Nets of Order $4m + 2$: Linear Dependence and Dimensions of Codes

by

Leah Howard
B.Sc. (Hons.), University of Toronto, 2003
M.Sc., University of Victoria, 2006

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in the Department of Mathematics & Statistics

© Leah Howard, 2009
University of Victoria

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopying or other means, without the permission of the author.
Nets of Order $4m + 2$: Linear Dependence and Dimensions of Codes

by

Leah Howard

B.Sc. (Hons.), University of Toronto, 2003
M.Sc., University of Victoria, 2006

Supervisory Committee

Dr. Peter Dukes, Supervisor
(Department of Mathematics & Statistics)

Dr. Kieka Mynhardt, Departmental Member
(Department of Mathematics & Statistics)

Dr. Wendy Myrvold, Outside Member
(Department of Computer Science)
Abstract

A $k$-net of order $n$, denoted $N_k$, is an incidence structure consisting of $n^2$ points and $nk$ lines. Two lines are said to be parallel if they do not intersect. A $k$-net of order $n$ satisfies the following four axioms: (i) every line contains $n$ points; (ii) parallelism is an equivalence relation on the set of lines; (iii) there are $k$ parallel classes, each consisting of $n$ lines and (iv) any two non-parallel lines meet exactly once.

A Latin square of order $n$ is an $n$ by $n$ array of symbols in which each row and column contains each symbol exactly once. Two Latin squares $L$ and $M$ are said to be orthogonal if the $n^2$ ordered pairs $(L_{i,j}, M_{i,j})$ are all distinct. A set of $t$ mutually orthogonal Latin squares is a collection of Latin squares, necessarily of the same order, that are pairwise orthogonal. A $k$-net of order $n$ is combinatorially equivalent to $k - 2$ mutually orthogonal Latin squares of order $n$. It is this equivalence that motivates much of the work in this thesis.

One of the most important open questions in the study of Latin squares is: given an order $n$ what is the maximum number of mutually orthogonal Latin squares of that order? This is a particularly interesting question when $n$ is congruent to two modulo four. A code is constructed from a net by defining the characteristic vectors of lines to be generators of the code over the finite field $F_2$. Codes allow the structure of nets to be profitably explored
using techniques from linear algebra.

In this dissertation a framework is developed to study linear dependence in the code of the net $N_6$ of order ten. A complete classification and combinatorial description of such dependencies is given. This classification could facilitate a computer search for a net or could be used in conjunction with more refined techniques to rule out the existence of these nets combinatorially. In more generality relations in 4-nets of order congruent to two modulo four are also characterized.

One type of dependency determined algebraically is shown not to be combinatorially feasible in a net $N_6$ of order ten. Some dependencies are shown to be related geometrically, allowing for a concise classification.

Using a modification of the dimension argument first introduced by Dougherty [19] new upper bounds are established on the dimension of codes of nets of order congruent to two modulo four. New lower bounds on some of these dimensions are found using a combinatorial argument. Certain constraints on the dimension of a code of a net are shown to imply the existence of specific combinatorial structures in the net.

The problem of packing points into lines in a prescribed way is related to packing problems in graphs and more general packing problems in combinatorics. This dissertation exploits the geometry of nets and symmetry of complete multipartite graphs and combinatorial designs to further unify these concepts in the context of the problems studied here.
Table of Contents

Supervisory Committee ii

Abstract iii

Table of Contents v

List of Tables vii

Acknowledgements viii

1 Introduction 1
   1.1 Latin Squares, Orthogonality and Nets 1
   1.2 Isotopy ........................................... 21
   1.3 Major Results ..................................... 26

2 The Dimension Argument 29
   2.1 The Dimension Argument ........................ 29

3 A Classification of the Relations 38
   3.1 Relations in $N_4$ ................................. 38
   3.2 Relations in $N_5$ ................................. 42
   3.3 Relations in $N_6$ ................................. 46
   3.4 Analogues .......................................... 55
   3.5 The Classification ................................. 58
   3.6 A Pairwise Balanced Design Approach .......... 63

4 The Structure of Relations 71
   4.1 Regularity Conditions ........................... 71
   4.2 Alpha and Beta .................................... 77
   4.3 Structure in Four or Five Classes ............... 80
## TABLE OF CONTENTS

4.4 Structure in Six Classes ........................................ 83  
4.5 Point Deletion for Local Structure ........................... 98  

5 Dimension and Minimum Weight .......................... 102  
5.1 Dimensions of Nets ............................................. 102  
5.2 Minimum Weight in Codes of Nets ......................... 118  

6 Packing Cliques .............................................. 124  
6.1 Packing Cliques in Four Classes ........................... 125  
6.2 Packing Cliques in Five Classes ........................... 129  
6.3 Packing Cliques in Six Classes ............................. 132  

7 Applications and Future Work ........................... 137  
7.1 Applications .................................................... 137  
7.2 Future Work .................................................... 140  

Bibliography .................................................... 142
List of Tables

Table 1  The dimension of $C_2(N_k)$  115
Acknowledgements

I would like to thank my supervisor, Dr. Peter Dukes, for giving me the freedom to explore my own ideas and for always being willing to discuss them with me. Thank you for encouraging me when I needed it the most.

Thank you to Dr. Kieka Mynhardt and Dr. Wendy Myrvold for investing time and effort as members of my committee and helping me to make this my best possible work.

Thank you to my parents and siblings for trying to understand my passion for mathematics and helping me to keep things in perspective. Thanks to my friends for keeping me sane, or at least never giving up the attempt. Thanks to my grandparents for relaxing afternoons and wonderful meals in Sidney. Finally a huge thanks to Ryan for being proud of me, and for cooking for me so many nights when I was too tired, or overwhelmed, or hopelessly trapped in piles of math papers.

Thank you to NSERC for generous financial support.
Chapter 1

Introduction

1.1 Latin Squares, Orthogonality and Nets

This work begins with some definitions and background material related to the problems studied here. For more theory, examples and historical context see [17] [18].

Definition 1.1 A Latin square of order $n$ is an $n \times n$ array of symbols in which each row and column contains each symbol exactly once.

Definition 1.2 Two Latin squares $L$ and $M$ are said to be orthogonal if the $n^2$ ordered pairs $(L_{i,j}, M_{i,j})$ are distinct.

It is well known that the maximum number of mutually orthogonal Latin squares of order $n$ is at most $n - 1$. A set of mutually orthogonal Latin squares of order $n$ is often referred to as a set of MOLS($n$).
Definition 1.3 A \((v,k,\lambda)\)-design is a set of \(v\) points \(V\) together with a set of blocks \(B\), each consisting of \(k\) distinct points chosen from \(V\), such that any pair of distinct points of \(V\) is contained in exactly \(\lambda\) blocks.

Definition 1.4 An affine plane of order \(n\) is an \((n^2, n, 1)\)-design.

In this context the blocks are called lines. Two lines that do not intersect are said to be parallel. A collection of lines that partitions the point set is called a parallel class. An important feature of an affine plane is that parallelism is an equivalence relation on the set of blocks. There are \(n + 1\) parallel classes, each consisting of \(n\) lines. It follows that any two non-parallel lines meet in exactly one point.

Definition 1.5 A projective plane of order \(n\) is an \((n^2 + n + 1, n + 1, 1)\)-design.

Again blocks are referred to as lines of a projective plane. It can be shown that any two distinct lines of a projective plane meet in exactly one point. This follows from the fact that a projective plane is a symmetric block design, which means that the number of blocks is equal to the number of points.

The existence of an affine plane of order \(n\), a projective plane of order \(n\) and a collection of \(n - 1\) mutually orthogonal Latin squares of order \(n\) are all equivalent. Such objects are known to exist for all prime powers \(n\) and are conjectured not to exist for any non-prime power order. For more background on this subject see [3].
Definition 1.6 A transversal design of group size \( n \) and block size \( k \), denoted 
\( TD(k,n) \), is a triple \( (\mathcal{V}, \mathcal{G}, \mathcal{B}) \) where:
(i) \( \mathcal{V} \) is a set of \( kn \) elements;
(ii) \( \mathcal{G} \) is a partition of \( \mathcal{V} \) into \( k \) classes (the groups), each of size \( n \);
(iii) \( \mathcal{B} \) is a collection of \( k \)-subsets of \( \mathcal{V} \) (the blocks);
(iv) each pair of distinct elements from \( \mathcal{V} \) is contained either in exactly one 
group or in exactly one block, but not both.

A \( TD(k,n) \) is equivalent to \( k-2 \) MOLS\( (n) \). The \( k \) groups of the transversal 
design index the rows, columns, and symbols in the \( k-2 \) squares respectively. Thus each ordered 
pair of row and column indices produces an ordered 
\( k \)-tuple of indices which can be turned into a \( k \)-set by assigning component 
\( i \) the symbols \( (i-1)n, (i-1)n+1, \ldots, (i-1)n+n-1 \). The process can 
be reversed to produce a set of \( k-2 \) MOLS\( (n) \) from the transversal design. 
Note that the process does not produce a unique transversal design or set of 
MOLS because the squares or groups can be taken in any order.

Definition 1.7 A group divisible design of order \( v \) \( (K\text{-GDD}) \) with block 
sizes from \( K \) and group sizes from \( G \) is a triple \( (\mathcal{V}, \mathcal{G}, \mathcal{B}) \) where \( \mathcal{V} \) is a set of 
cardinality \( v \), \( \mathcal{G} \) is a partition of \( \mathcal{V} \) into parts (groups) whose sizes lie in \( G \) 
and \( \mathcal{B} \) is a family of subsets (blocks) of \( \mathcal{V} \) that satisfy:
(i) if \( B \in \mathcal{B} \) then \( |B| \in K \);
(ii) every pair of distinct elements of \( \mathcal{V} \) occurs in exactly one block or group, 
but not both;
(iii) $|\mathcal{G}| > 1$.

Associated with a $K$-GDD is a type describing the structure of the groups. A $k$-GDD of type $m^n$ has blocks of size $k$ and $n$ groups of size $m$. If $n = k$ the GDD is a TD($k,m$).

**Definition 1.8** An orthogonal array $OA(k,s)$ is a $k \times s^2$ array with entries from an $s$-set $S$ having the property that in any two rows, each ordered pair of symbols from $S$ occurs exactly once.

An orthogonal array $OA(k,n)$ exists if and only if a TD($k,n$) exists.

**Example 1.9** The equivalence of 3 MOLS(4), an $OA(5,4)$ and a TD(5,4).

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
\end{array}
\]

The blocks of the TD(5,4) are as follows:
Taking the top two rows of the orthogonal array as row and column indices respectively, subsequent rows of the orthogonal array are filled with the corresponding entry from each of the three Latin squares. The blocks of the transversal design are constructed using the columns of the orthogonal array. Given a column, each symbol is paired with the row of the array in which it appears. For example, the block \( \{21, 12, 43, 34, 25\} \) of the transversal design corresponds to the fifth column of the orthogonal array. This is turn corresponds to the fact that in Row 2 and Column 1 the three Latin squares have symbols 4, 3 and 2 respectively.

A Latin square of order \( n \) is equivalent to the following objects: a TD(3, \( n \)); an OA(3, \( n \)); \( n^2 \) mutually non-attacking rooks on an \( n \times n \times n \) board; a single-error detecting code of length 3 having \( n^2 \) codewords from an \( n \)-symbol alphabet.

**Definition 1.10** Let \( K \) be a subset of positive integers. A pairwise balanced design \( PBD(v, K) \) of order \( v \) with block sizes from \( K \) is a pair \((\mathcal{V}, \mathcal{B})\), where \( \mathcal{V} \) is a finite set (the point set) of cardinality \( v \) and \( \mathcal{B} \) is a family of subsets
(blocks) of $V$ that satisfy:

(i) if $B \in \mathcal{B}$ then $|B| \in K$ and

(ii) every pair of distinct elements of $V$ occurs in exactly one block of $\mathcal{B}$.

Note that when the groups of a TD($k,n$) are added to the block set a PBD($kn,\{k,n\}$) is produced.

**Definition 1.11** A $k$-net of order $n$, which will be denoted $N_k$, is an incidence structure consisting of $n^2$ points and $nk$ lines satisfying the following four axioms:

(i) every line contains $n$ points;
(ii) parallelism is an equivalence relation on the set of lines;
(iii) there are $k$ parallel classes, each consisting of $n$ lines; and
(iv) any two non-parallel lines meet exactly once.

The number of parallel classes in a net is called its degree. The points of the net $N_k$ will often be described as ordered $k$-tuples from $\{0,1,\ldots,n-1\}^k$. In this way a point is considered as an intersection of $k$ lines of the net, one from each parallel class. Any pair of coordinates then uniquely identifies a point since any two non-parallel lines meet in exactly one point of the net.

**Definition 1.12** A subnet $N'$ of a net $N$ consists of the same points as $N$ and some subset of the parallel classes of $N$.

The existence of an $(n+1)$-net of order $n$ is equivalent to the existence of an affine plane of order $n$. A $k$-net of prime power order may be obtained
by taking \( k \) of the parallel classes of an affine plane of the same order. In general, a \( k \)-net of order \( n \) corresponds to a set of \( k - 2 \) MOLS(\( n \)). It should be noted that a \( k \)-net \( N_k \) is not always extendible to an affine plane when \( k < n \).

**Definition 1.13** A transversal in a net of order \( n \) is a set of \( n \) points containing exactly one point in common with each line of the net.

**Definition 1.14** Given a net \( N_k \) of order \( n \) whose points are labelled \( p_1, p_2, \ldots, p_{n^2} \), the characteristic function of a line of the net is the vector in \( F_2^{n^2} \) with \( i \)-th component equal to 1 if \( p_i \) lies on the line, and 0 otherwise. Let \( C_2(N_k) \) denote the vector space over \( F_2 \) generated by the characteristic functions of lines in \( N_k \). Let \( D_2(N_k) \) denote the vector space over \( F_2 \) generated by characteristic functions of vectors of the form \( l - m \) where \( l \) and \( m \) lie in the same parallel class.

**Definition 1.15** A linear \( q \)-ary code of length \( n \) is a subspace of the vector space \( F_q^n \). A codeword is any vector lying in the code.

The vector spaces \( C_2(N_k) \) and \( D_2(N_k) \) where the nets have order \( n \) are examples of linear binary codes of length \( n^2 \).

**Definition 1.16** The dimension of a linear code \( W \), denoted \( \dim W \), is the dimension of \( W \) as a vector space.

**Definition 1.17** The dual code \( W^\perp \) to a linear binary code \( W \) of length \( n^2 \) is the orthogonal complement \( W^\perp = \{ x \in F_2^{n^2} | [x, w] = 0 \text{ for all } w \in W \} \), where \( [x, w] \) is the dot product over \( F_2 \).
Definition 1.18 A relation in a net is a subset of the lines of the net so that each point of the net lies on an even number (possibly zero) of these lines. The parallel classes associated with a relation are those contributing a positive number of lines to the relation.

To rephrase this definition, a relation is a collection of lines from all but the last parallel class which sum over $F_2$ to a combination of lines from the last parallel class. Each relation in a net $N_k$ corresponds to a linear dependence in $C_2(N_k)$. The relations consisting of an even number of entire parallel classes will be called trivial relations.

Definition 1.19 A configuration is a relation with the lines from one parallel class deleted. The deleted parallel class must by definition contribute a positive number of lines to the original relation.

Definition 1.20 The type of a relation is a list of the number of lines that each parallel class contributes to the relation. These numbers are called the weights of the parallel classes.

If a parallel class contributes no lines to a relation it will not be listed in the type. Since relations are considered up to permutations of the parallel classes of the net the convention will be that relation types are listed in non-decreasing order.

Definition 1.21 The weight of a point with respect to a relation or a configuration is the number of lines on which it lies.
It is often useful to consider a configuration associated with a relation because it produces points of both odd and even weight, and provides information about how the lines from the last class intersect lines from the other classes.

**Notation 1.22** The variables $Z, S, D, T, Q, K, H$ represent the number of uncovered, weight one, weight two, weight three, weight four, weight five and weight six points in a configuration or relation. These points will be referred to as zeroes, singles, doubles, triples, quads, quints, and hexes.

**Notation 1.23** The parallel classes of a net $N_k$ (considered in a fixed ordering) will be labelled $P_1, P_2, \ldots, P_k$.

**Definition 1.24** The weight of a vector $(x_1, \ldots, x_m)$ in $F_2^m$ is the number of coordinates in which a 1 appears. In the context of a code the weight of a codeword is its weight as a vector.

**Example 1.25** A relation of type $\{4, 4, 4, 4\}$ in a net $N_4$ of order ten, corresponding to two orthogonal Latin squares of order ten. The relation consists of lines labelled 0, 1, 2 or 3 in each of the four parallel classes. Each of the 100 points of the net is represented by the ordered 4-tuple of lines which meet that point. The weight of each point is equal to the number of bold symbols in the corresponding 4-tuple.
In the work to follow it is often convenient to move between the geometry of nets, a graph-theoretic setting which simplifies packing problems, and a design-theoretic setting which has the advantage of being balanced with respect to pair coverage. A few more definitions are introduced and then a list of correspondences link the central ideas from the three perspectives.

**Definition 1.26** A graph $G = (V, E)$ is a set of vertices $V$ together with a set of edges $E$ where each edge is an unordered pair of distinct vertices of $V$.

The number of edges incident with a given vertex is called the degree of the vertex.

**Definition 1.27** A graph $G = (V, E)$ is said to be complete if $E$ consists of all pairs of distinct vertices of $V$. The order of a graph is the cardinality of its vertex set.
A complete graph is denoted $K_n$ where $|V| = n$. Complete graphs are often called *cliques* when they appear as a subgraph of a larger graph.

**Definition 1.28** A complete multipartite graph with parts $P_1, P_2, \ldots, P_r$ is a graph $G = (V, E)$ such that $V$ may be partitioned $P_1 \cup P_2 \cup \ldots \cup P_r$ and an edge $e = \{v_i, v_j\}$ is in $E$ if and only if $v_i \in P_i$ and $v_j \in P_j$ for some $i \neq j$.

If $|P_i| = s_i$ for $1 \leq s \leq r$ then the complete multipartite graph is unique up to isomorphism and will be denoted $K_{s_1, s_2, \ldots, s_r}$.

**Definition 1.29** A packing of graphs $G_1, G_2, \ldots, G_s$ into a graph $G$ is an assignment of a vertex of $V(G)$ to each vertex in $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_s)$ such that each edge in $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_s)$ is mapped to an edge in $E(G)$ and no edge in $E(G)$ is the image of more than one edge in $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_s)$.

**Definition 1.30** The leave of a packing is the set of edges in $E(G)$ which are not the image of an edge in $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_s)$.

**Definition 1.31** A packing into a graph $G$ is said to be a decomposition of $G$ if it uses all the edges of $G$, that is if the leave of the packing is empty.

Below several definitions associated with a net $N_k$ of order $n$ are listed together with the analogous concepts in the $k$-partite complete graph $K_{n, n, \ldots, n}$ and in a $\{k, n\}$-PBD. It is assumed that $k \neq n$ for simplicity of
description.

<table>
<thead>
<tr>
<th>Net</th>
<th>Graph</th>
<th>PBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>clique of order $k^*$</td>
<td>block of size $k$</td>
</tr>
<tr>
<td>line</td>
<td>vertex</td>
<td>point</td>
</tr>
<tr>
<td>parallel class</td>
<td>part</td>
<td>block of size $n$</td>
</tr>
<tr>
<td>pair of incident lines</td>
<td>edge</td>
<td>pair of distinct vertices in block of size $k$</td>
</tr>
</tbody>
</table>

*The analogy is not perfect because most cliques $K_k$ do not correspond to a point of the net.

The packing problem of fitting high weight points into the net while ensuring that no pair of distinct lines meet more than once can be rephrased in terms of packing large cliques into a complete multipartite graph or packing pairs of distinct vertices into blocks of prescribed sizes in a PBD so that no pair of distinct vertices occurs together more than once.

Occasionally notation and terminology will be abused to lend greater clarity. For instance in the graph-theoretic setting a clique corresponding to a point of weight four in the net may be referred to as a *quad* rather than a copy of $K_4$. 
CHAPTER 1. INTRODUCTION

One of the biggest open questions on the topic of orthogonal Latin squares is: What is the largest set of MOLS(n) when n is not a prime power? Some upper and lower bounds are presented below.

**Notation 1.32** Let $N(n)$ represent the maximum number of MOLS(n).

By MacNeish’s product construction (as cited in [13]) it is known that $N(n)$ is at least one less than the smallest prime power in the prime power expansion of $n$. This gives a lower bound of $N(6) \geq 1$ and $N(10) \geq 1$ for the first two non-prime powers.

The fact that $N(6) = 1$ was known to Euler and led him to conjecture that $N(n) = 1$ for all $n$ congruent to 2 modulo 4. It was Tarry [30] in 1900 who was the first to rigorously prove that $N(6) = 1$ and it was Bose and Shrikhande [7] in 1959 who disproved Euler’s more general conjecture by presenting two orthogonal Latin squares of order 22. They later showed that an infinite class of counterexamples exists [8]. Teaming up with Parker, the three [9] were then able to disprove the result for all $n$ congruent to 2 modulo 4 and larger than 6.

A short proof of the fact that $N(6) = 1$ was given by Stinson [29] in 1984. Stinson assumed the existence of TD(4, 6), combinatorially equivalent to two MOLS(6), and studied its point-line incidence matrix algebraically. He established three dependence relations on the rows and showed by a di-
mension argument that at least one more dependence relation should exist. Then by a combinatorial argument involving pair coverage in the blocks of the transversal design he showed that no such dependence relation exists, yielding a contradiction to the existence of two MOLS(6).

Stinson’s dimension argument motivated the work of Dougherty [19], who presented a new proof that \( N(6) = 1 \). Dougherty’s dimension argument generalized that of Stinson and raised the possibility of extending the methods to larger values of \( n \). Dougherty’s work in turn greatly influenced the work here, in particular by suggesting combinatorial methods that could be used to rule out certain configurations (which correspond to dependence relations in the incidence matrix).

Another short proof that \( N(6) = 1 \) was given by Betten [4] [5] in 1984. Betten showed that each equivalence class of Latin squares of order 6 contains a Latin square whose rows are associated with a specific type of permutation. This ultimately implies that a TD(3, 6) cannot have six mutually disjoint parallel classes, and therefore does not extend to a TD(4, 6).

While no \( N(n) \) value is known for non-prime power orders other than six, there are two types of results which narrow the gap between the trivial lower bound of 1 and the trivial upper bound of \( n - 1 \): combinatorial results and number-theoretical results. Combinatorial results produce a construc-
tive lower bound or an upper bound by contradiction. The landmark upper bound was by Lam, Thiel and Swiercz [22] in 1989 who demonstrated that $N(10)$ is less than nine.

A brief history of the work of Lam, Thiel and Swiercz is presented here as it motivates the study of dimensions of nets as well as the distribution of weights in the codes resulting from them. Assmus Jr. and Mattson Jr. [1] showed that the incidence matrix of a hypothetical projective plane of order 10 with a parity check column appended has rank 56. It follows that the binary code generated by the row vectors of this matrix would be self-dual. Much is known about self-dual codes, in particular the MacWilliams Equations or the Gleason Polynomials (in the case of a code over $F_2$ or $F_3$) may be used to compute the weight distribution of codewords. For more background on coding theory see [28].

The Gleason polynomials together with the number of codewords of weights 12, 15 and 16 would determine the weight distribution of codewords in the code resulting from the projective plane. In 1973, MacWilliams, Sloane, and Thompson [24] showed that there are no codewords of weight 15. In 1983 Lam, Thiel, Swiercz, and McKay [23] showed a similar result for codewords of weight 12. Finally in 1986 Lam, Thiel and Swiercz [21] showed that there are no codewords of weight 16 in the code. With the weight distribution determined, Lam, Crossfield and Thiel [20] announced a strategy to
investigate the codewords of weight 19 which were known to have interesting combinatorial properties. With the weight distribution known, the number of codewords of weight 19 was confirmed to equal 24675, so that a 19-point configuration was guaranteed to exist in any projective plane of order ten. A computer search was then conducted. Lam, Thiel and Swiercz [22] showed that no 19-point configuration can be extended to a projective plane of order 10. Their results imply the non-existence of a projective plane of order 10 and show that the upper bound of nine MOLS(10) cannot be achieved.

As Lam, Thiel and Swiercz themselves remarked, because of their reliance on the computer their result should perhaps not be termed a proof. While the probability of human or computer error is very small it remains a possibility. A new mathematical proof of their result would be a significant achievement for mathematics.

A result of Bruck [11] gives the current upper bound:

**Theorem 1.33** If $N(n) < n - 1$ then $N(n) < n - 1 - (2n)^{\frac{1}{2}}$.

Putting this together with the result of Lam, Thiel and Swiercz produces the current best upper bound of $N(10) \leq 6$.

