma \left( (B_{P,\lambda} + N(r + 1))\| P - R \| + \frac{q^N M^2}{1 - q} \right) \| f \|_1$$

$$\leq \Gamma(B_{P,\lambda} + B_P)(N\| P - R \| + q^N)\| f \|_1.$$  

for a suitable $B_P$. Then, we can choose $N \in \mathbb{N}$ such that $N\| P - R \| + q^N$ has the minimum value. Take the first derivative equal to zero, then

$$N = \frac{\ln \| P - R \| - \ln(\ln \frac{1}{q})}{\ln q} = \frac{\ln \| P - R \|}{\ln q} + \frac{\ln(\ln \frac{1}{q})}{\ln \frac{1}{q}}.$$  

Since $N \in \mathbb{N}$, take $N = \left[ \frac{\ln \| P - R \|}{\ln q} \right] + 1$ to complete the proof. \(\Box\)

By Lemma 5.15, $\| (\Phi(\lambda, P) - Id) \cdot \Phi(\lambda, P_\eta) \|_1 = O(\| P - P_\eta \| \cdot |\ln \| P - P_\eta \||)$. From this point of view, we get Theorem 5.16 and 5.17 from [24]. One can check the proof in the reference.

**Theorem 5.16.** [24] Let $P$ be an operator in $\mathcal{X}$. For a $\mathcal{X}$-bounded sequence $\{P_\eta\}_{\eta \in \mathbb{N}}$, let $\lim_{\eta \to \infty} \| P - P_\eta \| = 0$. Then

(a) $\| (\Phi(\lambda, P) - Id) \cdot \Phi(\lambda, P_\eta) \|_1 = O(\| P - P_\eta \| \cdot |\ln \| P - P_\eta \||) \to 0.$

(b) If $P$ is ergodic (i.e. $\dim \{ f : Pf = f \} = 1$), then

$$\| \Phi(1, P) - \Phi(1, P_\eta) \|_1 = O(\| P - P_\eta \| \cdot |\ln \| P - P_\eta \||) \to 0.$$  

Moreover, each $P_\eta$ is also ergodic when $\| (\Phi(1, P) - \Phi(1, P_\eta)) \|_1 < 1.$
Theorem 5.17. [24] Assume that $P \in \mathfrak{T}$, $\{P_\eta\}$ is $\mathfrak{T}$-bounded and
\[
\lim_{\eta \to \infty} \|P - P_\eta\| = 0.
\]
Set $d \equiv \dim (\Phi(1,P)(L^1))$, $d_\eta \equiv \dim (\Phi(1,P_\eta)(L^1))$. Then
(a) $\limsup_{\eta \to \infty} d_\eta \leq d$.

(b) The following are equivalent:
   (1) $\liminf_{\eta \to \infty} d_\eta \geq d$.
   (2) $\lim_{\eta \to \infty} d_\eta = d$.
   (3) $\lim_{\eta \to \infty} \|\Phi(1,P)f - \Phi(1,P_\eta)f\|_1 = 0$, $\forall f \in L^1$.
   (4) $\lim_{\eta \to \infty} \|\Phi(1,P) - \Phi(1,P_\eta)\|_1 = 0$.

We get the main theorem which follows from Theorem 5.16 and 5.17.

Theorem 5.18. [24] Let $P$ be an operator in $\mathfrak{T}$. Then $P$ is stochastically stable if and only if $\dim \{f : Pf = f\} = 1$. It means that $P$ has a unique invariant density.

Proof. If $\dim \{f : Pf = f\} = 1$, then $P$ is stochastically stable. This follows from Theorem 5.16 (b). On the other direction, assume $P \in \mathfrak{T}(\beta,C)$ is stochastically stable. Following Keller’s method, we construct a special sequence of operators $\{P_\eta\}$ such that $P_\eta \in \mathfrak{T}(\beta,C)$ and $\|P - P_\eta\| \to 0$. Without lost of generality, we can assume $C > 1$. Define
\[
P_\eta f = \frac{\eta - 1}{\eta} Pf + \frac{1}{\eta} \int f dm_n.
\]
Therefore,
\[
\|P_\eta f - Pf\|_1 = \left\| -\frac{1}{\eta} Pf + \frac{1}{\eta} \int f dm_n \right\|_1 \leq \frac{1}{\eta} (\|Pf\|_1 + \|f\|_1)
\leq \frac{2}{\eta} \|f\|_1 \leq \frac{2}{\eta} \|f\|_{BV}.
\Rightarrow \|P - P_\eta\| \leq \frac{2}{\eta} \to 0;
\]
\[
\|P_\eta f\|_{BV} \leq \frac{\eta - 1}{\eta} \|Pf\|_{BV} + \frac{1}{\eta} \left\| \int f dm_n \right\|_{BV}
\leq \left( 1 - \frac{1}{\eta} \right) (\beta \|f\|_{BV} + C \|f\|_1) + \frac{1}{\eta} \|f\|_1
\leq \beta \|f\|_{BV} + \left( \left( 1 - \frac{1}{\eta} \right) C + \frac{1}{\eta} \right) \|f\|_1 \leq \beta \|f\|_{BV} + C \|f\|_1.
\Rightarrow P_\eta \in \mathfrak{T}(\beta,C).
Now check \( P_\eta \) is ergodic. Suppose \( f_1 \) and \( f_2 \) are invariant densities of \( P_\eta \), then \( f_1 - f_2 \) is an invariant function. Since \( \int f_1 = 1 = \int f_2 \), \( \int (f_1 - f_2) = 0 \). Let \( h \equiv f_1 - f_2 \), then \( P_\eta h = h \) and \( \int h \, dm_n = 0 \). In this case, we have

\[
P_\eta h = \frac{\eta - 1}{\eta} Ph + \frac{1}{\eta} \int h \, dm_n = \frac{\eta - 1}{\eta} Ph + 0 = h.
\]

This implies

\[
h = \frac{\eta - 1}{\eta} Ph;
\]

\[
\|h\|_1 = \left\| \frac{\eta - 1}{\eta} Ph \right\|_1 \leq \frac{\eta - 1}{\eta} \|h\|_1 \implies \|h\|_1 = 0.
\]

Hence \( f_1 = f_2 \). (i.e. \( P_\eta \) is ergodic.) It means \( d_\eta = \dim (\Phi(1, P_\eta)(L^1)) = 1 \). Now we can apply Theorem 5.17 (b) so that stochastically stable \( P \) implies

\[
d = \lim_{\eta \to \infty} d_\eta = 1. \text{ (i.e. } P \text{ is ergodic).}
\]

Therefore,

\[
\dim (\Phi(1, P)(L^1)) = 1.
\]

5.3 Approximation by Piecewise Constant Functions

In this section, we consider the space of piecewise constant functions and use Ulam’s approximation, in [31, 37], to find an invariant density.

For any positive integer \( \eta \), let \( I^n \) be divided into \( \eta^n \) subsets, \( \{I_j\}_{j=1}^{\eta^n} \), of equal measure. Each \( I_j \) is defined by

\[
I_j = \left[ \frac{r_1}{\eta}, \frac{r_1 + 1}{\eta} \right] \times \cdots \times \left[ \frac{r_n}{\eta}, \frac{r_n + 1}{\eta} \right]
\]

for some \( r_1, \cdots, r_n = 0, 1, \cdots, \eta - 1 \) and every \( m_n(I_j) = \eta^{-n} \).

