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Abstract

The main object of the present paper is to introduce and study rather systematically two new subclasses $\mathcal{R}(p, \lambda, \alpha)$ and $\mathcal{C}(p, \lambda, \alpha)$ of meromorphically univalent functions with positive and negative coefficients, respectively. We first obtain a necessary and sufficient condition for a function to be in each of these classes. We then investigate the meromorphically starlikeness and meromorphically convexity of functions belonging to the classes $\mathcal{R}(p, \lambda, \alpha)$ and $\mathcal{C}(p, \lambda, \alpha)$. Several other properties and characteristics of functions in these classes are also derived.

1. Introduction and Definitions

Let $\mathcal{M}^{(A,B)}_p$ denote the class of functions $f(z)$ of the form:

$$f(z) := \frac{A}{z} + B \sum_{n=p}^{\infty} a_n z^n \quad (a_n \geq 0; \ AB \neq 0; \ p \in \mathbb{N} := \{1, 2, 3, \cdots\}),$$

(1.1)

which are analytic and univalent in the punctured unit disk

$$\mathcal{D} := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\},$$

and which have a simple pole at the origin ($z = 0$) with residue $A$ there. A function $f(z) \in \mathcal{M}^{(A,B)}_p$ is said to be in the class $\mathcal{M}^{(A,B)}_p(\alpha; \lambda, \mu)$ if it also satisfies the inequality:

$$\Re \left\{ \frac{(1 - 2\lambda) z f''(z) - \lambda z^2 f'''(z)}{(\mu - 1) f'(z) + \mu z f'(z)} \right\} > \alpha \quad (z \in \mathcal{D}; \ 0 \leq \alpha < 1)$$

(1.2)

for some suitably restricted real parameters $\lambda$ and $\mu$.

Two important subclasses of the class $\mathcal{M}^{(A,B)}_p(\alpha; \lambda, \mu)$ are worthy of mention. First of all, for $\lambda = \mu = 0$, the class $\mathcal{M}^{(A,B)}_p(\alpha; 0, 0)$ consists of meromorphically starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) (with positive or negative coefficients depending upon the value of the nonzero constant $B$). On the other hand, when $\lambda = \mu = 1$, the class $\mathcal{M}^{(A,B)}_p(\alpha; 1, 1)$ would consist of meromorphically convex functions of order $\alpha$ ($0 \leq \alpha < 1$) (with positive or
negative coefficients depending upon the value of the nonzero constant $B$ (cf., e.g., Duren [4] and Goodman [5]; see also Srivastava and Owa [6]). Some other subclasses of the class $M_p^{(A,B)}$ were studied recently by (for example) Cho et al. ([2] and [3]) and Altintaş et al. [1].

In the present paper we propose to investigate various interesting properties and characteristics of functions belonging to the following subclasses $\mathcal{R}(p, \lambda, \alpha)$ and $\mathcal{C}(p, \lambda, \alpha)$ of the general class $M_p^{(A,B)}(\alpha; \lambda, \mu)$ which we introduced above:

$$\mathcal{R}(p, \lambda, \alpha) := M_p^{(-1,1)}(\alpha; \lambda, \lambda)$$

$$\begin{align*}
(p \in \mathbb{N}; & \lambda \geq 1/(p + 1); 0 \leq \alpha < 1) \\
\end{align*}$$

and

$$\mathcal{C}(p, \lambda, \alpha) := M_p^{(1,-1)}(\alpha; 1, 1 - \lambda)$$

$$\begin{align*}
(p \in \mathbb{N}; & 0 \leq \lambda \leq p; 0 \leq \alpha < 1). \\
\end{align*}$$

Clearly, the class $\mathcal{R}(p, 1, \alpha)$ consists of meromorphically convex functions $f(z)$ of order $\alpha$ $(0 \leq \alpha < 1)$ with positive coefficients, given by

$$f(z) = -\frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \quad (a_n \geq 0; \ p \in \mathbb{N}).$$

(1.5)

Furthermore, since the condition $\lambda \geq 1/(p + 1)$ $(p \in \mathbb{N})$ is not actually a requirement for the definition (1.3), we may set $\lambda = 0$ in (1.3) and observe that the class $\mathcal{R}(p, 0, \alpha)$ consists of meromorphically starlike functions $f(z)$ of order $\alpha$ $(0 \leq \alpha < 1)$ with positive coefficients, given by (1.5). The class $\mathcal{C}(p, 0, \alpha)$, on the other hand, consists of meromorphically convex functions $f(z)$ of order $\alpha$ $(0 \leq \alpha < 1)$ with negative coefficients, given by

$$f(z) = \frac{1}{z} - \sum_{n=p}^{\infty} a_n z^n \quad (a_n \geq 0; \ p \in \mathbb{N}).$$

