

**SOME QUESTIONS AND CONJECTURES IN THE
THEORY OF UNIVALENT FUNCTIONS**

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Abstract

The main object of this paper is first to answer a question of Campbell and Singh in the affirmative, and then show that Komatu's conjecture and Thomas' conjecture are false at least in some cases.

1. Introduction

Let \mathcal{S} denote the class of *normalized* analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are univalent in the *open* unit disk \mathcal{U} . Also let \mathcal{S}_R denote its subclass consisting of functions with real coefficients. The set of all odd functions in \mathcal{S} is denoted by $\mathcal{S}^{(2)}$.

A function $f \in \mathcal{S}$ is said to be starlike of order α , denoted by $f \in \mathcal{S}^*(\alpha)$, if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < 1; \quad z \in \mathcal{U}).$$

Let

$$\mathcal{S}^*(0) = \mathcal{S}^* \quad \text{and} \quad \mathcal{K} = \{f : zf'(z) \in \mathcal{S}^*\}.$$

In this paper we first answer a question of Campbell and Singh [1] in the affirmative. We then show that Komatu's conjecture [5] and Thomas' conjecture [7, p. 166] are false at least in some cases.

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2. A Question of Campbell and Singh

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}.$$

Define their integral convolution $f \otimes g$ by

$$F(z) = (f \otimes g)(z) := z + \sum_{n=2}^{\infty} a_n b_n \frac{z^n}{n}.$$

Campbell and Singh [1] asked the following question:

Do there exist univalent functions $f(z)$ and $g(z)$ such that the coefficients of

$$F(z) = (f \otimes g)(z) = z + c_2 z^2 + c_3 z^3 + \dots$$

satisfy

$$|c_3 - c_2^2| > 1?$$

Here we give an explicit example which leads to an affirmative answer to this question.

Let α be a real number with $0 < |\alpha| < \frac{1}{2}\pi$. It is known that the function

$$f(z) = z(1 - z)^{-2e^{i\alpha} \cos \alpha} \tag{1}$$

is α -spirallike but not close-to-convex [2, p. 72].

Consider the rotation of the function (1), and set

$$f(z) = z(1 - e^{-i\theta} z)^{-2e^{i\alpha} \cos \alpha}. \tag{2}$$

Now choose

$$\alpha = \frac{1}{2} \arg(15 + 8i)$$

and

$$\theta = \arg(15 + 8i), \tag{3}$$

and expand the corresponding function (2) in powers of z ; we thus obtain

$$f(z) = z + \left(\frac{32}{17} - \frac{8}{17}i\right) z^2 + \left(\frac{688}{289} - \frac{444}{289}i\right) z^3 + \dots \quad (4)$$

We consider the integral convolution of the function (4) with itself, that is,

$$F(z) = (f \otimes f)(z) = z + c_2 z^2 + c_3 z^3 + \dots;$$

then

$$c_2 = \frac{32}{289} (15 - 8i)$$

and

$$c_3 = \frac{16}{250563} (17263 - 38184i). \quad (5)$$

Therefore

$$\begin{aligned} |c_3 - c_2^2| &= \frac{1}{250563} |218384 - 126336i| \\ &= 1.0069 \dots > 1. \end{aligned} \quad (6)$$

We denote the set of all α -spirallike functions by $\mathcal{SP}(\alpha)$ ($-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$). It is known that the class of close-to-convex functions and \mathcal{S}_R are not closed under the operation \otimes [2, p. 247]. In contrast with Ruscheweyh and Sheil-Small's proof (*cf.* [8]) of the Pólya-Schoenberg conjecture that $\mathcal{S}^* = \mathcal{SP}(0)$ is closed under the operation \otimes , we should like to point out that the above discussion yields the following result.

Theorem 1. *For each α ($\alpha \neq 0$; $|\alpha| < \alpha_0 = 0.3138 \dots (= 17.986 \dots \text{ degrees})$), there exist functions $f(z)$ and $g(z)$ in the class $\mathcal{SP}(\alpha)$ such that the coefficients of*

$$F(z) = (f \otimes g)(z) = z + c_2 z^2 + c_3 z^3 + \dots$$

satisfy

$$|c_3 - c_2^2| > 1,$$

where

$$(\tan \alpha_0)^2 = 0.1054 \dots$$

is the smallest positive root of the equation:

$$9x^3 + 36x^2 + 53x - 6 = 0. \quad (7)$$

In fact, if we let $f(z) = g(z)$ be the function given by (2) with $\theta = 2\alpha$ and $\gamma = \tan \alpha$, then

$$|c_3 - c_2^2| = \frac{\{4\gamma^2 + (1 + 3^{-1}\gamma^2)^2\}^{\frac{1}{2}}}{(1 + \gamma^2)^2}, \quad (8)$$

and the assertion of Theorem 1 is readily established by letting $\gamma^2 = x$.