**Definition 5.19.** Let \( \Delta_\eta \) be the \( \eta^n \)-dimensional linear subspace of \( L^1 \) generated by the characteristic functions of the \( I_j \). Thus, \( \Delta_\eta \) is the finite-dimensional space generated by \( \{\chi_j \equiv \chi_{I_j}\}_{j=1}^{\eta^n} \). That is,

\[
f \in \Delta_\eta \text{ if and only if } f = \sum_{j=1}^{\eta^n} a_j \chi_j \text{ for some constants } a_1, \cdots, a_{\eta^n}.
\]

**Definition 5.20.** Let \( T = \{\tau_k; p_k\}_{k=1,\ldots,q} \) be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise \( C^2 \) Jabłoński transformations, and \( P_T \) be its Perron-Frobenius operator. Define two sequences of operators \( \{Q_\eta\}_{\eta \in \mathbb{N}} \) and \( \{P_\eta\}_{\eta \in \mathbb{N}} \) as following.
(1) Let $Q_\eta : L^1 \to \Delta_\eta$, for all $f \in L^1$,

$$(Q_\eta f)(y) = \sum_{j=1}^{\eta^n} \left( \frac{1}{m_n(I_j)} \int_{I_j} f(x) dm_n(x) \right) \chi_j(y).$$

(2) Let $P_\eta : \Delta_\eta \to \Delta_\eta$, for every $\chi_t \in \Delta_\eta$,

$$(P_\eta \chi_t)(y) = \sum_{k=1}^{q} p_k \left( \sum_{j=1}^{\eta^n} \frac{m_n(\tau_k^{-1}(I_j) \cap I_t)}{m_n(I_j)} \chi_j(y) \right).$$

Lemma 5.21 is from [9, Lemma 2, 3] or Bose [4]. We will show the proof as following.

**Lemma 5.21.** Let sequences $\{P_\eta\}_{\eta \in \mathbb{N}}$ and $\{Q_\eta\}_{\eta \in \mathbb{N}}$ be defined as in Definition 5.20. Then we have:

(a) For every $f \in L^1$, the sequence $\{Q_\eta f\}_{\eta \in \mathbb{N}}$ converges to $f$ in $L^1$.

(b) For each $\eta \in \mathbb{N}$ and for all $f \in \Delta_\eta$,

$$P_\eta f = Q_\eta \circ P_T f.$$  

**Proof.** (a) By the definition of $Q_\eta f$, it is clear that $Q_\eta f$ converges to $f$ in $L^1$ by using the Martingale convergence theorem.

(b) Since $P_T = \sum_{k=1}^{q} p_k P_{\tau_k}$, for each $\chi_t \in \Delta_\eta$,

$$(Q_\eta P_T \chi_t)(y) = \sum_{k=1}^{q} \sum_{j=1}^{\eta^n} \left( \frac{p_k}{m_n(I_j)} \int_{I_j} P_{\tau_k} \chi_t(x) dm_n(x) \right) \chi_j(y)$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{\eta^n} \left( \frac{p_k}{m_n(I_j)} \int_{\tau_k^{-1}(I_j)} \chi_t(x) dm_n(x) \right) \chi_j(y)$$

$$= \sum_{k=1}^{q} \sum_{j=1}^{\eta^n} \left( \frac{p_k}{m_n(I_j)} \cdot m_n(\tau_k^{-1}(I_j) \cap I_t) \right) \chi_j(y)$$

Therefore, $P_\eta f = Q_\eta \circ P_T f$ on $\Delta_\eta$.

We also need to consider the effect of averaging by the operator $Q_\eta$ on variation. The result, that variation is not increased, is recorded in the following Proposition 5.25. The proof will require a number of preliminary computations which we now present.
Lemma 5.22. Suppose \( f : [a, b] \to (-\infty, \infty) \) and let \( V_{[a,b]} f \) denote the (regular, one-dimensional) variation of \( f \) on \([a, b]\). Then \( V_{[a,b]}(Q_\eta f) \leq V_{[a,b]} f \).

Proof. Since \( Q_\eta \) is piecewise constant, we easily compute
\[
V_{[a,b]}(Q_\eta f) = \sum_{j=1}^{\eta-1} |C_{j+1} - C_j|,
\]
and we need to estimate the terms in the sum. We first divide the \( C_j \) into increasing and decreasing runs as follows
\[
C_1 \leq C_2 \cdots \leq C_{j_1} \geq C_{j_1+1} \geq \cdots \geq C_{j_2} \leq \cdots
\]
Note, it may happen \( j_1 = 1 \) if there is no increase to start the run. We set \( j_0 = 1 \) and \( j_k = \eta \) in any event. Now define quantities \( m_i, M_i \in I_{j_i}, \ i = 0, 1, 2, \ldots k \) so that
\[
f(m_i) \leq C_{j_i} \leq f(M_i).
\]
Now it is easy to see that
\[
\sum_{j=j_0}^{j_1-1} |C_{j+1} - C_j| \leq f(M_1) - f(m_0) = |f(M_1) - f(m_0)| := \Delta_1
\]
\[
\sum_{j=j_1}^{j_2-1} |C_{j+1} - C_j| \leq f(M_1) - f(m_2) = |f(m_2) - f(M_1)| := \Delta_2
\]
\[:
\]
and so on. It follows that
\[
\sum_{j=1}^{\eta-1} |C_{j+1} - C_j| \leq \sum_{i=1}^{k} \Delta_i \leq V_{[a,b]} f.
\]
\[\square\]

Our next lemma concerns the connection between one-dimensional variation and averaging in orthogonal coordinates.

Lemma 5.23. Let \( x = (x, y) \in I^n \) with \( x \) representing the \( i^{th} \) coordinate and \( y = \pi_i x \) representing the remaining coordinates. Consider a rectangle \([0, 1] \times A\) where \( A \) is an \((n-1)\) dimensional rectangle in the \( y\)-coordinate. Let \( V_x \) denote the variation in the \( x\)-direction. Then
\[
V_x \left( \frac{1}{m_{n-1}(A)} \int_A f(x, y) dm_{n-1}(y) \right) \leq \frac{1}{m_{n-1}(A)} \int_A V_x f(x, y) dm_{n-1}(y).
\]
Proof. Fix \( 0 = x_0 < x_1 < \ldots x_m = 1 \) to be a partition with respect to which we will compute the discrete variation. Then

\[
S = \sum_{i=1}^{m} \frac{1}{m_{n-1}(A)} \left| \int_{A} f(x_i, y) dm_{n-1}(y) - \frac{1}{m_{n-1}(A)} \int_{A} f(x_{i-1}, y) dm_{n-1}(y) \right|
\]

\[
\leq \sum_{i=1}^{m} \frac{1}{m_{n-1}(A)} \int_{A} |f(x_i, y) - f(x_{i-1}, y)| dm_{n-1}(y)
\]

\[
= \frac{1}{m_{n-1}(A)} \int_{A} \sum_{i=1}^{m} |f(x_i, y) - f(x_{i-1}, y)| dm_{n-1}(y)
\]

\[
\leq \frac{1}{m_{n-1}(A)} \int_{A} V_x f(x, y) dm_{n-1}(y)
\]

as claimed. \( \square \)

