SPECTRUM PRESERVING LINEAR MAPS ON THE
SPACE OF SELF ADJOINT OPERATORS

BY

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§0. Introduction. Spectrum-preserving or invertibility-preserving linear maps between certain Banach algebras have been studied by several authors. See for example [10] and [11] for maps on algebras of matrices; [1, pp. 29-30] for related results; [8] for maps between commutative Banach algebras; [3] for certain positive maps between C*-algebras; and [7] for maps between algebras of operators on Banach spaces.

In §1 of this note we characterize spectrum preserving surjective linear maps between the real linear spaces of all self-adjoint operators on (real or complex) Hilbert spaces. We show that such a map \( \phi \) takes one of the forms \( \phi(A) = UAU^* \) or \( \phi(A) = UA^tU^* \) for some unitary operator \( U \), where \( A^t \) denotes the transpose of \( A \) with respect to an arbitrary, but fixed, orthonormal basis. For finite-dimensional complex Hilbert spaces, the same result was obtained in [10] under the additional assumption that \( \phi \) preserves the multiplicity of the eigenvalues. In §2 we give a characterization of invertibility-preserving linear maps on the space of self-adjoint operators on finite dimensional real or complex Hilbert space.

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In the sequel, we use the following notation and terminology. The letters $H, H_1$ and $H_2$ denote real or complex Hilbert spaces. The algebra of all bounded linear operators on $H$ is denoted by $B(H)$. The real linear space of all self-adjoint operators on $H$ will be denoted by $S(H)$. By a subspace we always mean a closed linear manifold. The lattice of all closed subspaces of $H$ will be denoted by $C(H)$. If $M \in C(H)$, then $P_m$ denotes the orthogonal projection on $M$, and $M^\perp$ denotes the orthogonal complement of $M$ in $H$. By a projection we always mean a self-adjoint idempotent. If $P$ and $Q$ are two projections, then we write $P \perp Q$ if $PQ = QP = 0$. If $T \in B(H)$, then $\sigma(T)$, $N(T)$ and $R(T)$ denote the spectrum, the null space and the range of $T$ respectively. The transpose of $T$ with respect to a fixed, but arbitrary, orthonormal basis will be denoted by $T^t$. A linear map $\phi$ from a subspace of $B(H_1)$ to a subspace of $B(H_2)$ is said to be invertibility-preserving if the image of every invertible operator is invertible. It is said to be spectrum-preserving if $\sigma(\phi(T)) = \sigma(T)$, for every $T$. For a non-zero $x \in H$, the rank-one operator on $H$ defined by $u \rightarrow (u,x)x$ will be denoted by $x \otimes x$. Note that $x \otimes x$ is a projection if and only if $\|x\| = 1$.

§1. Spectrum-preserving maps. Our main result of this section is the following.

THEOREM 1. Let $\phi: S(H_1) \rightarrow S(H_2)$ be a surjective spectrum-preserving linear map. Then there exists a unitary operator $U: H_1 \rightarrow H_2$ such that $\phi$ has one of the following forms:
(i) \( \phi(A) = UAU^* \), \text{ for every } A \in S(H_1), \\
or

(ii) \( \phi(A) = U^t U^* \), \text{ for every } A \in S(H_1).

Note. In the real case, the two forms coincide.

Before we proceed further we should point out that the surjectivity assumption on \( \phi \) cannot be removed as can be seen by considering the map \( \phi: S(H) \rightarrow S(H \oplus H) \) given by \( \phi(A) = A \oplus A \).

We begin with the following lemma.

**Lemma 1.** The linear map \( \phi \) has the following properties:

(i) \( \phi \) is injective,

(ii) \( \phi(I) = I \),

(iii) \( \phi(P) \) is a projection if and only if \( P \) is a projection,

(iv) if \( P \) and \( Q \) are two projections, then \( \phi(P) \perp \phi(Q) \) if and only if \( P \perp Q \),

(v) \( \phi(A^2) = (\phi(A))^2 \) for every finite rank \( A \in S(H_1) \),

(vi) \( \phi(P) \) is a rank-one projection if and only if \( P \) is a rank-one projection.