3. A Conjecture of Komatu

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$$

and define the function $f_\lambda(z)$ by

$$f_\lambda(z) = z + \sum_{n=2}^{\infty} a_n n^{-\lambda} z^n. \quad (9)$$

Komatu [5] proved that, if $f \in \mathcal{S}$, then $f_\lambda \in \mathcal{S}^*$ at least for $\lambda \geq \lambda_0$, where $\lambda_0 \in (3, 4)$ is the unique root of the equation $\zeta(\lambda - 2) = 2$ (ζ being the Riemann zeta function), and conjectured that

- (i) If $f \in \mathcal{S}$, then $f_\lambda \in \mathcal{S}$ at least for $\lambda \geq 1$;
- (ii) If $f \in \mathcal{K}$ (or, more generally, $f \in \mathcal{S}^*$), then $f_\lambda \in \mathcal{K}$ at least for $\lambda \geq 1$.

Lewis [6] showed that Conjecture (ii) is true. In the case $\lambda = 1$, Conjecture (i) reduces to the Biernacki conjecture which is false [2, p. 257]. We shall show that Conjecture (i) is also false in the case $\lambda = 2$. Further remarks on Komatu's conjectures can be found in the work of Srivastava and Owa [9, p. 82].

Theorem 2. *There exists a function $f \in \mathcal{S}$ such that $f_1(z)$ and $f_2(z) \notin \mathcal{S}$.*

Proof. For the function (1), one can obtain

$$\begin{aligned} f_1(z) &= e^{-2i\alpha} \left\{ (1-z)^{-e^{2i\alpha}} - 1 \right\} \\ &= e^{-2i\alpha} \left\{ \exp(-e^{2i\alpha} \log(1-z)) - 1 \right\}. \end{aligned} \quad (10)$$

The geometric property of $\exp(\beta \log(1-z))$ shows that $f_1(z)$ is univalent in \mathcal{U} if and only if β is either in the closed disk $|\beta - 1| \leq 1$ or in the closed disk $|\beta + 1| \leq 1$ (cf. [3, Vol. II, p. 150]).

Since $\beta = -e^{2i\alpha}$, we deduce that, if

$$\frac{\pi}{6} < |\alpha| < \frac{\pi}{3}, \quad (11)$$

then $f_1(z) \notin \mathcal{S}$.

From the definition (9), we have

$$z f_2'(z) = f_1(z). \quad (12)$$

For the function (1) with $\alpha = \frac{\pi}{4}$, we have

$$z f_2'(z) = f_1(z) = -i\{(1-z)^{-i} - 1\}.$$

Hence

$$z_0 f_2'(z_0) = 0 \quad \text{at} \quad z_0 = 1 - e^{-2\pi} \in \mathcal{U}.$$

This shows that $f_2(z) \notin \mathcal{S}$, which completes the proof of Theorem 2.

4. A Conjecture of Thomas

Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}.$$

The Hankel determinant of the coefficients of $f(z)$ is denoted by $H_q(n)$, where

$$H_1(n) = a_n a_{n+2} - a_{n+1}^2.$$

For $f \in \mathcal{S}$, Thomas [7, p. 166] conjectured that

$$|H_1(n)| \leq 1 \quad (n = 2, 3, 4, \dots).$$

It follows from this conjecture that, if

$$h(z) = z + \sum_{k=1}^{\infty} c_{2k+1} z^{2k+1} \in \mathcal{S}_R \cap \mathcal{S}^{(2)}, \quad (13)$$

then

$$|c_{2k+1}| \leq 1 \quad \text{and} \quad |c_{2k+1} c_{2k+3}| \leq 1 \quad (k = 1, 2, 3, \dots).$$

Recently, Jakubowski [4] proved that, if $h \in \mathcal{S}_R \cap \mathcal{S}^{(2)}$, then $|c_3 c_5| \leq 1$. However, since the elementary counterexample of Shaeffer and Spencer [2, p. 107] shows that, for $h \in \mathcal{S}_R \cap \mathcal{S}^{(2)}$, the above conclusion is false in the cases when $k \geq 2$. Therefore, Thomas' conjecture is also false in the cases when $n \geq 4$.

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