It is convenient at this point to introduce a temporary notation expressing \( Q_\eta \) as a product of lower-dimensional projections. Fix a coordinate index \( i \). Note that the partition \( \pi_i(I_j) \) of \( I^{n-1} \) obtained by projection along the \( i^{th} \) coordinate contains \( \eta^{n-1} \) cells of equal measure \( \eta^{-(n-1)} \). Let us denote these cells by \( A_j, j = 1, 2, \ldots, \eta^{(n-1)} \) and let \( Q_\eta^{(n-1)} \) denote the projection obtained by averaging over cells \( A_j \) leaving a function of \( x \), while \( Q_\eta^{1} \) denotes averages with respect to cells \( [\frac{j-1}{\eta}, \frac{j}{\eta}] \) partitioning the cube according to the \( i^{th} \) coordinate and leaving a function of \( y = \pi_i x \), where \( x = (x, y) \).

As an example of the use of this notation, it follows from Fubini’s Theorem that

\[
Q_\eta f = Q_\eta^{1} Q_\eta^{(n-1)} f = Q_\eta^{(n-1)} Q_\eta^{1} f.
\]

**Lemma 5.24.** In the notation above and with \( x = x_i \) representing the \( i^{th} \) coordinate, we have

\[
V_x Q_\eta^{(n-1)} f(y) \leq Q_\eta^{(n-1)} V_x f(y)
\]

pointwise over the range of \( y \in \pi_i(I^n) \).

**Proof.** Fix \( y \in \pi_i(I^n) \) and unique \( k \) so \( y \in A_k \).

\[
V_x (Q_\eta^{(n-1)} f)(y) = V_x \left( \sum_{j=1}^{\eta^{n-1}} \frac{1}{m_{n-1}(A_j)} \int_{A_j} f(x, z) dm_{n-1}(z) \cdot \chi_{[0,1]\times A_j}(x, y) \right)
\]

\[
= V_x \left( \frac{1}{m_{n-1}(A_k)} \int_{A_k} f(x, z) dm_{n-1}(z) \right)
\]

\[
\leq \frac{1}{m_{n-1}(A_k)} \int_{A_k} V_x f(x, z) dm_{n-1}(z)
\]

\[
= \sum_{j=1}^{\eta^{n-1}} \frac{1}{m_{n-1}(A_j)} \int_{A_j} V_x f(x, z) dm_{n-1}(z) \cdot \chi_{[0,1]\times A_j}(x, y)
\]

\[
= Q_\eta^{(n-1)} V_x f(y),
\]
where we have used Lemma 5.23 to obtain the inequality.

\[ \square \]

**Proposition 5.25.** [4] Let \( f : I^n \to (-\infty, \infty) \) with \( f \in BV(I^n) \) with respect to Tonelli Variation. Then
\[
V_{I^n}(Q_\eta f) \leq V_{I^n} f.
\]

**Proof.** Let \( \epsilon \) be given and fix a coordinate index \( i \). Once again denote the coordinates \((x, y) \in I \times I^{(n-1)}\) where \( x \) denotes the \( i - th \) coordinate. Choose \( \tilde{f} \) so
\[
\int_{\pi_i(I^n)} V_x \tilde{f}(x, y) \, dm_{n-1}(y) \leq V_{I^n, i} f + \epsilon.
\]

Note that \( Q_\eta \tilde{f} = Q_\eta f \). Then
\[
V_{I^n, i}(Q_\eta f) \leq \int_{\pi_i(I^n)} V_x (Q_\eta \tilde{f}(x, y)) \, dm_{n-1}(y)
\]
\[
= \int_{\pi_i(I^n)} V_x (Q_\eta \tilde{f}(x, y)) \, dm_{n-1}(y)
\]
\[
= \int_{\pi_i(I^n)} V_x (Q_\eta^{(n-1)} \tilde{f}(x, y)) \, dm_{n-1}(y)
\]
\[
\leq \int_{\pi_i(I^n)} V_x (Q_\eta^{(n-1)} \tilde{f}(x, y)) \, dm_{n-1}(y)
\]
by an application of Lemma 5.22. Finally we use Lemma 5.24 to continue the estimate
\[
\int_{\pi_i(I^n)} V_x Q_\eta^{(n-1)} \tilde{f}(x, y) \, dm_{n-1}(y) \leq \int_{\pi_i(I^n)} Q_\eta^{(n-1)} V_x \tilde{f}(x, y) \, dm_{n-1}(y)
\]
\[
= \int_{\pi_i(I^n)} V_x \tilde{f}(x, y) \, dm_{n-1}(y)
\]
\[
\leq V_{I^n, i} f + \epsilon.
\]

Since \( \epsilon \) is arbitrary, this completes the proof. \( \square \)

Finally, we need to consider the sense in which \( Q_\eta f \to f \) as \( \eta \to \infty \). This is provided by the following.

**Lemma 5.26.** For each \( f \in BV(I^n) \) and \( \eta \geq 0 \) we have
\[
\|Q_\eta f - f\|_1 \leq \frac{1}{\eta} n^{3/2} V_{I^n} f
\]
in other words, \( \|\iota - Q_\eta\| = O(\frac{1}{\eta}) \), where \( \iota : BV \hookrightarrow L^1 \).
Proof. For $f \in BV_I^n$ we know, from Lemma 2.18 that $GV_I^n(f) \leq nV_I^n(f)$. It is well-known that with respect to generalized variation (see Murray [35, Lemma A.1])

$$\|Q_\eta f - f\|_1 \leq \frac{1}{\eta} \sqrt{n} GV_I^n f.$$  

These two results provide the estimate. \hfill \Box

Now we need extra condition, $0 < \alpha < \frac{1}{2}$, to get the following lemma, see [9, Lemma 7].

Lemma 5.27. Let $T = \{\tau_k; p_k\}_{k=1,\ldots,q}$ be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise $C^2$ Jabłoński transformations, and $P_T$ be its Perron-Frobenius operator. For the sequences $\{P_\eta\}_{\eta \in \mathbb{N}}$ and $\{Q_\eta\}_{\eta \in \mathbb{N}}$ defined in Definition 5.20, if $0 < \alpha < \frac{1}{2}$, then

(a) $P_T \in \mathfrak{T}$.

(b) The sequence $\{P_\eta\}_{\eta \in \mathbb{N}}$ is $\mathfrak{T}$-bounded.

(c) For each invariant density, $f_\eta; P_\eta f_\eta = f_\eta$, the sequence $\{V_I^n f_\eta\}_{\eta \in \mathbb{N}}$ is bounded.

Proof. (a) By weak Lasota-Yorke inequality, Equation (3.8), for $0 < \alpha < \frac{1}{2}$, the following inequality

$$V_I^n P_T f \leq 2\alpha V_I^n f + \gamma \|f\|_1$$

implies

$$\|P_T f\|_{BV} = V_I^n P_T f + \|P_T f\|_1 \leq 2\alpha V_I^n f + (\gamma + 1)\|f\|_1 \leq 2\alpha \|f\|_{BV} + (\gamma + 1)\|f\|_1.$$  

Therefore, $P_T \in \mathfrak{T}$; in particular, $P_T \in \mathfrak{T}(2\alpha, \gamma + 1)$.

