ON A CERTAIN LINEAR OPERATOR DEFINED BY USING FRACTIONAL CALCULUS

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ABSTRACT

For a real number $\lambda \neq 1,2,3,\ldots,$ we define, by using fractional calculus, a linear operator $\Lambda(\lambda)$ on the class $\mathcal{A}$ of analytic functions $f(z)$ in the unit disk $|z| < 1$ satisfying the usual normalization conditions $f(0) = 0$ and $f'(0) = 1$. This operator is related to the linear operator $\mathcal{L}(a,c)$ introduced by B.C. Carlson and D.B. Shaffer in their study of starlike and prestarlike hypergeometric functions. We prove several interesting starlike and convex properties of the function $\Lambda(\lambda)f(z)$.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 = 1) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all univalent functions in the unit disk $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{1.2}$$

for all $z \in \mathcal{U}$. We denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of all starlike functions in the unit disk $\mathcal{U}$.

A function $f(z)$ belonging to $\mathcal{S}$ is said to be convex if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \tag{1.3}$$

for all $z \in \mathcal{U}$. We use $\mathcal{K}$ to denote the subclass of $\mathcal{S}$ consisting of all convex functions in the unit disk $\mathcal{U}$. Note that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$, and that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$.

Many essentially equivalent definitions of fractional calculus (fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [4], [6], [9], and [11]). For convenience, we recall here the following definitions which were used recently by Owa [5] (and by Srivastava and Owa [10]).

DEFINITION 1. The fractional integral of order λ is defined, for a function $f(z)$, by

(1.4) $$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta,$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

DEFINITION 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

(1.5) $$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta,$$

where $0 \leqq \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $n + \lambda$ is defined by

(1.6) $$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda}(z),$$

where $0 \leqq \lambda < 1$, $n \in \mathcal{N}_0 = \{0,1,2,\ldots\}$.

By using the above definitions of fractional calculus, we now introduce the linear operator $\Lambda(\lambda)$ given by

(1.7) $$\Lambda(\lambda) f \equiv \Lambda(\lambda) f(z) = \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)$$

for $f(z) \in \mathcal{A}$ and $\lambda \notin \mathcal{N} = \{1,2,3,\ldots\} = \mathcal{N}_0 - \{0\}$.

Let the functions $f_j(z)$, $j = 1$ or 2, be defined by

(1.8) $$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1}.$$

We denote the Hadamard product (or convolution) of two functions $f_1(z)$ and $f_2(z)$ by

$$(1.9)\qquad f_1*f_2(z) = \sum_{n=0}^{\infty} a_{1,n+1}a_{2,n+1}z^{n+1}.$$

Also, following Carlson and Shaffer [1], we define a linear operator $\mathcal{L}(a,c)$ on $\mathcal{A}$ by

$$(1.10)\qquad \mathcal{L}(a,c)f(z) = \left(\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}\right) * f(z)$$

for $f(z) \in \mathcal{A}$ and $c \neq 0, -1, -2, \ldots,$ where $(\alpha)_n$ is the Pochhammer symbol defined by

$$(1.11)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1)\ldots(\alpha+n-1), & \text{if } n \in \mathcal{N}. \end{cases}$$

We note that $\mathcal{L}(a,c)$ maps $\mathcal{A}$ onto itself, and that if $a \neq 0, -1, -2, \ldots,$ then $\mathcal{L}(c,a)$ is the inverse of $\mathcal{L}(a,c)$. This linear operator $\mathcal{L}(a,c)$ was recently employed by Carlson and Shaffer [1] in their systematic study of starlike and prestarlike hypergeometric functions.

By using the linear operator $\mathcal{L}(a,c)$, we observe from (1.7) and (1.1) that

$$(1.12)\qquad \begin{aligned} \Lambda(\lambda)f &= \Gamma(1-\lambda)z^{1+\lambda}\, D_z^{1+\lambda}\, f(z) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+2)\Gamma(1-\lambda)}{\Gamma(n+1-\lambda)} a_{n+1}z^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(2)_n}{(1-\lambda)_n} a_{n+1}z^{n+1} \\ &= \left(\sum_{n=0}^{\infty} \frac{(2)_n}{(1-\lambda)_n} z^{n+1}\right) * f(z) \\ &= \mathcal{L}(2,1-\lambda)f(z). \end{aligned}$$

which indeed provides an interesting relationship between the operators $\Lambda(\lambda)$ and $\mathcal{L}(a,c)$.