(1.6)
2. Coefficient Inequalities and Inclusion Properties

A necessary and sufficient condition for a function \( f(z) \), given by (1.5), to be in the class \( \mathcal{R}(p, \lambda, \alpha) \) is provided by

**Theorem 1.** Let a function \( f(z) \) be in the class \( \mathcal{M}_p^{(-1,1)} \). Then the function \( f(z) \) belongs to the class \( \mathcal{R}(p, \lambda, \alpha) \) if and only if

\[
\sum_{n=p}^{\infty} (n + \alpha)(n\lambda + \lambda - 1) a_n \leq 1 - \alpha \quad (0 \leq \alpha < 1; \ \lambda \geq 1/(p + 1); \ p \in \mathbb{N}). \tag{2.1}
\]

The result is sharp.

**Proof.** First of all, suppose that the function \( f(z) \), given by (1.5), is in the class \( \mathcal{R}(p, \lambda, \alpha) \). Then it is easily seen from (1.5), (1.3), and (1.2) that

\[
\Re \left\{ \frac{1 - \sum_{n=p}^{\infty} n(n\lambda + \lambda - 1) a_n z^{n+1}}{1 + \sum_{n=p}^{\infty} (n\lambda + \lambda - 1) a_n z^{n+1}} \right\} > \alpha \tag{2.2}
\]

\((z \in \mathcal{D}; \ 0 \leq \alpha < 1; \ \lambda \geq 1/(p + 1); \ p \in \mathbb{N}).\)

If we choose values of \( z \) on the real axis and let \( z \to 1^- \) through real values, we obtain the inequality:

\[
1 - \sum_{n=p}^{\infty} n(n\lambda + \lambda - 1) a_n \geq \alpha \tag{2.3}
\]

\((0 \leq \alpha < 1; \ \lambda \geq 1/(p + 1); \ p \in \mathbb{N}),\)

which readily yields the assertion (2.1) of Theorem 1.

Conversely, let us suppose that the inequality (2.1) holds true and let

\[
z \in \partial \mathcal{D} := \{z : z \in \mathbb{C} \text{ and } |z| = 1\}. \tag{2.4}
\]
Then we find from the definition (1.5) that

\[
\left| \frac{(1 - 2\lambda) z f'(z) - \lambda z^2 f''(z)}{(\lambda - 1) f(z) + \lambda z f'(z)} - 1 \right|
\]

\[
\leq \frac{\sum_{n=p}^{\infty} (n + 1)(n\lambda + \lambda - 1) a_n |z|^{n+1}}{1 + \sum_{n=p}^{\infty} (n\lambda + \lambda - 1) a_n |z|^{n+1}}
\]

\[
= \frac{\sum_{n=p}^{\infty} (n + 1)(n\lambda + \lambda - 1) a_n}{1 + \sum_{n=p}^{\infty} (n\lambda + \lambda - 1) a_n} \quad \text{(since } z \in \partial D) \quad (2.5)
\]

\[
\leq 1 - \alpha \quad (0 \leq \alpha < 1; \lambda \geq 1/(p + 1); p \in \mathbb{N}),
\]

where we have made use of the inequality (2.1). Thus, by the maximum modulus theorem, we conclude from (2.5) that

\[f(z) \in \mathcal{R}(p, \lambda, \alpha).\]

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being given by

\[f(z) = \frac{1}{z} + \frac{1 - \alpha}{p + \alpha(p\lambda + \lambda - 1)} z^p \quad (p \in \mathbb{N}). \quad (2.6)\]

In precisely the same manner, we can prove

**Theorem 2.** Let a function \( f(z) \) be in the class \( \mathcal{M}_p^{1,-1} \). Then the function \( f(z) \) belongs to the class \( \mathcal{C}(p, \lambda, \alpha) \) if and only if

\[
\sum_{n=p}^{\infty} \{n(n - \lambda \alpha) + \alpha(n - \lambda)\} a_n \leq 1 - \alpha \quad (0 \leq \alpha < 1; \ 0 \leq \lambda \leq p; \ p \in \mathbb{N}). \quad (2.7)
\]

The result is sharp, the extremal function being given by

\[f(z) = \frac{1}{z} - \frac{1 - \alpha}{p(p - \lambda \alpha) + \alpha(p - \lambda)} z^p \quad (p \in \mathbb{N}). \quad (2.8)\]