(b) By Proposition 5.25, for each $f$ in $\Delta_\eta$, $V_I^n P_\eta f = V_I^n (Q_\eta \circ P_T f) \leq V_I^n P_T f$. By weak Lasota-Yorke inequality, for each $\eta \in \mathbb{N},$

$$\|P_\eta f\|_{BV} = V_I^n P_\eta f + \|P_\eta f\|_1 \leq V_I^n P_T f + \|f\|_1 \leq 2\alpha V_I^n f + (\gamma + 1)\|f\|_1 \leq 2\alpha \|f\|_{BV} + (\gamma + 1)\|f\|_1.$$  

Let $\beta = 2\alpha$ and $C = \gamma + 1$ so that $\{P_\eta\}_{\eta \in \mathbb{N}} \subseteq \mathfrak{T}(\beta, C)$.

(c) By part (b), each $P_\eta \in \mathfrak{T}(\beta, C)$. By Equation (5.1), for all $f \in \Delta_\eta,$

$$\|P_\eta f\|_{BV} \leq \beta \|f\|_{BV} + C\|f\|_1.$$
Let $P_\eta f_\eta = f_\eta$,

$$
\|f_\eta\|_{BV} = \|P_\eta f_\eta\|_{BV} \leq \beta \|f_\eta\|_{BV} + C \|f_\eta\|_1;
$$

$$
V_{r^n} f_\eta + \|f_\eta\|_1 \leq \beta V_{r^n} f_\eta + \|f_\eta\|_1 + C \|f_\eta\|_1
$$

$$
V_{r^n} f_\eta \leq \beta V_{r^n} f_\eta + C \|f_\eta\|_1
$$

which implies $V_{r^n} f_\eta \leq \frac{C}{1 - \beta} \|f_\eta\|_1 < \infty$. Therefore, the sequence $\{V_{r^n} f_\eta\}_{\eta \in \mathbb{N}}$ is bounded.

\[\square\]

**Note.** The condition $0 < \alpha < \frac{1}{2}$ is essential. It allows us to get the bounded sequence $\{V_{r^n} f_\eta\}_{\eta \in \mathbb{N}}$. From Lemma 5.27 and 2.26, $\{f_\eta\}_{\eta \in \mathbb{N}} \subset BV$ is a precompact subset in $BV$. Therefore, for any $f$ in $BV$, there exists a convergent subsequence of $\{f_\eta\}_{\eta \in \mathbb{N}}$ so that we have the following theorem (see [22, Theorem 7.1]).

**Theorem 5.28.** Let $T = \{\tau_k; \ p_k\}_{k=1,\ldots,q}$ be a random dynamical system satisfying Equation (3.5) which is constructed from piecewise $C^2$ Jabłoński transformations, and $P_T$ be its Perron-Frobenius operator. Let sequence $\{P_\eta\}_{\eta \in \mathbb{N}}$ be defined in Definition 5.20, and each $f_\eta$ be an invariant density of $P_\eta$. If $0 < \alpha < \frac{1}{2}$, then

(a) For all $f \in \mathcal{D}$, the sequence $\{P_\eta f\}_{\eta \in \mathbb{N}}$ converges to $P_T f$ in $L^1$.

(b) Any convergent subsequence of $\{f_\eta\}_{\eta \in \mathbb{N}}$ converges to an invariant density of $P_T$.

(c) If the random dynamical system has a unique ACIM $\mu$ with respect to the invariant function $f^*$ (i.e. $d\mu = f^* d\mu_n$), then

$$
\|f_\eta - f^*\|_1 = O\left(\frac{\ln \eta}{\eta}\right).
$$

**Proof.** (a) By Lemma 5.21 (a), for every $h$ in $L^1$, given $\epsilon > 0$, there exists $N_\epsilon > 0$ such that for all $\eta \geq N_\epsilon$, $\|Q_\eta h - h\|_1 < \epsilon$. By Lemma 5.21 (b), for all $f \in \mathcal{D}$,

$$
P_\eta f = Q_\eta \circ P_T f,
$$

$$
\|P_\eta f - P_T f\|_1 = \|Q_\eta \circ P_T f - P_T f\|_1 < \epsilon.
$$

Thus, $P_\eta f$ converges to $P_T f$ in $L^1$.

(b) Since $\{f_\eta\}_{\eta \in \mathbb{N}}$ is precompact in $BV$ by Lemma 2.26, we assume a subsequence $\{f_{\eta_k}\}_{k \in \mathbb{N}}$ of $\{f_\eta\}_{\eta \in \mathbb{N}}$ converges to $f^*$ (i.e. $f_{\eta_k} \to f^*$). Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\eta \geq N$, $\|f^* - f_{\eta_k}\|_1 \leq \frac{\epsilon}{3}$. By (a), $\|P_\eta f^* - P_T f^*\|_1 \leq \frac{\epsilon}{3}$. Hence,

$$
\|P_T f^* - f^*\|_1 \leq \|P_T f^* - P_{\eta_k} f^*\|_1 + \|P_{\eta_k} f^* - P_{\eta_k} f_{\eta_k}\|_1 + \|P_{\eta_k} f_{\eta_k} - f^*\|_1
$$

$$
\leq \frac{\epsilon}{3} + \|P_{\eta_k}\|_1 \|f^* - f_{\eta_k}\|_1 + \|f_{\eta_k} - f^*\|_1
$$

$$
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
$$
Hence, $P_T f^* = f^*$.

d (c) Since $P_T$ is ergodic we can apply the results of Lemma 5.16 (b). First, since
\[
\|\Phi(1, P_\eta) - \Phi(1, P_T)\|_1 = O(-\|P_\eta - P_T\| \cdot \ln \|P_\eta - P_T\|),
\]
and
\[
\|P_\eta - P_T\| = \|Q_\eta P_T - P_T\| = \|(Q_\eta - Id)P_T\| = O\left(\frac{1}{\eta}\right)
\]
by Lemma 5.26. We also have
\[
\|\Phi(1, P_\eta) f^* - f^*\|_1 = \|\Phi(1, P_\eta) f^* - \Phi(1, P_T) f^*\|_1 = O\left(\frac{\ln \eta}{\eta}\right).
\]
It follows that for sufficiently large $\eta$, $\Phi(1, P_\eta) f^* \neq 0$ and we can identify
\[
f_\eta = \Phi(1, P_\eta) f^*/\|\Phi(1, P_\eta) f^*\|_1.
\]
Moreover,
\[
\|f_\eta - \Phi(1, P_\eta) f^*\|_1 = \left\|\frac{\Phi(1, P_\eta) f^*}{\|\Phi(1, P_\eta) f^*\|_1} (1 - \|\Phi(1, P_\eta) f^*\|_1)\right\|_1
\]
\[
= \frac{|1 - \|\Phi(1, P_\eta) f^*\|_1|}{\|\Phi(1, P_\eta) f^*\|_1} \|\Phi(1, P_\eta) f^*\|_1
\]
\[
= |1 - \|\Phi(1, P_\eta) f^*\|_1| = |\|f^*\|_1 - \|\Phi(1, P_\eta) f^*\|_1|
\]
\[
\leq \|f^* - \Phi(1, P_\eta) f^*\|_1 = O\left(\frac{\ln \eta}{\eta}\right).
\]
Putting these two estimates together with the triangle inequality yields
\[
\|f_\eta - f^*\|_1 = O\left(\frac{\ln \eta}{\eta}\right)
\]
as required.