**Proof.** (i) If \( \phi(A) = 0 \), then \( \sigma(A) = \{0\} \). Since \( A \) is self-adjoint, it follows that \( A = 0 \).

(ii) Since \( \phi(I) \) is self-adjoint with spectrum \( \{1\} \), we get \( \phi(I) = I \).

(iii) This follows from the fact that a self-adjoint operator is a projection if and only if its spectrum is a subset of \( \{0,1\} \).
(iv) For two projections $P$ and $Q$, the operator $P + Q$ is a projection if and only if $P \perp Q$. Since $\phi(P+Q) = \phi(P) + \phi(Q)$, (iv) follows from (iii).

(v) If $A \in S(H_1)$ is of finite rank, then $A = \sum_{i=1}^{k} c_i E_i$, where the $E_i$'s are projections with $E_i \perp E_j$, $i \neq j$. Now using (iv) we have,

$$\phi(A^2) = \phi \left( \sum_{i=1}^{k} c_i^2 E_i \right) = \sum_{i=1}^{k} c_i^2 \phi(E_i) = \left( \sum_{i=1}^{k} c_i \phi(E_i) \right)^2 = (\phi(A))^2.$$

(vi) A projection has rank one if and only if it cannot be written as the sum of two nonzero mutually perpendicular projections. This together with (i), (iii) and (iv) proves (vi). ■

We also need the following version of Lemma 4 of [7]. For completeness, we give the proof.

**Lemma 2.** Let $T \in B(H)$, $x \in H$ and $\lambda \notin \sigma(T)$. Then $\lambda \in \sigma(T \oplus x \otimes x)$ if and only if $\langle (\lambda-T)^{-1}x, x \rangle = 1$.

**Proof.** If $\langle (\lambda-T)^{-1}x, x \rangle = 1$, then

$$(T + x \otimes x)(\lambda-T)^{-1}x = T(\lambda-T)^{-1}x + x = \lambda(\lambda-T)^{-1}x,$$

and hence $\lambda$ is an eigenvalue of $T + x \otimes x$. Conversely, if $\lambda \in \sigma(T + x \otimes x)$, then by a variant of the Fredholm alternative, $\lambda$ is an eigenvalue of $T + x \otimes x$ and so there exists a nonzero vector $y \in H$ such that $(T + x \otimes x)y = \lambda y$. Thus $y = (y,x)(\lambda-T)^{-1}x$. This implies that $\langle (\lambda-T)^{-1}x, x \rangle = 1$. ■

**Proof of Theorem 1.** We treat the real and complex cases separately.
Case I. Complex Hilbert spaces. We first consider finite dimensional spaces. (It is obvious that $H_1$ and $H_2$ have the same dimension.) We extend $\phi$ to a map from $\mathcal{B}(H_1)$ onto $\mathcal{B}(H_2)$ by the formula $\phi(A+iB) = \phi(A) + i\phi(B)$ for $A, B \in S(H_1)$. It follows from Lemma 1(v), that $\phi$ is a Jordan isomorphism from $\mathcal{B}(H_1)$ to $\mathcal{B}(H_2)$. By a result from ring theory (see [6], p. 50) $\phi$ is either an algebra isomorphism or an algebra anti-isomorphism, and hence there exists a bijective linear map $S: H_1 \rightarrow H_2$ such that $\phi$ takes one of the forms $\phi(A) = SAS^{-1}$ or $\phi(A) = SA^tS^{-1}$. Since $\phi(A^*) = \phi(A)^*$, we have $S = \alpha U$ for a scalar $\alpha$ and a unitary $U$. It follows that $\phi(A) = UAU^*$ for every $A \in S(H_1)$ or $\phi(A) = UA^tU^*$ for every $A \in S(H_1)$.

We now consider the infinite-dimensional case. Define $L: C(H_1) \rightarrow C(H_2)$ by $L(M) = \phi(M)H_2$. Using Lemma 1(iv), it is easy to see that $L$ is an order-isomorphism between $C(H_1)$ and $C(H_2)$, i.e. $L(M) \subseteq L(N)$ if and only if $M \subseteq N$. It follows easily that $L$ is a lattice-isomorphism. By a result of Fillmore and Longstaff [5, Theorem 1], there exists a bicontinuous linear or conjugate linear bijection $S: H_1 \rightarrow H_2$ such that $L(M) = SM$, for every $M \in C(H_1)$. This fact, together with Lemma 1(iii, iv), imply that $\phi(P) = SPS^{-1}$ for every projection $P \in S(H_1)$. But every self-adjoint operator is a real-linear combination of a finite number of projections (see [4] and [12]), therefore $\phi(A) = SAS^{-1}$ for every $A \in S(H_1)$.

