SOME INEQUALITIES FOR STARLIKE AND CONVEX
FUNCTIONS OF ORDER $\alpha^+$

by

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ABSTRACT

For certain normalized starlike and normalized convex functions of order \( \alpha \) in the unit disk, the authors prove four general theorems which give upper estimates for \( \frac{zf'(z)}{f(z)} \) in terms of \( |f(z)| \). The results presented here provide interesting generalizations of the estimates derived earlier by J.B. Twomey, S. Ruscheweyh and V. Singh. The methods used depend, among other things, on the concept of subordination between analytic functions.

1. INTRODUCTION AND PRELIMINARIES

Let \( \mathcal{S} \) denote the class of functions normalized in the form

\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \). Further, let \( \mathcal{S} \) denote the class of all functions in \( \mathcal{S} \) which are univalent in the unit disk \( \mathbb{D} \).

Then a function \( f(z) \) belonging to the class \( \mathcal{S} \) is said to be starlike of order \( \alpha \) if and only if (cf. [1], [4], and [13]; see also [11])

\[
(1.2) \quad \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{D})
\]

for some \( \alpha \) \( (0 \leq \alpha < 1) \). We denote by \( \mathcal{S}_\alpha^* \) the class of all functions in \( \mathcal{S} \) which are starlike of order \( \alpha \). Throughout this paper, it should be understood that functions such as \( zf'(z)/f(z) \), which have removable singularities as \( z = 0 \), have had these singularities removed in statements like (1.2).

A function \( f(z) \) belonging to the class \( \mathcal{S} \) is said to be convex of order \( \alpha \) if and only if

\[
(1.3) \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{D})
\]

for some \( \alpha \) \( (0 \leq \alpha < 1) \). We denote by \( \mathcal{K}(\alpha) \) the class of all functions in \( \mathcal{S} \) which are convex of order \( \alpha \). Note that \( f(z) \in \mathcal{K}(\alpha) \) if and only if
$zf'(z) \prec \mathcal{S}^*(\alpha)$, and that

\[(1.4) \quad \mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*, \quad \mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K}, \quad \text{and} \quad \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}
\]

for $0 \leq \alpha < 1$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [13], and were studied subsequently by Schild [17], MacGregor [8], Pinchuk [12], Jack [4], and others (cf., e.g., [1] and [11]). The object of the present paper is to continue our study of these classes by deriving various new inequalities for $\left|zf'(z)/f(z)\right|^j$ ($j = 1$ or $2$) in terms of $|f(z)|$. Our methods are based, among other things, upon the concept of subordination between analytic functions.

In order to recall here the aforementioned concept of subordination, let the functions $f(z)$ and $g(z)$ be analytic in the unit disk $\mathbb{U}$. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $h(z)$ analytic in the unit disk $\mathbb{U}$, with $h(0) = 0$ and $|h(z)| < 1$, such that

\[(1.5) \quad f(z) = g(h(z))\]

for $z \in \mathbb{U}$. We denote this subordination by

\[(1.6) \quad f(z) \prec g(z).\]

In particular, if $g(z)$ is univalent in the unit disk $\mathbb{U}$, the subordination (1.6) is equivalent to (cf. [2], p. 190)

\[(1.7) \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).\]

The concept of subordination can be traced back to Lindelöf [5], although Littlewood ([6], [7]) and Rogosinski ([14], [15]) introduced the term and established the basic results involving subordination. Furthermore, Suffridge [20], Hallenbeck and Ruscheweyh [3], and Owa [10], studied various families of subordinate functions. More recently, Srivastava and Owa [19] investigated various interesting properties of the generalized hypergeometric functions by using the concept of subordination.

In our present investigation we shall also need the following results:
LEMMA 1 (Singh [18]). Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha)$. Then
\begin{equation}
\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha + (1-\alpha) \frac{1 - r^2}{r^{1/(1-\alpha)}} |f(z)|^{1/(1-\alpha)}
\end{equation}
and
\begin{equation}
\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \leq \frac{1 + (1-2\alpha)r}{1 - r} + \frac{2r \log \left(\frac{(1-r)^{2(1-\alpha)}}{r^2} |f(z)|\right)}{(1-r)^2 \log \left(\frac{1+r}{1-r}\right)}
\end{equation}
for $|z| = r < 1$.

LEMMA 2 (MacGregor [9]). Let the function $f(z)$ defined by (1.1) be in the class $H(\alpha)$. Then $f(z) \in S^*\left(\beta(\alpha)\right)$, where
\begin{equation}
\beta(\alpha) = \begin{cases} 
\frac{2\alpha - 1}{2(1-2^{1-2\alpha})} & (\alpha \neq \frac{1}{2}) \\
\frac{1}{2 \log 2} & (\alpha = \frac{1}{2}).
\end{cases}
\end{equation}

2. INEQUALITIES FOR THE CLASS $S^*(\alpha)$

We begin by stating our first result as

THEOREM 1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha)$. Then
\begin{equation}
\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{r \log \left(\frac{(1+r)^{2(1-\alpha)}}{r^2} |f(z)|\right)}{(1-r) \log \left(\frac{1+r}{1-r}\right)} + \alpha + 1
\end{equation}
for $|z| = r < 1$. Equality in (2.1) holds true for the function $f(z)$ defined by

\[ f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \]

with $z = r$.