**Definition 1.34** A graph is said to be strongly regular if there exist integers $\rho, \lambda$ and $\mu$ such that:

(i) each vertex of the graph has degree $\rho$;
(ii) the number of vertices mutually adjacent to two adjacent vertices is \( \lambda \);
(iii) the number of vertices mutually adjacent to two non-adjacent vertices is \( \mu \).

Peeters [27] points out that the upper bound \( N(10) \leq 6 \) can also be derived by considering net graphs. The collinearity graph for a net is called a net graph and is always strongly regular. A strongly regular graph having parameters of a net graph, but which may or may not correspond to a net, is called a pseudo-net graph. Whenever \( n \neq 4 \), pseudo-net graphs of degree two are unique, and therefore always correspond to a net. So when \( n \neq 4 \), a net of degree \( n - 1 \) can always be completed to a net of degree \( n + 1 \) via the complement of its net graph. Thus by contrapositive if there are fewer than \( n - 1 \) MOLS\((n)\) there can be at most \( n - 4 \) MOLS\((n)\). In the exceptional case \( n = 4 \) the two pseudo-net graphs of degree two are the lattice graph \( L_2(4) \) and the Shrikhande graph.

A result by McKay, Meynert and Myrvold [25] heavily restricts the possibilities for a triple of mutually orthogonal Latin squares of order 10. They showed that any square in such a triple must have a trivial symmetry group (this symmetry group is actually the autoparatopy group defined in the next section).

For completeness some of the known lower bounds for \( N(n) \) are listed below. Of particular interest is the case of \( n \) congruent to two modulo four:
$N(10) \geq 2; \ N(12) \geq 5; \ N(14) \geq 3; \ N(15) \geq 4; \ N(18) \geq 3; \ N(20) \geq 4.$

It should also be noted that Chowla, Erdős and Straus [13] established that $N(n)$ tends to infinity asymptotically by showing $N(n) > \frac{1}{3}n^{\frac{1}{2}}$ for sufficiently large $n$. Since then, much work has been done to improve the exponent in this bound.

The second type of result leading to improved upper bounds for $N(n)$ is number-theoretical:

**Theorem 1.35** [3] (Bruck-Ryser-Chowla) If a symmetric $(v, k, \lambda)$-design exists and $v$ is even then $n = k - \lambda$ is a square. If $v$ is odd then there is an integer solution $(x, y, z) \neq (0, 0, 0)$ to the equation $z^2 = (k - \lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$.

A corollary to this is:

**Theorem 1.36** [12] (Bruck-Ryser) If $n \equiv 1, 2 \pmod{4}$ and the square-free part of $n$ contains at least one prime factor of the form $4k + 3$ then there is no projective plane of order $n$.

For instance, it follows from the Bruck-Ryser Theorem that $N(n) < n - 1$ for $n = 6, 14, 21, 22, 30, 33, 38$ and an infinity of other orders.

There are many algebraic techniques for constructing Latin squares. Below are two generalizations of the group concept which provide useful constructions:
CHAPTER 1. INTRODUCTION

Definition 1.37 A set $S$ is called a quasigroup if there is a binary operation $\ast$ defined on $S$ and if, when any two elements $a$ and $b$ of $S$ are given, the equations $a \ast x = b$ and $y \ast a = b$ each has a unique solution in $S$.

Definition 1.38 A loop is a quasigroup $L$ with an identity element: that is a quasigroup in which there exists an element $e$ of $L$ with $e \ast x = x \ast e = x$ for each $x$ in $L$.

The multiplication table of a quasigroup is a Latin square. It is easily checked that the uniqueness of solutions to the two equations above ensure the Latin property in each row and column.

Example 1.39 The set of integers modulo $3$ with the binary operation $a \ast b = 2a + b + 1$ yields the Latin square below (with rows and columns indexed in the natural order):

$$
\begin{array}{ccc}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1 \\
\end{array}
$$

The multiplication table of a quasigroup or loop produces a Latin square whose structure permits a study of distribution of symbols within the square, the position of transversals, possibilities for finding an orthogonal mate for the square, and symmetries of the square. Unfortunately most Latin squares do not come from such an algebraic structure.
A brief construction for projective planes of prime power order is presented next. For \( q \) a prime power let \( \mathcal{V} = F_q^3 \). Let \( V_1 \) and \( V_2 \) be the set of all one-dimensional subspaces of \( \mathcal{V} \) and the set of all two-dimensional subspaces of \( \mathcal{V} \) respectively. For each \( B \) in \( V_2 \) define a block \( A_B = \{ L \in V_1 : L \subset B \} \). Let \( \mathcal{B} = \{ A_B : B \in V_2 \} \). Then \((\mathcal{V}, \mathcal{B})\) is a projective plane of order \( q \). It is usually described as the finite geometry \( \text{PG}(2,q) \). It is also known as the Galois plane of order \( q \).

The work of Moorhouse, which will be discussed in more detail later on, requires the following definitions:

**Definition 1.40** A triangle in a net is a set of three mutually non-parallel lines which intersect pairwise in three distinct points. The three lines are called the sides of the triangle. A projective or affine plane is said to be desarguesian provided that the following holds: for any two triangles with vertices \( A_i, B_i, C_i \) and opposite sides \( a_i, b_i, c_i, \ i = 1, 2, \) if the lines \( A_1A_2, B_1B_2 \) and \( C_1C_2 \) are concurrent (intersect in a single point) then the points \( a_1 \cap a_2, b_1 \cap a_2 \) and \( c_1 \cap c_2 \) are collinear.

**Definition 1.41** A \( k \)-net of order \( n \) is said to be desarguesian if it can be extended to a desarguesian affine plane of order \( n \).

It is known that the only finite desarguesian projective planes are the projective planes \( \text{PG}(2,q) \) where \( q \) is a prime power [14]. Non-desarguesian pro-
jective planes do exist; in particular there are three known non-desarguesian projective planes of order nine.

\section*{1.2 Isotopy}

In this section the types of transformations which can be applied to a Latin square to produce another Latin square of the same order are described. These transformations will be used later to greatly simplify the combinatorial structures under consideration.

\textbf{Definition 1.42} Two Latin squares are said to be isotopic if one can be obtained from the other by some composition of row, column, and symbol permutations.

It is standard to represent row permutations with \( \theta \), column permutations with \( \phi \) and symbol permutations with \( \psi \). The symbol set will be taken to be the set of integers \( \{0, 1, \ldots, n-1\} \), where \( n \) is the order of the Latin square. Rows and columns will be indexed by the symbol set in the natural way.

A further notational convention involves permutations: a cycle within a permutation is read from left to right whereas cycles in a permutation act from right to left. For example, the permutation \((01)(012)\) sends the element 2 to the element 1.
Example 1.43 The Latin squares with rows indexed by $R_0, R_1$ and $R_2$, and columns indexed by $C_0, C_1$ and $C_2$:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0
\end{array}
\quad
\begin{array}{ccc}
2 & 0 & 1 \\
0 & 1 & 2
\end{array}
\quad
\begin{array}{ccc}
2 & 0 & 1 \\
1 & 2 & 0
\end{array}
\]

are isotopic. Given $\theta = (R_1 R_2)$, $\phi = (C_1 C_2)$ and $\psi = (02)$, the mapping $\psi \phi \theta$ sends the first square to the second. The mapping is not unique: $\theta = (R_0 R_1 R_2)$ also sends the first square to the second.

While expressing $\theta$ and $\phi$ in terms of rows and columns lends clarity, it is sometimes convenient to think of $\theta$ and $\phi$ as permutations of row or column indices. In Example 1.43 it could be said that $\theta = (12) = \phi$ with no loss of information. It is this notation that is used in the following definition.

Definition 1.44 Two Latin squares are said to be isomorphic if they are isotopic with $\theta = \phi = \psi$.

Definition 1.45 A given Latin square $L$ has six conjugates, defined as follows:

$L$ the Latin square itself;

$-1L$ each column permutation is replaced with its inverse;

$L^{-1}$ each row permutation is replaced with its inverse;

$L^*$ rows and columns are interchanged ($L$ is transposed);
\((-1)L^*\) the result of taking \(L \mapsto -1 L\) and then transposing;
\((L^{-1})^*\) the result of taking \(L \mapsto L^{-1}\) and then transposing.

Stated in a different way, the six conjugates correspond to permutations of row, column and symbol sets, or permutations of the three groups of the corresponding transversal design. The six conjugates are not necessarily distinct; because they form a subgroup of the symmetric group on three symbols there are either one, two, three or six distinct conjugates.

**Definition 1.46** A set of Latin squares which comprises all the members of an isotopy class together with their conjugates is called a **main class**. Two squares lying in the same main class are said to be **paratopic**.

The set of all Latin squares of a given order partitions into main classes, each of which is a union of isotopy classes. Each isotopy class in turn partitions into isomorphism classes. Main class equivalence preserves the number of subsquares of a given order, the number of transversals and the number of partitions into transversals [17].

Let \(T\) represent the group of conjugate mappings and \(\langle S^3_n, T \rangle\) represent the group of mappings sending a Latin square to an element of its main class.

**Definition 1.47** The autoparatopy group of a Latin square \(L\) is the stabilizer \(\text{Par}(L) = \{ \sigma \in \langle S^3_n, T \rangle \mid \sigma(L) = L \}\).
Example 1.48 Consider the row permutation $\theta = (021)$ acting on the first square only of a pair of orthogonal Latin squares. Rows and columns are indexed by symbols 0, 1 and 2.

$$
\begin{array}{ccc}
00 & 11 & 22 \\
12 & 20 & 01 \\
21 & 02 & 10 \\
\end{array}
\Rightarrow
\begin{array}{ccc}
10 & 21 & 02 \\
22 & 00 & 11 \\
01 & 12 & 20 \\
\end{array}
$$

so that orthogonality is preserved.

Example 1.49 Now consider the column permutation $\phi = (12)$ acting on the first square only of a pair of orthogonal Latin squares:

$$
\begin{array}{ccc}
00 & 11 & 22 \\
12 & 20 & 01 \\
21 & 02 & 10 \\
\end{array}
\Rightarrow
\begin{array}{ccc}
00 & 21 & 12 \\
12 & 00 & 21 \\
21 & 12 & 00 \\
\end{array}
$$

If a Latin square is replaced with one isotopic to it, orthogonality to other Latin squares will not necessarily be preserved.

Given a Latin square of order $n$ define the $n^2$ ordered pairs of row and column coordinates to be the point set of a net of order $n$. Then $3n$ lines of cardinality $n$ are formed so that each line is a collection of points with a common row index ($n$ of these), a common column index ($n$ of these) or a common symbol in the indexed position of the Latin square ($n$ of these). This produces a net of degree three and order $n$. An isomorphism of nets is a
bijection between point sets which carries lines to lines. Two nets are *isomorphic* if and only if the two equivalent Latin squares are main class equivalent.

In the work to follow an important question is whether a set of lines can be completed to a net of given order and degree. To simplify the question some symmetries can be considered: any (independent) permutation of the symbol set of each parallel class; any permutation of the parallel classes; any independent permutation of lines of the two classes coordinatizing the net (corresponding to simultaneous row and column permutations of the Latin squares formed by all but the two coordinatizing parallel classes). None of these transformations affects the embeddability of a set of points and lines into a net.

Much of the work to follow involves establishing bounds on the dimensions of codes $C_2(N_k)$ of various nets of order $4m + 2$ and characterizing or restricting the structures which lead to dependence relations in these codes. A detailed list of major results is presented in the next section. This study of nets is motivated by the following questions: What is the maximum number of MOLS(10)? Can the results of Dougherty [19] be extended to show that there are at most three MOLS(10)? If there are four MOLS(10) what structures would be forced to occur in the corresponding net? Could a characterization of these structures provide the basis for a systematic search for four MOLS(10)?
1.3 Major Results

The major results contained in this thesis are summarized below.

Proposition 2.12 If a net $N_6$ of order ten exists then $\dim C_2(N_6) \leq 53$.

Proposition 2.13 Let $N_k$ be a net of order $n$ for even $n > 2$ with $k \geq \frac{n}{2} + 1$.
If $k$ is even or $N_k$ has a transversal then a non-trivial upper bound is $\dim C_2(N_k) \leq \frac{n^2 + k}{2}$.

Propositions 3.3, 3.12 The identification of analogues and complement analogues to produce 32 different types of relations in a 6-net of order ten.

Theorem 3.13 A classification of the types of relations in a 6-net of order ten.

Section 3.6 A classification of the pairwise balanced designs corresponding to the relations in a 6-net of order ten.

Proposition 4.1 Given a relation in a net $N_6$ of order ten and its representation as an edge-decomposed complete multipartite graph, for each $i$ the number of $K_i$ incident with a vertex of each part of the graph depends only on the size of the part. These are known as regularity conditions.
Proposition 4.2 Given a relation in a net $N_6$ of order ten and its representation as an edge-decomposed complete multipartite graph, the values of $Q_{iL}$ and $D_{iL}$ depend only on the relation in question, the cardinality $i$ of the part, and the number of $K_6$ incident with the vertex. These are known as local regularity conditions.

Proposition 4.3 There is no packing of five copies of $K_4$ and eight copies of $K_3$ into the complete multipartite graph $K_{2,2,2,2,4}$ when the regularity conditions of a net $N_6$ of order ten are obeyed.

Proposition 4.4 The relation corresponding to the configuration (C6) in a net of order ten is ruled out.

Section 4.3 A structural characterization of the relations in four or five classes of a net of order ten.

Section 4.4 Structural descriptions of the relations in six classes of a net of order ten and upper bounds on the maximum number of structures.

Proposition 5.1 If $N_6$ is a 6-net of order ten and $\dim C_2(N_6) \leq 50$ then the net contains a relation that is not complementable to a relation of type \{4, 4, 4, 4, 4, 4\}.
Proposition 5.2 If $N_4$ is a 4-net of order ten then $\dim C_2(N_4) \geq 33$.

Proposition 5.3 If $N_4$ is a 4-net of order fourteen then $\dim C_2(N_4) \geq 49$.

Proposition 5.4 A relation in a 4-net of order $n \equiv 2 \pmod{4}$ with at most $\frac{n}{2}$ lines in each of the first three classes must be of type $\{k, k, k, k\}$ for $k$ an even integer satisfying $\frac{n}{3} \leq k < \frac{n}{2}$.

Corollary 5.6 Any nontrivial relation in a 4-net of order $n \equiv 2 \pmod{4}$ is complementable to one of type $\{k, k, k, k\}$ with parameters

$$(Z, S, D, T) = \left(\frac{2n^2 + 3k^2 - 5nk}{2}, \frac{3nk - 3k^2}{2}, \frac{3nk - 3k^2}{2}, \frac{3k^2 - nk}{2}\right).$$

Proposition 5.8 Let $N_k$ be a net of even order. If $k$ is even or $N_k$ has a transversal, then $\dim C_2(N_k) > \dim C_2(N_{k-1})$.

Proposition 6.10 An optimal packing of $K_4$ into $K_{a,b,c,d}$ contains exactly $ab$ copies of $K_4$ except for the following $(a, b, c, d, \text{number of } K_4)$:

$$(2, 2, 2, 2), (2, 2, 3, 3) \text{ and } (6, 6, 6, 34).$$
Chapter 2

The Dimension Argument

2.1 The Dimension Argument

The original goal of this work was to use a dimension argument on codes of nets to obtain an upper bound on the number of mutually orthogonal Latin squares of order 10. Although this goal is still a long way off, the dimension argument can be profitably applied to other questions. The most important of these include obtaining new bounds on dimensions of nets and characterizing the relations that the codes of certain nets admit. The work in this section follows the set-up of S. T. Dougherty’s paper [19] which gave a similar result for mutually orthogonal Latin squares of order 6.

Lemma 2.1 A net $N_1$ has:

$$\dim C_2(N_1) = n \text{ and } \dim D_2(N_1) = n - 1.$$
CHAPTER 2. THE DIMENSION ARGUMENT

Proof: After row reduction, the characteristic vectors of the lines of $N_1$ produce an $n \times n$ matrix in reduced row echelon form with $n$ leading ones. For the second equality, take a parallel class of $N_1$, say $\{b_1, b_2, \ldots, b_n\}$, and consider $D_2(N_1) \subseteq C_2(N_1)$. Any element of $D_2(N_1)$ is a sum of an even number of lines of $N_1$. This means that equality cannot hold and $\dim D_2(N_1) < \dim C_2(N_1) = n$. On the other hand, the following set of vectors is linearly independent: $\{b_1 + b_2, b_1 + b_3, \ldots, b_1 + b_n\}$. Thus $\dim D_2(N_1) \geq n - 1$, giving the desired equality. □

Notation 2.2 Given two $n$-vectors $x$ and $y$ the dot product of $x$ and $y$ is represented by $[x, y]$.

This notation is both consistent with the notion of the dot product as an inner product and convenient when dealing with vectors that are expressed as linear combinations of other vectors.

Proposition 2.3 [19] If $k$ is even or if $N_k$ has a transversal for a net of even order then $\dim C_2(N_k) - \dim D_2(N_k) = k$.

Proof: Since $C_2(N_k) = \langle m_1, \ldots, m_k, D_2(N_k) \rangle$ where $m_i$ lies in the $i$th parallel class, it follows that $\dim C_2(N_k) - \dim D_2(N_k) \leq k$. The result is true for $k = 1$ since $\dim C_2(N_1) = n$ and $\dim D_2(N_1) = n - 1$ by Lemma 2.1. Now for $k > 1$ it will be shown that $\{m_1, \ldots, m_k\}$ are linearly independent over $D_2(N_k)$. Observe that since $n$ is even, $D_2(N_k) \subseteq C_2(N_k)^\perp$. Suppose that $w = a_1m_1 + \ldots + a_km_k \in D_2(N_k) \subseteq C_2(N_k) \cap C_2(N_k)^\perp$. If $l_j$ is in the $j$th
parallel class then \( 0 \equiv [l_j, w] \equiv \sum_{i \neq j} a_i \) (mod 2). Thus \((k - 1) \sum_{i=1}^{n} a_i \equiv 0 \) (mod 2), and if \( k \) is not congruent to 1 modulo 2 it follows that \( \sum_{i=1}^{n} a_i \equiv 0 \) (mod 2). This together with \( \sum_{i \neq j} a_i \equiv 0 \) (mod 2) yields \( a_1 \equiv \cdots \equiv a_k \equiv 0 \) (mod 2) as desired.

Now suppose \( N_k \) has a transversal, \( t \). Because \([t, m - l] \equiv 0 \) (mod 2) for \( l \) and \( m \) in the same parallel class, \( t \in D_2(N_k)^\perp \). Thus \([t, w] \equiv 0 \) (mod 2) and \( \sum a_i \equiv 0 \) (mod 2). The argument follows as above. \( \square \)

**Proposition 2.4** [19] Suppose \( N_k \) has \( k \)th parallel class \( \{l_1^k, \ldots, l_n^k\} \), and that a non-trivial relation \( \sum_{i=1}^{n} a_{ij}^k l_i^k + \sum_{i=1}^{n} a_{ij}^{k-1} l_i^{k-1} + \cdots + \sum_{i=1}^{n} a_{ij}^1 l_i^1 = \vec{0} \) exists over \( F_2 \), where each \( a_{ij}^j \) is either 0 or 1. Let \( a^j = |\{a_{ij}^j | a_{ij}^j = 1\}| \). If \( a^j \) is odd for any \( j \) then \( \dim C_2(N_k) - \dim D_2(N_k) < k \).

**Proof:** Take one line with non-zero coefficient from each parallel class having odd weight, label it \( l_j^j \) and re-express the relation (over \( F_2 \)) \( \sum_{a_{ij} \text{ odd}} l_i^j = \sum a_{ij}^k l_i^k + \sum a_{ij}^{k-1} l_i^{k-1} + \cdots + \sum a_{ij}^1 l_i^1 \). Summation runs through \( 1 \leq i \leq n \) for classes having even weight originally and \( 2 \leq i \leq n \) for classes which had odd weight. Now consider the right-hand side. The weight of each parallel class is now even, so that the lines of each class may be paired up. Since the relation is over \( F_2 \) each of these pairs is an element of \( D_2(N_k) \). The right-hand side therefore belongs to \( D_2(N_k) \). By equality the left-hand side is also in \( D_2(N_k) \) so that \( \dim C_2(N_k) - \dim D_2(N_k) < k \). \( \square \)

**Corollary 2.5** Let \( n \) be even. If \( k \) is even or if \( N_k \) has a transversal then any relation as in Proposition 2.4 has all \( a^j \) even.
CHAPTER 2. THE DIMENSION ARGUMENT

Proof: By Proposition 2.3 these conditions ensure that \( \dim C_2(N_k) - \dim D_2(N_k) = k \). By Proposition 2.4 the left-hand side of the equation
\[
\sum_{a_{i1}, \ldots, a_{ik} \text{ odd}} l_i^1 = \sum_{i=1}^n a_{i1}k^k + \sum_{i=1}^n a_{i1}^{k-1}l_i^{k-1} + \cdots + \sum_{i=1}^n a_{i1}l_i^1
\]
must be zero. \( \square \)

Proposition 2.6 The dimension \( \dim C_2(N_2) = 2n - 1 \).

Proof: Let \( N_2 \) have parallel classes \( \{l_i\}_{i=1}^n \) and \( \{m_i\}_{i=1}^n \). It will be shown that any relation \( \sum_{i=1}^n a_il_i = \sum_{i=1}^n b_im_i \) must have \( a_i = 0 = b_i \) for all \( i \) or \( a_i = 1 = b_i \) for all \( i \) so that \( \{l_1, \ldots, l_n, m_1, \ldots, m_{n-1}\} \) is a basis for \( C_2(N_2) \).

Suppose some \( a_i = 1 \), say \( a_1 = 1 \). The line \( l_1 \) contains \( n \) points and the lines of \( \{m_i\}_{i=1}^n \) intersect \( l_1 \) in distinct points. So \( b_i = 1 \) for all \( i \), which in turn forces \( a_i = 1 \) for all \( i \). The alternative is that \( a_i = 0 \) for all \( i \), which forces \( b_i = 0 \) for all \( i \). \( \square \)

Lemma 2.7 Let \( N_3 \) have parallel classes \( \{l_i\}_{i=1}^n \), \( \{m_i\}_{i=1}^n \), \( \{t_i\}_{i=1}^n \). If \( \sum_{i=1}^n a_it_i = \sum_{i=1}^n b_il_i + \sum_{i=1}^n c_im_i \) then the weight of \((a_1, \ldots, a_n)\) is one of: 0, \( \frac{n}{2} \), \( n \).

Proof: Assume that the weight of \((a_1, \ldots, a_n)\) is neither 0 nor \( n \). Then without loss of generality relabel \( \{t_i\}_{i=1}^n \) so that \( a_1 = 1 \) and \( a_2 = 0 \). By considering coverage of the points in \( t_1 \), the weight of \((b_1, \ldots, b_n)\) plus the weight of \((c_1, \ldots, c_n)\) must equal \( n \). However, because the lines of \( \{l_i\}_{i=1}^n \) must yield zero coverage of the points of \( t_2 \), the two weights above must also be equal and therefore both equal \( \frac{n}{2} \). Interchanging the roles of \( \{t_i\}_{i=1}^n \) and \( \{l_i\}_{i=1}^n \) shows that the weight of \((a_1, \ldots, a_n)\) is also \( \frac{n}{2} \). \( \square \)
CHAPTER 2. THE DIMENSION ARGUMENT

Proposition 2.8 \[19\] Let \( n \equiv 2 \pmod{4} \). If \( N_3 \) has a transversal then
\[
\dim C_2(N_3) - \dim C_2(N_2) = n - 1 \quad \text{for any } N_2 \subset N_3.
\]

Proof: It suffices to show that all relations in \( N_3 \) are trivial when \( n \equiv 2 \pmod{4} \). If \( N_3 \) has a transversal then Proposition 2.3 implies that
\[
\dim C_2(N_3) - \dim D_2(N_3) = 3.
\]
Corollary 2.5 then implies that weight of any parallel class is even in any relation in \( N_3 \). Lemma 2.7 implies that the weights in any relation in \( N_3 \) are 0 or \( n \) since \( \frac{n}{2} \) is odd when \( n \equiv 2 \pmod{4} \). This establishes that all relations in \( N_3 \) are trivial. \( \square \)

Corollary 2.9 If \( N_3 \) of order \( n \equiv 2 \pmod{4} \) has a transversal then
\[
\dim C_2(N_3) = 3n - 2.
\]

Proof: By Proposition 2.6, \( \dim C_2(N_2) = 2n - 1 \) for any subnet \( N_2 \) of \( N_6 \).
The previous result gives \( \dim C_2(N_3) = 2n - 1 + n - 1 = 3n - 2 \), since \( N_3 \) has a transversal. \( \square \)

Before discussing some ideas for restricting the structure of a 6-net \( N_6 \) of order ten one more result is needed:

Proposition 2.10 If there is any relation involving some, or all, of the parallel classes of a net \( N_6 \) of order ten then the weight of each parallel class in the relation must be even.