In this paper we aim at presenting a systematic study of the various interesting properties and applications of the operator $\Lambda(\lambda)$. We prove several characterization theorems involving starlikeness and convexity of the function $\Lambda(\lambda)f(z)$.

2. STARLIKENESS AND CONVEXITY OF $\Lambda(\lambda)f(z)$

In order to prove our main results (Theorem 1 and Theorem 2 below) depicting starlike and convex properties of the function $\Lambda(\lambda)f(z)$, we shall need the following lemmas.

LEMMA 1 (Ruscheweyh and Sheil-Small [7, p. 126, Lemma 2.4]). *Let* $h(z)$ *and* $g(z)$ *be analytic in the unit disk* $\mathcal{U}$ *and satisfy*

$$h(0) = g(0) = 0,\ h'(0) \neq 0,\ g'(0) \neq 0.$$

Suppose that, for each $\sigma(|\sigma|=1)$ *and* $\rho(|\rho|=1)$, *we have*

$$h(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) g(z) \neq 0 \qquad (z \in \mathcal{U} - \{0\}). \tag{2.1}$$

Then, for each function $F(z)$ *analytic in the unit disk* $\mathcal{U}$ *and satisfying the inequality* $\mathrm{Re}\{F(z)\} > 0 \quad (z \in \mathcal{U})$,

$$\mathrm{Re}\left(\frac{h(z)*G(z)}{h(z)*g(z)}\right) > 0 \qquad (z \in \mathcal{U}), \tag{2.2}$$

where $G(z) = F(z)\ g(z)$.

LEMMA 2 (Twomey [12, p. 95, Equation (3)]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$. *Then*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq 1 + \frac{|z| \log\left(\dfrac{(1+|z|)^2 |f(z)|}{|z|}\right)}{(1-|z|) \log\left(\dfrac{1+|z|}{1-|z|}\right)} \tag{2.3}$$

for $z \in \mathcal{U}$. *Equality in* (2.3) *holds true for the Köebe function*

$$f(z) = \frac{z}{(1-z)^2}. \tag{2.4}$$

LEMMA 3 (Singh [8, p. 133, Theorem IV]). *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$. *Then*

$$(2.5)\qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \frac{1-|z|^2}{|z|}\,|f(z)|$$

and

$$(2.6)\qquad \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \leqq \frac{1+|z|}{1-|z|} + \frac{2|z|\log\left(\dfrac{(1-|z|)^2|f(z)|}{|z|}\right)}{(1-|z|^2)\log\left(\dfrac{1+|z|}{1-|z|}\right)}$$

for $z \in \mathcal{U}$. *Equality in* (2.5) *is attained for a function of the form*

$$(2.7)\qquad f(z) = \frac{z}{(1-ze^{i\gamma})^{2\delta}(1-ze^{-i\gamma})^{2(1-\delta)}} \qquad (0 \leqq \delta \leqq 1;\ \gamma\ \textit{real})$$

and equality in (2.6) *is attained for a function of the form*

$$(2.8)\qquad f(z) = \frac{z}{(1-z)^{2\delta}(1+z)^{2(1-\delta)}} \qquad (0 \leqq \delta \leqq 1),$$

where δ *satisfies*

$$(2.9)\qquad 2\delta\log\left(\frac{1+|z|}{1-|z|}\right) = \log\left(\frac{(1+|z|)^2|f(z)|}{|z|}\right).$$

Applying Lemma 1, we now prove

THEOREM 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$ *and let*

$$(2.10)\qquad \mathcal{L}(2,1-\lambda)f(z) \neq 0 \qquad (z \in \mathcal{U} - \{0\})$$

for $\lambda \notin \mathcal{N}$. *Then* $\Lambda(\lambda)f(z)$ *is also in the class* $\mathcal{S}^*$.