Corollary 1 and Corollary 2 below are rather immediate consequences of Theorem 1 and Theorem 2, respectively:
Corollary 1. If \( f(z) \in \mathcal{R}(p, \lambda, \alpha) \), then

\[
a_n \leq \frac{1 - \alpha}{(n + \alpha)(n\lambda + \lambda - 1)} \quad (n \geq p; \ p \in \mathbb{N}). \tag{2.9}
\]

Corollary 2. If \( f(z) \in \mathcal{C}(p, \lambda, \alpha) \), then

\[
a_n \leq \frac{1 - \alpha}{n(n - \lambda\alpha) + \alpha(n - \lambda)} \quad (n \geq p; \ p \in \mathbb{N}). \tag{2.10}
\]

Next we prove

**Theorem 3.** Let the function \( f(z) \) defined by (1.5) and the function \( g(z) \) defined by

\[
g(z) = -\frac{1}{z} + \sum_{n=p}^{\infty} b_n z^n \quad (b_n \geq 0; \ p \in \mathbb{N}) \tag{2.11}
\]

be in the same class \( \mathcal{R}(p, \lambda, \alpha) \). Then the function \( h(z) \) defined by

\[
h(z) := (1 - \xi) f(z) + \xi g(z) = -\frac{1}{z} + \sum_{n=p}^{\infty} c_n z^n \tag{2.12}
\]

\[
(c_n := (1 - \xi) a_n + \xi b_n \geq 0; \ 0 \leq \xi \leq 1)
\]

is also in the class \( \mathcal{R}(p, \lambda, \alpha) \).

**Proof.** Suppose that each of the functions \( f(z) \) and \( g(z) \), involved in Theorem 3, is in the class \( \mathcal{R}(p, \lambda, \alpha) \). Then, making use of (2.1) and (2.12), we observe that

\[
\sum_{n=p}^{\infty} (n + \alpha)(n\lambda + \lambda - 1) c_k
\]

\[
= (1 - \xi) \sum_{n=p}^{\infty} (n + \alpha)(n\lambda + \lambda - 1) a_n + \xi \sum_{n=p}^{\infty} (n + \alpha)(n\lambda + \lambda - 1) b_n \tag{2.13}
\]

\[
\leq (1 - \xi)(1 - \alpha) + \xi(1 - \alpha)
\]

\[
= 1 - \alpha \quad (0 \leq \alpha < 1; \ \lambda \geq 1/(p + 1); \ p \in \mathbb{N}; \ 0 \leq \xi \leq 1),
\]

which evidently completes the proof of Theorem 3.

Similarly, by employing Theorem 2 in place of Theorem 1, we obtain
Theorem 4. Let the function $f(z)$ defined by (1.6) and the function $g(z)$ defined by

$$g(z) = \frac{1}{z} - \sum_{n=p}^{\infty} b_n z^n \quad (b_n \geq 0; \ p \in \mathbb{N})$$  \hspace{1cm} (2.14)

be in the same class $C(p, \lambda, \alpha)$. Then the function $h(z)$ defined by

$$h(z) := (1 - \xi) f(z) + \xi g(z) = \frac{1}{z} - \sum_{n=p}^{\infty} c_n z^n$$

$$ (c_n := (1 - \xi) a_n + \xi b_n \geq 0; \ 0 \leq \xi \leq 1)$$

is also in the class $C(p, \lambda, \alpha)$.

The following results (Theorem 5 and Theorem 6) involve the quasi-Hadamard product (or convolution) of functions belonging to the classes $R(p, \lambda, \alpha)$ and $C(p, \lambda, \alpha)$, respectively. We first state

Theorem 5. Let the function $f(z)$ defined by (1.5) and the function $g(z)$ defined by (2.11) be in the same class $R(p, \lambda, \alpha)$. Then

$$(f \ast g)(z) \in R(p, \lambda, \beta),$$

where $(f \ast g)(z)$ denotes the quasi-Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by

$$(f \ast g)(z) := -\frac{1}{z} + \sum_{n=p}^{\infty} a_n b_n z^n \quad (a_n \geq 0; \ b_n \geq 0; \ p \in \mathbb{N})$$  \hspace{1cm} (2.16)

and

$$\beta \leq 1 - \frac{(p + 1)(1 - \alpha)^2}{(p \lambda + \lambda - 1)(p + \alpha)^2 + (1 - \alpha)^2} \quad (0 \leq \alpha < 1; \ p \in \mathbb{N}).$$  \hspace{1cm} (2.17)