\[\Box\]

**Remark 5.29.** The rate of convergence of approximations to the invariant density via Ulam’s method which is presented in part (c) of the previous theorem is known to be sharp, even for the case of a single transformation on a 1-dimensional space. See, for example Bose and Murray [5].
Chapter 6

Conclusion

In this article, we focus on the Perron-Frobenius operator $P_T$ with respect to the given random dynamical system $T$. We discuss the existence and uniqueness of invariant densities under $P_T$ and the approximation to $P_T$. Two main results we get here: using the weak Lasota-Yorke inequality to prove that any invariant density is also in $BV$ and using the spectral decomposition theorem to get a unique ACIM under $T$.

We extend Lasota-Yorke inequality, in [30], into a weak form so that every invariant density of the given Markov operator is still in $BV$ space. We give the proof for the existence of absolutely continuous invariant measure (ACIM) by using weak Lasota-Yorke inequality in Chapter 3.

The given constrictive Markov operator $P$ has a unique invariant density if and only if the permutation in the spectral decomposition theorem is cyclic, in [29]. The support of an invariant density is an open set $a.e.$, in Corollary 4.32. This is an important concept so that without lost of generality, we can consider an $L^1$ function with an open support. We discuss the uniqueness, between $T$ and its individual transformations, in [8]. In the end of Chapter 4, we give an example to illustrate that it is possible to have a unique ACIM under $T$, but there are two ACIMs under each individual transformation.

For the asymptotic behavior, we consider two models. One model is about the behavior of infinite iterations under a single Markov operator in Section 5.1. The other model is about the approximation by a sequence of perturbation operators for the given Markov operator $P$, see Sections 5.2 and 5.3. In Keller’s method, [24], we can find a sequence of $\{P_\eta\}_{\eta \in \mathbb{N}}$ such that $P_\eta$ approximates to the given $P$, and the dimensions of invariant densities under $P_\eta$ is also approximating to the dimensions of invariant densities under $P$. In Section 5.3, we consider the approximation by piecewise constant functions. If the given operator has a unique invariant density, then we can use Ulam’s method, [31, 37], to approximate the invariant density.
Appendix A

A.1 Radon-Nikodym Theorem

The Lebesgue-Radon-Nikodym Theorem is from [16]. Some usual definitions in Real Analysis are listed below. Let \((X, \mathcal{M})\) be a measure space.

**Definition A.1. (signed measure)**
A signed measure \(\nu\) on \((X, \mathcal{M})\) is a function \(\nu : \mathcal{M} \rightarrow [-\infty, \infty]\) such that

(a) \(\nu(\emptyset) = 0\).

(b) \(\nu\) assumes at most one of the values \(\pm \infty\).

(c) If \(\{E_j\}\) is a sequence of disjoint sets in \(\mathcal{M}\), then

\[
\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j),
\]

where the latter sum converges absolutely if \(\nu(\bigcup_{j=1}^{\infty} E_j)\) is finite.

**Definition A.2. (mutually singular)**
Two signed measures \(\mu\) and \(\nu\) on \((X, \mathcal{M})\) are mutually singular if there exist \(E, F \in \mathcal{M}\) such that \(E \cap F = \emptyset\), \(E \cup F = X\); \(E\) is null for \(\mu\) and \(F\) is null for \(\nu\) and write

\(\mu \perp \nu\).

**Definition A.3. (absolutely continuous)**
Suppose \(\nu\) is a signed measure and \(\mu\) is a positive measure on a measurable space \((X, \mathcal{M})\). \(\nu\) is absolutely continuous with respect to \(\mu\) and we write

\(\nu << \mu\),

if \(\nu(E) = 0\) for every \(E \in \mathcal{M}\), for which \(\mu(E) = 0\).

**Theorem A.4. (Lebesgue-Radon-Nikodym Theorem)**
Let \(\nu\) be a \(\sigma\)-finite signed measure and \(\mu\) a \(\sigma\)-finite positive measure on \((X, \mathcal{M})\). There exist unique \(\sigma\)-finite signed measures \(\lambda, \rho\) on \((X, \mathcal{M})\) such that
Moreover, there is an extended $\mu$-integrable function $f : X \rightarrow \mathbb{R}$ such that $d\rho = f d\mu$ and any two such functions are equal $\mu$-a.e. The second part of the result is usually known as the Radon-Nikodym theorem, and $f$ is called Radon-Nikodym derivative of $\nu$ with respect to $\mu$.

### A.2 Perron-Frobenius Theorem

The Perron-Frobenius Theorem is from Walters [38]. We give a general setting as follows: Consider a $k \times k$ matrix, let $A = [a_{ij}]$.

**Definition A.5.** (1) $A$ is non-negative if $a_{ij} \geq 0$ for all $i, j$.

(2) Let $a_{i,j}^{(n)}$ be the $(i,j)$-th element of $A^n$. Then $A$ is called irreducible if for any pair of $i, j$ there is some $n > 0$ such that $a_{i,j}^{(n)} > 0$.

(3) $A$ is called irreducible and aperiodic if there exists $n > 0$ such that $a_{i,j}^{(n)} > 0$ for all $i, j$.

**Theorem A.6.** (Perron-Frobenius Theorem)

Let $A = [a_{ij}]$ be a non-negative $k \times k$ matrix. Then

(a) There is a non-negative eigenvalue $\lambda$ such that no eigenvalue of $A$ has absolute value greater than $\lambda$.

(b) We have

$$\min_i \left( \sum_{j=1}^k a_{ij} \right) \leq \lambda \leq \max_i \left( \sum_{j=1}^k a_{ij} \right).$$

(c) Corresponding to the eigenvalue $\lambda$ there is a non-negative left (row) eigenvector $u = (u_1, \ldots, u_k)$ and a non-negative right (column) eigenvector

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}.$$  

(d) If $A$ is irreducible, then $\lambda$ is a simple eigenvalue and the corresponding eigenvectors are strictly positive (i.e. for each $i = 1, \cdots, k$, $u_i > 0$, $v_i > 0$).

(e) If $A$ is irreducible, then $\lambda$ is the only one eigenvalue of $A$ with a non-negative eigenvector.