PROOF. Making use of the Herglotz representation (cf. [16], p. 326), we may write

\[ \frac{zf'(z)}{f(z)} - \alpha = (1-\alpha) \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \]

with a certain probability measure $\mu(t)$ on $[0, 2\pi]$. By integration and the addition of $\log(1+r)^{2(1-\alpha)}$ to the resulting equation, we obtain

\[ \log \left( \frac{(1+r)^{2(1-\alpha)}}{r} \right) |f(z)| = 2(1-\alpha) \int_0^{2\pi} \log \left( \frac{1 + \frac{r}{1-r}}{|1-ze^{-it}|} \right) d\mu(t) \]

on taking real parts. Furthermore, we note that (see [21], p. 96)

\[ \frac{1 + z}{1 - z} \leq \frac{2r \log \left( \frac{1 + r}{|1-z|} \right)}{(1-r)\log \left( \frac{1+r}{1-r} \right)} + 1 \]

for $|z| = r < 1$. Consequently, by using (2.3), (2.4) and (2.5), we have

\[ \left| \frac{zf'(z)}{f(z)} \right| \leq (1-\alpha) \int_0^{2\pi} \left| \frac{1 + ze^{-it}}{1 - ze^{-it}} \right| d\mu(t) + \alpha \]

\[ \leq (1-\alpha) \int_0^{2\pi} \left[ \frac{2r \log \left( \frac{1 + r}{|1-ze^{-it}|} \right)}{(1-r)\log \left( \frac{1+r}{1-r} \right)} + 1 \right] d\mu(t) + \alpha \]
r \log \left( \frac{(1+r)^2(1-\alpha)}{r} |f(z)| \right) \\
= \frac{(1-r) \log \left( \frac{1+r}{1-r} \right)}{\log \left( \frac{1-r}{1-r} \right)} + \alpha + 1,

which immediately gives the assertion (2.1).

Finally, it is easily seen that the equality in (2.1) does hold true for the function \( f(z) \) defined by (2.2), and the proof of Theorem 1 is thus completed.

Next we prove

**Theorem 2.** Under the hypotheses of Theorem 1,

\[
(2.7) \quad \left| \frac{zf'(z)}{f(z)} \right|^2 \leq A(\alpha, r) + \frac{4r[1+(1-2\alpha)r^2]}{(1-r^2)^2 \log \left( \frac{1+r}{1-r} \right)} \log \left( \frac{(1-r)^2(1-\alpha)}{r} |f(z)| \right)
\]

for \( |z| = r < 1 \), where

\[
(2.8) \quad A(\alpha, r) = \frac{1}{(1-r^2)^2} \left[ 1 + 4(1-\alpha)r + 6(1-2\alpha)r^2 \\
+ 4(1-\alpha)(1-2\alpha)r^3 + (1-2\alpha)^2 r^4 \right].
\]

**Proof.** For \( f(z) \in \mathfrak{S}^*(\alpha) \), we have

\[
(2.9) \quad \frac{zf'(z)}{f(z)} < \frac{1 - (1-2\alpha)z}{1 - z} \quad (z \in \mathbb{D}),
\]

whence we can write

\[
(2.10) \quad \frac{zf'(z)}{f(z)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} \quad [w(z) \neq 1],
\]

where the function \( w(z) \) is analytic in the unit disk \( \mathbb{D} \), with

\[
(2.11) \quad w(0) = 0 \text{ and } |w(z)| < 1.
\]

Furthermore, we find from (2.10) that
(2.12) \[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1-2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1-\alpha)r}{1 - r^2},
\]
is, that

(2.13) \[
\left| \frac{zf'(z)}{f(z)} \right|^2 \leq \frac{2[1+(1-2\alpha)r^2]}{1 - r^2} \text{Re} \left( \frac{zf'(z)}{f(z)} \right) - \left( \frac{1 - (1-2\alpha)r^2}{1 - r^2} \right)^2.
\]

Now the inequality (2.7) follows from (2.13) in view of Lemma 1.

REMARK 1. For \( \alpha = 0 \), Theorem 1 reduces to a result due to Twomey ([21], p. 95, Equation (3)), while Theorem 2 yields another result proved recently by Ruscheweyh and Singh ([16], p. 328, Theorem 4).

3. INEQUALITIES FOR THE CLASS \( \mathcal{K}(\alpha) \)

By combining Theorem 1 and Lemma 2 in a straightforward manner, we can establish

THEOREM 3. Let the function \( f(z) \) defined by (1.1) be in the class \( \mathcal{K}(\alpha) \). Then

(3.1) \[
\left| \frac{zf'(z)}{f(z)} \right| \leq \frac{r \log \left( \frac{(1+r)^2[1-\beta(\alpha)]}{r} \left| f(z) \right| \right)}{(1-r)\log \left( \frac{1+r}{1-r} \right)} + \beta(\alpha) + 1
\]

for \( |z| = r < 1 \), where \( \beta(\alpha) \) is given by (1.10). Equality in (3.1) holds true for the function \( f(z) \) defined by

(3.2) \[
f(z) = \frac{z}{(1-z)^2[1-\beta(\alpha)]}
\]

with \( z = r \).
From Theorem 2 and Lemma 2, we similarly deduce

THEOREM 4. Under the hypotheses of Theorem 3,

\[ \left| \frac{zf'(z)}{f(z)} \right|^2 \leq A(\beta(\alpha), r) \]

\[ + \frac{4r[1+2\beta(\alpha)]r^2}{(1-r^2)^2} \left[ \log\frac{1+r}{1-r} \right] \]

\[ \leq \log \left( \frac{(1-r)^2[1-\beta(\alpha)]}{r} \right) \left\{ \frac{1}{|f(z)|} \right\} \]

for \(|z| = r < 1\), where \(\beta(\alpha)\) is defined by (1.10), and \(A(\beta(\alpha), r)\) is given by (2.8) with, of course, \(\beta(\alpha)\) in place of \(\alpha\).

REMARK 2. In view of the last relationship in (1.4), Theorem 3 and Theorem 4 provide interesting improvements over the inequalities derivable for the class \(\mathcal{K}(\alpha)\) from Theorem 1 and Theorem 2, respectively.

REFERENCES


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