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = -\frac{1}{z} + \frac{1 - \alpha}{(p + \alpha)(p \lambda + \lambda - 1)} z^p \quad (p \in \mathbb{N}).$$  \hspace{1cm} (2.18)

Proof. Applying Theorem 1 to the functions $f(z)$ and $g(z)$, we obtain

$$\sum_{n=p}^{\infty} \frac{(n + \alpha)(n \lambda + \lambda - 1)}{1 - \alpha} a_n \leq 1 \quad (0 \leq \alpha < 1; \ \lambda \geq 1/(p + 1); \ p \in \mathbb{N})$$  \hspace{1cm} (2.19)
and
\[ \sum_{n=p}^{\infty} \frac{(n + \alpha)(n\lambda + \lambda - 1)}{1 - \alpha} b_n \leq 1 \quad (0 \leq \alpha < 1; \lambda \geq 1/(p+1); p \in \mathbb{N}), \] (2.20)
respectively.

In order to prove Theorem 5, it is sufficient to find the largest \( \beta \) such that
\[ \sum_{n=p}^{\infty} \frac{(n + \beta)(n\lambda + \lambda - 1)}{1 - \beta} a_n b_n \leq 1 \quad (0 \leq \beta < 1; \lambda \geq 1/(p+1); n \in \mathbb{N}). \] (2.21)

Indeed, in view of the Cauchy-Schwarz inequality, we find from (2.19) and (2.20) that
\[ \sum_{n=p}^{\infty} \frac{(n + \alpha)(n\lambda + \lambda - 1)}{1 - \alpha} \sqrt{a_n b_n} \leq 1 \quad (0 \leq \alpha < 1; \lambda \geq 1/(p+1); p \in \mathbb{N}). \] (2.22)

Therefore, the inequality (2.21) holds true if
\[ \sqrt{a_n b_n} \leq \frac{(1 - \beta)(n + \alpha)}{(1 - \alpha)(n + \beta)} \quad (n \geq p; p \in \mathbb{N}), \] (2.23)
that is, if
\[ \frac{1 - \alpha}{(n + \alpha)(n\lambda + \lambda - 1)} \leq \frac{(1 - \beta)(n + \alpha)}{(1 - \alpha)(n + \beta)} \quad (n \geq p; p \in \mathbb{N}), \] (2.24)
which readily yields
\[ \beta \leq 1 - \frac{(n + 1)(1 - \alpha)^2}{(n\lambda + \lambda - 1)(n + \alpha)^2 + (1 - \alpha)^2} \quad (n \geq p; p \in \mathbb{N}). \] (2.25)

Finally, letting
\[ \Phi(n) := 1 - \frac{(n + 1)(1 - \alpha)^2}{(n\lambda + \lambda - 1)(n + \alpha)^2 + (1 - \alpha)^2} \quad (n \geq p; p \in \mathbb{N}), \] (2.26)
we see that \( \Phi(n) \) is an increasing function of \( n \). This shows, in conjunction with the inequality (2.25), that
\[ \beta \leq \Phi(p) = 1 - \frac{(p + 1)(1 - \alpha)^2}{(p\lambda + \lambda - 1)(p + \alpha)^2 + (1 - \alpha)^2} \quad (0 \leq \alpha < 1; \lambda \geq 1/(p+1); p \in \mathbb{N}), \] (2.27)
which completes the proof of Theorem 5.
A similar consequence of Theorem 2 may be stated as

**Theorem 6.** Let the function \( f(z) \) defined by (1.6) and the function \( g(z) \) defined by (2.14) be in the same class \( C(p, \lambda, \alpha) \). Then

\[
(f \ast g)(z) \in C(p, \lambda, \beta),
\]

where \((f \ast g)(z)\) denotes the quasi-Hadamard product (or convolution) of \( f(z) \) and \( g(z) \), defined by

\[
(f \ast g)(z) := \frac{1}{z} - \sum_{n=p}^{\infty} a_n b_n z^n \quad (a_n \geq 0; \; b_n \geq 0; \; p \in \mathbb{N}) \quad (2.28)
\]

and

\[
\beta \leq 1 - \frac{(p - \lambda - p\lambda + p^2)(1 - \alpha)^2}{[p(p - \lambda\alpha) + \alpha(p - \lambda)]^2 + (p - \lambda - p\lambda)(1 - \alpha)^2} \quad (0 \leq \alpha < 1; \; p \in \mathbb{N}). \quad (2.29)
\]