Proof: Because 6 is even, \( \dim C_2(N_6) - \dim D_2(N_6) = 6 \) by Proposition 2.3. Similarly, because every proper subnet \( N_i \) of \( N_6 \) for \( 1 \leq i \leq 5 \) contains a
transversal (a necessary condition for the existence of another parallel class that could be added to $N_i$), \( \dim C_2(N_i) - \dim D_2(N_i) = i \). Proposition 2.4 with \( n = 10 \) then shows that all weights must be even.

\[ \square \]

**Proposition 2.11** The weights of all but the last class may be assumed to be zero, two or four in any relation in a net of order ten.

**Proof:** Suppose some weight is equal to six, eight or ten. By complementing the lines in the class in question and in the last class, another relation results. This is true because the parity of the weight of each point is preserved. The weight in question is thus reduced to zero, two or four.

\[ \square \]

The weight of the last class may therefore be two, four, six or eight. If the weight of the last class is zero or ten then a relation in fewer parallel classes may be considered, possibly by another application of the complementation argument above. Instead of studying individual relations in a net, the objects of interest are the equivalence classes of relations under complementation in an even number of parallel classes.

The idea is to study \( \dim C_2(N_{i+1}) - \dim C_2(N_i) \) for any subnet \( N_i \) contained in \( N_{i+1} \) and for \( i = 3, 4, 5 \). Proposition 2.11 will be used extensively to classify the possible relations that may exist in \( N_6 \). Then combinatorial arguments will be used to disprove the existence of some of these relations.

**Proposition 2.12** If a net \( N_6 \) of order ten exists then \( \dim C_2(N_6) \leq 53 \).
Proof: Suppose an $N_6$ of order ten exists. Let $\dim C_2(N_6) = 53 + \alpha$ for some integer $\alpha$. Then $\dim D_2(N_6) = 47 + \alpha$ by Proposition 2.3 and $\dim D_2(N_6)^\perp = 53 - \alpha$. Now $C_2(N_6) \subseteq D_2(N_6)^\perp$ implies $53 + \alpha \leq 53 - \alpha$ so that $\alpha \leq 0$. \hfill \Box

This gives an upper bound of at most 3 MOLS(10) provided $\dim C_2(N_6)$ can be shown to be at least 54.

It should be noted that the best possible bound arising from this method is that the maximum number of MOLS($n$) is $\leq \frac{n^2}{2} - 2$ when $n \equiv 2 \pmod{4}$. Assuming that $\dim C_2(N_k) - \dim D_2(N_k) = k$ and $\dim C_2(N_k) = n + (n - 1)(k - 1)$ gives $\dim D_2(N_k) = n + (n - 1)(k - 1) - k$. This means $\dim D_2(N_k)^\perp = n^2 - n - (n - 1)(k - 1) + k$. A contradiction results if $n + (n - 1)(k - 1) > n^2 - n - (n - 1)(k - 1) + k$, which happens when $k \geq \frac{n}{2} + 1$. So $k \leq \frac{n}{2}$ is necessary for the existence of $N_k$ provided that the hypothesis of Proposition 2.3 and the linear independence requirements are satisfied.

The condition that $n \equiv 2 \pmod{4}$ is necessary to ensure that $\dim C_2(N_3) = 3n - 2$. Without this the dimension argument fails immediately. However, the fact that $C_2(N_k) \subseteq D_2^\perp(N_k)$ for any even order $n$ can still be used to get an upper bound on the possible dimension of $\dim C_2(N_k)$ when $k$ is sufficiently large.
Proposition 2.13 Let $N_k$ be a net of order $n$ for even $n > 2$ with $k \geq \frac{n^2}{2} + 1$. If $k$ is even or $N_k$ has a transversal then a non-trivial upper bound $\dim C_2(N_k) \leq \frac{n^2+k}{2}$ results.

Proof: Since $n$ is even $C_2(N_k) \subseteq D_2^\perp(N_k)$. Let $\dim C_2(N_k) = \alpha$. Because $k$ is even or $N_k$ has a transversal, $\dim D_2(N_k) = \alpha - k$ by Proposition 2.3. Note that this proposition requires only that $n$ is even. This implies that $\dim D_2^\perp(N_k) = n^2+k-\alpha$. Then $C_2(N_k) \subseteq D_2^\perp(N_k)$ gives $\alpha \leq n^2+k-\alpha$ or $\alpha \leq \frac{n^2+k}{2}$. This upper bound is non-trivial precisely when $\frac{n^2+k}{2} < n + (k-1)(n-1)$. This is equivalent to $k > \frac{n}{2} + \frac{3}{4} + \frac{1}{4(2n-3)}$ or $k \geq \frac{n}{2} + 1$ when $n > 2$. □

There are other possibilities for a weakened version of the dimension argument. Two of these are presented below. Weakened versions of the dimension argument could prove useful if the embedding of certain types of relations into nets $N_6$ of order ten cannot be easily ruled out.

Proposition 2.14 If $\dim C_2(N) \geq 54$ for a collection of lines $N$ from any net $N_7$ of order ten then there there are at most five MOLS(10).

Proof: If the net $N_7$ has no transversal then there can be no $N_8$ containing this net, and therefore the five MOLS of order ten corresponding to the net are a maximal set of MOLS(10). If the net $N_7$ has a transversal then $\dim C_2(N_7) - \dim D_2(N_7) = 7$ and $\dim D_2(N_7) \geq 47$ so that $\dim D_2(N_7)^\perp \leq 53$, a contradiction to the existence of $N_7$ since $C_2(N_7) \subseteq D_2(N_7)^\perp$. In this case there can be at most four MOLS(10). □
The most useful application of this proposition would probably involve a partition of 54 lines into classes of sizes 10, 9, 7, 7, 7, 7 so that only configurations with at most two parallel classes contributing eight lines need to be considered.

**Proposition 2.15** If $\dim C_2(N) \geq 55$ for a collection of lines $N$ from any net $N_8$ of order ten then there are at most five MOLS(10).

**Proof:** Since 8 is even, by an argument similar to the one above $\dim C_2(N_8) - \dim D_2(N_8) = 8$ and $\dim D_2(N_8) \geq 47$ so that $\dim D_2(N_8)^\perp \leq 53$, a contradiction to the existence of $N_8$ since $C_2(N_8) \subseteq D_2(N_8)^\perp$. In this case there can be at most five MOLS(10). \hfill $\square$

The most useful application of this proposition would probably involve a partition of 55 lines into classes of sizes 10, 7, 7, 7, 7, 7, 5, 5 so that only configurations with at most one parallel class contributing eight lines need to be considered. The previous result could be sharpened if it were known that $C_2(N_8)$ and $D_2(N_8)^\perp$ were not the same vector space.
Chapter 3

A Classification of the Relations

In this chapter a classification is presented of the relations over $F_2$ in nets of order ten with degree at most six. Some observations simplify the cases to be considered. Further properties of nets of order ten will be used in later sections to rule out some of these cases and to understand their structure. The pairwise balanced design approach to the problem is presented and is shown to produce the same classification.

3.1 Relations in $N_4$

In this section, relations in four classes in nets of order ten are described. Recall from the introduction that $Z, S, D, T, Q, K$ represent the number of
zeroes, singles, doubles, triples, quads and quints in a configuration. The number $H$ represents the number of hexes in a relation in six classes and $H = K$ necessarily, though normally $K$ will be used in the context of a configuration and $H$ in the context of a relation. A lemma determining a relationship between $S, T$ and $K$ is presented. This lemma will be used in classifying relations in nets of degree four, five and six.

**Lemma 3.1** Given a relation in at most six parallel classes in a net of order ten, let $s, t$ and $k$ represent the number of weight one, three and five points respectively that meet a line from the last parallel class. Then $s + t + k = 10$ and $s + 3t + 5k = l$, where $l$ is the number of lines in the configuration formed by deleting the last parallel class of the relation. Furthermore, $k = 0$ for relations in nets of degree four or five.

**Proof:** The total number of points on any line is ten. This gives the first equation. A line from the last parallel class of a relation must meet each line of the relation lying in a different parallel class. This gives the second equation. Relations in nets of degree four or five have no weight six points, therefore $k = 0$. □

The two equations in the lemma determine a relationship between $S, T$ and $K$ for any configuration in a net of degree at most six and order ten.
CHAPTER 3. A CLASSIFICATION OF THE RELATIONS

The general linear system corresponding to a configuration coming from a relation in $N_4$ is of the form:

$$Z + S + D + T = 100$$
$$D + 3T = p$$
$$3Z + S = c$$

where $p$ is the number of ways to select a pair of lines in two different parallel classes so that each line is in the configuration. The quantity $c$ is the number of ways to select a pair of lines in two different parallel classes so that each line is not in the configuration. For example, for a configuration of type $\{x, y, z\}$, $p = xy + xz + yz$ and $c = (10 - x)(10 - y) + (10 - x)(10 - z) + (10 - y)(10 - z)$.

The first three restrictions are based on: total number of points, coverage of pairs of lines in the configuration, coverage of pairs of lines not in the configuration. A fourth restriction comes from Lemma 3.1 and depends on the type of the configuration.

**Proposition 3.2** A relation in a net $N_4$ of order ten consists of at least ten lines from any three of the four parallel classes.

**Proof:** If adding the fourth parallel class produces no new odd weight points then there must be at least ten lines in the union of the first three classes of the configuration. $\square$
By the previous proposition there are two types of configurations coming from relations in four classes to consider: types \(\{2, 4, 4\}\) and \(\{4, 4, 4\}\). Lemma 3.1 implies that \(s = 10\) for a configuration of type \(\{2, 4, 4\}\). Therefore \(T = 0\) in this case. This type of configuration yields the following system:

\[
\begin{align*}
Z + S + D &= 100 \\
D &= 32 \\
3Z + S &= 132
\end{align*}
\]

The unique solution is \((Z, S, D) = (32, 36, 32)\). Because \(S = 36\) is not divisible by 10, there is no combination of lines in the fourth parallel class that can produce a zero sum.

Lemma 3.1 implies that \(s = 9\) and \(t = 1\) for a configuration of type \(\{4, 4, 4\}\). This implies that \(S = 9T\) in this case. A configuration of type \(\{4, 4, 4\}\) yields the following system:

\[
\begin{align*}
Z + S + D + T &= 100 \\
D + 3T &= 48 \\
3Z + S &= 108 \\
S &= 9T
\end{align*}
\]
This system has the unique solution \((Z, S, D, T) = (24, 36, 36, 4)\).

A lot of structure is forced by the above relation, if it exists. The singles have to partition into four sets of points, with no two collinear points belonging to the same set. Furthermore, no two of the triples may be collinear.

### 3.2 Relations in \(N_5\)

The general linear system corresponding to a configuration coming from a relation in a net \(N_5\) of order ten is of the form:

\[
\begin{align*}
Z + S + D + T + Q &= 100 \\
D + 3T + 6Q &= p \\
6Z + 3S + D &= c
\end{align*}
\]

where \(p\) is the number of pairs of intersecting lines in the configuration and \(c\) is the number of pairs of intersecting lines not in the configuration. The first restriction comes from the total number of points in the net. A fourth restriction comes from Lemma 3.1 and determines a relationship between \(S\) and \(T\). This last restriction depends on the type of the configuration.

The assumptions on weights carry over from the previous section: here it is assumed that the weights of the first four parallel classes are either two or
four. Since $\{2, 2, 2, 2\}$ has too few lines, the first interesting case is $\{2, 2, 2, 4\}$.

The systems will be formed and solutions will be parametrized in terms of $Q$. Then explicit solutions in terms of $(Z, S, D, T, Q)$ will be found. The non-negative integrality condition will be used extensively to rule out certain types of configurations.

In the case $\{2, 2, 2, 4\}$, $Q$ is bounded above by four since there are two parallel classes with two lines each. Lemma 3.1 implies that $s = 10$. This implies that $T = 0$. The following system results:

\[
\begin{align*}
Z + S + D &= 100 - Q \\
D &= 36 - 6Q \\
6Z + 3S + D &= 336
\end{align*}
\]

The various solutions are:

\[
\begin{align*}
(Z, S, D, T, Q) &= (24, 60, 12, 0, 4); \\
(Z, S, D, T, Q) &= (27, 52, 18, 0, 3); \\
(Z, S, D, T, Q) &= (30, 44, 24, 0, 2); \\
(Z, S, D, T, Q) &= (33, 36, 30, 0, 1); \\
(Z, S, D, T, Q) &= (36, 28, 36, 0, 0).
\end{align*}
\]

When $Q = 0, 1, 2, 3$ the number of singles is not divisible by 10. Only the case $Q = 4$ requires more study. This gives a new configuration $(Z, S, D, T, Q) =
In the case \{2, 2, 4, 4\} \(Q\) is again at most 4. Lemma 3.1 implies that \(s = 9\) and \(t = 1\). This implies that \(S = 9T\). The system becomes:

\[
\begin{align*}
Z + S + D + T &= 100 - Q \\
D + 3T &= 52 - 6Q \\
6Z + 3S + D &= 292 \\
S &= 9T
\end{align*}
\]

There are two distinct solutions in non-negative integers:

\[
(Z, S, D, T, Q) = (12, 72, 4, 8, 4); \\
(Z, S, D, T, Q) = (25, 36, 34, 4, 1).
\]

In the case \{2, 4, 4, 4\} Lemma 3.1 implies that \(s = 8\) and \(t = 2\). This implies that \(S = 4T\). The following system results:

\[
\begin{align*}
Z + S + D + T &= 100 - Q \\
D + 3T &= 72 - 6Q \\
6Z + 3S + D &= 252 \\
S &= 4T
\end{align*}
\]

Here \(Q \leq 8\) and again there are only two non-negative integer solutions:
Finally, in the case \( \{4, 4, 4, 4\} \) Lemma 3.1 implies that \( s = 7 \) and \( t = 3 \). This implies that \( 3S = 7T \). The system becomes:

\[
\begin{align*}
Z + S + D + T &= 100 - Q \\
D + 3T &= 96 - 6Q \\
6Z + 3S + D &= 216 \\
3S &= 7T
\end{align*}
\]

Substituting \( S = \frac{7}{3}T \) into the first three equations gives:

\[
\begin{align*}
Z + D + \frac{10}{3}T &= 100 - Q \tag{3.1} \\
D + 3T &= 96 - 6Q \tag{3.2} \\
6Z + D + 7T &= 216 \tag{3.3}
\end{align*}
\]

Then \( 5 \times (3.2) + (3.3) - 6 \times (3.1) \) yields \( 2T = 96 - 24Q \) which implies that \( 96 - 24Q \geq 0 \) is necessary for a non-negative solution. So \( Q \leq 4 \). This time there are three non-negative integer solutions:

\[
\begin{align*}
(Z, S, D, T, Q) &= (24, 0, 72, 0, 4); \\
(Z, S, D, T, Q) &= (15, 28, 42, 12, 3); \\
(Z, S, D, T, Q) &= (6, 56, 12, 24, 2).
\end{align*}
\]
The solution with $Q = 4$ is eliminated because the absence of odd weight points means that there are no lines in the last class. This corresponds to the relation of type $\{4, 4, 4, 4\}$ in four classes.

### 3.3 Relations in $N_6$

The general linear system corresponding to a configuration coming from a relation in a net $N_6$ of order ten is of the form:

$$Z + S + D + T + Q + K = 100$$
$$D + 3T + 6Q + 10K = p$$
$$10Z + 6S + 3D + T = c$$

where $p$ is the number of pairs of intersecting lines in the configuration and $c$ is the number of pairs of intersecting lines not in the configuration. The first restriction comes from the total number of points in the net. A fourth restriction comes from Lemma 3.1 and depends on the type of the configuration. This last restriction determines a relationship between $S, T$ and $K$.

Before analyzing the cases in detail, the consequences of Lemma 3.1 are explored when $l$, the number of lines in the configuration, is 10, 12, 14, 16, 18 or 20.
Lemma 3.1 with $l = 10$ implies that $s = 10$. This implies that $T = 0 = K$.

Lemma 3.1 with $l = 12$ implies that $s = 9$ and $t = 1$. This implies that $S = 9T$ and $K = 0$.

Lemma 3.1 with $l = 14$ yields two possibilities. Either $s = 9$ and $k = 1$ or $s = 8$ and $t = 2$. This implies that $S = 9K + 4T$.

When $l = 16$, Lemma 3.1 again yields two possibilities. Either $s = 8$ and $t = 1 = k$ or $s = 7$ and $t = 3$. This implies that $S = 8K + \frac{7}{3}(T - K)$, which simplifies to $3S = 17K + 7T$.

Several possibilities arise with $l = 18$ in Lemma 3.1: (i) $s = 8, k = 2$; (ii) $s = 7, t = 2, k = 1$; (iii) $s = 6, t = 4$. Let $m$ be the number of lines of type (i) and $n$ be the number of lines of type (ii). Then $K = 2m + n$. Furthermore, $6(T - 2n) = 4(S - 8m - 7n)$, which simplifies to $6T + 16K = 4S$.

Again several possibilities arise with $l = 20$ in Lemma 3.1: (i) $s = 7, t = 1, k = 2$; (ii) $s = 6, t = 3, k = 1$; (iii) $s = 5 = t$. Let $m$ be the number of lines of type (i) and $n$ be the number of lines of type (ii). Then $K = 2m + n$. Furthermore, $(S - 7m - 6n) = (T - m - 3n)$, which simplifies to $S = T + 3K$.

The assumption that weights in the first five classes are two or four still
holds. The first potential relation in $N_6$ arises from choosing two lines from each of the first five parallel classes. Lemma 3.1 implies that $T = 0 = K$. Furthermore the number of weight one points must be divisible by ten. The system therefore reduces to:

$$Z + S + D + Q = 100 \quad (3.4)$$
$$D + 6Q = 40 \quad (3.5)$$
$$10Z + 6S + 3D = 640 \quad (3.6)$$

From (3.6) the fact that $S$ is a multiple of 10 implies that $D$ is a multiple of 10. Equation (3.5) then implies that $Q$ is a multiple of 5. So (3.5) can be rewritten:

$$\frac{D}{10} + \frac{3Q}{5} = 4$$

which in turn implies two possible solutions:

$$(Z, S, D, T, Q, K) = (25, 60, 10, 0, 5, 0);$$
$$(Z, S, D, T, Q, K) = (40, 20, 40, 0, 0, 0)$$

Next consider the case $\{2, 2, 2, 2, 4\}$. Lemma 3.1 implies that $S = 9T$ and
$K = 0$. The following system results:

\[ Z + S + D + T + Q = 100 \]
\[ D + 3T + 6Q = 56 \]
\[ 10Z + 6S + 3D + T = 576 \]
\[ S = 9T \]

Two non-negative integer solutions result:

Firstly $(Z, S, D, T, Q, K) = (26, 36, 32, 4, 2, 0)$, corresponding to four lines in the last class; secondly $(Z, S, D, T, Q, K) = (13, 72, 2, 8, 5, 0)$, corresponding to eight lines in the last class.

Next consider the case \{2, 2, 4, 4\}. Lemma 3.1 implies that $S = 9K + 4T$. The following system results:

\[ Z + S + D + T + Q + K = 100 \]
\[ D + 3T + 6Q + 10K = 76 \]
\[ 10Z + 6S + 3D + T = 516 \]
\[ S = 9K + 4T \]
The solution of this system is given as follows, parametrized by $K$ and $Q$:

\[ T = -12 + 8Q + 6K \]
\[ D = 112 - 30Q - 28K \]
\[ Z = 48 - 11Q - 12K \]
\[ S = -48 + 32Q + 33K \]

The quantity $K$ is at most 4 since the first two classes have only two lines each (and therefore intersect in only four points). Now condition on $K$ to determine the possibilities for $Q$:

When $K = 0$ it follows that $S = -48 + 32Q$ and non-negative integral solutions result for $Q = 2$ and $Q = 3$. These look like $(Z, S, D, T, Q, K) = (26, 16, 52, 4, 2, 0)$ and $(Z, S, D, T, Q, K) = (15, 48, 22, 12, 3, 0)$, with two and six lines respectively in the last class.

When $K = 1$ it follows that $S = -15 + 32Q$ and non-negative integral solutions result for $Q = 1$ and $Q = 2$. These look like $(Z, S, D, T, Q, K) = (25, 17, 54, 2, 1, 1)$ and $(Z, S, D, T, Q, K) = (14, 49, 24, 10, 2, 1)$, with two and six lines respectively in the last class.

When $K = 2$ it follows that $S = 18 + 32Q$ and non-negative integral solutions result for $Q = 1$ and $Q = 0$. These look like $(Z, S, D, T, Q, K) =
CHAPTER 3. A CLASSIFICATION OF THE RELATIONS

(13, 50, 26, 8, 1, 2) and \((Z, S, D, T, Q, K) = (24, 18, 56, 0, 0, 2)\), with six and two lines respectively in the last class.

When \(K = 3\) it follows that \(S = 51 + 32Q\) and non-negative integral solutions result for \(Q = 0\) only. The solution is \((Z, S, D, T, Q, K) = (12, 51, 28, 6, 0, 3)\), with six lines in the last class.

When \(K = 4\) it follows that \(S = 84 + 32Q\) and non-negative integral solutions result for \(Q = 0\) only. The solution is \((Z, S, D, T, Q, K) = (0, 84, 0, 12, 0, 4)\), which results in ten lines in the last class. This case is ruled because the last parallel class would have weight ten. This corresponds to a relation in five classes by complementing two classes.

Next consider the case \(\{2, 2, 4, 4, 4\}\). Lemma 3.1 implies that \(3S = 17K + 7T\). The following system results:

\[
\begin{align*}
Z + S + D + T + Q + K &= 100 \\
D + 3T + 6Q + 10K &= 100 \\
10Z + 6S + 3D + T &= 460 \\
3S &= 17K + 7T
\end{align*}
\]

This solution may be parametrized in terms of \(Q\) and \(K\). Then solutions will exists for \(0 \leq K \leq 4\) since two of the classes contribute only two lines:
Each possibility for $K$ gives rise to one or two admissible $Q$-values. The nine solutions $(Z, S, D, T, Q, K)$ are listed below:

- $(7, 56, 10, 24, 3, 0)$ corresponding to eight lines in the last class;
- $(16, 28, 40, 12, 4, 0)$ corresponding to four lines in the last class;
- $(6, 57, 12, 22, 2, 1)$ corresponding to eight lines in the last class;
- $(15, 29, 42, 10, 3, 1)$ corresponding to four lines in the last class;
- $(5, 58, 14, 20, 1, 2)$ corresponding to eight lines in the last class;
- $(14, 30, 44, 8, 2, 2)$ corresponding to four lines in the last class;
- $(4, 59, 16, 18, 0, 3)$ corresponding to eight lines in the last class;
- $(13, 31, 46, 6, 1, 3)$ corresponding to four lines in the last class;
- $(12, 32, 48, 4, 0, 4)$ corresponding to four lines in the last class.

The next case to consider is $\{2, 4, 4, 4, 4\}$. Lemma 3.1 implies that $6T + 16K = 4S$.

The following system results:

\[
\begin{align*}
Z + S + D + T + Q + K &= 100 \\
D + 3T + 6Q + 10K &= 128 \\
10Z + 6S + 3D + T &= 408 \\
6T + 16K &= 4S
\end{align*}
\]

Since $K$ is at most eight (the first class contains only two lines and the second
class contains four lines), there are only eleven solutions \((Z, S, D, T, Q, K)\):

\[(9, 36, 26, 24, 5, 0)\] corresponding to six lines in the last class;
\[(16, 12, 56, 8, 8, 0)\] corresponding to two lines in the last class;
\[(8, 37, 28, 22, 4, 1)\] corresponding to six lines in the last class;
\[(15, 13, 58, 6, 7, 1)\] corresponding to two lines in the last class;
\[(7, 38, 30, 20, 3, 2)\] corresponding to six lines in the last class;
\[(14, 14, 60, 4, 6, 2)\] corresponding to two lines in the last class;
\[(6, 39, 32, 18, 2, 3)\] corresponding to six lines in the last class;
\[(13, 15, 62, 2, 5, 3)\] corresponding to two lines in the last class;
\[(5, 40, 34, 16, 1, 4)\] corresponding to six lines in the last class;
\[(12, 16, 64, 0, 4, 4)\] corresponding to two lines in the last class;
\[(4, 41, 36, 14, 0, 5)\] corresponding to six lines in the last class.