(f) Let $u$ and $v$ defined as in part (c) be the strictly positive eigenvectors corresponding to the largest eigenvalue $\lambda$. If $A$ is irreducible and aperiodic, then for each pair $i, j$,

$$\lim_{n \to \infty} \lambda^n a_{i,j}^{(n)} = u_j v_i.$$
A.3 Spectral Decomposition Theorem

The proof of the Spectral Decomposition Theorem is mainly following from Lasota, Li and Yorke [28]. We outline the proof under the condition \( P1 = 1 \) and the whole space with finite measure. In the general case, assuming only \( Pf^* = f^* \) for some density \( f^* \), one can check the reference [28]. Here we use the same notation as in Section (4.1) unless specified. Let \( P : L^1(X, \mathcal{B}_X, \mu) \rightarrow L^1(X, \mathcal{B}_X, \mu) \) be a strongly constrictive Markov operator on an arbitrarily measurable space \( (X, \mathcal{B}_X, \mu) \) with \( \mu(X) < \infty \). For a measurable set \( A \), we denote its characteristic function by \( \chi_A \) and set \( \chi_X \equiv 1 \). Some definitions and useful lemmas are listed as below. For the next few pages we assume \( P1 = 1 \).

Definition A.7. (nice set and nice function)

Any measurable set \( A \) is called a nice set if \( P^N\chi_A \) is also a characteristic function for each positive integer \( N \), and its characteristic function \( \chi_A \) is called a nice function.

Lemma A.8. (a) If \( A \) is a nice set, then \( X \setminus A \) is a nice set.

(b) If \( A_1, A_2 \) are nice sets, then \( A_1 \cup A_2 \) is a nice set.

From the above, the collection of nice sets forms an algebra of subsets. In fact it is a measure algebra since a nice set, perturbed by a set of measure zero is a nice set. In addition, if \( A \) is a nice set, then \( B = \text{supp } P\chi_A \) is a nice set, by definition.

Lemma A.9. Let \( A_1, A_2 \) be two nice sets with disjoint supports. Assume \( P\chi_{A_1} = \chi_{B_1} \) and \( P\chi_{A_2} = \chi_{B_2} \), then \( B_1 \cap B_2 = \emptyset \).

Definition A.10. Let \( \mathcal{A} \) denote the (measure) algebra of nice subsets of \( X \).

Lemma A.11. For every \( A \in \mathcal{A} \), let \( f = \frac{1}{\mu(A)}\chi_A \) and \( A_N = \text{supp } (P^N f) \). Then for all \( N \in \mathbb{N} \)

\[
\mu(A_N) = \mu(A). \tag{A.1}
\]

Proof. For \( A \in \mathcal{A} \) and each \( N \in \mathbb{N} \), there exists \( B_N \) such that \( P^N\chi_A = \chi_{B_N} \). Hence,

\[
\mu(B_N) = \int_X P^N\chi_A d\mu = \int_X \chi_{B_N} d\mu = \mu(A).
\]

Since \( \text{supp } (P^N f) = \text{supp } (P^N\chi_A) \), \( \mu(A_N) = \mu(B_N) = \mu(A) \) for all \( N \in \mathbb{N} \).

Lemma A.12. There exists a \( \delta > 0 \) such that for every nonempty nice set \( A \) satisfying,

\[
\mu(A) > \delta.
\]

Proof. We use two steps to prove this lemma. The first part, we know that Equation (A.2) is always true when \( P \) is constrictive. The second part, we use a contradiction to get \( \mu(A) > \delta \).
(1) Let $A \in \mathcal{A}$ and $f = \frac{1}{\mu(A)}\chi_A$. Since $P$ is constrictive and $f \in \mathcal{D}$, we have

$$\lim_{N \to \infty} \left( \inf_{g \in \mathcal{F}} \|P^N f - g\|_1 \right) = 0.$$ 

Given $\epsilon \in (0, 1)$, there exist a sequence $\{g_N\}_{N \in \mathbb{N}} \subset \mathcal{F}$ and a positive integer $N_0$ such that for all $N \geq N_0$,

$$\|P^N f - g_N\|_1 < 1 - \epsilon. \quad (A.2)$$

Since $\mathcal{F}$ is weakly compact, for the same $\epsilon$, there exists $\delta > 0$ such that

$$\mu(B) < \delta \implies \int_B g_N d\mu < \epsilon. \quad (A.3)$$

(2) For each $N \in \mathbb{N}$, let $A_N = \text{supp} (P^N f)$. Then $\mu(A_N) = \mu(A)$ by Equation (A.1). Hence, we only have to prove $\mu(A_N) > \delta$. Suppose $\mu(A_N) < \delta$. By Equation (A.3), $\int_{A_N} g_N d\mu < \epsilon$. Therefore,

$$\|P^N f - g_N\|_1 = \int_X |P^N f - g_N| d\mu \geq \int_{A_N} |P^N f - g_N| d\mu \geq \int_{A_N} P^N f d\mu - \int_{A_N} g_N d\mu = \int_X P^N f d\mu - \int_{A_N} g_N d\mu = \int_X f d\mu - \int_{A_N} g_N d\mu > 1 - \epsilon.$$ 

This is a contradiction to Equation (A.2). Thus, we get the desired result.

\[\square\]

**Definition A.13. (atom)**

For any algebra $\mathcal{B}$, a nonempty set $B \in \mathcal{B}$ is called an atom if the only subsets of $B$ are $B$ and $\phi$ mod zero. i.e. $B$ has no any other subset with positive measure except $B$ itself.

**Remark A.14. (a)** An atom in $\mathcal{A}$ is a special form of a nice set.

(b) Distinct atoms in $\mathcal{A}$ are disjoint up to measure zero.

(c) For $\mu(X) < \infty$, there are only finitely many atoms in $\mathcal{A}$.

**Lemma A.15. (a)** Every nice subset $A \in \mathcal{A}$ with $\mu(A) > 0$ contains an atom.

(b) Every nice subset $A \in \mathcal{A}$ is a disjoint union of finitely many atoms. In particular, $X$ is equal to the union of all atoms.

(c) There exists a permutation $\sigma$ of the integers $\{1, 2, \cdots, r\}$ such that

$$P\chi_{A_i} = \chi_{A_{\sigma(i)}}.$$
Proof. (a) For $\mathcal{A}$ being an algebra, if $A_1 \subset A_2$ are both nice, then $A_2 \setminus A_1$ is also nice. Suppose $A \in \mathcal{A}$ with $\mu(A) > 0$. If $A$ is an atom, we are done. Otherwise, there is a nice $B \subset A$ with $0 < \mu(B) < \mu(A)$. Let $b = \inf_{B \subset A, B \in \mathcal{A}} \mu(B)$, where $b > \delta$ from Lemma A.12. Choose $B_0 \subset A$ to be a nice set with measure $b < \mu(B_0) < b + \frac{\delta}{2}$. If $B_0$ is an atom, we are done. Otherwise, $B_0$ contains a nice $B_1 \subset A$ with $\mu(B_1) = \mu(B_0) = b$. Since $\mathcal{A}$ is an algebra, both $B_1$ and $B_0 \setminus B_1$ are in $\mathcal{A}$. Hence, on the right hand side both $\mu(B_1) \geq b$ and $\mu(B_0 \setminus B_1) \geq b$. However, set $B_0$ is chosen to have the measure less than $b + \frac{\delta}{2}$. It is a contradiction. Therefore, $B_0$ must be an atom.

(b) Let $B$ be nice. If $B$ is an atom, we are done. If not, by part (a), we can find an atom $A_1$ in $B$. Then, consider $B \setminus A_1$ (another nice set). Repeat the same argument to find $A_2, A_3, \ldots$. This must stop after finitely many steps since each $\mu(A_i) > \delta$. Moreover, $X$ is also a nice set, so $X$ is equal to the union of all atoms.