The result is sharp for the functions \( f(z) \) and \( g(z) \) given by

\[
f(z) = g(z) = \frac{1}{z} - \frac{1 - \alpha}{p(p - \lambda\alpha) + \alpha(p - \lambda)} z^p \quad (p \in \mathbb{N}). \quad (2.30)
\]

3. Growth and Distortion Theorems

The assertion (2.1) of Theorem 1 readily yields the following coefficient inequalities for a function \( f(z) \) belonging to the class \( \mathcal{R}(p, \lambda, \alpha) \):

\[
\sum_{n=p}^{\infty} a_n \leq \frac{1 - \alpha}{(p + \alpha)(p\lambda + \lambda - 1)} \quad (0 \leq \alpha < 1; \; \lambda > 1/(p + 1); \; p \in \mathbb{N}) \quad (3.1)
\]

and

\[
\sum_{n=p}^{\infty} n a_n \leq \frac{p(1 - \alpha)}{(p + \alpha)(p\lambda + \lambda - 1)} \quad (0 \leq \alpha < 1; \; \lambda > 1/(p + 1); \; p \in \mathbb{N}). \quad (3.2)
\]

By applying the coefficient inequalities (3.1) and (3.2), it is not difficult to prove
Theorem 7. If \( f(z) \in \mathcal{R}(p, \lambda, \alpha) \), then
\[
\frac{1}{|z|} - \frac{1 - \alpha}{(p + \alpha)(p\lambda + \lambda - 1)} |z|^p \leq |f(z)|
\]
\[
\leq \frac{1}{|z|} + \frac{1 - \alpha}{(p + \alpha)(p\lambda + \lambda - 1)} |z|^p
\]
(\( z \in \mathcal{D}; \ 0 \leq \alpha < 1; \ \lambda > 1/(p + 1); \ p \in \mathbb{N} \))

and
\[
\frac{1}{|z|^2} - \frac{p(1 - \alpha)}{(p + \alpha)(p\lambda + \lambda - 1)} |z|^{p-1} \leq |f'(z)|
\]
\[
\leq \frac{1}{|z|^2} + \frac{p(1 - \alpha)}{(p + \alpha)(p\lambda + \lambda - 1)} |z|^{p-1}
\]
(\( z \in \mathcal{D}; \ 0 \leq \alpha < 1; \ \lambda > 1/(p + 1); \ p \in \mathbb{N} \)).

Each of these results is sharp for the function \( f(z) \) given by (2.6).

Similarly, by applying analogous coefficient inequalities derivable from the assertion (2.7) of Theorem 2, we obtain

Theorem 8. If \( f(z) \in \mathcal{C}(p, \lambda, \alpha) \), then
\[
\frac{1}{|z|} - \frac{1 - \alpha}{p(p - \lambda\alpha) + \alpha(p - \lambda)} |z|^p \leq |f(z)|
\]
\[
\leq \frac{1}{|z|} + \frac{1 - \alpha}{p(p - \lambda\alpha) + \alpha(p - \lambda)} |z|^p
\]
(\( z \in \mathcal{D}; \ 0 \leq \alpha < 1; \ 0 \leq \lambda \leq p; \ p \in \mathbb{N} \))

and
\[
\frac{1}{|z|^2} - \frac{p(1 - \alpha)}{p(p - \lambda\alpha) + \alpha(p - \lambda)} |z|^{p-1} \leq |f'(z)|
\]
\[
\leq \frac{1}{|z|^2} + \frac{p(1 - \alpha)}{p(p - \lambda\alpha) + \alpha(p - \lambda)} |z|^{p-1}
\]
(\( z \in \mathcal{D}; \ 0 \leq \alpha < 1; \ 0 \leq \lambda \leq p; \ p \in \mathbb{N} \)).

Each of these results is sharp for the function \( f(z) \) given by (2.8).