The last case to consider is \(\{4, 4, 4, 4, 4\}\). Lemma 3.1 implies that \(S = T + 3K\). The following system results:

\[
\begin{align*}
Z + S + D + T &= 100 - K - Q \\
D + 3T &= 160 - 10K - 6Q \\
10Z + 6S + 3D + T &= 360 \\
S &= T + 3K
\end{align*}
\]

The restriction that \(K\) is at most sixteen follows from the fact that the first
two classes contribute only four lines each. Lemma 3.1 further implies that \( K \leq \frac{7}{2} S \). This reduces the number of admissible solutions \((Z, S, D, T, Q, K)\). The resulting number of lines in the last class is checked as a further feasibility test. (The number of lines should be a positive even integer less than ten by independence.) There are fifteen solutions:

- \((5, 40, 10, 40, 5, 0)\) corresponding to eight lines in the last class;
- \((10, 20, 40, 20, 10, 0)\) corresponding to four lines in the last class;
- \((4, 41, 12, 38, 4, 1)\) corresponding to eight lines in the last class;
- \((9, 21, 42, 18, 9, 1)\) corresponding to four lines in the last class;
- \((3, 42, 14, 36, 3, 2)\) corresponding to eight lines in the last class;
- \((8, 22, 44, 16, 8, 2)\) corresponding to four lines in the last class;
- \((2, 43, 16, 34, 2, 3)\) corresponding to eight lines in the last class;
- \((7, 23, 46, 14, 7, 3)\) corresponding to four lines in the last class;
- \((1, 44, 18, 32, 1, 4)\) corresponding to eight lines in the last class;
- \((6, 24, 48, 12, 6, 4)\) corresponding to four lines in the last class;
- \((0, 45, 20, 30, 0, 5)\) corresponding to eight lines in the last class;
- \((5, 25, 50, 10, 5, 5)\) corresponding to four lines in the last class;
- \((4, 26, 52, 8, 4, 6)\) corresponding to four lines in the last class;
- \((3, 27, 54, 6, 3, 7)\) corresponding to four lines in the last class;
- \((2, 28, 56, 4, 2, 8)\) corresponding to four lines in the last class.
3.4 Analogues

In this section it is shown that the previous list of configurations can be grouped by the relation they generate. A relation is said to be algebraically feasible if it is one of the solutions found previously.

**Proposition 3.3** Any algebraically feasible relation in at most six parallel classes in a net of order ten consisting of both two and four lines and no other class cardinality is repeated at least twice in the previous list.

**Proof:** Deleting any class from a relation produces a configuration. At least two distinct configurations will be produced depending on whether a class with two or four lines is chosen for deletion. □

**Corollary 3.4** An admissible relation in at most six parallel classes in a net of order ten consisting of both two and four lines and no other class cardinality must be repeated at least twice in the previous list.

**Proof:** If such a relation produces only one configuration in the list then the configuration of the second type is not algebraically feasible. This means that the relation itself does not exist; otherwise deleting an appropriate class would produce both types of configurations. □

**Definition 3.5** Any two configurations which correspond to the same relation will be called analogues.
Example 3.6 The \( \{2, 2, 4, 4\} \) configuration \((Z, S, D, T, Q) = (25, 36, 34, 4, 1)\) corresponding to four lines in the fifth class is an analogue to the \( \{2, 4, 4, 4\} \) configuration \((Z, S, D, T, Q) = (25, 16, 54, 4, 1)\) with two lines in the fifth class. When the fifth class is added they both produce the relation \((Z, D, Q) = (25, 70, 5)\).

Corollary 3.7 Neither of the two \( \{2, 2, 2, 4, 4\} \) configurations

\((Z, S, D, T, Q, K) = (25, 17, 54, 2, 1, 1)\) and \((24, 18, 56, 0, 0, 2)\) with two lines in the last class produces a relation in a net of order ten.

Proof: Neither configuration has an analogue of type \( \{2, 2, 2, 2, 4\} \). □

Proposition 3.8 Any relation in at most six parallel classes having all classes of the same cardinality (necessarily two or four lines each) produces a single configuration in a net of order ten.

Proof: There is only one choice of cardinality for the deleted class. Thus one type of configuration is produced. A priori there may be two or more analogous configurations within a given type. For type \( \{4, 4, 4\} \) there is only one configuration. For type \( \{4, 4, 4, 4\} \) with four lines in the last class there is only one configuration. For type \( \{2, 2, 2, 2, 2\} \) with two lines in the last class there is only one configuration. Lastly, for type \( \{4, 4, 4, 4, 4\} \) with four lines in the last class there are nine configurations. Each produces a distinct relation because the number of weight six points are all distinct after the last class is added. □
Definition 3.9 Configurations having themselves as analogues will be called self-analogues.

Configurations that are self-analogues come from relations in which each parallel class has the same weight. If the number of lines in the last class is six or eight, then the configuration has no analogue. However, once the corresponding relation is constructed the lines in any two classes may be complemented to achieve another relation. Classes of cardinalities four and eight can be complemented to produce two new classes of cardinalities two and six.

Definition 3.10 Two configurations coming from two relations related by such a complementation will be called complement analogues.

The two relations will also be referred to as complement analogues.

Example 3.11 A configuration of type \( \{2, 2, 2, 4\} \) with six lines in the last class that embeds in a net \( N_5 \) of order ten has a complement analogue of type \( \{2, 2, 4, 4\} \) with eight lines in the last class.

Every relation with six or eight lines in one class has at least one candidate complement analogue, as the classification will illustrate.

Proposition 3.12 If a relation having a class containing six or eight lines is embeddable in a net \( N_6 \) of order ten then it has at least one complement analogue.
**Proof:** First assume the relation has a class containing six lines. By inspection of the previous list of configurations each such relation also contains a class with two lines. Embed the relation in a net $N_6$ and complement the relation in the class containing six lines and a class containing two lines. The result is still a relation. Now assume the relation has a class containing eight lines. By inspection of the previous list of configurations each such relation also contains a class with four lines. Embed this relation in a net $N_6$ and complement the relation in the class containing eight lines and a class containing four lines. The result is still a relation. □

In general embedding the relation in a net of $N_6$ is not necessary to complete the proof above. An embedding in a net $N_k$ for $k$ the number of classes in the relation is sufficient. Note that any such relation may have more than one candidate complement analogue. This will be shown in the classification to follow.

### 3.5 The Classification

Here those configurations coming from relations in at most six classes in a net of order ten that need further examination are listed. Some of the techniques that might be used to show combinatorially that they cannot be embedded in a net $N_6$ are discussed.

(C1) the $\{4,4,4\}$ configuration with $(Z,S,D,T) = (24,36,36,4)$ cor-
responding to four lines in the fourth class. This configuration is a self-analogue.

(C2) the \{2, 2, 2, 4\} configuration with \((Z, S, D, T, Q) = (24, 60, 12, 0, 4)\) corresponding to six lines in the fifth class. This configuration is a complement analogue to the \{2, 2, 4, 4\} configuration \((Z, S, D, T, Q) = (12, 72, 4, 8, 4)\) with eight lines in the fifth class.

(C3) the \{2, 2, 4, 4\} configuration \((Z, S, D, T, Q) = (25, 36, 34, 4, 1)\) corresponding to four lines in the fifth class. This configuration is an analogue to the \{2, 4, 4, 4\} configuration \((Z, S, D, T, Q) = (25, 16, 54, 4, 1)\) with two lines in the fifth class.

(C4) the \{2, 4, 4, 4\} configuration with \((Z, S, D, T, Q) = (14, 48, 24, 12, 2)\) corresponding to six lines in the fifth class. This configuration is a complement analogue to the \{4, 4, 4, 4\} configuration \((Z, S, D, T, Q) = (6, 56, 12, 24, 2)\) with eight lines in the fifth class.

(C5) the \{4, 4, 4, 4\} configuration with \((Z, S, D, T, Q) = (15, 28, 42, 12, 3)\) corresponding to four lines in the fifth class. This configuration is a self-analogue.

(C6) the \{2, 2, 2, 2\} configuration with \((Z, S, D, T, Q, K) = \)
(25, 60, 10, 0, 5, 0) corresponding to six lines in the sixth class. This configuration is a complement analogue to the \( \{2, 2, 2, 2, 4\} \) configuration

\[(Z, S, D, T, Q, K) = (13, 72, 2, 8, 5, 0) \] corresponding to eight lines in the last class.

(C7) the \( \{2, 2, 2, 2, 2\} \) configuration with \((Z, S, D, T, Q, K) = (40, 20, 40, 0, 0, 0)\) corresponding to two lines in the last class. This configuration is a self-analogue.

(C8) the \( \{2, 2, 2, 2, 4\} \) configuration with \((Z, S, D, T, Q, K) = (26, 36, 32, 4, 2, 0)\) and four lines in the last class. This configuration is an analogue to the \( \{2, 2, 2, 4, 4\} \) configuration \((Z, S, D, T, Q, K) = (26, 16, 52, 4, 2, 0)\) with two lines in the last class.

(C9-C12) the \( \{2, 2, 2, 4, 4\} \) configurations as follows: \((Z, S, D, T, Q, K) = (15, 48, 22, 12, 3, 0), (14, 49, 24, 10, 2, 1), (13, 50, 26, 8, 1, 2)\) and \((12, 51, 28, 6, 0, 3)\), each with six lines in the last class. Each of these configurations has as a complement analogue one or more of: \((7, 56, 10, 24, 3, 0), (6, 57, 12, 22, 2, 1), (5, 58, 14, 20, 1, 2), (4, 59, 16, 18, 0, 3)\), each corresponding to eight lines in the last class.

(C13) the \( \{2, 2, 4, 4, 4\} \) configuration \((Z, S, D, T, Q, K) = (16, 28, 40, 12, 4, 0)\) corresponding to four lines in the last class. This con-
figuration is an analogue to the \( \{2, 4, 4, 4, 4\} \) configuration \((16, 12, 56, 8, 8, 0)\) with two lines in the last class.

(C14) the \( \{2, 2, 4, 4, 4\} \) configuration \((Z, S, D, T, Q, K) = (15, 29, 42, 10, 3, 1)\) corresponding to four lines in the last class. This configuration is an analogue to the \( \{2, 4, 4, 4, 4\} \) configuration \((15, 13, 58, 6, 7, 1)\) with two lines in the last class.

(C15) the \( \{2, 2, 4, 4, 4\} \) configuration \((Z, S, D, T, Q, K) = (14, 30, 44, 8, 2, 2)\) corresponding to four lines in the last class. This configuration is an analogue to the \( \{2, 4, 4, 4, 4\} \) configuration \((14, 14, 60, 4, 6, 2)\) with two lines in the last class.

(C16) the \( \{2, 2, 4, 4, 4\} \) configuration \((Z, S, D, T, Q, K) = (13, 31, 46, 6, 1, 3)\) corresponding to four lines in the last class. This configuration is an analogue to the \( \{2, 4, 4, 4, 4\} \) configuration \((13, 15, 62, 2, 5, 3)\) with two lines in the last class.

(C17) the \( \{2, 2, 4, 4, 4\} \) configuration \((Z, S, D, T, Q, K) = (12, 32, 48, 4, 0, 4)\) corresponding to four lines in the last class. This configuration is an analogue to the \( \{2, 4, 4, 4, 4\} \) configuration \((12, 16, 64, 0, 4, 4)\) with two lines in the last class.
(C18-C23) the six \( \{2, 4, 4, 4, 4\} \) configurations with six lines in the last class: \((Z, S, D, T, Q, K) = (9, 36, 26, 24, 5, 0), (8, 37, 28, 22, 4, 1), (7, 38, 30, 20, 3, 2), (6, 39, 32, 18, 2, 3), (5, 40, 34, 16, 1, 4), (4, 41, 36, 14, 0, 5).\)

Each of these has as a complement analogue one or more of the following \( \{4, 4, 4, 4, 4\} \) configurations with eight lines in the last class:

\((Z, S, D, T, Q, K) = (5, 40, 10, 40, 5, 0), (4, 41, 12, 38, 4, 1), (3, 42, 14, 36, 3, 2), (2, 43, 16, 34, 2, 3), (1, 44, 18, 32, 1, 4), (0, 45, 20, 30, 0, 5).\)

(C24-C32) the nine \( \{4, 4, 4, 4, 4\} \) configurations with four lines in the last class: \((Z, S, D, T, Q, K) = (10, 20, 40, 20, 10, 0), (9, 21, 42, 18, 9, 1), (8, 22, 44, 16, 8, 2), (7, 23, 46, 14, 7, 3), (6, 24, 48, 12, 6, 4), (5, 25, 50, 10, 5, 5), (4, 26, 52, 8, 4, 6), (3, 27, 54, 6, 3, 7), (2, 28, 56, 4, 2, 8).\) Each of these configurations is a self-analogue.

The results from the previous five sections are summarized below.

**Theorem 3.13** A relation in a net of degree six and order ten is of type \( \{4, 4, 4, 4\}, \{2, 2, 2, 4, 6\}, \{2, 2, 4, 4, 4\}, \{2, 4, 4, 4, 6\}, \{4, 4, 4, 4, 4\}, \{2, 2, 2, 2, 2\}, \{2, 2, 2, 2, 6\}, \{2, 2, 2, 2, 4, 4\}, \{2, 2, 2, 4, 4, 6\}, \{2, 2, 4, 4, 4, 4\}, \{2, 4, 4, 4, 4, 6\} \text{ or } \{4, 4, 4, 4, 4, 4\} \) with \( Z, S, D, T, Q, K \) as specified in the classification above.

Relations of type \( \{2, 2, 2, 2, 2, 6\} \) will be ruled out in Chapter 4. There remains the possibility of further ruling out some of these relations by more
sophisticated counting techniques.

### 3.6 A Pairwise Balanced Design Approach

In this section an alternative approach is presented that yields the same list of relations. The techniques were developed by Stinson [29] to disprove the existence of a 4-net of order six. This version of the classification has the advantage that several relations in the previous classification now correspond to the same combinatorial object. Pairwise balanced designs have been well studied and may lend more powerful techniques to the problem of reducing the list of feasible configurations.

Recall from the Introduction that a TD($k, n$) is equivalent to $k - 2$ MOLS($n$). In this section the object TD($6, 10$) is studied instead of working directly with four MOLS($10$) or a net $N_6$. Given a TD($6, 10$) $(X, G, B)$ produce a pairwise balanced design $(X, G \cup B)$ with 60 points $x_i$ ($1 \leq i \leq 60$) and 106 blocks $b_j$ ($1 \leq j \leq 106$). In particular there are six blocks of size ten and 100 blocks of size six. The $60 \times 106$ incidence matrix $A$ is defined by $A_{i,j} = 1$ if $x_i \in b_j$ and 0 otherwise.

If each row vector $r_i$ is considered to be an element of $F_2^{106}$ then the $r_i$'s span a subspace $C$ called the code of the pairwise balanced design.

**Lemma 3.14** The dimension $\dim C \leq 53$. 
Proof: Recall from Chapter 2 that \([u, v]\) is the dot product over \(F_2\). Then \([r_i, r_j] = 1\) for any distinct \(i, j\). So \([r_i, r_j + r_k] = 0\) for any distinct \(i, j, k\).

Now consider \(C^\perp = \{u : [u, r_i] = 0\ \text{for all} \ i\}\). Let \(\dim C = d\) with basis \(\{r_1, \ldots, r_d\}\). Then \(C^\perp\) contains \(r_1 + r_2, \ldots, r_1 + r_d\) and these are linearly independent. So \(\dim C^\perp \geq \dim C - 1\). However \(\dim C^\perp + \dim C = 106\) which implies that \(\dim C \leq 53\).

Now \(A\) consists of 60 rows and has a rowspace of dimension at most 53. Thus there are at least seven (linearly independent) dependencies in the rows. Each of these is a set of rows such that each of the 106 columns has column sum zero in \(F_2\). Five of these dependencies are already known: if \(G_1, \ldots, G_6\) are the groups of the original transversal design then the rows corresponding to \(G_1 \cup G_2, G_1 \cup G_3, G_1 \cup G_4, G_1 \cup G_5, G_1 \cup G_6\) produce linearly independent dependencies. These dependencies will be referred to as trivial dependencies.

In order for a TD(6, 10) to exist there must be at least two further dependencies. How these dependencies correspond to the configurations classified earlier will now be examined. The groups of the transversal design correspond to the six parallel classes of lines forming the configurations. The blocks of the transversal design come from individual points in the configuration. Each block is formed by listing the lines that meet the point of the net. The number of points \(m\) in the PBD is equal to the total number of lines in the relation, meaning the number of lines when the last class is included.
CHAPTER 3. A CLASSIFICATION OF THE RELATIONS

Note that any relation being embeddable in an $N_6$ corresponds to two dependencies in the matrix $A$. The two dependencies come from taking the rows corresponding to the set of lines forming the relation and also the rows from the complement of this set. These two dependencies are not linearly independent because the five trivial dependencies also sum to $r_1 + \ldots + r_{60}$. It will be shown later when the structure of the relations is described that there may be more than one way in which a configuration can occur.

Because each pair of distinct groups produces a dependency, the points in any two groups may be complemented without forming a linearly independent dependency. This justifies the assumption that each group contributes at most four points to a dependency, with the possible exception of a single group that could contribute up to eight points. It will also be shown that the number of points in a dependency is divisible by four. Thus there is no dependency with $0, 0, 2, 2, 2, 8$ points from the six groups nor one with $2, 2, 2, 2, 2, 8$ points from the six groups. If a dependency of type $0, 2, 2, 2, 2, 8$ can be ruled out then the assumption that there is no group contributing eight points is justified (by complementing in two groups if necessary). Phrasing the argument in the language of the PBD, a point from the last group must be added to precisely ten blocks each having odd size. This requires at least ten distinct points in the point set belonging to the first five groups which is not the case. Therefore it suffices to look for dependencies involving at most 24 points.
The dependencies (corresponding to configurations already classified) can be represented as a list of group sizes (the type of the relation), the partition of blocks into block sizes (corresponding to $Z, D, Q, H$), the largest group size, and the number of blocks of odd size that are left when the points of this group are deleted from the PBD. This contains precisely the same information as in the previous classification but the configurations are now produced in a much more efficient way. Because the group of largest size (up to size six) is chosen to act as the last group, no analogues or complement analogues are generated.

Let $b_0, b_2, b_4, b_6$ be the number of blocks of appropriate size in the PBD. Note that groups of the transversal design are counted as blocks. Let $g_0, g_2,$ $g_4, g_6$ be the number of groups that contribute a given number of points to the point set of the PBD. The number $m$ is the number of points in the PBD, corresponding to the total number of lines in the relation. The numbers $Z, S, D, T, Q, K$ are as before. Then:

\[
\begin{align*}
  b_0 + b_2 + b_4 + b_6 & = 106 \\
  2b_2 + 4b_4 + 6b_6 & = 11m \\
  b_2 + 6b_4 + 15b_6 & = \frac{m(m-1)}{2}
\end{align*}
\]
Taking two times (3.9) minus (3.8) gives \( b_4 + 3b_6 = \frac{m(m-12)}{8} \) and it follows that \( m \) must be divisible by four. This explains why the number of lines in each relation is a multiple of four. For each solution consider \( g_0, g_2, g_4, g_6 \) ranging from 0 to 6, where \( g_6 \) is either 0 or 1, satisfying the following restrictions:

\[
\begin{align*}
g_0 + g_2 + g_4 + g_6 &= 6 \\
b_6 - g_6 &\geq 0 \\
b_4 - g_4 &\geq 0 \\
g_0 &\leq 2
\end{align*}
\]

\[
45g_0 + 15(b_0 - g_0) + 28g_2 + 6(b_2 - g_2) + 15g_4 + (b_4 - g_4) + 6g_6 = \frac{(60 - m)(59 - m)}{2}
\]

\[
2(b_2 - g_2)(4 - g_0) + 4(b_4 - g_4)(2 - g_0) + 2 \cdot 8g_2 + 4 \cdot 6(g_4 + g_6) = m(60 - 10g_0 - m)
\]

The last two equations count pairs in the 60 – \( m \) points not in the PBD and cross-pairs between points in the PBD and points not in the PBD respectively. These must hold if the PBD is embeddable in a TD(6,10).

Given a solution \((b_0, b_2, b_4, b_6, m, g_0, g_2, g_4, g_6)\) all that remains to specify a configuration is to determine \( S \) and \( T \). The quantities \( Z \) and \( K \) are given by: \( Z = b_0 - g_0 \) and \( K = b_6 - g_6 \). Then \( D = b_2 - g_2 - S \) and \( Q = b_4 - g_4 - T \).
Take $p$ to be the largest group size in the PBD. Then $S$ and $T$ range between 0 and 80 and satisfy:

\[ S + T + (b_6 - g_6) = 10p \]
\[ S + 3T + 5(b_6 - g_6) = (m - p)p \]

These conditions correspond to the last two conditions used in finding the configurations. There are three further restrictions necessary to rule out configurations that are not feasible:

If $g_4 = 6$ then $(b_6 - g_6) \leq \frac{3}{7}S$. This is because when the configuration has twenty lines the only ways to partition the odd-weight points on lines of the last class are: $5T, 5S; K, 3T, 6S; 2K, T, 7S$. This yields $K \leq \frac{3}{7}S$. In terms of the PBD this restriction corresponds to considering the ten block sizes in which a point of the distinguished group appears.

The case $g_2 = 4$ and $g_4 = 2$ corresponds to a configuration with 12 lines. The only partition is $T, 9S$. Thus $K = 0$. In terms of the PBD, if a point is to appear in ten blocks and appear in the same block as twelve other points the only possibility is for it to appear in one block of size four and nine of size two.
If $g_0 = 2, m = 16$ and $p = 6$ (corresponding to a relation of type \{2,4,4,6\}) the following restriction holds: $3(b_0 - g_0) + S = \frac{1}{2}[(30 - m + p)^2 - 6^2g_4 - 8^2g_2]$. This is a re-phrasing of the fact that for a relation in four classes $3Z + S$ must equal the number of pairs generated by lines not in the relation.

There are 32 solutions $(b_0, b_2, b_4, b_6, m, g_0, g_2, g_4, g_6, S, T, K)$ corresponding to relations in a net $N_6$ of order ten. These solutions correspond to the seventeen PBD’s:

\[
\begin{align*}
  v &= 12 \quad b_2 = 66 \\
  v &= 16 \quad b_2 = 72 \quad b_4 = 8 \\
  v &= 16 \quad b_2 = 75 \quad b_4 = 5 \quad b_6 = 1 \\
  v &= 20 \quad b_2 = 82 \quad b_4 = 8 \quad b_6 = 4 \\
  v &= 20 \quad b_2 = 79 \quad b_4 = 11 \quad b_6 = 3 \\
  v &= 20 \quad b_2 = 76 \quad b_4 = 14 \quad b_6 = 2 \\
  v &= 20 \quad b_2 = 73 \quad b_4 = 17 \quad b_6 = 1 \\
  v &= 20 \quad b_2 = 70 \quad b_4 = 20
\end{align*}
\]

\[
\begin{align*}
  v &= 24 \quad b_2 = 84 \quad b_4 = 12 \quad b_6 = 8 \\
  v &= 24 \quad b_2 = 81 \quad b_4 = 15 \quad b_6 = 7 \\
  v &= 24 \quad b_2 = 78 \quad b_4 = 18 \quad b_6 = 6 \\
  v &= 24 \quad b_2 = 75 \quad b_4 = 21 \quad b_6 = 5 \\
  v &= 24 \quad b_2 = 72 \quad b_4 = 24 \quad b_6 = 4 \\
  v &= 24 \quad b_2 = 69 \quad b_4 = 27 \quad b_6 = 3 \\
  v &= 24 \quad b_2 = 66 \quad b_4 = 30 \quad b_6 = 2 \\
  v &= 24 \quad b_2 = 63 \quad b_4 = 33 \quad b_6 = 1 \\
  v &= 24 \quad b_2 = 60 \quad b_4 = 36
\end{align*}
\]

The benefit of this approach is that in some cases more than one relation corresponds to the same PBD. For instance, the relations of type \{4,4,4,4\}, \{2,2,4,4,4\} and \{2,2,2,2,4,4\} all correspond to the PBD with $v = 16, b_2 = 72, b_4 = 8$. 
A drawback of the PBD approach is that the ad-hoc arguments that ruled out specific configurations earlier are not as obvious in this setting.

It may be possible to rule out some configurations if the PBD fails to exist. The packing problem to be discussed later in the language of complete multipartite graphs corresponds to packing pairs into the blocks of sizes four and six.
Chapter 4

The Structure of Relations

In this chapter, the structure of the relations in nets $N_6$ of order ten is described in terms of which parallel classes contain lines incident with each point of the net. It is shown that one of the relations in Theorem 3.13 does not embed in a net $N_6$ of order ten.