(c) Suppose $\{A_i\}_{i=1}^r$ are atoms in $\mathcal{A}$. Let $\chi_{B_i} = P\chi_{A_i}$. Then, $B_i$’s are disjoint from Lemma A.9, and $\mu(A_i) = \mu(B_i)$ from Lemma A.11. Besides,

$$\sum_{i=1}^r \mu(B_i) = \sum_{i=1}^r \mu(A_i).$$

By part (b), each $B_i$ can be written as a disjoint union of the $A_i$’s. Hence, it is clear to see the existence of a permutation. For example, consider any $A_i$ with minimal measure. Then $B_i$ must have the same (minimal) measure, so $B_i$ cannot be a non-trivial union of atoms. Therefore, $B_i = A_{\sigma(i)}$. Do this for all $i$ corresponding to minimal measure for a matching. Next, move to the next smallest value of $\mu(A_i)$. Continue in this way to construct $\sigma$.

Before we get into the main theorem, Spectral Decomposition Theorem A.24, we need some more technical lemmas and an important theorem, Theorem A.22, which shows that there exists a subspace $\mathcal{G}$ of $L^1$ with some properties as defined in Definition A.16. Such subspace $\mathcal{G}$ has finite dimensions. From now on, we use $A_i$’s to be the atoms defined as above.

**Definition A.16.** Let $P$ be a constrictive Markov operator, which implies that for each $f \in L^1$, $\{P^N f\}_{N \in \mathbb{N}}$ is precompact. Let $\Omega(f)$ be the set of limit points of $\{P^N f\}$ and 

$$\mathcal{G} = \bigcup_{f \in L^1} \Omega(f).$$

**Lemma A.17.** If $f \in \mathcal{G}$, then $f \in \Omega(f)$.

**Lemma A.18.** If $f \in \mathcal{G}$, then $\|Pf\|_1 = \|f\|_1$. 


Lemma A.19. If \( \|Pf\|_1 = \|f\|_1 \), then \( \text{supp} \ (Pf)^+ \) and \( \text{supp} \ (Pf)^- \) are disjoint. In particular, this holds for all \( f \in \mathcal{G} \).

Lemma A.20. If \( f_1 \) and \( f_2 \) are nonnegative and have the same support, then \( Pf_1 \) and \( Pf_2 \) have the same support.

Note: From Lemma A.17 to A.19, it is not difficult to verify. For Lemma A.20, one can check the reference [28] in the section four.

The following proposition shows that every \( f \in \mathcal{G} \) is \( \mathcal{A} \)-measurable. Here we still need the condition \( P_1 = 1 \).

Proposition A.21. If \( f \in \Omega(g) \) for some \( g \in L^1 \), then for each \( v \in \mathbb{R} \),

\[ f^{-1}(-\infty, v) \text{ is a nice set.} \]

Proof. We prove this in two cases. Case 1: \( \mu(f^{-1}(v)) = 0 \).

\[ h_1 = \chi_{f^{-1}(-\infty, v)}, \quad h_2 = \chi_{f^{-1}[v, \infty)}. \]

Note: If \( \mu(f^{-1}(v)) \neq 0 \), then by the definition of the support of a function, we get \( (\text{supp} \ h_1)^c \neq \text{supp} \ h_2 \).

Let \( g_o = f - v \). Then \( g_o \in \mathcal{G} \) and

\[ \text{supp} \ g_o^+ = \{ x : f(x) > v \} = \text{supp} \ h_2 \]
\[ \text{supp} \ g_o^- = \{ x : f(x) < v \} = \text{supp} \ h_1. \]

From Lemma A.18 to A.20, we get

\[ \text{supp} \ (P^N h_1) \cap \text{supp} \ (P^N h_2) = \emptyset \text{ mod zero.} \]

Since \( P1 = 1 \) implies \( P^N h_1 + P^N h_2 = 1 \), \( P^N h_1 \) is also a characteristic function for each \( N \in \mathbb{N} \). Thus, \( f^{-1}(-\infty, v) \) is a nice set.

For the other case, \( \mu(f^{-1}(v)) \neq 0 \). Since \( \mu(X) < \infty \), there are finite \( u \)'s such that \( u > 0 \) and \( \mu(f^{-1}(u)) \neq 0 \). Thus, all \( f^{-1}(u) \)'s are disjoint from different \( u > 0 \). We can pick an increasing sequence \( \{v_i\}_{i=1}^\infty \) with \( v_i \neq u \) and \( v_i \uparrow v \) such that for each \( i, \mu(f^{-1}(v_i)) = 0 \). Therefore,

\[ f^{-1}(-\infty, v) = \bigcup_{i=1}^{\infty} f^{-1}(-\infty, v_i). \]

Since each \( f^{-1}(-\infty, v_i) \in \mathcal{A} \), \( f^{-1}(-\infty, v) \) is also in \( \mathcal{A} \).

Theorem A.22. Let \( \mathcal{G} \) be defined as in Definition A.16. Then,

(a) \( \mathcal{G} \) is a linear subspace of \( L^1(X, \mathcal{B}_X, \mu) \).
(b) $\mathcal{G}$ is finite dimensional with basis $\{\chi_{A_1}, \chi_{A_2}, \cdots, \chi_{A_r}\}$.

**Proof.** (a) Clearly, when $f_1, f_2 \in \mathcal{G}$, then for any $a, b \in \mathbb{R}$, $af_1 + bf_2$ is also in $\mathcal{G}$.

(b) Since $f$ is $\mathcal{A}$-measurable by Proposition A.21, it can be written as $f = \sum_i \lambda_i \chi_{A_i}$ for some constants $\lambda_i$, where $A_i$ is the finite set of atoms generating $\mathcal{A}$. Hence, $\mathcal{G}$ is finite. \hfill $\square$

From Theorem A.22 (b) and Lemma A.15 (c), we get the following conclusion. Let $P : L^1(X, \mathcal{B}_X, \mu) \to L^1(X, \mathcal{B}_X, \mu)$ be a constrictive Markov operator with $P1 = 1$ and give a sequence of densities $\{g_i\}_{i=1}^r$ with disjoint supports. Assume a permutation $\sigma$ on $\{1, \cdots, r\}$ such that for each $i = 1, \cdots, r$,

$$Pg_i = g_{\sigma(i)}.$$

**Lemma A.23.** Let $g_i$’s are densities defined as above and a sequence of linear functionals $\{\lambda_i\}_{i=1}^r$ on $L^1(X, \mathcal{B}_X, \mu)$ such that for every density $f \in \mathcal{D}$,

$$\lim_{N \to \infty} \left\| P^N \left( f - \sum_{i=1}^r \lambda_i(f)g_i \right) \right\|_1 = 0. \quad \text{(A.4)}$$

