LIMITING PROBABILITIES FOR REPEATED ROLLS OF A DIE

(Mathematics Magazine Problem 1217)

by

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1217. A die is rolled repeatedly. Let $p_n$ be the probability that the accumulated score is at some time equal to $n$. Find $\lim_{n \to \infty} p_n$. [David Callan, Lafayette College.]

Solution by Bruce R. Johnson, University of Victoria.

More generally, we will solve the problem using a fair $r$-sided die with equally likely possible outcomes $1, 2, \ldots, r$. We will give three different solutions; the first relies on probabilistic intuition, the second uses Markov Chain Theory, and the third uses Renewal Theory.

**Solution 1** (Intuitive). For positive integer $n$ and $j \in \{1, 2, \ldots, r\}$, we let $A_{n,j}$ denote the event that $j$ occurs on the roll that causes the accumulated sum to become $\geq n$ for the first time. Since in the long run the outcomes $1, 2, \ldots, r$ will occur in equal proportions and since the accumulated sum will advance $j$ units each time $j$ is rolled, it follows that, for large $n$, $P(A_{n,j})$ will be approximately proportional to $j$. That is,

$$\lim_{n \to \infty} P(A_{n,j}) = j/(1+2+\ldots+r), \quad j = 1, 2, \ldots, r.$$**

Therefore,

$$\lim_{n \to \infty} P(\text{accumulated sum ever equals } n)$$

$$= \lim_{n \to \infty} \prod_{j=1}^{r} P(A_{n,j}) P(\text{accumulated sum ever equals } n | A_{n,j})$$

$$= \lim_{n \to \infty} \prod_{j=1}^{r} P(A_{n,j}) \cdot (1/j)$$
\[ \sum_{j=1}^{r} \left\{ \frac{j}{1+2+\ldots+r} \right\} \left( \frac{1}{j} \right) = \frac{r}{1+2+\ldots+r} = \frac{2}{(1+r)}. \]

Hence, for a standard die the answer is \( \lim_{n \to \infty} p_n = \frac{2}{7} \).

Solution 2 (Markov Chain Theory). For every nonnegative integer \( n \) we define

\[ X_n = n - (\text{largest accumulated sum that is } \leq n). \]

Then \( \{X_n\}_{n=0}^{\infty} \) is an ergodic Markov chain with state space \( \{0,1,\ldots,r-1\} \) and transition matrix \( \mathcal{P} \) given by

\[
\mathcal{P} = \begin{bmatrix}
\frac{1}{r} & \frac{r-1}{r} & 0 & 0 & 0 & \ldots & 0 \\
\frac{1}{r-1} & 0 & \frac{r-2}{r-1} & 0 & 0 & \ldots & 0 \\
\frac{1}{r-2} & 0 & 0 & \frac{r-3}{r-2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{2} & 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} \\
1 & 0 & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}.
\]

To find the stationary distribution \( (\pi_0, \pi_1, \ldots, \pi_{r-1}) \) for this Markov chain we solve the linear system

\[ \sum_{j=0}^{r} \pi_j = 1, \quad (\pi_0, \pi_1, \ldots, \pi_{r-1}) \mathcal{P} = (\pi_0, \pi_1, \ldots, \pi_{r-1}), \]

and obtain

\[ \pi_j = \frac{2(r-j)}{r(1+r)}, \quad j = 0, 1, \ldots, r-1. \]
Therefore,

\[
\lim_{n \to \infty} P(\text{accumulated sum ever equals n}) = \lim_{n \to \infty} P(X_n = 0) = \pi_0 = 2/(1+r).
\]

**Solution 3** (Renewal Theory). For each positive integer \(j\) we let \(T_j\) denote the outcome of the \(j\)th roll. Then \(T_1, T_2, \ldots\) are independent random variables with common distribution \(F\), where \(F\) denotes the discrete uniform distribution on \(1, 2, \ldots, r\). We let \(\{N(t): t \geq 0\}\) be the renewal process determined by the interarrival times \(T_1, T_2, \ldots\). That is,

\[
N(t) = \text{number of indices } n \text{ for which } T_1 + T_2 + \ldots + T_n \leq t.
\]

For \(t \geq 0\) we consider the "current life" random variable

\[
\delta_t = t - (T_1 + T_2 + \ldots + T_{N(t)}).
\]

(Of course, \(\delta_t = t\) if \(N(t) = 0\).)

For fixed \(y \geq 0\) we let \(A_y(t) = P(\delta_t \leq y)\). Since

\[
P(\delta_t \leq y \mid T_1 = x) = \begin{cases} 
A_y(t-x) & \text{if } x \leq t \\
1 & \text{if } x > t \text{ and } y \geq t \\
0 & \text{if } x > t \text{ and } y < t,
\end{cases}
\]

it follows that \(A_y(t)\) satisfies the renewal equation

\[
A_y(t) = \int_{[0, \infty]} \left[ \int_{[0,t]} A_y(t-x) \, dF(x) \right] dF(x) \\
= a_y(t) + \int_{[0,t]} A_y(t-x) \, dF(x), \text{ where}
\]

\[
a_y(t) = \begin{cases} 
1 - F(t) & \text{if } 0 \leq t \leq y \\
0 & \text{if } t > y.
\end{cases}
\]

Therefore, from the Basic Renewal Theorem (see [1], page 191), we obtain
\[
\lim_{n \to \infty} A_y(n) = \frac{2}{1 + \frac{1}{r}} \sum_{k=0}^{\infty} a_y(k) = \frac{2}{1 + \frac{1}{r}} \sum_{k=0}^{[y]} (1 - F(k)).
\]

Hence,

\[
\lim_{n \to \infty} P(\text{accumulated sum ever equals } n) = \lim_{n \to \infty} \frac{P(\delta_n = 0)}{1 + \frac{1}{r}} = \lim_{n \to \infty} A_0(n) = 2/(1+r).
\]

REFERENCES