4.1 Regularity Conditions

In this section, local and global conditions on the distribution of points of various weights are established. Complete multipartite graphs will be used to represent each configuration or relation. This approach is motivated by the fact that configurations and relations are edge-decompositions of a complete multipartite graph into cliques. For example, a configuration of type $\{2, 4, 4, 4\}$ coming from a relation of type $\{2, 4, 4, 4, 6\}$ can be viewed as an
edge-decomposition of the complete multipartite graph $K_{2,4,4,4}$. Each line of
the configuration or relation will be represented by a vertex of the graph.
The cardinality of the part in which each line appears corresponds to the
number of lines from its parallel class appearing in the configuration or re-
lation. A weighted point in the configuration or relation will be represented
by a clique in the graph whose order corresponds to the weight of the point.

Given an edge-decomposed complete multipartite graph corresponding to
a relation, let $Q_i, D_i$ be the number of $K_4$ or $K_2$ incident with some vertex of
a part of cardinality $i$. Only cliques contributing to the edge-decomposition
are counted. The quantities $P$ and $C$ will be defined with respect to this part
of the graph. Let $P$ (pairs) be the number of edges of the graph incident
with a vertex of the part. In net terminology, this corresponds to the number
of pairs of lines in the relation that include one member of the given parallel
class. Let $C$ (cross-pairs) be the number of pairs of lines formed by one line
not in the relation and one line in the given parallel class. For example, with
respect to the $\{2, 4, 4, 4, 6\}$ relation and the corresponding graph $K_{2,4,4,4,6}$, for
the part of cardinality two the calculation gives $P = 2 \times (4 + 4 + 4 + 6) = 36$
and $C = 2 \times (6 + 6 + 6 + 4) = 44$.

**Proposition 4.1** Given a relation in a net $N_6$ of order ten and its repre-
sentation as an edge-decomposed complete multipartite graph, for each $i$ the
number of $K_i$ incident with a vertex of each part of the graph depends only
on the size of the part.
Proof: The only relation with four classes satisfies $3Q_4 + D_4 = 48$ and $2D_4 = 72$. This gives $(Q_4, D_4) = (4, 36)$. When the relation has five classes the following equations hold for any part of cardinality $i$: $3Q_i + D_i = P$ and $Q_i + 3D_i = C$. This gives $(Q_i, D_i) = \left(\frac{3P-C}{8}, \frac{3C-P}{8}\right)$. Note that $P$ and $C$ depend only on the relation in question and $i$, the cardinality of the part. When the relation has six classes the following equations hold for any part of cardinality $i$: $5H + 3Q_i + D_i = P$ and $2Q_i + 4D_i = C$. This gives $(Q_i, D_i) = \left(\frac{4P-C}{10} - 2H, \frac{3C-2P}{10} + H\right)$. Since $H$ is constant with respect to any relation, this solution depends only on the relation in question and $i$, the cardinality of the part. □

The solutions are given explicitly at the end of this section. The next proposition introduces $Q_{iL}$ and $D_{iL}$, the number of $K_4$ and $K_2$ incident with a given vertex of a part of size $i$.

**Proposition 4.2** Given a relation in a net $N_6$ of order ten and its representation as an edge-decomposed complete multipartite graph, the values of $Q_{iL}$ and $D_{iL}$ depend only on the relation in question, the cardinality $i$ of the part, and the number of $K_6$ incident with the vertex.

**Proof:** In the relations with four and five classes, there are no points of weight six. The local conditions on $Q_{iL}$ and $D_{iL}$ are as follows: $3Q_{iL} + D_{iL} = \frac{P}{7}$ and $Q_{iL} + D_{iL} = 10$. Since $P + C = 10 \times i \times \alpha$, where $\alpha$ is the number of classes in the configuration, the last equation can be rewritten $Q_{iL} + D_{iL} = \frac{P+C}{i \times \alpha}$. From the solution in the Proposition 4.1 it follows that
\( (i \times Q_{iL}, i \times D_{iL}) = (Q_i, D_i) \). In other words, \( (Q_{iL}, D_{iL}) = (\frac{Q_i}{i}, \frac{D_i}{i}) \). When the relation has six classes consider a specific vertex of the graph and let \( h \) be the number of \( K_6 \) incident with that vertex (corresponding to the number of weight six point lying on that line). The following two equations result:

\[
5h + 3Q_{iL} + D_{iL} = \frac{P}{i} \quad \text{and} \quad h + Q_{iL} + D_{iL} = 10.
\]

Since these equations are linearly independent there is a unique solution and the proposition has been established.

\[\Box\]

In terms of the net, Proposition 4.2 says that the number of weight \( i \) points on a line of a relation depends on the relation, the weight of the parallel class to which the line belongs, and the number of weight six points on the line.

The solutions are now given explicitly, depending on the configuration, the cardinality of the part, and the value of \( h \). Unless explicitly stated otherwise, \( (Q_{iL}, D_{iL}) = (\frac{Q_i}{i}, \frac{D_i}{i}) \).

(C1) Relation of type \{4, 4, 4, 4\}: \( (Q_4, D_4) = (4, 36) \).

(C2) Relation of type \{2, 2, 2, 4, 6\}: \( (Q_6, D_6) = (0, 60); (Q_4, D_4) = (4, 36); (Q_2, D_2) = (4, 16) \). Complement Analogue of type \{2, 2, 4, 4, 8\}: \( (Q_8, D_8) = (8, 72); (Q_4, D_4) = (12, 28); (Q_2, D_2) = (8, 12) \).
(C3) Relation of type $\{2, 2, 4, 4, 4\}$: $(Q_4, D_4) = (4, 36)$; $(Q_2, D_2) = (4, 16)$.

(C4) Relation of type $\{2, 4, 4, 4, 6\}$: $(Q_6, D_6) = (12, 48)$; $(Q_4, D_4) = (12, 28)$; $(Q_2, D_2) = (8, 12)$. Complement Analogue of type $\{4, 4, 4, 4, 8\}$: $(Q_8, D_8) = (24, 56)$; $(Q_4, D_4) = (20, 20)$.

(C5) Relation of type $\{4, 4, 4, 4, 4\}$: $(Q_4, D_4) = (12, 28)$.

(C6) Relation of type $\{2, 2, 2, 2, 2, 6\}$: $(Q_6, D_6) = (0, 60)$; $(Q_2, D_2) = (4, 16)$. Complement Analogue of type $\{2, 2, 2, 2, 4, 8\}$: $(Q_8, D_8) = (8, 72)$; $(Q_4, D_4) = (12, 28)$; $(Q_2, D_2) = (8, 12)$.

(C7) Relation of type $\{2, 2, 2, 2, 2, 2\}$: $(Q_2, D_2) = (0, 20)$.

(C8) Relation of type $\{2, 2, 2, 2, 4, 4\}$: $(Q_4, D_4) = (4, 36)$; $(Q_2, D_2) = (4, 16)$.

(C9)-(C12) Relations of type $\{2, 2, 2, 4, 4, 6\}$: Here $0 \leq H \leq 3$ and $(Q_6, D_6) = (12 - 2H, 48 + H)$; $(Q_4, D_4) = (12 - 2H, 28 + H)$; $(Q_2, D_2) = (8 - 2H, 12 + H)$. The local conditions are $(Q_{6L}, D_{6L}) = (2 - 2h, 8 + h)$; $(Q_{4L}, D_{4L}) = (3 - 2h, 7 + h)$; $(Q_{2L}, D_{2L}) = (4 - 2h, 6 + h)$, where these are non-negative. Each relation has one or more complement analogues of type $\{2, 2, 4, 4, 4, 8\}$: Here $0 \leq H \leq 3$ and $(Q_8, D_8) = (24 - 2H, 56 + H)$; $(Q_4, D_4) = (20 - 2H, 20 + H)$; $(Q_2, D_2) = (12 - 2H, 8 + H)$. The local conditions are $(Q_{8L}, D_{8L}) = (3 - 2h, 14 + h)$; $(Q_{4L}, D_{4L}) = (9 - 2h, 7 + h)$; $(Q_{2L}, D_{2L}) = (14 - 2h, 5 + h)$, where these are non-negative.
$2h, 7 + h); (Q_{4L}, D_{4L}) = (5 - 2h, 5 + h); (Q_{2L}, D_{2L}) = (6 - 2h, 4 + h)$, where these are non-negative.

(C13)-(C17) Relations of type $\{2, 2, 4, 4, 4\}$: Here $0 \leq H \leq 4$ and $(Q_4, D_4) = (12 - 2H, 28 + H); (Q_2, D_2) = (8 - 2H, 12 + H)$. The local conditions are $(Q_{4L}, D_{4L}) = (3 - 2h, 7 + h); (Q_{2L}, D_{2L}) = (4 - 2h, 6 + h)$, where these are non-negative.

(C18)-(C23) Relations of type $\{2, 4, 4, 4, 4, 4\}$: Here $0 \leq H \leq 5$ and $(Q_6, D_6) = (24 - 2H, 36 + H); (Q_4, D_4) = (20 - 2H, 20 + H); (Q_2, D_2) = (12 - 2H, 8 + H)$. The local conditions are $(Q_{6L}, D_{6L}) = (4 - 2h, 6 + h); (Q_{4L}, D_{4L}) = (5 - 2h, 5 + h); (Q_{2L}, D_{2L}) = (6 - 2h, 4 + h)$, where these are non-negative. Each relation has one or more complement analogues of type $\{4, 4, 4, 4, 4, 8\}$: Here $0 \leq H \leq 5$ and $(Q_8, D_8) = (40 - 2H, 40 + H); (Q_4, D_4) = (28 - 2H, 12 + H)$. The local conditions are $(Q_{8L}, D_{8L}) = (5 - 2h, 5 + h); (Q_{4L}, D_{4L}) = (7 - 2h, 3 + h)$, where these are non-negative.

(C24)-(C32) Relations of type $\{4, 4, 4, 4, 4, 4\}$: Here $0 \leq H \leq 8$ and $(Q_4, D_4) = (20 - 2H, 20 + H)$. The local conditions are $(Q_{4L}, D_{4L}) = (5 - 2h, 5 + h)$, where these are non-negative.

Nonnegativity of $Q_{2L}, Q_{4L}, Q_{6L}$ and $Q_{8L}$ give upper bounds on $h$. These may differ with the cardinality of the part of the graph.
4.2 Alpha and Beta

In this section regularity conditions are used to give more information about the structure of the relations in six classes. The configurations are again viewed as edge-decompositions of complete multipartite graphs into cliques.

Given a relation in six classes label the parts of a complete multipartite graph $P_1, P_2, P_3, P_4, P_5, P_6$. The cardinalities of the parts correspond to the cardinalities of the classes (the type) of the relation. Let $a$ be the number of edges between two of the first four parts and let $b$ be the number of edges incident with $P_5$ and one of the first four parts. Consider the configuration produced when $P_6$ is deleted. Let $\alpha$ be the number of quads in the configuration incident with $P_5$ and let $\beta$ be the number of triples in the configuration incident with $P_5$. Then:

$$3\alpha + 6(Q - \alpha) + \beta + 3(T - \beta) + 6K \leq a$$
$$3\alpha + 2\beta + 4K \leq b$$

This results in the string of inequalities $6Q + 3T + 6K - a \leq 3\alpha + 2\beta \leq b - 4K$ for $\alpha \leq Q$ and $\beta \leq T$. If there are no weight five points in the configuration these two inequalities simplify to $6Q + 3T - a \leq 3\alpha + 2\beta \leq b$ for $\alpha \leq Q$ and $\beta \leq T$. Because the triples incident with $P_5$ must form quads when the last class is added in, there is a pair restriction $\beta + K \leq l_5 \times l_6$ where $l_5$ and $l_6$ are the cardinalities of $P_5$ and $P_6$ respectively. If $\alpha$ quads are incident
with $P_5$ then necessarily $Q - \alpha$ quads are incident with all four of the first four classes. Because $\alpha + \beta$ is the number of quads incident with $P_5$ in the relation it follows that $\alpha + \beta = Q_i$ where $i$ is the size of $P_5$. Thus the number of quads passing through $P_5$ and $P_6$ minus the number of quads incident with all four of the first four classes must equal $Q_i - Q$. Of course $Q$ varies with the choice of $P_6$. It is also necessary that $Q - \alpha$ is at most the number of hexes in the type of relation produced by complementing the lines in $P_5, P_6$.

The configuration (C6) leads to the question: Is it possible to pack five copies of $K_4$ and eight copies of $K_3$ into the complete multipartite graph $K_{2,2,2,2,4}$? It will be shown that if the regularity conditions established previously are obeyed then the answer is in the negative, ruling out configuration (C6).

**Proposition 4.3** There is no packing of five copies of $K_4$ and eight copies of $K_3$ into the complete multipartite graph $K_{2,2,2,2,4}$ when the regularity conditions of a net $N_6$ of order ten are obeyed.

**Proof:** Label the parts of the graph $P_1, P_2, P_3, P_4, P_5$ so that $P_5$ is the part of cardinality four. Recall that $Q_4 = 12$ for any packing obeying the regularity conditions. This means that the total number of $K_4$ and $K_3$ incident with $P_5$ must equal twelve. Note that there are 24 edges between the first four parts and 32 between the first four parts and the fifth part. Suppose for contradiction that there exists a packing. Let $\alpha$ be the number of $K_4$ incident
with $P_5$ and let $\beta$ be the number of $K_3$ incident with $P_5$. The packing uses 
$3\alpha + 6(5 - \alpha) + \beta + 3(8 - \beta) = 54 - 3\alpha - 2\beta$ edges between the first four 
parts and $3\alpha + 2\beta$ edges incident with $P_5$. This implies $30 \leq 3\alpha + 2\beta \leq 32$. 
Since $\alpha \leq 5$ and $\beta \leq 8$ the lower bound is false unless $\alpha = 5$ and $\beta = 8$. 
Thus every clique is incident with $P_5$ and $Q_4 = 13$, a contradiction. \hfill \square

**Proposition 4.4** It is impossible to construct configuration (C6) and embed it in a net $N_5$ of order ten.

**Proof:** It is convenient to view a point of the configuration as a 5-tuple 
with the $i$th component equal to the index of the line in the $i$th class meeting 
the point. Lines belonging to one of the first four classes and appearing in 
the relation will be indexed with the set $\{0, 1\}$. Lines from the fifth class 
appearing in the relation will be indexed with the set $\{0, 1, 2, 3\}$. Since any 
pair of lines from distinct classes meet exactly once in a net, any pair of 
entries from different components determine a point. That is, each ordered 
pair of symbols from $\{0, 1, \ldots, 9\}$ must occur exactly once among any two 
distinct components in the points of a net, and at most once among any two 
distinct components in the points of the configuration. The packing imposes 
a weaker restriction on the points of the configuration by indexing the lines 
not in the relation with the symbol *. The nonexistence of a packing of eight 
$K_3$ and five $K_4$ into $K_{2,2,2,4}$ now implies the nonexistence of the configuration 
$(Z, S, D, T, Q, K) = (13, 72, 2, 8, 5, 0)$ embeddable in a net $N_5$ of order ten.\hfill \square

**Corollary 4.5** It is impossible to construct configuration (C6) and extend it
to a net $N_6$ of order ten.

**Proof:** Suppose for contradiction that it were possible to embed configuration (C6) in a net $N_6$ of order ten. Then there is some parallel class which contains no lines of the configuration. By removing this parallel class an $N_5$ remains and the configuration is preserved within it, contradicting the previous proposition. □

Thus the configuration (C6) is ruled out.

### 4.3 Structure in Four or Five Classes

In the structural description to follow consider the classes to be ordered and represent each even-weight point as a string of numbers and stars. A number indicates that the point is incident with the class of appropriate size; a * indicates that the point is not incident with this class. This structure comes from the regularity conditions found previously. Once this structure is determined the possibilities for forming transversals in the remaining class(es) may be examined. This structure relies on the fact that a transversal must meet each line in the other classes exactly once.

(C1) Four Q of type 4444; the remaining pairs are covered by D. Each line in the fifth or sixth class meets one of: (4Q, 6Z); (3Q, 2D, 5Z); (2Q, 4D, 4Z); (1Q, 6D, 3Z); (8D, 2Z).
(C2) Four Q of type 2224*; the remaining pairs are covered by D. Each line in the sixth class meets one of: (1Q, 6D, 3Z); (8D, 2Z). No line may be incident with more than two Q because each Q meets a point of the first class. The case (2Q, 4D, 4Z) is ruled out because the required four pairs cannot be formed. Thus transversal structure for configuration (C2) is determined. The structure of the complement analogue is as follows: twelve Q with four each of types 2244*, *2448, 2*448. Each line in the sixth class meets one of: (2Q, 6D, 2Z); (1Q, 8D, 1Z); (10D). Other cases are ruled out by a high Z value.

(C3) Five Q of types 2244*, 224*4, 22*44, 2*444, *2444; the remaining pairs are covered by D. Each line in the sixth class meets one of: (3Q,2D,5Z); (2Q,4D,4Z); (1Q,6D,3Z); (8D,2Z). Note that the possibility of (4Q, 6Z) is ruled out because $Q_2 = 4$ and at most two of these four may be incident with a given line in the last class.

(C4) Fourteen Q with the following types and multiplicities: two each of types 2444*, 2*446, 24*46, 244*6; six of type *4446; the remaining pairs are covered by D. Each line in the sixth class meets one of: (4Q, 2D, 4Z); (3Q, 4D, 3Z); (2Q, 6D, 2Z); (1Q, 8D, 1Z); (10D). The case (5Q, 5Z) is ruled out because there are only four lines in the complement of the fifth class. This prevents 5Z from meeting a single line in the sixth class.
The complement analogue has a more restricted structure. It has 26 Q with the following types and multiplicities: two of type 4444* and six each of types 4*448, 44*48, 444*8; the remaining pairs are covered by D. Each line in the sixth class meets one of: (4Q, 4D, 2Z); (3Q, 6D, Z); (2Q, 8D). The other cases are ruled out by a high Z value.

(C5) Fifteen Q with the following types and multiplicities:
three each of types *4444, 4*444, 44*44, 444*4, 4444*; the remaining pairs are covered by D. Each line in the sixth class meets one of: (5Q, 5Z); (4Q, 2D, 4Z); (3Q, 4D, 3Z); (2Q, 6D, 2Z); (1Q, 8D, 1Z); (10D).

(C6) has been ruled out.

(C7) is easily seen to consist of all possible pairs covered in D.

**Proposition 4.6** Deleting one or more classes from a relation in a net $N_6$ of order ten does not produce another relation.

**Proof:** If a relation is produced by deleting a collection of classes from a relation then the deleted classes also form a relation. Since all relations have at least four classes this is impossible unless the original relation has at least eight classes. □

The possibility still exists that one relation is contained inside another.
For example, it is possible that two disjoint \(\{2,2,2,2,2\}\) relations lie inside a \(\{4,4,4,4,4\}\) relation. Since the \(\{2,2,2,2,2\}\) relation contains no quads or hexes, \(H = 0\) is necessary in any candidate \(\{4,4,4,4,4\}\) relation. Furthermore the 30 quads in the \(\{4,4,4,4,4\}\) relation use all of the 120 cross-pairs between the two \(\{2,2,2,2,2\}\) relations. It would greatly simplify the problem to reduce all the relations to those that are *minimal* in some sense, for instance those that do not contain another relation with fewer lines.

### 4.4 Structure in Six Classes

Again, the goal is a structural description of the relations. The structures of the quads determine the structure of the relation as a whole. The structures will be described in terms of which classes occur together in quads and in which combinations. The primary goal is to find a rough upper bound on the number of possible structures. The fact that the maximum number of different structures admitted by each relation is not too large makes the previous classification of relations a useful one.

Assume that the six classes are distinguished and ordered. Let \(a, b, \ldots, o\) represent the number of each type of quad, where the correspondence is as
follows. A 1 represents an incidence with the appropriate class:

\[ a: 1111** \quad b: 111*1* \quad c: 111**1 \quad d: 11*11* \quad e: 11*1*1 \]
\[ f: 11**11 \quad g: 1*111* \quad h: 1*11*11 \quad i: 1*1*11 \quad j: 1***111 \]
\[ k: *1111* \quad l: *111*1 \quad m: *11*11 \quad n: *1111* \quad o: **11111 \]

For each relation the possibilities for \( a, b, \ldots, o \) will be examined, keeping the restrictions in mind. Firstly, these must sum to \( Q \). Secondly, the regularity conditions must be obeyed in each class. Furthermore, small upper bounds exist for each of the fifteen quantities. For instance when the last two classes of the net are complemented, the \( a \) quads of type \( 1111** \) each become an \( H \) in the new type of relation. By studying the analogues an upper bound on each variable can be determined. It will then be possible to list the \( a, b, \ldots, o \) that satisfy the restrictions.

First consider the relation of type \( \{2, 2, 2, 4, 4\} \). The only configuration of this type has \( H = 0 \). The quantities \( Q = 6 \) and \( Q_2 = 4 = Q_4 \) come from the regularity conditions. The analogues are of types \( \{2, 2, 4, 4, 8, 8\} \), \( \{2, 2, 4, 6, 8\} \) and \( \{2, 2, 2, 6, 6\} \). The configurations corresponding to each type of relation are listed below:

\[ (Z, S, D, T, Q, K) = (2 - H, 40 + H, 16 + 2H, 40 - 2H, 2 - H, H) \text{ for } 0 \leq H \leq 2; \]
\[ (Z, S, D, T, Q, K) = (6 - H, 56 + H, 12 + 2H, 24 - 2H, 2 - H, H) \text{ for } 0 \leq H \leq 2; \]
\[ (Z, S, D, T, Q, K) = (14 - H, 48 + H, 24 + 2H, 12 - 2H, 2 - H, H) \text{ for } 0 \leq H \leq 2. \]
Solutions must have each of $a, b, \ldots, o$ at most two.

It will be desirable to avoid generating multiple structures that can be obtained from one another by permuting classes of the net. Recall that the classes of the net are labelled $P_1, P_2, P_3, P_4, P_5, P_6$. Some of the classes are already distinguished from one another by the number of lines they contribute to the relation. Requiring that $j \geq n \geq o$ and $b \geq c$ prevents the same structure from being listed in many embeddings that differ only by a permutation of parallel classes. This potentially distinguishes $P_1, P_2, P_3$ from one another as well as $P_5, P_6$ from one another. Note that quads of type $b$ and $c$ pass through $P_1, P_2$ and $P_3$ so the restrictions are made without loss of generality. There are 46 possibilities.

It may be possible to rule out some of the specific structures above using more refined combinatorial arguments. The structural description could also provide a starting point for a search for a net containing the relation. Constructing the net from the structural description involves assigning a line label to each incidence of a hex, quad or double with a parallel class. The label is chosen from $\{0, 1\}$ in the first four classes or from $\{0, 1, 2, 3\}$ in the last two classes. Finally the incidences of quads, doubles and zeros with lines not in the relation are labelled with $\{2, 3, 4, 5, 6, 7, 8, 9\}$ or $\{4, 5, 6, 7, 8, 9\}$ depending on the cardinality of the parallel class. Local techniques are called for to restrict these packings of line labels into cliques.
Next some upper bounds will be calculated on the number of structural possibilities for the remaining relations. The purpose of this is to show that, in most cases, the number is fairly small. A systematic search for these structures could be possible if some of them could be ruled out quickly.

The relations of type \( \{2, 2, 2, 4, 4, 6\} \) with \( 0 \leq H \leq 3 \) are complementable to configurations of types \( \{2, 4, 4, 6, 8, 8\}, \{2, 2, 4, 6, 6, 8\}, \{2, 2, 4, 4, 8\}, \{2, 2, 6, 6, 6\} \) and \( \{2, 2, 2, 4, 4, 6\} \). The configurations corresponding to each type of relation are listed below:

\[
(Z, S, D, T, Q, K) = (3 - H, 24 + H, 10 + 2H, 56 - 2H, 7 - H, H) \text{ for } 0 \leq H \leq 3;
\]
\[
(Z, S, D, T, Q, K) = (3 - H, 40 + H, 14 + 2H, 40 - 2H, 3 - H, H) \text{ for } 0 \leq H \leq 3;
\]
\[
(Z, S, D, T, Q, K) = (7 - H, 56 + H, 10 + 2H, 24 - 2H, 3 - H, H) \text{ for } 0 \leq H \leq 3;
\]
\[
(Z, S, D, T, Q, K) = (7 - H, 36 + H, 30 + 2H, 24 - 2H, 3 - H, H) \text{ for } 0 \leq H \leq 3;
\]
\[
(Z, S, D, T, Q, K) = (15 - H, 48 + H, 22 + 2H, 12 - 2H, 3 - H, H) \text{ for } 0 \leq H \leq 3.
\]

Thus it suffices to search for solutions with each of \( a, b, \ldots, o \) at most three.