Then this is equivalent to say that for every density $f$ and $\epsilon > 0$, there are constants $C_1, \cdots, C_r$ and $N_o \in \mathbb{N}$ such that

$$\left\| P^{N_o} f - \sum_{i=1}^r C_i g_i \right\|_1 \leq \epsilon.$$

**Proof.** We only have to prove one direction, and the other direction is trivial. Assume for every density $f$ and $\epsilon > 0$, there are constants $C_1, \cdots, C_r$ and $N_o \in \mathbb{N}$ such that

$$\left\| P^{N_o} f - \sum_{i=1}^r C_i g_i \right\|_1 \leq \epsilon.$$

Let $\epsilon_k \searrow 0$ be given. Choose $N_k$ and $C_i(k)$ for $i = 1, 2, \cdots, r$ such that

$$\left\| P^{N_k} f - \sum_{i=1}^r C_i(k) g_{\sigma^{N_k}(i)} \right\|_1 \leq \epsilon_k. \quad \text{(A.5)}$$

Then, we have

$$\sum_{i=1}^r |C_i(k)| = \sum_{i=1}^r |C_i(k)||g_{\sigma^{N_k}(i)}|_1 = \left\| \sum_{i=1}^r |C_i(k)| g_{\sigma^{N_k}(i)} \right\|_1.$$ 

It implies $\sum_{i=1}^r |C_i(k)| \leq \| P^{N_k} f \|_1 + \epsilon_k \leq \| f \|_1 + \epsilon_k$. Therefore, for $i = 1, \cdots, r$, each $\{C_i(k)\}_{k \in \mathbb{N}}$ is a bounded sequence. Hence, define $\lambda_i(f) = \lim_{k \to \infty} C_i(k)$.
Now we show \( \{\lambda_i\}_{i=1,\ldots,r} \) are linear. Observe first that the \( \lambda_i = \lambda_i(f) \) are unique since the \( g_i \)'s are linearly independent. For \( f_1, f_2 \in \mathcal{D} \) and by Equation (A.4), we get

\[
\lim_{N \to \infty} \left\| P^N \left( f_1 - \sum_{i=1}^{r} \lambda_i(f_1)g_i \right) \right\|_1 = 0,
\]

\[
\lim_{N \to \infty} \left\| P^N \left( f_2 - \sum_{i=1}^{r} \lambda_i(f_2)g_i \right) \right\|_1 = 0.
\]

Therefore, we have

\[
\lim_{N \to \infty} \left\| P^N \left( (f_1 + f_2) - \sum_{i=1}^{r} (\lambda_i(f_1) + \lambda_i(f_2))g_i \right) \right\|_1 = 0.
\]

Thus, \( \lambda_i(f_1) + \lambda_i(f_2) = \lambda_i(f_1 + f_2) \) for each \( i = 1, \ldots, r \).

So far we get all the tools we need. We state the theorem again and begin the proof as follows.

**Theorem A.24. (spectral decomposition theorem)**

Let \( P \) be a constrictive Markov operator, \( P : L^1(X, \mathcal{B}_X, \mu) \to L^1(X, \mathcal{B}_X, \mu) \). Then there are two sequences of nonnegative functions \( g_i \in \mathcal{D} \) and \( h_i \in L^\infty, \ i = 1, \ldots, r \) for some \( r \in \mathbb{N} \), and an operator \( Q : L^1(X, \mathcal{B}_X, \mu) \to L^1(X, \mathcal{B}_X, \mu) \) such that

\[
Pf = \sum_{i=1}^{r} \lambda_i(f)g_{\sigma(i)} + PQf,
\]

where \( \lambda_i(f) = \int f h_i \, d\mu \). The functions \( g_i \) and the operator \( Q \) have the following properties:

(a) For all \( i \neq j, \ i, j = 1, \ldots, r \), \( g_i(x)g_j(x) = 0 \) a.e. Hence, all \( g_i \)'s in Equation (4.1) have disjoint support up to measure zero.

(b) For each \( i = 1, \ldots, r \), there exists a unique integer \( \sigma(i) \) such that \( P g_i = g_{\sigma(i)} \), where \( \sigma \) is a permutation on the numbers \( \{1, 2, \ldots, r\} \). Furthermore, \( \sigma(i) \neq \sigma(j) \) for \( i \neq j \).

(c) \( \|P^NQf\|_1 \to 0 \) as \( N \to \infty \), for each density \( f \).

**Proof.** Define \( \mathcal{G} \) as in Definition A.16. Since \( \mathcal{G} \) is a finite dimensional subspace of \( L^1(X, \mathcal{B}_X, \mu) \) by Theorem A.22 and \( P \) is a constrictive Markov operator, we have for each density \( f \),

\[
\lim_{N \to \infty} \left( \inf_{\xi \in \mathcal{G}} \|P^N f - \xi\|_1 \right) = 0.
\]

This is if and only if there exist \( \xi_N \in \mathcal{G} \) such that

\[
\lim_{N \to \infty} \|P^N f - \xi_N\|_1 = 0.
\]
For $A$ being finite from Remark A.14 (c), we assume $\{A_i\}_{i=1}^r$ are atoms in $A$. Define each $g_i = \frac{\chi_{A_i}}{m_n(A_i)}$. Since each $\xi_N \in G$, there exist $\{C_i(N)\}_{i=1}^r$ such that $\xi_N = \sum_{i=1}^r C_i(N)g_i$. Thus, given $\epsilon > 0$ then $\exists N_0 \in \mathbb{N}$ and

$$\left\| P_{N_0} f - \sum_{i=1}^r C_i(N)g_i \right\|_1 \leq \epsilon.$$ 

By Lemma A.23, there exists a sequence of linear functionals $\{\lambda_i\}_{i=1}^r$ such that $\lambda_i : L^1(X, \mathcal{B}_X, \mu) \to L^1(X, \mathcal{B}_X, \mu)$, and we have $C_i(N_k) \to \lambda_i(f)$ as $k \to \infty$. That is, for every $f \in \mathcal{D}$,

$$\lim_{N \to \infty} \left\| P^N \left( f - \sum_{i=1}^r \lambda_i(f)g_i \right) \right\|_1 = 0. \tag{A.6}$$

Therefore, we can define an operator on $L^1(X, \mathcal{B}_X, \mu)$ to itself as

$$Q(f) = f - \sum_{i=1}^r \lambda_i(f)g_i. \tag{A.7}$$

(a) Since $\{g_i\}_{i=1}^r$ is a basis of $G$ with disjoint supports, it is clear that

$$g_i(x)g_j(x) = 0, \text{ for all } i \neq j, \ i, j = 1, \cdots, r.$$

(b) By Lemma A.15 (c), there exists a permutation $\sigma$ on $\{1, 2, \cdots, r\}$ and

$$Pg_i = g_{\sigma(i)}, \ \forall i = 1, \cdots, r.$$

(c) By the definition of the operator $Q$ on Equation (A.7), we can substitute this into Equation (A.6). Then, for all $f \in \mathcal{D}$,

$$\lim_{N \to \infty} \left\| P^N Qf \right\|_1 = 0.$$

Remark A.25. If $Pf^* = f^*$, for all $g \in L^1 \cap L^\infty$, we can define

$$\hat{P}(g) = \begin{cases} \frac{1}{f^*} P(f^*g) & \text{if } f^* \neq 0 \\ 0 & \text{if } f^* = 0. \end{cases} \tag{A.8}$$

Then $\hat{P}1 = 1$. Thus, we can extend the condition from $P1 = 1$ into $Pf^* = f^*$. 
Bibliography