If $S$ is linear, the fact that $SAS^{-1}$ is a self-adjoint operator for every $A \in S(H_1)$ implies that $S = \alpha U$ for a scalar $\alpha$ and a unitary $U$. Therefore $\phi(A) = UAU^*$ for every $A \in S(H_1)$.

If $S$ is conjugate linear, then we can write $S = RJ$ where $R$ is linear and $J$ is the conjugation operator with respect to an orthonormal basis $\mathcal{B}$ for $H_1$; i.e. $(Jx,e) = (e,x)$ for every $e \in \mathcal{B}$. It follows that $\phi(A) = RA^tR^{-1}$
where $A^t$ is the transpose of $A$ with respect to $B$. As before, $R$ is a scalar multiple of a unitary operator $V$ and $\phi(A) = V A^t V^*$ for every $A \in S(H_1)$. This ends the proof for the complex case.

Case II. Real Hilbert spaces. We need to consider two-dimensional spaces separately. In this case, let $B_1 = \{x_1, x_2\}$ be an orthonormal basis for $H_1$. Then $\phi(x_i \otimes x_i), i = 1, 2,$ is a rank-one projection by Lemma 1(vi). Since $(x_1 \otimes x_1) \perp (x_2 \otimes x_2)$, we have $\phi(x_1 \otimes x_1) \perp \phi(x_2 \otimes x_2)$. Thus we can choose an orthonormal basis $B_2 = \{y_1, y_2\}$ for $H_2$ such that $\phi(x_i \otimes x_i) = y_i \otimes y_i, i = 1, 2$. Replacing operators by their matrices relative to the bases $B_1$ and $B_2$ we have

$$\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By part (v) of Lemma 1, we have that, $\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ is a symmetric matrix of the form $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta^2 = 1$. For every $c \in \mathbb{R}$, we have

$$\phi\left(c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} c+\alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

and so the matrices $\begin{bmatrix} c & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} c+\alpha & \beta \\ \beta & -\alpha \end{bmatrix}$ have the same spectrum. This implies that $\alpha = 0$ and $\beta = \pm 1$. Replacing $y_1$ by $-y_1$ if necessary, we may assume that $\phi\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$ Since $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ is a basis for $S(H_1)$ we have $\phi(A) = UAU^*$ for every $A \in S(H_1)$, where
U is the unitary operator mapping $B_1$ to $B_2$.

Now assume that $\dim H_1$ (and hence $\dim H_2$) $\geq 3$. As in case I, the equation $L(M) = \phi(P_M)H_2$ defines a lattice-isomorphism $L$ between $C(H_1)$ and $C(H_2)$. A result of Mackey, implicit in [9], asserts that, except in dimension 2, every such lattice-isomorphism $L$ is induced by an invertible linear operator $S: H_1 \to H_2$ via the equation $L(M) = SM$. (In finite-dimensional spaces, Mackey's result is simply the Fundamental Theorem of Projective Geometry [2, p. 44]). As in case I, we get that $S$ is a scalar multiple of a unitary operator $U$ and that $\phi(P) = UPU^*$ for every projection $P$ in $S(H_1)$.

At this point, the proof diverges from that of Case I; we don't know whether every self-adjoint operator on a real Hilbert space is a linear combination of projections. Instead we use a technique which we used in [7].