The restrictions on the solutions are that \( a + b + \ldots + o = 15 - 3H \) and that the regularity conditions \( Q_2 = 8 - 2H, Q_4 = 12 - 2H, Q_6 = 12 - 2H \) are obeyed. Additionally the restrictions \( j \geq n \geq o \) and \( a \geq b \) potentially distinguish \( P_1, P_2, P_3 \) from one another as well as \( P_4 \) and \( P_5 \).
The total number of structures which meet these requirements is 183 + 71 + 11 + 1 = 266, corresponding to $H = 0, 1, 2, 3$ respectively.

Similarly, the relations of type $\{2, 2, 4, 4, 4\}$ with $0 \leq H \leq 4$ are complementable to configurations of types $\{4, 4, 4, 8, 8\}, \{2, 4, 4, 6, 8\}, \{2, 2, 4, 6, 6\}$. The corresponding configurations are listed below:

\[(Z, S, D, T, Q, K) = (4-H, 24+H, 8+2H, 56-2H, 8-H, H)\] for $0 \leq H \leq 4$;
\[(Z, S, D, T, Q, K) = (4-H, 40+H, 12+2H, 40-2H, 4-H, H)\] for $0 \leq H \leq 4$;
\[(Z, S, D, T, Q, K) = (8-H, 36+H, 28+2H, 24-2H, 4-H, H)\] for $0 \leq H \leq 4$.

It is sufficient to search for solutions with each of $a, b, \ldots, o$ at most four.

The restrictions are that $a + b + \ldots + o = 16 - 3H$ and that the regularity conditions $Q_2 = 8 - 2H, Q_4 = 12 - 2H$ are obeyed. Additionally the restrictions $a \geq b \geq c$ and $j \geq n$ potentially distinguish $P_1$ and $P_2$ as well as $P_4, P_5, P_6$ from one another.

The total number of structures which meet these requirements is $1156 + 311 + 68 + 10 + 1 = 1546$, corresponding to $H = 0, 1, 2, 3, 4$ respectively.

Now consider the relations of type $\{2, 4, 4, 4, 4, 6\}$ with $0 \leq H \leq 5$. These are complementable to configurations of types $\{4, 4, 4, 6, 6, 8\}, \{4, 4, 4, 4, 8\}, \{2, 4, 4, 6, 6, 6\}$ and $\{2, 4, 4, 4, 6\}$. The corresponding con-
figurations are listed below:

\[(Z, S, D, T, Q, K) = (5 - H, 24 + H, 6 + 2H, 56 - 2H, 9 - H, H)\] for \(0 \leq H \leq 5\);

\[(Z, S, D, T, Q, K) = (5 - H, 40 + H, 10 + 2H, 40 - 2H, 5 - H, H)\] for \(0 \leq H \leq 5\);

\[(Z, S, D, T, Q, K) = (5 - H, 24 + H, 26 + 2H, 36 - 2H, 9 - H, H)\] for \(0 \leq H \leq 5\);

\[(Z, S, D, T, Q, K) = (9 - H, 36 + H, 26 + 2H, 24 - 2H, 5 - H, H)\] for \(0 \leq H \leq 5\).

It is sufficient to search for solutions with each of \(a, b, \ldots, o\) at most five.

Consider the configurations of type \{2, 4, 4, 4, 4, 6\} with \(a, b, \ldots, o\) at most five and \(Q, Q_2, Q_4, Q_6\) as prescribed. In addition to the restrictions \(f \geq i \geq j\) it can also be assumed that \(a, b, d, g\) are at most \(H\) for each \(0 \leq H \leq 5\). Each of these four quad types produces an \(H\) when a class of size four and one of size six are complemented. Overcounting is prevented by assuming that a complementation has been made in the classes that give a configuration of type \{2, 4, 4, 4, 4, 6\} with maximum \(H\). This gives \(384 + 2302 + 2105 + 706 + 150 + 11 = 5658\) structures.

Finally consider the configurations of type \{4, 4, 4, 4, 4, 4\} with \(0 \leq H \leq 8\). These are complementable only to configurations of type \{4, 4, 4, 6, 6\} with \((Z, S, D, T, Q, K) = (6 - H, 24 + H, 24 + 2H, 36 - 2H, 10 - H, H)\) for \(0 \leq H \leq 6\). This means that it suffices to consider \(a, b, \ldots, o\) at most six. The restrictions that \(Q = 30 - 3H\) and \(Q_4 = 20 - 2H\) as well as the conditions \(j \geq n \geq o\) are all imposed.
CHAPTER 4. THE STRUCTURE OF RELATIONS

The total number of structures which meet these requirements is $234163 + 129037 + 63562 + 27857 + 27857 + 10910 + 3794 + 1150 + 290 + 58 = 470821$. This bound is very rough and presents a worst-case scenario. Additional class permutation restrictions could be introduced but this must be done carefully to ensure no loss of generality.

The total number of structures corresponding to relations in six classes is then at most $46 + 266 + 1546 + 5658 + 470821 = 478337$. Including the six structures corresponding to configurations (C1), (C2), (C3), (C4), (C5) and (C7) there are at most 478343 different structures that could lie in a 6-net of order ten. However, if it were known that some relation other than one completable to one of type \{4, 4, 4, 4, 4\} had to exist in any 6-net of order ten there would be at most 7522 possible structures, one of which would have to occur in the net. This reduction of the number of possible structures motivates much of the work in this thesis.

In order to embed these structures in a net, line labels must be assigned to each intersection of a quad or a hex with a class. These packings correspond to the choice of a vertex from a given part in the graph-theoretic edge packing problem. The structures described above incorporate all of the algebraic restrictions on a packing. These restrictions are not, however, generally sufficient to guarantee that the packing exists.
Next explicit examples of packings for each type of relation are presented. Each example is chosen to have maximum $H$ value because intuitively these packings should be less likely to exist. Some indication is given of when packings are unlikely to exist. The packing corresponding to any relation is determined by the assignment of line labels to quads and hexes.

**Example 4.7** The relation from $(C1)$ is of type $\{4,4,4,4\}$ and is a packing of four quads. The packing of the quads determines the doubles that appear since all pairs of line labels in different classes must occur. This may be done uniquely up to labelling of lines:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

**Example 4.8** The relation from $(C2)$ is of type $\{2,2,2,4,6\}$ and is a packing of four quads. By the structure previously determined this may also be done uniquely up to line labelling:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 & * \\
0 & 1 & 1 & 1 & * \\
1 & 0 & 1 & 2 & * \\
1 & 1 & 0 & 3 & * \\
\end{array}
\]
Example 4.9  The relation from (C3) is of type \{2, 2, 4, 4, 4\} and is a packing of five quads. By the structure previously determined this may be done uniquely up to line labelling and permutation of the last three classes:

\[
\begin{align*}
0 & 0 & 0 & 0 & \ast \\
0 & 1 & 1 & \ast & 0 \\
1 & 0 & \ast & 1 & 1 \\
1 & \ast & 2 & 2 & 2 \\
\ast & 1 & 3 & 3 & 3
\end{align*}
\]

Example 4.10  The relation from (C4) is of type \{2, 4, 4, 4, 6\} and is a packing of fourteen quads. Here is an example that respects the structure previously determined and the local regularity conditions:
Example 4.11  The relation from (C5) is of type \{4, 4, 4, 4\} and is a packing of 15 quads. Here is one of the packings coming from the structure previously determined together with the local regularity conditions:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 2 & 3 & 4 \\
1 & 3 & 3 & 5 \\
0 & 1 & * & 12 \\
1 & 1 & * & 23 \\
0 & * & 120 \\
1 & * & 201 \\
* & 0 & 330 \\
* & 1 & 031 \\
* & 2 & 232 \\
* & 3 & 103 \\
* & 3 & 224 \\
* & 2 & 015 \\
\end{array}
\]
The relation of type (C6) has been ruled out. The relation of type (C7) consists of all possible doubles, which is a trivial packing.

Example 4.12 The relation from (C8) is of type \{2, 2, 2, 2, 4, 4\}. The only possibility is \(H = 0\) for this type of relation. This is a packing of six quads. Although the packing is not determined uniquely up to line labelling, there are very few packings that respect the structure and local regularity conditions. Here is one example:
Example 4.13 The relations from \((C9)-(C12)\) are of type \(\{2, 2, 4, 4, 6\}\). The possibilities are \(0 \leq H \leq 3\) for this type of relation. With \(H = 3\) this is a packing of six quads. Although the packing is not determined uniquely up to line labelling, there are very few packings that respect the structure and local regularity conditions. Here is one example:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & * & * \\
1 & 1 & 1 & 1 & * & * \\
0 & 1 & * & * & 0 & 0 \\
1 & 0 & * & * & 1 & 1 \\
* & * & 0 & 1 & 2 & 2 \\
* & * & 1 & 0 & 3 & 3 \\
\end{array}
\]

Example 4.14 The relations from \((C13)-(C17)\) are of type \(\{2, 2, 4, 4, 4\}\). The possibilities are \(0 \leq H \leq 4\) for this type of relation. With \(H = 4\) this is
a packing of four quads. Although the packing is not determined uniquely up to line labelling, there are again very few packings that respect the structure and local regularity conditions. In this case the packing of the line labels into the hexes is unique up to labelling. Here is one example:

```
 0 0 0 0 0 0
 0 1 1 1 1 1
 1 0 2 2 2 2
 1 1 3 3 3 3
 * * 0 1 2 3
 * * 1 2 3 0
 * * 2 3 0 1
 * * 3 0 1 2
```

**Example 4.15** The relations from \((C18)-(C23)\) are of type \(\{2, 4, 4, 4, 4, 6\}\). The possibilities are \(0 \leq H \leq 5\) for this type of relation. With \(H = 5\) this is a packing of fourteen quads. Here is a packing that respects the structure and local regularity conditions:
Example 4.16 The relations from \((C24)-(C32)\) are of type \(\{4,4,4,4,4\}\). The possibilities are \(0 \leq H \leq 8\) for this type of relation. With \(H = 8\) this is a packing of six quads. Here is an example of a packing that respects the
structure and local regularity conditions:

```
0 0 0 0 0 0
0 1 1 1 1 1
1 0 2 2 2 1
1 2 3 3 1 2
2 3 0 1 2 2
2 1 3 0 3 3
3 2 1 2 3 0
3 3 2 3 0 3
1 3 1 0  *  *
3 0 3 1  *  *
0 2  *  * 2 3
2  * 2  * 1 0
* 1  * 2 0 2
*  * 0 3 3 1
```

In the above example, not only was the packing not unique up to line labelling but there were also many possibilities for the packing into the hexes.

These examples show that packings corresponding to each type of relation do indeed exist. However more refined techniques are called for to show that some of their analogues can be ruled out or that some of these packings do not embed in a 6-net of order ten.
4.5 Point Deletion for Local Structure

In some cases the deletion of lines from a relation may help to determine the local structure of the relation. This technique is applied to the relation of type \(\{4, 4, 4, 4, 4, 4\}\) with \(H = 8\) in this section. The relation will be viewed as a GDD in which the lines of the relation are the points of the design. Blocks of size two, four and six will be referred to as doubles, quads and hexes respectively.

Consider a relation of type \(\{6, 6, 6, 6, 6, 6\}\) with \(H = 2\), which is obtained by complementing a relation of type \(\{4, 4, 4, 4, 4, 4\}\) with \(H = 8\). There are two possibilities for the distribution of points in the two hexes. Either there are 12 distinct points contained in the two hexes or, without loss of generality, the points in the two hexes may be labelled:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

First assume all twelve points in the two hexes are distinct. Upon deleting these twelve points, a configuration of type \(\{4, 4, 4, 4, 4, 4\}\) is produced. In this sense a configuration is simply an arrangement of points in blocks. This is a more general definition that the one used in the context of the classification because this configuration does not necessarily complete to a relation. All the points in the hexes have been deleted so that all pairs in the resulting configuration are covered by quads and doubles of the original relation. The
original quads may have remained quads (if the points did not intersect the hexes); or may have become triples (if a quad intersected precisely one of the hexes); or doubles if a quad intersected both hexes.

There were initially six doubles in the relation of type \{6, 6, 6, 6, 6, 6\}. There were 30 cross-pairs between the two hexes covered in the original relation. This means that between 24 and 30 of these cross-pairs were covered by quads in the original relation. Let this number be called \(D'\) since it is the number of doubles coming from quads in the original relation.

The number of triples \(T'\) in the configuration of type \{4, 4, 4, 4, 4\} coming from quads in the original relation can be calculated using the local structure. Each deleted point of the original relation had eight quads passing through it. Thus \(T' = 12 \times 8 - 2 \times D'\). Finally the number of quads \(Q'\) which are preserved after deleting the twelve points is \(84 - D' - T'\).

The configuration of type \{4, 4, 4, 4, 4, 4\} consists of 240 pairs to be covered. Necessarily \(6Q' + 3T' + D' \leq 240\). This is an inequality rather than an equality since there may be some doubles that were not destroyed by the point deletion. As \(D'\) ranges between 24 and 30 the quantity \(6Q' + 3T' + D'\) ranges between 240 and 246, implying that \(D' = 24, T' = 48, Q' = 12\) is the only possibility. All six of the doubles are used in covering cross-pairs between the hexes.
Now consider the possibility that a point appears in both hexes. Deleting the eleven points appearing in hexes produces a configuration of type \{5, 4, 4, 4, 4, 4\}. In the original relation there are eight quads through each of the eleven points, except for the point appearing in both hexes. This point has six quads passing through it.

There are now 20 cross-pairs to be covered between deleted points. Let \(D'\) now represent the number of these pairs covered by quads, i.e. the number of doubles that are produced when the eleven points are deleted. Since the number of doubles in the original relation is still six, \(D'\) may range between 14 and 20.

Let \(T'\) be the number of triples produced when the eleven points are deleted. By counting quads incident with the points in the hexes, \(T' = 6 + 10 \times 8 - 2 \times D'\). Finally let \(Q'\) be the number of quads that are preserved after the eleven points are deleted. Since the number of quads in the original relation is still 84, \(Q' = 84 - T' - D'\). Now \(6Q' + 3T' + D' \leq 260\). The seven possibilities \(14 \leq D' \leq 20\) generate values ranging from 260 to 266. This implies again that all six of the doubles are used in covering cross-pairs between the hexes.

The local regularity conditions for a relation of type \{6, 6, 6, 6, 6\} are
$Q_{6L} = 10 - 2h, D_{6L} = h$. This means that the point common to the two hexes has two doubles incident with it. This is impossible by the conclusion above because all pairs through the common point are already covered.

Thus point deletion has determined that all six of the doubles in the relation join cross-pairs between the two hexes. Furthermore it has determined that the two hexes are formed by 12 distinct points. This technique may be useful in refining the local structure of some of the other relations, particular those troublesome relations of type \{4, 4, 4, 4, 4, 4\} which at first seem to have many different structural possibilities.
Chapter 5

Dimension and Minimum Weight

In the first section, some new results on nets of order ten and fourteen are given. The dimensions of the codes of nets over $F_2$ (the vector spaces $C_2(N_k)$), or the ranks of the corresponding incidence matrices, are studied. In the second section, the minimum weight of codewords in codes of projective structures are examined where the projective structures are closely related to nets of even order.

5.1 Dimensions of Nets

Recall that the even-class-sum relations will be referred to as the *trivial* relations. In the 6-nets of order ten the basis elements for the five even-class-sum
relations were named $B_1 \cup B_2 = \vec{0}$, $B_1 \cup B_3 = \vec{0}$, $B_1 \cup B_4 = \vec{0}$, $B_1 \cup B_5 = \vec{0}$, $B_1 \cup B_6 = \vec{0}$, where $\vec{0}$ is the zero vector in $F_{100}^2$ and the $B_i$ are the sums of the row vectors corresponding to the lines of the parallel classes.

Suppose there are five other linearly independent relations $S_1 = \vec{0}$, $S_2 = \vec{0}$, $S_3 = \vec{0}$, $S_4 = \vec{0}$, $S_5 = \vec{0}$ in a net of order ten. Then these five relations together with the five relations above form a basis for the dependencies in the rowspace of the incidence matrix for the net.

Given the number of lines contributed to each $S_i$ from each parallel class, the goal is to understand which types of relations exist in the net. Choose an arbitrary parallel class of the net and let $s_i$ equal the number of lines in this parallel class contained in the set $S_i$, for $1 \leq i \leq 5$.

Let $T_i$ be a set of cardinality $s_i$, for $1 \leq i \leq 5$, where:

$T_1 = A \cup A5 \cup B \cup B5 \cup C \cup C5 \cup D \cup D5 \cup E \cup E5 \cup F \cup F5 \cup G \cup G5 \cup H \cup H5;$
$T_2 = C \cup C5 \cup D \cup D5 \cup G \cup G5 \cup H \cup H5 \cup K \cup K5 \cup L \cup L5 \cup M \cup M5 \cup N \cup N5;$
$T_3 = E \cup E5 \cup F \cup F5 \cup G \cup G5 \cup H \cup H5 \cup I \cup I5 \cup J \cup J5 \cup K \cup K5 \cup L \cup L5;$
$T_4 = A \cup A5 \cup C \cup C5 \cup E \cup E5 \cup G \cup G5 \cup I \cup I5 \cup K \cup K5 \cup M \cup M5 \cup O \cup O5;$
$T_5 = A5 \cup B5 \cup C5 \cup D5 \cup E5 \cup F5 \cup G5 \cup H5 \cup I5 \cup J5 \cup K5 \cup L5 \cup M5 \cup N5 \cup O5 \cup X5.$

The situation can be understood as a Venn diagram with five sets rep-
resenting which lines of the parallel class are contained in which relations. These five relations generate \( \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 26 \) further relations over \( F_2 \). This leads to a system of linear equations. Let the lower-case symbols represent the number of lines contained in the region labelled with the upper-case symbol; for instance \( a \) represents the number of lines in \( A \) while \( c5 \) represents the number of lines in \( C5 \).

The 26 new relations have the following number of lines in the given class:

\[
\begin{align*}
  a + b + e + f + k + l + m + n + a5 + b5 + e5 + f5 + k5 + l5 + m5 + n5; \\
  a + b + c + d + i + j + k + l + a5 + b5 + c5 + d5 + i5 + j5 + k5 + l5; \\
  b + d + f + h + i + k + m + o + b5 + d5 + f5 + h5 + i5 + k5 + m5 + o5; \\
  a + b + c + d + e + f + g + h + i5 + j5 + k5 + l5 + m5 + n5 + o5 + x5; \\
  c + d + m + n + e + f + i + j + c5 + d5 + m5 + n5 + e5 + f5 + i5 + j5; \\
  d + h + l + n + a + e + i + o + d5 + h5 + l5 + n5 + a5 + e5 + i5 + o5; \\
  a5 + b5 + c + d + e5 + f5 + g + h + i5 + j5 + k + l + m + n + o5 + x5; \\
  f + h + j + l + a + c + m + o + f5 + h5 + j5 + l5 + a5 + e5 + m5 + o5; \\
  a5 + b5 + c5 + d5 + e + f + g + h + i + j + k + l + m5 + n5 + o5 + x5; \\
  a + b5 + c + d5 + e + f5 + g + h5 + i + j5 + k + l5 + m + n5 + o + x5; \\
  a + b + i + j + m + n + g + h + a5 + b5 + i5 + j5 + m5 + n5 + g5 + h5; \\
  b + c + f + l + i + n + g + o + b5 + c5 + f5 + l5 + i5 + n5 + g5 + o5; \\
  a + b + e + f + k + l + m + n + c5 + d5 + g5 + h5 + i5 + j5 + o5 + x5; \\
  b + d + j + l + e + g + m + o + b5 + d5 + j5 + l5 + e5 + g5 + m5 + o5;
\end{align*}
\]
\[a + b + c + d + i + j + k + l + e5 + f5 + g5 + h5 + m5 + n5 + o5 + x5;\]
\[b + d + f + h + i + k + m + o + a5 + c5 + e5 + g5 + j5 + l5 + n5 + x5;\]
\[d + n + f + j + g + k + a + o + d5 + n5 + f5 + j5 + g5 + k5 + a5 + o5;\]
\[c + d + m + n + e + f + i + a5 + b5 + g5 + h5 + k5 + l5 + o5 + x5;\]
\[d + h + l + n + a + e + i + o + b5 + c5 + f5 + g5 + j5 + k5 + m5 + x5;\]
\[f + h + j + l + a + c + m + o + b5 + d5 + e5 + g5 + i5 + k5 + n5 + x5;\]
\[b + c + e + h + j + k + n + o + b5 + c5 + e5 + h5 + j5 + k5 + n5 + o5;\]
\[a + b + i + j + m + n + g + h + c5 + d5 + e5 + f5 + k5 + l5 + o5 + x5;\]
\[b + c + f + l + i + n + g + o + a5 + d5 + e5 + h5 + j5 + k5 + m5 + x5;\]
\[b + d + j + l + e + g + m + o + a5 + c5 + f5 + h5 + i5 + k5 + n5 + x5;\]
\[d + n + f + j + g + k + a + o + b5 + c5 + e5 + h5 + i5 + l5 + m5 + x5;\]
\[b + c + e + h + j + k + n + o + a5 + d5 + f5 + g5 + i5 + l5 + m5 + x5.\]

Combinatorial feasibility of these five relations existing in the net requires that each quantity above lies in the set \(\{0, 2, 4, 6, 8, 10\}\), provided the evenness condition on relations holds in the net (for instance if the net has an even number of parallel classes or at least one transversal). It also requires that the thirty-two regions \(A, A5, \ldots, X, X5\) partition the parallel classes so that the sum \(a + a5 + \ldots + x + x5 = 10.\)

**Proposition 5.1** If \(N_6\) is a 6-net of order ten and \(\dim C_2(N_6) \leq 50\) then the net contains a relation that is not complementable to a relation of type \(\{4, 4, 4, 4, 4, 4\}\).
CHAPTER 5. DIMENSION AND MINIMUM WEIGHT

Proof: Consider all systems of equations where \( s_i = 6 \) for \( 1 \leq i \leq 5 \), each of the 26 quantities above lies in \( \{0, 2, 4, 6, 8, 10\} \) and \( a + a5 + b + b5 + \ldots + o + o5 + x + x5 = 10 \). For each solution there is at least one region of size either zero or ten or there are at most 23 regions out of the 26 with sizes either four or six. Linear independence of the relations means that none of these relations is the zero relation. So a relation with zero, two, eight, or ten lines in at least one class is guaranteed. None of these relations is complementable to one of type \( \{4, 4, 4, 4, 4\} \).

One implication of the above result is that if \( \dim C_2(N_6) \leq 50 \) for a 6-net of order ten then the net must contain one of the configurations (C1)-(C23) other than (C6) (which has been shown not to exist).

This line of reasoning can also be used to establish some lower bounds on the dimension of \( C_2(N_4) \) for nets of order ten and fourteen. Recall that Proposition 2.8 implies that if \( n \equiv 2 \pmod{4} \) and \( N_3 \) is a 3-net of order \( n \) then \( \dim C_2(N_3) = 3n - 2 \). However, nothing is yet known about the possible dimension of \( C_2(N_4) \).

**Proposition 5.2** If \( N_4 \) is a 4-net of order ten then \( \dim C_2(N_4) \geq 33 \).

Proof: Suppose for contradiction that \( N_4 \) contains at least five linearly independent relations \( S_i, 1 \leq i \leq 5 \). Then, by the above argument, \( N_4 \) also contains a nontrivial relation with zero, two, eight, or ten lines in some parallel class. This is a contradiction because it has been shown that the
only type of relation in $N_4$ is complementable to one of type $\{4, 4, 4, 4\}$. So $N_4$ contains at most four linearly independent relations. These relations, together with the three trivial relations, generate the dependencies of $C_2(N_4)$. So $\dim C_2(N_4) \geq 4 \times 10 - 4 - 3 = 33$. □

**Proposition 5.3** If $N_4$ is a 4-net of order fourteen then $\dim C_2(N_4) \geq 49$.

Before the proof of Proposition 5.3 is presented, the possible relations in a 4-net of order fourteen will be examined. Consider the systems corresponding to configuration types $\{2, 6, 6\}, \{4, 4, 6\}, \{4, 6, 6\}$ and $\{6, 6, 6\}$, where $L$ represents the number of lines in the configuration and $l$ represents the number of lines in the last class:

\[
\begin{align*}
Z + S + D + T &= 196 \\
D + 3T &= P \\
3Z + S &= C \\
S + T &= 14l \\
S + 3T &= Ll
\end{align*}
\]

where $P = 60, 64, 84, 108$ and $C = 256, 260, 224, 192$ respectively. The first three systems have no solution in non-negative integers with $l$ even. The last system has the unique solution $(Z, S, D, T, l) = (40, 72, 72, 12, 6)$. Therefore the only relation in a 4-net of order 14 is of type $\{6, 6, 6, 6\}$.
Proof: (Proposition 5.3) The only relation in a 4-net of order fourteen is of type \{6,6,6,6\}. Consider all systems of equations where \(s_i = 6\) for \(1 \leq i \leq 5\), each of the 26 quantities above lies in \{0, 2, 4, 6, 8, 10, 12, 14\} and \(a + a5 + \ldots + x + x5 = 14\). For each solution there is at least one region of size either zero or fourteen or there are at most 24 regions out of the 26 with sizes either six or eight. Linear independence of the relations means that none of these relations is the zero relation. So a relation with zero, two, four, ten, twelve, or fourteen lines in at least one class is guaranteed. None of these relations is complementable to one of type \{6,6,6,6\}. This contradicts the existence of five linearly independent relations in the net. So \(\dim C_2(N_4) \geq 4 \times 14 - 3 - 4 = 49\).