A number of simpler distortion properties can easily be deduced from Theorem 7 and Theorem 8 by suitably specializing the parameter \( \lambda \) occurring in the assertions (3.3) to (3.6).
4. Meromorphically Starlikeness and Meromorphically Convexity of Functions in the Classes $R(p, \lambda, \alpha)$ and $C(p, \lambda, \alpha)$

We begin by proving

**Theorem 9.** If $f(z) \in R(p, \lambda, \alpha)$, then $f(z)$ is meromorphically starlike of order $\gamma$ ($0 \leq \gamma < 1$) in

$$0 < |z| < r_1(p, \lambda, \alpha, \gamma) := \inf_n \left\{ \frac{(1 - \gamma)(n + \alpha)(n\lambda + \lambda - 1)}{(1 - \alpha)(n - \gamma + 2)} \right\}^{1/(n+1)} \quad (n \geq p; \ p \in \mathbb{N}).$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = -\frac{1}{z} + \frac{1 - \alpha}{(n + \alpha)(n\lambda + \lambda - 1)} z^n \quad (n \geq p; \ p \in \mathbb{N}).$$

**Proof.** We must show that

$$\left| -\frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \gamma$$

$$(0 < |z| < r_1(p, \lambda, \alpha, \gamma); \ f(z) \in R(p, \lambda, \alpha); \ 0 \leq \gamma < 1).$$

Indeed we find from the definition (1.5) that

$$\left| -\frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{n=p}^{\infty} \frac{(n + 1) a_n |z|^{n+1}}{1 - \frac{(n - \gamma + 2)|z|^{n+1}}{1 - \gamma}} \quad (n \geq p)$$

$$\leq 1 - \gamma,$$

provided that

$$\frac{(n - \gamma + 2)|z|^{n+1}}{1 - \gamma} \leq \frac{(n + \alpha)(n\lambda + \lambda - 1)}{1 - \alpha} \quad (0 \leq \alpha < 1; \ 0 \leq \gamma < 1; \ \lambda > 1/(p + 1); \ p \in \mathbb{N}).$$

Solving this last equation (4.5) for $|z|$, we obtain the inequality (4.1), and the proof of Theorem 9 is thus completed.
Since a function \( f(z) \in \mathcal{M}_p^{(-1,1)} \) is meromorphically convex of order \( \gamma \) (0 \( \leq \gamma < 1 \)) if and only if \( zf'(z) \) is meromorphically starlike of order \( \gamma \) (0 \( \leq \gamma < 1 \)), an immediate consequence of Theorem 9 may be stated as

**Theorem 10.** If \( f(z) \in \mathcal{R}(p, \lambda, \alpha) \), then \( f(z) \) is meromorphically convex of order \( \gamma \) (0 \( \leq \gamma < 1 \)) in

\[
0 < |z| < r_2(p, \lambda, \alpha, \gamma) := \inf_n \left\{ \frac{(1 - \gamma)(n + \alpha)(n\lambda + \lambda - 1)}{n(1 - \alpha)(n - \gamma + 2)} \right\}^{1/(n+1)} \quad (n \geq p; \ p \in \mathbb{N}).
\]

The result is sharp for the functions \( f(z) \) given by (4.2).

The derivations of Theorem 9 and Theorem 10 can be applied *mutatis mutandis* in order to obtain the following results for the class \( \mathcal{C}(p, \lambda, \alpha) \).

**Theorem 11.** If \( f(z) \in \mathcal{C}(p, \lambda, \alpha) \), then \( f(z) \) is meromorphically starlike of order \( \delta \) (0 \( \leq \delta < 1 \)) in

\[
0 < |z| < r_3(p, \lambda, \alpha, \delta) := \inf_n \left\{ \frac{(1 - \delta)[n(n - \lambda\alpha) + \alpha(n - \lambda)]}{(1 - \alpha)(n - \delta + 2)} \right\}^{1/(n+1)} \quad (n \geq p; \ p \in \mathbb{N}).
\]

The result is sharp for the functions \( f(z) \) given by

\[
f(z) = \frac{1}{z} - \frac{1 - \alpha}{n(n - \lambda\alpha) + \alpha(n - \lambda)} z^n \quad (n \geq p; \ p \in \mathbb{N}).
\]

**Theorem 12.** If \( f(z) \in \mathcal{C}(p, \lambda, \alpha) \), then \( f(z) \) is meromorphically convex of order \( \delta \) (0 \( \leq \delta < 1 \)) in

\[
0 < |z| < r_4(p, \lambda, \alpha, \delta) := \inf_n \left\{ \frac{(1 - \delta)[n(n - \lambda\alpha) + \alpha(n - \lambda)]}{n(1 - \alpha)(n - \delta + 2)} \right\}^{1/(n+1)} \quad (n \geq p; \ p \in \mathbb{N}).
\]

The result is sharp for the functions \( f(z) \) given by (4.8).
Meromorphically starlikeness and meromorphically convexity of functions belonging to several simpler classes can indeed be deduced by suitably specializing the parameter $\lambda$ occurring in Theorems 9 to 12 above.

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