For a unit vector $h \in H_1$ we have $\phi(h \otimes h) = U(h \otimes h)U^* = Uh \otimes Uh$, and hence $\phi(x \otimes x) = Ux \otimes Ux$, for every $x \in H_1$. Let $A \in S(H_1)$ and let $\lambda > \|A\|$. Since $\phi$ preserves the spectrum, we have $\lambda \in \sigma(A + x \otimes x)$ if and only if $\lambda \in \sigma(\phi(A) + Ux \otimes Ux)$ and so by Lemma 2, we have that $((\lambda - A)^{-1}x, x) = 1$ if and only if $((\lambda - \phi(A))^{-1}Ux, Ux) = 1$. Since both $(\lambda - A)^{-1}$ and $(\lambda - \phi(A))^{-1}$ are positive operators, it follows that for every $x \in H_1$ we have

$$((\lambda - A)^{-1}x, x) = ((\lambda - \phi(A))^{-1}Ux, Ux).$$

Replacing $\lambda$ by $\frac{1}{t}$, we get

$$((1 - tA)^{-1}x, x) = ((1 - t\phi(A))^{-1}Ux, Ux)$$

for $0 < t < \|A\|^{-1}$. The same equation holds for $t = 0$ since $U$ is unitary.

Taking the derivative from the right at $t = 0$ we get $(Ax, x) = (\phi(A)Ux, Ux)$ for every $x \in H_1$. By polarization, we get $\phi(A) = UAU^*$. \n
We have the following immediate corollary.

COROLLARY 1. Let $H_1$ and $H_2$ be two complex Hilbert spaces and $\phi: B(H_1) \rightarrow B(H_2)$ a surjective, adjoint-preserving linear map. If $\phi$ preserves the spectrum for self-adjoint operators, then $\phi$ takes one of the forms given in Theorem 1. In particular $\phi$ preserves the spectrum of every operator.

§2. Invertibility-preserving maps. Now we consider invertibility-preserving linear maps $\phi: S(H) \rightarrow S(H)$. For finite-dimensional spaces, we give a characterization of such maps similar to the result of Theorem 1. We note, in passing, that under the assumption that $\phi(I) = I$, the map $\phi$ is invertibility-preserving if and only if $\sigma(\phi(T)) \subseteq \sigma(T)$ for every $T \in S(H)$. Our result, below, shows that in finite-dimensional spaces, the above conditions imply that $\phi$ preserves the spectrum.

THEOREM 2. Let $H$ be a real or complex finite-dimensional Hilbert space and let $\phi: S(H) \rightarrow S(H)$ be an invertibility preserving linear map. Then there exists an invertible operator $S$ on $H$ such that $\phi$ has one of the following forms.

(1) $\phi(A) = \pm SAS^*$, for every $A \in S(H)$,

or

(2) $\phi(A) = \pm SA^*S^*$, for every $A \in S(H)$.

Before we start the proof, we require the following lemma.
Lemma 3. Let $A \in S(H)$, $H$ finite-dimensional. Then $\sigma(A+B) \cap \sigma(B)$ is nonempty for every $B \in S(H)$ if and only if $A = 0$.

Proof. The result is obvious if $A = 0$. If $A \neq 0$ then by the spectral theorem $A = A_1 \oplus A_2$ where $A_1$ is of rank-one and $A_2$ is invertible. (The second direct summand may be absent). We will construct a self-adjoint operator $B$ of the form $B_1 \oplus kI$ such that $\sigma(A+B) \cap \sigma(B)$ is empty. We first construct $B_1$ such that $\sigma(A_1+B_1) \cap \sigma(B_1) = \emptyset$. We have $A_1 = \pm x \otimes x$ for a nonzero vector $x$. Choose a self-adjoint operator $B_1$ such that none of its eigenvectors is orthogonal to $x$. (This is equivalent to choosing an orthonormal basis such that every entry of the matrix of $A_1$ is nonzero and then choosing $B_1$ to be an operator whose matrix is diagonal with distinct eigenvalues). If $\lambda \in \sigma(A_1+B_1) \cap \sigma(B_1)$, then there exists a nonzero vector $y$ such that $(A_1+B_1)y = \lambda y$, i.e. $(y,x) = \pm (\lambda-B_1)y$. If $(y,x) = 0$, then $y$ is an eigenvector of $B_1$ orthogonal to $x$, a contradiction. If $(y,x) \neq 0$, then $x \in \mathbb{R}(\lambda-B_1)$ and so $x$ is orthogonal to $N(\lambda-B_1)$, again contradicting the choice of $B_1$. We conclude that $\sigma(A_1+B_1) \cap \sigma(B_1)$ is empty. Now, by taking $k$ large enough, it is easy to see that $\sigma(A+B) \cap \sigma(B)$ is empty. □

Proof of Theorem 2. The proof is divided into several steps. We will write $S$ for $S(H)$.