An analogous result for order eighteen is not immediate because more than one type of relation is possible. However these results lead to a more general observation about relations in 4-nets.

**Proposition 5.4** A relation in a 4-net of order \(n \equiv 2 \pmod{4}\) with at most \(\frac{n}{2}\) lines in each of the first three classes must be of type \(\{k, k, k, k\}\) for \(k\) an even integer satisfying \(\frac{n}{3} \leq k < \frac{n}{2}\).

**Proof:** By Proposition 2.8 there is no relation in three classes other than those involving an even number of entire parallel classes. Suppose a nontrivial relation of type \(\{p, q, r, s\}\) exists in the net. The \(p + q\) lines from the first two classes produce \(np + nq - 2pq\) odd weight points because there are \(pq\) disjoint points of intersection. Similarly the \(r + s\) lines from the last two classes
produce \( nr + ns - 2rs \) odd weight points. This implies \( np + nq - 2pq = nr + ns - 2rs \) for any such relation. This holds for any partition of the four classes into two pairs of classes. This gives:

\[
np + nq - nr - ns = 2pq - 2rs \\
np + nr - nq - ns = 2pr - 2qs
\]

where the first equation comes from the partition of lines \( \{p, q\} \cup \{r, s\} \) and the second comes from the partition of lines \( \{p, r\} \cup \{q, s\} \). Adding the two equations gives

\[
2np - 2ns = 2pq + 2pr - 2rs - 2qs
\]

which can be rewritten

\[
2n(p - s) = 2(p - s)(q + r).
\]

It follows that \( q + r = n \) or \( p - s = 0 \). Since four is even the relations in the 4-net have even weight in each class so \( q \) and \( r \) are both strictly less than \( \frac{n}{2} \).

Thus \( p - s = 0 \) and \( p = s \). By a similar argument \( p = q \) and \( p = r \). Thus \( p = q = r = s = k \) for \( k \) an even number. If \( p + q + r < n \) then the lines in the fourth class will create new odd weight points, contradicting the existence of the relation. So \( p + q + r \geq n \) and \( k \geq \frac{n}{3} \).

\[\square\]

**Corollary 5.5** Any nontrivial relation in a 4-net of order \( n \equiv 2 \pmod{4} \) is
complementable to one of type \( \{k, k, k, k\} \).

This immediately rules out many relations in 4-nets of order congruent to two modulo four. It is possible to describe the potential relations of type \( \{k, k, k, k\} \) further by solving the following system:

\[
\begin{align*}
Z + S + D + T &= n^2 \\
D + 3T &= 3k^2 \\
3Z + S &= 3(n - k)^2 \\
S + T &= nk \\
S + 3T &= 3k^2
\end{align*}
\]

The unique solution in terms of \( k \) and \( n \) is

\[
(Z, S, D, T) = \left( \frac{2n^2 + 3k^2 - 5nk}{2}, \frac{3nk - 3k^2}{2}, \frac{3nk - 3k^2}{2}, \frac{3k^2 - nk}{2} \right).
\]

**Corollary 5.6** Any nontrivial relation in a 4-net of order \( n \equiv 2 \pmod{4} \) is complementable to one of type \( \{k, k, k, k\} \) with the above parameters.

**Example 5.7** By Corollary 5.5 a relation of type \( \{4, 4, 6, l\} \) for any even \( l \) can immediately be ruled out in a 4-net of order 14. This relation is not complementable to one of type \( \{k, k, k, k\} \). However each of the first three classes contributes a positive even number of lines and the first three classes sum to fourteen lines, so the two obvious requirements on the type of a relation in four classes are met.
Ultimately it would be useful to show that a relation other than one of type \{2, 4, 4, 4, 4, 6\} or \{4, 4, 4, 4, 4\} occurs in any 6-net of order ten. If \(|S_i| = 6\) for \(1 \leq i \leq 5\) and there is no relation with zero or ten lines in some class, then there are at least three relations having two or eight lines in some class. But because \(3 \times 6 < 26 + 1\) there is no obvious way of guaranteeing that there exists a relation having zero or ten lines in some class or two or eight lines in at least two classes.

The discussion of dimension of nets of order ten will now be resumed.

**Proposition 5.8** If \(k\) is even or \(N_k\) has a transversal for a net of even order, then
\[
\dim C_2(N_k) > \dim C_2(N_{k-1}).
\]

**Proof:** Since \(k\) is even or \(N_k\) has a transversal, the evenness condition applies to weights of parallel classes in any relation. This means that the row vector corresponding to any transversal of \(N_{k-1}\) lying in the \(k\)th parallel class must be linearly independent over \(C_2(N_{k-1})\).

**Corollary 5.9** If 6-nets of order ten and fourteen exist then
\[
\dim C_2(N_6) \geq 35 \text{ for a net of order ten and } \dim C_2(N_6) \geq 51 \text{ for a net of order fourteen.}
\]

**Proof:** Suppose a 6-net of order ten or fourteen exists. Then any \(N_5\) contained in it has a transversal. Thus Proposition 5.8 applies with \(k = 5\).
and \( k = 6 \) to give \( \dim C_2(N_6) \geq 33 + 1 + 1 = 35 \) for a net of order ten and \( \dim C_2(N_6) \geq 49 + 1 + 1 = 51 \) for a net of order fourteen.

**Corollary 5.10** If \( \dim C_2(N_5) < 46 \) or \( \dim C_2(N_6) < 47 \) for a net of order ten then there is a nontrivial relation in four or five classes in the net.

**Proof:** If \( \dim C_2(N_5) < 46 = 5 \times 10 - 4 \) then there is a nontrivial relation in the net, which is necessarily a relation in at most five classes. If \( \dim C_2(N_6) < 47 \) then any subnet \( N_5 \) has a transversal by the existence of \( N_6 \) and Proposition 5.8 applies to give \( \dim C_2(N_5) < 47 - 1 = 46 \).

This means that one of the configurations (C1)-(C5) must appear in any such net. Thus a proof of the non-existence of a 6-net of order ten could happen in two ways: by showing that the dimension of \( C_2(N_6) \) is too high and applying the dimension argument or by showing that the dimension of \( C_2(N_6) \) is low and showing that (C1)-(C5) cannot extend to 6-nets.

A famous conjecture of Moorhouse [26] gives a lower bound on the differences in the dimensions of \( C_p(N_k) \) and \( C_p(N_{k-1}) \). The codes are now vector spaces over the field \( F_p \) for \( p \) a prime. This setting is more general than the case \( p = 2 \) that was previously discussed.

**Conjecture 5.11** (Moorhouse) Let \( N_k \) be any \( k \)-net of order \( n \) and let \( N_{k-1} \) be any \((k-1)\)-subnet thereof. If \( p \) is any prime dividing \( n \) exactly once, then \( \dim C_p(N_k) - \dim C_p(N_{k-1}) \geq n - k + 1 \).
Support for this conjecture is computational. The examples studied suggest that dimensions of nets tend to be smaller when there is algebraic structure, for example when the net comes from a Latin square coordinatized by a cyclic group [26].

Consider the statement of Moorhouse’s Conjecture with $p = 2$ and $n = 10$ or $n = 14$. If a 4-net of order ten exists then $\dim C_2(N_3) = 28$ for any subnet and Moorhouse’s Conjecture hypothesizes that $\dim C_2(N_4) \geq 35$. If a 4-net of order fourteen exists then $\dim C_2(N_3) = 40$ for any subnet and Moorhouse’s Conjecture hypothesizes that $\dim C_2(N_4) \geq 51$.

A counterexample to Moorhouse’s Conjecture in the case $p = 2$ and $n = 10$ was recently found by Myrvold (private communication). The following net $N_4$ of order ten has dimension 34. Each of the 100 points of the net is represented by the ordered 4-tuple of lines which meet that point. The three linearly independent nontrivial relations are listed following the net, where the parallel classes of the relation are listed in the same order as in the net.
A subsequent computer search by Myrvold showed that nets $N_4$ of order ten with dimension 34, 35, 36 and 37 all exist (private communication). Whether nets $N_4$ of order ten exist with dimension 33, the lower bound established here, is still an open question.

The truth of Moorhouse’s Conjecture for $p = 2$ and any order $4m + 2 \geq 6$ would imply that there is no affine plane of that order. To see this let $m \geq 1$ and $n = 4m + 2$. Suppose $N_n$ exists. (So that an affine plane of order $n$ exists by Bruck’s extension of nets [11].) The dimension of $C_2(N_n)$ is at least
CHAPTER 5. DIMENSION AND MINIMUM WEIGHT

\[ 3n - 2 + \frac{1}{2}(n - 3)(n - 2) = \frac{n^2}{2} + \frac{n}{2} + 1. \]

Since \( n \) is even \( \dim D_2 (N_n) \geq \frac{n^2}{2} - \frac{n}{2} + 1 \).

This gives \( \dim D_2^\perp (N_n) \leq \frac{n^2}{2} + \frac{n}{2} - 1 \), a contradiction since \( C_2 (N_n) \subseteq D_2^\perp (N_n) \).

Below the various upper bounds (UB) and lower bounds (LB) on \( \dim C_2 (N_k) \) are summarized for nets of degree \( k \) and order \( n \) which have been presented in this thesis. Moorhouse’s conjectured lower bounds are given as a reference although the conjectured lower bound on \( \dim C_2 (N_4) \) is known to be false for order ten. It is possible that Moorhouse’s Conjecture still holds over certain fields \( F_p \), or for other orders of the net.

**Table 1. The dimension of \( C_2 (N_k) \)**

<table>
<thead>
<tr>
<th>OBJECT ( N_k(n) )</th>
<th>MOORHOUSE LB</th>
<th>NEW LB</th>
<th>NEW UB</th>
<th>TRIVIAL UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_4(10) )</td>
<td>35</td>
<td>33</td>
<td>(-)</td>
<td>37</td>
</tr>
<tr>
<td>( N_4(14) )</td>
<td>51</td>
<td>49</td>
<td>(-)</td>
<td>53</td>
</tr>
<tr>
<td>( N_6(10) )</td>
<td>46</td>
<td>35</td>
<td>53</td>
<td>55</td>
</tr>
<tr>
<td>( N_6(14) )</td>
<td>70</td>
<td>51</td>
<td>(-)</td>
<td>79</td>
</tr>
<tr>
<td>( N_8(14) )</td>
<td>85</td>
<td>(-)</td>
<td>102</td>
<td>105</td>
</tr>
<tr>
<td>( N_{2+1}(n)^* )</td>
<td>( \frac{3n^2}{8} + \frac{3n}{4} + 1 )</td>
<td>(-)</td>
<td>( \frac{n^2}{2} + \frac{n}{4} + \frac{1}{2} )</td>
<td>( \frac{n^2}{2} + \frac{n}{2} )</td>
</tr>
<tr>
<td>( N_k(n)^{**} )</td>
<td>(-)</td>
<td>(-)</td>
<td>( \frac{n^{2+k}}{2} )</td>
<td>( n + (k - 1)(n - 1) )</td>
</tr>
</tbody>
</table>

*For \( n \equiv 2 \mod 4 \) with \( n > 2 \). Moorhouse’s Conjecture only applies when \( n \equiv 2 \mod 4 \). In calculating the bound it is assumed that \( N_3 \) has a transversal and therefore \( \dim C_2 (N_3) = 3n - 2 \).
For even $n > 2$ and $k \geq \frac{n}{2} + 1$. The line above is a special case of this result.

Now the situations where $\dim C_2(N_6) = 51, 52$ or 53 are examined.

**Proposition 5.12** If $\dim C_2(N_6) = 53$ for a net of order ten then any $N_7$ containing this net has no transversal.

**Proof:** If $\dim C_2(N_6) = 53$ then since 6 is even, $\dim D_2(N_6) = 47$ and $\dim D_2^+(N_6) = 53$. Since $C_2(N_6) \subseteq D_2^+(N_6)$ this implies that $C_2(N_6) = D_2^+(N_6)$. Any transversal of $N_6$ belongs to $D_2^+(N_6)$ and therefore lies in $C_2(N_6)$. A relation in $N_7$ with odd weight in the seventh parallel class is thus produced. This means that $N_7$ has no transversal. □

**Corollary 5.13** If $\dim C_2(N_6) = 53$ for a net of order ten then this net cannot be extended to more than seven parallel classes.

**Proof:** By Proposition 5.12 any $N_7$ containing $N_6$ has no transversal and certainly the net cannot extend to an eighth parallel class. □

To phrase this in the contrapositive if there exists a net $N_8$ of order ten then $\dim C_2(N_6) \leq 52$ for any subnet $N_6$.

**Proposition 5.14** If there exists a net $N_8$ of order ten and $\dim C_2(N_6) = 52$ for a subnet $N_6$ then $D_2^+(N_6) = D_2^+(N_7)$ for any $N_7$ with $N_6 \subset N_7 \subset N_8$.

**Proof:** If $\dim C_2(N_6) = 52$ then $\dim D_2(N_6) = 46$ since 6 is even. This means that $\dim D_2^+(N_6) = 54$. If a net $N_8$ exists then $\dim C_2(N_8) \leq 54$ by
The dimension argument so that \( \dim C_2(N_7) \leq 53 \) and \( \dim C_2(N_6) \leq 52 \) for these subnets by Proposition 5.8. By the hypothesis, equality must hold and \( \dim C_2(N_7) = 53 \), meaning that \( \dim D_2(N_7) = 46 \) since \( N_7 \) has a transversal and \( \dim D^1_2(N_7) = 54 \). Now \( D^1_2(N_7) \subseteq D^1_2(N_6) \) for \( N_6 \) a subnet of \( N_7 \) and since they both have dimension 54, \( D^1_2(N_7) = D^1_2(N_6) \). \( \square \)

The cases \( \dim C_2(N_6) = 52 \) and \( \dim C_2(N_6) = 51 \) are now examined in more detail.

**Proposition 5.15** If \( \dim C_2(N_6) = 51 \) for a net of order ten then \( N_6 \) contains a nontrivial relation that is not complementable to one of type \( \{ 2, 2, 2, 2, 2 \} \).

**Proof:** If \( \dim C_2(N_6) = 51 \) then there are four nontrivial relations in \( N_6 \). Consider the four relations \( S_1 = \vec{0}, S_2 = \vec{0}, S_3 = \vec{0}, S_4 = \vec{0} \) with the \( T_i \) defined as in the previous discussion except that \( A5 = A, B5 = B, \ldots, X = X5 \). Let \( s_1 = s_2 = s_3 = s_4 = 2 \) and impose the conditions that the eleven quantities not involving \( T5 \) all take on values in \( \{ 0, 2, 4, 6, 8, 10 \} \) and that \( a + b + c + d + e + f + g + h + i + j + k + l + m + n + o + x = 10 \). Then any solution has either at least one of the eleven quantities equal to zero or ten or at most six of the eleven quantities equal to two or eight. Since the relations corresponding to these regions are all nontrivial a relation not complementable to one of type \( \{ 2, 2, 2, 2, 2 \} \) is guaranteed. \( \square \)

**Proposition 5.16** If \( \dim C_2(N_6) = 52 \) for a net of order ten then \( N_6 \)
contains a nontrivial relation that is not complementable to one of type \{2,2,2,2,2\}.

Proof: If \( \dim C_2(N_6) = 52 \) then there are three nontrivial relations in \( N_6 \).
Consider the three relations \( S_1 = \vec{0}, S_2 = \vec{0}, S_3 = \vec{0} \) and define three sets
\[ T_1 = A \cup B \cup C \cup D; \quad T_2 = B \cup D \cup F \cup G; \quad T_3 = C \cup D \cup E \cup F, \]
where \( a+b+c+d = b+d+f+g = c+d+e+f = 2 \) and \( a+b+c+d+e+f+g+x = 10 \).
If the conditions are imposed that the four quantities \( a+c+f+g, a+b+e+f, b+g+c+e, a+d+e+g \) all take on values in \( \{0, 2, 4, 6, 8, 10\} \) then any solution has at most three of \( a+c+f+g, a+b+e+f, b+g+c+e, a+d+e+g \) equal to two or eight. Since the relations corresponding to these regions are all nontrivial a relation not complementable to one of type \{2,2,2,2,2\} is guaranteed. \( \square \)

These results show that further knowledge about the possible dimension of \( C_2(N_6) \) could narrow down the list of configurations that must be contained in the net.

5.2 Minimum Weight in Codes of Nets

The distribution of weights of codewords generated by the hypothesized projective plane of order ten eventually led to Lam, Thiel and Swiercz’s result [22] that such a projective plane does not exist. This motivates an understanding of the distribution of weights in the code of any hypothesized net.
This question is computationally difficult in most instances but it can easily be modified to obtain some nontrivial lower bounds on minimum weight in a code.

Given a net $N_k$ of even order $n$ and degree $k$, add $k$ points $\infty_1, \infty_2, \ldots, \infty_k$ to the point set so that lines in the $i$th parallel class now contain $\infty_i$ for $1 \leq i \leq k$. The new lines will now be called the *extended lines* of the net. Let $C_2(P_k)$ be the code over $F_2$ generated by the extended lines of the net $N_k$. The notation $P_k$ comes from the fact that every pair of distinct extended lines intersect in precisely one point, giving $P_k$ some projective structure. If in addition to $\infty_1, \infty_2, \ldots, \infty_k$ a parity check point $p$ is also added so that lines in the $i$th parallel class contain $\infty_i$ and $p$ for $1 \leq i \leq k$ then the code generated over $F_2$ by the new lines will be called $C_2(P_k^*)$.

**Lemma 5.17** Let $N_k$ be a net of even order. The code $C_2(P_k^*)$ is contained in $C_2^\perp(P_k^*)$.

**Proof:** It suffices to show that any two generators have dot product zero over $F_2$. Any generator $l$ of $C_2(P_k^*)$ has weight $n + 2$ so that $l \cdot l = 0$. If $l_1$ and $l_2$ lie in the $i$th parallel class then they intersect in precisely $\infty_i$ and $p$ so that $l_1 \cdot l_2 = 0$. Finally if $l_1$ and $l_2$ lie in different parallel classes then $l_1$ and $l_2$ intersect in $p$ and one point of the net $N_k$. Again $l_1 \cdot l_2 = 0$. □

**Corollary 5.18** Let $N_k$ be a net of even order. Any codeword of odd (even) weight in $C_2(P_k)$ meets every extended line in an odd (even) number of points.
Proposition 5.19 Suppose $N_k$ is a net of even order $n$ with $k \leq n$. The minimum weight of a nonzero codeword in $C_2(P_k)$ is at least $k$ if $k$ is odd and at least $k + 1$ if $k$ is even.

Proof: Suppose the weight of $a$ is odd for some $a \in C_2(P_k)$. Since $a$ has odd weight it meets each of the $nk$ extended lines of $N_k$ in at least one point. Each point of $P_k$ lies on at most $n$ lines. (A point lies on $k$ lines if it is a point of the net $N_k$ and on $n$ lines if it is one of the $\infty_i$.) Thus the weight of $a$ is at least $k$. Suppose now that the weight of $a$ is even for some $a \in C_2(P_k)$. Then $a$ does not correspond to an extended line. Let $q$ be a point lying on $a$. There are at least $k$ extended lines of the net through the point $q$. (Again whether this quantity is $k$ or $n$ depends on the choice of $q$.) Each of these lines meets $a$ at some other point, and these second points must all be distinct. This means that the weight of $a$ is at least $k + 1$. □

Proposition 5.20 Suppose $N_k$ is a net of even order $n$ with $k = n + 1$. The minimum weight of a nonzero codeword in $C_2(P_k)$ is equal to $n + 1$.

Proof: Suppose the weight of $a$ is odd for some $a \in C_2(P_k)$. Since $a$ has odd weight it meets each of the $n(n + 1)$ extended lines of $N_k$ in at least one point. Each point of $P_k$ lies on at most $n + 1$ lines. Thus the weight of $a$ is at least $n$. Because the weight of $a$ is odd and $n$ is even the weight of $a$ is at least $n + 1$. Suppose now that the weight of $a$ is even for some $a \in C_2(P_k)$. Then $a$ does not correspond to an extended line. Let $q$ be a point lying on $a$. There are at least $n$ lines of the net through the point $q$. (Again whether
this quantity is \( n \) or \( k = n + 1 \) depends on the choice of \( q \).) Each of these
lines meets \( a \) at some other point, and each of these second points must all
be distinct. This means that the weight of \( a \) is at least \( n + 1 \). Equality holds
because any extended line has weight \( n + 1 \).

\[ \square \]

**Corollary 5.21** Let \( N_k \) be a net of even order. The code \( C_2(P_k) \) corrects at
least \( \left\lfloor \frac{k}{2} \right\rfloor \) errors.

**Proof:** Since \( C_2(P_k) \) is linear its minimum distance is equal to the minimum
of the weights of its codewords. The latter is at least \( k \) if \( k \) is odd and \( k + 1 \)
if \( k \) is even. Thus the code can correct at least \( \frac{k-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor \) errors if \( k \) is odd
and \( \frac{k+1-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor \) errors if \( k \) is even.

\[ \square \]

These lower bounds are weak for small \( k \). The minimum weight of a
codeword in \( C_2(P_k) \) could be as large as \( n+1 \). The next proposition improves
the results for \( k = 1 \) and \( k = 2 \).

**Proposition 5.22** Suppose \( N_k \) is a net of even order \( n \) with \( k \leq 2 \). The
minimum weight of a nonzero codeword in \( C_2(P_k) \) is \( n + 1 \).

**Proof:** If only one parallel class contributes a line then any weight is of
the form \((n+1) + na\) for a non-negative integer \( a \) and the result follows.
If two parallel classes contribute lines then any weight is of the form \( na + 1 + nb + 1 - 2ab \) for \( a, b \) the number of lines contributed by each class. If
\( a \leq \frac{n}{2} \) or \( b \leq \frac{n}{2} \) then without loss of generality assume that \( b \leq \frac{n}{2} \). Then
\( na + 1 + nb + 1 - 2ab \geq na + 2 + nb - 2a\frac{n}{2} = nb + 2 \geq n + 2 \). Otherwise
CHAPTER 5. DIMENSION AND MINIMUM WEIGHT

If \( a = n = b \) then the weight is zero because every point is covered an even number of times. If one parallel class contributes \( n \) lines and another \( n - 1 \), call the class contributing \( n \) lines the first class and the other the second class. Each line from the second class intersects \( n \) other lines. There is no weight contributed from these points. The lines from the first class each intersect \( n - 1 \) lines, so that each of these lines contributes a point covered once. One of the points \( \infty_1, \infty_2 \) is covered an even number of times and one an odd number of times. The total weight is therefore \( n + 1 \).

Now it may be assumed that \( n > a > \frac{n}{2} \) and \( n > b > \frac{n}{2} \). The weight of any codeword is \( na + nb + 1 - 2ab \geq (a + 1)a + (b + 1)b + 2 - 2ab = a + b + 2 + (a - b)^2 \geq a + b + 2 > n + 1 \). Equality holds by taking the codeword corresponding to a single line.

For small \( k \) the next result gives a lower bound on minimum weight in the net \( N_k \).

**Proposition 5.23** Suppose \( N_k \) is a net of order \( n \) with \( k \leq 2 \). The minimum weight of a nonzero codeword in \( C_2(N_k) \) is \( n \).

**Proof:** If only one parallel class contributes a line then any nonzero weight is a positive multiple of \( n \). If two parallel classes contribute lines then let the number of contributed lines be \( a \) and \( b \). If \( a = n = b \) then the weight is zero. If \( a = n \) and \( b = n - 1 \) then the weight is \( n \). It may be assumed that \( 1 \leq a, b < n \). If one of \( a \) or \( b \) is at most \( \frac{n}{2} \), without loss of generality assume that \( b \leq \frac{n}{2} \). The weight is \( na + nb - 2ab \geq na + nb - na = nb \geq n \). Otherwise
\( \frac{n}{2} < a, b < n \). The weight is \( na + nb - 2ab \geq a + b + (a - b)^2 \geq a + b \geq n \).

