SOME GENERALIZATIONS OF A COMBINATORIAL
IDENTITY OF L. VIETORIS

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where \( m, n, k \) are integers, with

\[
0 \leq k \leq m - 1 \quad \text{and} \quad n \geq 0.
\]  

(5)

Making use of the definition (3), Vietoris's identity (4) can readily be re-written in its equivalent form:

\[
\begin{pmatrix} m+n \\ n \end{pmatrix} = \sum_{i=0}^{n} \binom{k+i}{i} \binom{m+n-k-i-1}{n-i},
\]

(6)

where, as before, \( m, n, k \) are integers constrained by (5).

A closer examination of the combinatorial identity (6) would suggest the existence of an interesting generalization of Vietoris's result (4) in the form:

\[
\begin{pmatrix} \mu+n \\ n \end{pmatrix} = \sum_{i=0}^{n} \binom{\lambda+i}{i} \binom{\mu-\lambda+n-i-1}{n-i},
\]

(7)

where \( \lambda \) and \( \mu \) are arbitrary complex numbers, and \( n = 0, 1, 2, \ldots \).

Formula (7) can indeed be rewritten in a form analogous to (4) by using Gamma functions.

\[ \underline{2. \ Derivation \ of \ the \ Identity \ (7)} \]

In view of the elementary relationship (2), we have

\[
\begin{pmatrix} \lambda+i \\ i \end{pmatrix} = (-1)^i \binom{-\lambda-1}{i}, \quad i = 0, 1, 2, \ldots,
\]

(8)

and

\[
\begin{pmatrix} \mu-\lambda+n-i-1 \\ n-i \end{pmatrix} = (-1)^{n-i} \binom{\lambda-\mu}{n-i}, \quad 0 \leq i \leq n,
\]

(9)
the right-hand side of the general identity (7) equals

\( (\mu - \lambda)_n F(-n; \lambda + 1; -\mu - n + 1; 1) \), \hspace{1cm} (16)

where the hypergeometric series is finite because \( n \) is a nonnegative integer.

Now apply a special case of Gauss's summation theorem ([3], p. 19) in the form:

\[ F(-n, b; c; 1) = \frac{(c-b)^n}{(c)^n}, \quad n = 0, 1, 2, \ldots, \] \hspace{1cm} (17)

and note from (14) that

\[ \frac{(-\mu-n)_n}{(\lambda-\mu-n+1)_n} = \frac{\lambda+1}{\mu-\lambda} \frac{n}{n}, \quad n = 0, 1, 2, \ldots, \] \hspace{1cm} (18)

\[ \lambda - \mu + 1 \neq 1, 2, 3, \ldots, \]

and (16) immediately yields the left-hand side of the general identity (7) under the (easily removable) constraint that \( \lambda - \mu + 1 \) is not a positive integer.

3. A Basic (or \( q \)-) Extension of the Identity (7)

In terms of the basic (or \( q \)-) number \( [\lambda] \) and basic (or \( q \)-) factorial \( [n]! \) defined by

\[ [\lambda] = \frac{1 - q^\lambda}{1 - q}; \quad [n]! = [1][2][3] \ldots [n], \; [0]! = 1, \] \hspace{1cm} (19)

let the basic (or \( q \)-) binomial coefficient be given by [cf. Equation (1) et seq.]
REFERENCES


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