Step 1. $\phi$ is injective (and hence bijective). To prove this assume that $\phi(A) = 0$. For any $T \in S$, if $\lambda I - T$ is invertible, then $\phi(I)(\lambda - \phi(I)^{-1} \phi(T))$ is also invertible. It follows that $\sigma(\phi(I)^{-1} \phi(T)) \subseteq \sigma(T)$. Using this for $T = A + B$ and $T = B$, we get

$$\sigma(\phi(I)^{-1} \phi(B)) \subseteq \sigma(A+B) \cap \sigma(B)$$
for every \( B \in S \). By Lemma 3, we get that \( A = 0 \) and \( \phi \) is injective.

**Step 2.** \( \phi(I) = \pm D^2 \) for an invertible positive operator \( D \), i.e. \( \phi(I) \) is either strictly positive or strictly negative. The invertibility of \( \phi(I) \) is obvious. Let \( C = \phi^{-1}(I) \), so \( \phi(I - \lambda C) = \phi(I) - \lambda I \). It follows that if \( \lambda \in \sigma(\phi(I)) \), then \( \lambda \neq 0 \) and \( \lambda^{-1} \in \sigma(C) \). Therefore it suffices to show that \( C \succeq 0 \) or \( C \preceq 0 \). Assume to the contrary that \( C \) has two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) with \( \lambda_1 \lambda_2 < 0 \). Identifying operators with matrices, we may write \( C \) in the form

\[
\begin{bmatrix}
\lambda_2 & 0 \\
0 & \lambda_1
\end{bmatrix} \oplus C_1 \oplus 0 \quad \text{where} \quad C_1 \quad \text{is invertible. (The second or the third direct summands may be absent.)}
\]

Let \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0 \oplus I \). We have that, for every nonzero real \( \lambda \), the operator \( C + \lambda A \) is invertible. It follows that \( I + \lambda \phi(A) \) is invertible for every real \( \lambda \). This implies that \( \phi(A) = 0 \), contradicting the fact that \( \phi \) is injective.

**Step 3.** Define \( \psi : S \to S \) by \( \psi(A) = \pm D^{-1} \phi(A) D^{-1} \). The map \( \psi \) is an invertibility-preserving bijective linear map and \( \psi(I) = I \). It follows easily that \( \sigma(\psi(A)) \subseteq \sigma(A) \) for every \( A \in S \).

**Step 4.** \( \psi \) preserves the spectrum. In view of Step 3, it suffices to show that \( \psi^{-1} \) preserves invertibility. To prove this, assume that \( A \in S \) and \( \psi(A) \) is invertible. Let \( A = \sum_{i=1}^{n} c_i E_i \) be the spectral representation of \( A \). By examining the proof of Lemma 1(iii), (iv), we see that the "if parts" are valid under the weaker assumptions that \( \sigma(\phi(A)) \subseteq \sigma(A) \) for every \( A \in S \).
We conclude that \( \psi(E_i), i = 1, 2, \ldots, n \) are mutually orthogonal nonzero projections. We have \( \psi(A) = \sum_{i=1}^{n} c_i \psi(E_i) \), and since \( \psi(A) \) is invertible, we have \( c_i \neq 0, 1 \leq i \leq n \), and hence \( A \) is invertible.

**Step 5.** Now we apply Theorem 1 to \( \psi \) to get a unitary operator \( U: H \to H \) such that \( \psi(A) = UAU^* \) for every \( A \in S \) or \( \psi(A) = UA^tU^* \) for every \( A \in S \). If \( S = DU \), then \( \phi(A) = \pm SAS^* \) or \( \phi(A) = \pm SA^tS^* \). ■

**COROLLARY 2.** Let \( H \) be a complex finite dimensional Hilbert space and \( \phi: \mathcal{B}(H) \to \mathcal{B}(H) \) an adjoint preserving linear map. If \( \phi \) preserves invertibility for self-adjoint operators, then \( \phi \) takes one of the forms given by Theorem 2. In particular \( \phi \) preserves invertibility on \( \mathcal{B}(H) \).

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