Equality holds by taking the codeword corresponding to a single line. \( \square \)

**Corollary 5.24** For \( k \leq 2 \) the code \( C_2(P_k) \) corrects \( \frac{n}{2} \) errors, where \( n \) is the even order of the net \( N_k \).

**Corollary 5.25** For \( k \leq 2 \) the code \( C_2(N_k) \) corrects \( \left\lfloor \frac{n-1}{2} \right\rfloor \) errors, where \( n \) is the order of the net \( N_k \).
Chapter 6

Packing Cliques

The problem of packing cliques into complete multipartite graphs is considered here. This is motivated by the problem of packing high-weight points into configurations in a net. Given a complete multipartite graph upper bounds will be established on the number of cliques of largest order that can be contained in a packing. The focus will be on packings where all cliques have even order, or odd order in the special case of a graph with an odd number of parts. Regularity conditions will be imposed in five and six classes to study those packings that potentially embed in a net of order fourteen. Techniques to rule out the embeddability of specific packings will be presented.

Notation 6.1 For $2 \leq i \leq 6$ let $n_i$ represent the number of $K_i$ in a packing of cliques into a given complete multipartite graph.
6.1 Packing Cliques in Four Classes

The maximum number of edge-disjoint $K_4$ that can be contained in the complete multipartite graph $K_{a,b,c,d}$, where $a \leq b \leq c \leq d$, will be determined.

Theorem 6.2 A pair of orthogonal Latin squares of order $n$ exists for every $n \geq 1$, except for $n = 2$ or $n = 6$.

Proof: When $n = 1$, the unique Latin square is self-orthogonal. MacNeish’s product construction [14] shows that $N(mn)$ is at least the minimum of $N(m)$ and $N(n)$ for $m,n$ greater than one. Thus when $n$ is larger than one and not congruent to two modulo four, the smallest prime power factor of $n$ is at least three, yielding at least two (mutually) orthogonal Latin squares of order $n$. The work of Bose, Shrikhande and Parker [9] showed that $OLS(n)$ exist whenever $n$ is congruent to two modulo four, except in the cases $n = 2$ or $n = 6$.

Corollary 6.3 An optimal packing of $K_4$ into $K_{a,b,c,d}$, where $a \leq b \leq c \leq d$ and $b$ is not equal to 2 or 6, contains exactly $ab$ copies of $K_4$.

Proof: Given a pair of orthogonal Latin squares of order $b$ label the rows, columns, and symbol sets of the two squares with symbol set $\{0,1,\ldots,b-1\}$. Each of the $b^2$ positions then corresponds to a $K_4$. The $b^2$ copies of $K_4$ produced in this way decompose $K_{b,b,b,b}$ because orthogonality, the row and column Latin properties of both squares, and the orthogonality of row and column indices ensure that each edge is covered precisely once.
CHAPTER 6. PACKING CLIQUES

Since $b$ is not equal to 2 or 6, there exists a pair of orthogonal Latin squares of order $b$. Restricting both squares to their first $a$ rows yields the desired packing.

\[\Box\]

**Proposition 6.4** An optimal packing of $K_4$ into $K_{1,2,c,d}$, where $2 \leq c \leq d$ contains exactly two copies of $K_4$.

**Proof:** Take the two cliques formed by $(0,0,0,0)$ and $(0,1,1,1)$. \[\Box\]

When $a = 2$ the answer depends upon the values of $c$ and $d$.

**Proposition 6.5** An optimal packing of $K_4$ into $K_{2,2,2,d}$, where $d \geq 2$, contains exactly $d$ copies of $K_4$ if $d < 4$, or four copies of $K_4$ if $d \geq 4$.

**Proof:** The answer is equal to the maximum number of cells that may be filled in a $2 \times 2$ array without repeating a symbol in a row or column and without repeating an ordered pair when taken together with the unique (up to isomorphism) Latin square of order 2. Assume the first square is the reduced Latin square of order 2. Any of the six pairs of distinct cells in the second square require different symbols. Thus an optimal packing contains precisely $d$ copies of $K_4$ if $d < 4$, or four copies of $K_4$ if $d \geq 4$. \[\Box\]

The next case to consider is $c$ is at least three, which forces $d$ to be at least three as well.

**Proposition 6.6** An optimal packing of $K_4$ into $K_{2,2,c,d}$, where $c \geq 3$, contains exactly four copies of $K_4$. 
CHAPTER 6. PACKING CLIQUES

Proof: It suffices to show that four is possible. Consider the four ordered 4-tuples: \((0, 0, 0, 0), (0, 1, 1, 1), (1, 0, 2, 1), (1, 1, 0, 2)\). They respect the values of \(a, b, c, d\) and do not cover any pair more than once. \(\square\)

Finally consider the case \(b = 6\). A Latin square of order 6 is presented together with a \(6 \times 6\) array missing only two cells. The latter array has the Latin property in both rows and columns; furthermore no ordered pair is repeated between the two squares:

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 3 & 2 & 5 & 4 \\
2 & 3 & 4 & 5 & 0 & 1 \\
3 & 2 & 5 & 4 & 1 & 0 \\
4 & 5 & 1 & 0 & 3 & 2 \\
5 & 4 & 0 & 1 & 2 & 3 \\
\end{array}
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 4 & 0 & 1 \\
1 & 0 & 3 & 2 & 5 & 4 \\
4 & 5 & 1 & 0 & 3 & 2 \\
5 & 4 & 0 & 1 & 2 & 3 \\
3 & 2 & 4 & 5 & * & * \\
\end{array}
\]

Proposition 6.7 An optimal packing of \(K_4\) into \(K_{a,6,6,6}\), where \(a \leq 5\) contains exactly \(6a\) copies of \(K_4\).

Proof: The upper bound comes from the number of edges between the first two parts. This can be achieved by using the first part to index rows, the second part to index columns and the third and fourth parts respectively to represent the symbols in the appropriate cell of the first and second arrays above respectively. \(\square\)

Proposition 6.8 An optimal packing of \(K_4\) into \(K_{6,6,6,6}\) contains exactly 34 copies of \(K_4\).
Proof: The lower bound is given by the pair of arrays above. Since there is no pair of OLS(6), the optimal packing in this case contains either 34 or 35 copies of $K_4$. Suppose it contains 35 copies of $K_4$. Represent this packing by a $6 \times 6$ array of 35 ordered pairs. Since there is no packing of 36 $K_4$ either adding the missing symbols causes a repeated pair or at least one of the coordinates in the empty position must have a different symbol demanded by the row and column in which it appears. By counting occurrences of symbols in the array, the missing ordered pair is precisely the one determined by the two missing symbols. So it must be the case that at least one of the coordinates in the empty position must have a different symbol demanded by the row and column in which it appears. This is a contradiction because all but one of the symbols already appears six times in this coordinate of the array. □

Proposition 6.9 An optimal packing of $K_4$ into $K_{6,6,6,d}$, where $d \geq 7$, contains exactly 36 copies of $K_4$.

Proof: The upper bound follows from the number of edges between the first two parts. Consider the second array illustrated above. Replacing the symbol 0 in Row 2 Column 1 with a 6 and filling in the last row with 6 followed by 0 produces a pair of arrays corresponding to the required packing. □

These results can be summarized as follows:

Proposition 6.10 An optimal packing of $K_4$ into $K_{a,b,c,d}$ contains exactly $ab$ copies of $K_4$ except for the following ($a, b, c, d$, number of $K_4$):
This result completely describes the pairs \((n_2, n_4)\) so that \(n_2\) copies of \(K_2\) and \(n_4\) copies of \(K_4\) can be packed into a complete multipartite graph with four parts. The packing is possible if and only if \(n_4\) is at most the maximum given above and \(n_2 + 6 \cdot n_4\) is at most the number of edges of the graph.

\section*{6.2 Packing Cliques in Five Classes}

Some additional techniques are presented to rule out relations in five classes in a net of order fourteen. This is motivated by the fact that there are 23 potential types of such relations. Many of these relations must be ruled out before pursuing further structural analysis. The new techniques, illustrated below in examples, are Contradiction of Regularity, Pair Contradiction by Quad Structure and \(Q < Q_i\).

\textbf{Example 6.11} Contradiction of Regularity.

The relation of type \(\{2, 2, 4, 6, 6\}\) in a net of order 14 has \(Q_2 = 2 \cdot 2 = 4, Q_4 = 4 \cdot 1 = 4\) and \(Q_6 = 0\). This is a contradiction because a quad must be incident with four classes.

\textbf{Example 6.12} Pair Contradiction by Quad Structure.

The relation of type \(\{2, 2, 4, 6, 10\}\) in a net of order 14 has \(Q_2 = 2 \cdot 4 = 8, Q_4 = 4 \cdot 3 = 12, Q_6 = 6 \cdot 2 = 12, Q_{10} = 0\). This gives \(Q = \frac{1}{4}(8 + 8 + 12 + 12 + 0) = 10\). Since \(Q_{10} = 0\) these ten quads pass through the first four classes, a
contradiction because at most four quads can pass through both the first two classes.

Example 6.13 Pair Contradiction by Quad Structure.
The relation of type \( \{2, 4, 4, 4, 10\} \) in a net of order 14 has \( Q_2 = 2 \cdot 4 = 8 \), \( Q_4 = 4 \cdot 3 = 12 \), \( Q_{10} = 0 \). This gives \( Q = \frac{1}{4}(8 + 12 + 12 + 12 + 0) = 11 \). Since \( Q_{10} = 0 \) these eleven quads pass through the first four classes, a contradiction because at most eight quads can pass through both the first two classes.

Example 6.14 Pair Contradiction by Quad Structure.
The relation of type \( \{4, 4, 4, 4, 12\} \) in a net of order 14 has \( Q_4 = 4 \cdot 5 = 20 \) and \( Q_{12} = 12 \cdot 1 = 12 \). This gives \( Q = \frac{1}{4}(20 + 20 + 20 + 20 + 12) = 23 \). There are twelve quads of type \( \{4, 4, 4, 12\} \) and the remaining eleven are of type \( \{4, 4, 4, 4\} \). The number of edges between two of the first four parts that are contained in quads is \( 12 \cdot 3 + 11 \cdot 6 = 102 \). This is a contradiction because there are only \( 16 \cdot 6 = 96 \) edges between two of the first four parts.

Example 6.15 \( Q < Q_6 \).
The relation of type \( \{2, 2, 6, 6, 12\} \) in a net of order 14 has \( Q_2 = 2 \cdot 6 = 12 \), \( Q_6 = 6 \cdot 4 = 24 \), \( Q_{12} = 12 \cdot 1 = 12 \). This gives \( Q = \frac{1}{4}(12 + 12 + 24 + 24 + 12) = 21 \). This is a contradiction since \( Q < Q_6 \).

Example 6.16 \( Q < Q_6 \).
The relation of type \( \{2, 4, 4, 6, 12\} \) in a net of order 14 has \( Q_2 = 2 \cdot 6 = 12 \), \( Q_4 = 4 \cdot 5 = 20 \), \( Q_6 = 6 \cdot 4 = 24 \), \( Q_{12} = 12 \cdot 1 = 12 \). This gives \( Q = \frac{1}{4}(12 + 20 + 20 + 24 + 12) = 22 \). This is a contradiction since \( Q < Q_6 \).
The relations in nets $N_5$ of order ten and fourteen tend to have $T + Q$, the number of quads in the relation, small compared to the maximum possible $n_4$ in a packing. A packing of copies of $K_2$ and $K_4$ that embeds in a net corresponds to a packing of $K_3$ and $K_5$ on the complementary set of lines. Furthermore, these packings often have $n_5$ close to the maximum possible value. This motivates the study of packings of $K_3$ and $K_5$ into complete multipartite graphs.

The maximum number of $K_5$ which can be packed into a complete multipartite graph $G$ is at most the minimum number of edges between two parts. In the case of embeddability in a net, the packing is a decomposition and this imposes a further necessary condition. Since $\binom{5}{2} \equiv 1 \pmod{3}$ the condition $n_5 \equiv |E(G)| \pmod{3}$ is also necessary. It may be possible to rule out further configurations in a 5-net of order 14 if some of these packings can be shown not to exist.

Packings of $K_3$ into complete multipartite graphs have been studied in special cases. Billington and Lindner [6] gave the optimal leaves when the parts all have equal size. Colbourn, Hoffman and Rees [15] showed that the necessary conditions for a $K_3$-decomposition are sufficient for complete multipartite graphs with any number of parts where at most one part is of a different size.
Some interesting questions arise from this research: What are the best possible leaves for packings of $K_3$ in special cases of complete multipartite graphs, for example $K_{a,a,b,b}$ and $K_{a,b,b,b}$ with $a$ not equal to $b$? In the first case the minimal leave is always nontrivial because $a^2 + b^2 = 2ab$ is necessary to cover all pairs using $K_3$ from $K_{a,a,b}$ and $K_{a,b,b}$.

An open question is whether there exists a $K_3$-decomposition of $K_{a,b,c,d}$ for any $a \leq b \leq c \leq d$ whenever the necessary conditions are satisfied. The necessary conditions are: the number of edges is divisible by three; $a, b, c$ and $d$ are all even (so that when any part is deleted an even number of vertices remain); and $d \leq a+b$ so that all necessary triples through the third part may be formed. The more general necessary conditions for a $K_3$-decomposition of a complete multipartite graph (a 3-GDD) can be found in [16]. Packings of $K_3$ in bounded degree graphs also has applications to genome rearrangements in computational biology [2].

6.3 Packing Cliques in Six Classes

Consider a relation in a net $N_6$ of order $n$. Let $L$ be the number of lines in the relation and let $l$ be the maximum number of lines contributed by a parallel class. Consider the configuration produced when a class of cardinality $l$ is deleted. Let $P$ be the number of pairs of lines in the configuration that intersect in the net and let $C$ be the number of pairs of lines not in the
configuration that intersect in the net. Then a relation can be described by a solution to:

\[
\begin{align*}
Z + S + D + T + Q + K &= n^2 \\
D + 3T + 6Q + 10K &= P \\
10Z + 6S + 3D + T &= C \\
S + T + K &= n \cdot l \\
S + 3T + 5K &= (L - l)l
\end{align*}
\]

Let \( r \) be an integer. Then \((Z, S, D, T, Q, K)\) is an integral solution to the system above if and only if \((Z - r, S + r, D + 2r, T - 2r, Q - r, K + r)\) is an integral solution. Given any relation let \(Z_0, T_0, Q_0\) be the \(Z, T\) and \(Q\) values associated with the solution \(K = 0\). Then the maximum \(H\)-value for this type of relation is at most the minimum of \(Z_0, \frac{1}{2}T_0\) and \(Q_0\).

Next some packings of \(K_2, K_4, K_6\) corresponding to potential relations in the net \(N_6\) of order fourteen are ruled out by an upper bound on \(n_6\). There are two other quantities that contribute to this upper bound. Consider the complete multipartite graph corresponding to the relation. The quantity \(n_6\) is necessarily at most the number of edges between any two distinct parts in the graph. Let \(p\) represent this quantity for a specific type of relation. A different upper bound comes from considering a part of cardinality \(l\) in the graph. Let \(w_i\) be the number of cliques \(K_i\) through a vertex in this part. If
the relation is embeddable in a net of order fourteen then \( w_2 + w_4 + w_6 = 14 \) and \( w_2 + 3w_4 + 5w_5 = L - l \), where \( L \) is the total number of lines in the relation. This means that \( w_6 \leq \frac{L-l-14}{4} \). Let \( l^* \) be the minimum of \( l \cdot \lfloor \frac{L-l-14}{4} \rfloor \) over all the possible sizes of parts \( l \).

**Proposition 6.17** Given a potential relation in a net \( N_6 \) of order fourteen, the value of \( K \) in such a relation is at most \( \min\{p, Z_0, \frac{1}{2}T_0, Q_0, l^*\} \).

Since \( K = H \) in any relation in six classes from a packing perspective this is an upper bound on \( n_6 \) in the complete multipartite graph corresponding to the type of the relation.

**Example 6.18** Proposition 6.17 gives the following upper bounds on \( K \) for relations of given type in a net \( N_6 \) of order fourteen:

Type \( \{2, 2, 2, 4, 4, 6\} \) has \( K = 0 \) as the only possibility because \( T = 0 \).

Type \( \{2, 2, 4, 4, 4, 4\} \) has \( K \leq 2 \) because \( Q_0 = 2 \).

Type \( \{6, 6, 6, 6, 6, 10\} \) has \( K \leq 11 \) because \( Z_0 = 11 \).

Type \( \{2, 2, 4, 4, 4, 8\} \) has \( K = 0 \) as the only possibility by taking \( l = 8 \).

Finally some additional techniques are presented to rule out relations in six classes having \( K = 0 \) in a net of order fourteen. This is motivated by the fact that there are forty potential types of such relations. Many of these relations must be ruled out before pursuing further structural analysis. The new techniques, illustrated below in examples, are Structural Contradiction from \( K_4 \) Packing, \( Q < Q_i \), and Pair Contradiction by Alpha and Beta.
Example 6.19 Structural Contradiction from $K_4$ Packing.

The relation of type $\{2, 2, 2, 2, 6, 6\}$ in a net of order 14 has $Q_6 = 0$ and $Q_2 = 2 \cdot 2 = 4$. This means that all four quads are of type $\{2, 2, 2, 2\}$, a contradiction to Proposition 6.5.

Example 6.20 $Q < Q_6$.

The relation of type $\{2, 2, 2, 6, 10\}$ in a net of order 14 has $Q_2 = 2 \cdot 4 = 8, Q_6 = 6 \cdot 2 = 12, Q_{10} = 0$. This gives $Q = \frac{1}{4}(8 + 8 + 8 + 12 + 0) = 11$. This is a contraction since $Q < Q_6$.

Example 6.21 $Q < Q_6$.

The relation of type $\{2, 2, 2, 6, 12\}$ in a net of order 14 has $Q_2 = 2 \cdot 6 = 12, Q_4 = 4 \cdot 5 = 20, Q_6 = 6 \cdot 4 = 24, Q_{12} = 12 \cdot 1 = 12$. This gives $Q = \frac{1}{4}(12 + 12 + 12 + 20 + 24 + 12) = 23$. This is a contraction since $Q < Q_6$.

Example 6.22 Pair Contradiction by Alpha and Beta.

The relation of type $\{2, 2, 4, 4, 4, 12\}$ in a net of order 14 has $Q_2 = 2 \cdot 6 = 12, Q_4 = 4 \cdot 5 = 20, Q_{12} = 12 \cdot 1 = 12$. This gives $Q = \frac{1}{4}(12 + 12 + 20 + 20 + 20 + 12) = 24$. Using the Alpha and Beta technique from Proposition 4.3, let $\alpha$ and $\beta$ be the number of quads and triples respectively incident with a vertex of the fifth part after the sixth part is deleted. The number of edges incident with the fifth part and one of the first four parts gives $3\alpha + 2\beta \leq 48$. The regularity condition implies that $\alpha + \beta = 20$. Together these imply that $\alpha \leq 8$ and therefore that $\beta \geq 12$. Each of these $\beta$ triples is necessarily incident with a vertex of the sixth part which has $Q_{12} = 12$ so $\beta = 12$. This means
that there are 12 quads of type \{4, 4, 4, 12\}. The remaining twelve quads are incident with both of the parts of size two to ensure that \(Q_2 = 12\). This is a contradiction since there may be at most four quads simultaneously incident with both parts of size two.
Chapter 7

Applications and Future Work

Some applications of the results in this dissertation are discussed here. A list of future research problems is also presented.

7.1 Applications

This work draws on theory from combinatorics and finite geometry, elementary number theory as well as the algebra of finite fields and vector spaces. The most immediate application of this work is that it builds theory towards a mathematical proof of an upper bound on the number of MOLS(10). This work could lead to a proof that four MOLS(10) do not exist either by ruling out more of the configurations and using the dimension argument, or by narrowing the range of possible dimensions of a 6-net of order ten and using some of the structural results presented here. As discussed in the Introduction the
possibility also exists of weakening the dimension argument and achieving a larger upper bound on the number of MOLS(10). This weakened version of the dimension argument might be useful if ruling out the embedding of specific relations into a net $N_6$ of order ten proves to be difficult. An investigation of the dimension argument as applied to MOLS of other orders congruent to two modulo four is already simplified by the characterization of the relations possible in any 4-net.

This work also has applications to computer science in that it lays the groundwork for a possible computer search for four MOLS(10). Such a search might be conducted if more of the configurations could be ruled out, particularly some of those of type $\{4,4,4,4,4\}$. The theory of point deletion may be developed further to decrease the number of structures possible in this case.

Chapter 5 presents some preliminary results that connect minimum weight in nets with lower bounds on the error-correction in the codes they generate. This work generates additional research questions because of the combinatorial equivalence of nets to MOLS, orthogonal arrays, transversal and pairwise balanced designs as well as to complete multipartite graphs with added structure. Do dimension bounds produce any useful or interesting results in these contexts? Are structural arguments simplified? The goal of improving the upper bound on the number of MOLS(10) motivates these questions.
The predominant questions in the literature on nets relate to whether a given net may be completed to an affine plane, or even extended to a net with more parallel classes. Much of the early work in this area is due to Bruck [10], [11]. However most of it applies when \( n \) is large compared to \( n + 1 - k \). The embeddability questions studied in this thesis are more general. With the introduction of the Alpha and Beta approach and the proof that (C6) cannot be embedded in a net \( N_6 \) of order ten, some new techniques have been added to the theory of completion and embeddability of nets.

The major motivation of this work, to explore the maximum number of MOLS(\( 4m + 2 \)), has applications to statistical experimental design. In medical, industrial or manufacturing applications MOLS are used in the scheduling of experimental trials. The rows and columns of the Latin squares correspond to blocking factors (for instance the patient or machine involved versus the day or location of the trial). The entries of the Latin squares correspond to treatments (for instance the medication or product to be tested). The combinatorial orthogonality of the Latin squares corresponds to statistical orthogonality of various treatments and blocking factors. MOLS are particularly useful when treatment interactions need to be gauged, for instance in medical or software testing applications. In these instances an experimental design based on MOLS uses a relatively small amount of material or resources for a given number of treatments (number of squares) and levels of factors.
7.2 Future Work

Below are a list of problems that stem from the work presented here.

- Develop techniques for showing that a configuration and one of its analogues in nets of $N_6$ of order ten are incompatible.

- Apply the PBD approach to complement analogues to rule out some configurations in nets $N_6$ of order ten.

- Refine techniques to rule out embeddings of configurations in a net $N_6$ of order ten. The structure of transversals may be useful.

- Restrict the local structure of packings in nets $N_6$ of order ten, especially those of type $\{4,4,4,4,4,4\}$ and $\{2,4,4,4,4,6\}$. The point deletion approach has some promise.

- Improve the dimension bounds on codes of the nets discussed here.

- What structure is implied if $D_2^+(N_6) = C_2(N_6)$ or $D_2^+(N_6) = D_2^+(N_7)$ for nets of order ten?

- What are the best possible leaves for packings of $K_3$ in $K_{a,a,b,b}$ and $K_{a,b,b,b}$ with $a$ not equal to $b$?

- Are there special cases of Moorhouse’s Conjecture in which a coding approach is useful?
Further work towards special cases of Moorhouse’s Conjecture is motivated by some results due to Moorhouse.

**Theorem 7.1** [26] Suppose that $\Pi$ is a projective plane of order $n$, where $n$ is squarefree or $n \equiv 2 \pmod{4}$. If Conjecture 5.11 holds for $n$, then $n$ is prime and $\Pi$ is desarguesian.

**Corollary 7.2** Suppose that $n \equiv 2 \pmod{4}$ and that Conjecture 5.11 holds for $n$. Then the existence of a projective plane of order $n$ implies that $n = 2$.

Desarguesian nets are of particular interest in the context of Moorhouse’s Conjecture because of the following result:

**Theorem 7.3** [26] Conjecture 5.11 holds with equality in the case of desarguesian nets, that is subnets of desarguesian affine planes, of prime order.
Bibliography


(1992)

ford (1999)


[19] Dougherty, S.T.: A coding theoretic solution to the 36 officer problem,


[21] Lam, C., Thiel, L., Swiercz, S.: The nonexistence of code words of
weight 16 in a projective plane of order 10, *J. of Comb. Theory, Ser. A*
42, 207-214 (1986)

[22] Lam, C., Thiel, L., Swiercz, S.: The non-existence of finite projective

[23] Lam, C., Thiel, L., Swiercz, S., McKay, J.: The nonexistence of ovals in