PROOF. It is sufficient to show that

$$(2.11)\qquad \mathrm{Re}\left\{\frac{z(\Lambda(\lambda)f)'}{\Lambda(\lambda)f}\right\} > 0$$

for $z \in \mathcal{U}$. By virtue of (1.12), we have

$$\text{(2.12)} \qquad \mathrm{Re}\left\{\frac{z(\Lambda(\lambda)f)'}{\Lambda(\lambda)f}\right\} = \mathrm{Re}\left\{\frac{z(\mathcal{L}(2,1-\lambda)f(z))'}{\mathcal{L}(2,1-\lambda)\ f(z)}\right\}$$

$$= \mathrm{Re}\left\{\frac{\mathcal{L}(2,1-\lambda)(zf'(z))}{\mathcal{L}(2,1-\lambda)\ f(z)}\right\}$$

$$= \mathrm{Re}\left\{\frac{\left(\sum_{n=0}^{\infty}\frac{(2)_n}{(1-\lambda)_n}z^{n+1}\right) * (zf'(z))}{\left(\sum_{n=0}^{\infty}\frac{(2)_n}{(1-\lambda)_n}z^{n+1}\right) * f(z)}\right\}.$$

Setting $\sigma = 1$, $\rho = -1$,

$$h(z) = \sum_{n=0}^{\infty}\frac{(2)_n}{(1-\lambda)_n}z^{n+1},$$

$$F(z) = \frac{zf'(z)}{f(z)},$$

and $g(z) = f(z)$ in Lemma 1, we conclude from (2.12) that $\Lambda(\lambda)f(z)$ satisfies the inequality (2.11), that is, that $\Lambda(\lambda)f(z) \in \mathcal{S}^*$.

This evidently completes the proof of Theorem 1.

COROLLARY 1. Under the hypotheses of Theorem 1,

$$\text{(2.13)} \qquad \left|\frac{\Lambda(\lambda)zf'(z)}{\Lambda(\lambda)f(z)}\right| \leqq 1 + \frac{|z|\ \log\left(\frac{(1+|z|)^2|\Lambda(\lambda)f(z)|}{|z|}\right)}{(1-|z|)\ \log\left(\frac{1+|z|}{1-|z|}\right)}$$

for $z \in \mathcal{U}$. Equality in (2.13) holds true for the function $f(z)$ given by

$$\text{(2.14)} \qquad f(z) = \mathcal{L}(1-\lambda,2)\left(\frac{z}{(1-z)^2}\right).$$

PROOF. The assertion of Corollary 1 follows immediately upon applying Lemma 2 to Theorem 1.

Furthermore, by applying Lemma 3 to Theorem 1, we have

COROLLARY 2. Under the hypotheses of Theorem 1,

$$\text{(2.15)} \qquad \operatorname{Re}\left\{\frac{\Lambda(\lambda) z f'(z)}{\Lambda(\lambda) f(z)}\right\} \geqq \frac{1-|z|^2}{|z|} |\Lambda(\lambda) f(z)|$$

and

$$\text{(2.16)} \qquad \operatorname{Re}\left\{\frac{\Lambda(\lambda) z f'(z)}{\Lambda(\lambda) f(z)}\right\} \leqq \frac{1+|z|}{1-|z|} + \frac{2|z| \log\left(\dfrac{(1-|z|)^2 |\Lambda(\lambda) f(z)|}{|z|}\right)}{(1-|z|^2) \log\left(\dfrac{1+|z|}{1-|z|}\right)}$$

for $z \in \mathcal{U}$. Equality in (2.15) holds true for the function $f(z)$ given by

$$\text{(2.17)} \qquad f(z) = \mathcal{L}(1-\lambda, 2)\left(\frac{z}{(1-ze^{i\gamma})^{2\delta}(1-ze^{-i\gamma})^{2(1-\delta)}}\right),$$

and equality in (2.16) holds true for the function $f(z)$ given by

$$\text{(2.18)} \qquad f(z) = \mathcal{L}(1-\lambda, 2)\left(\frac{z}{(1-z)^{2\delta}(1+z)^{2(1-\delta)}}\right),$$

where $0 \leqq \delta \leqq 1$, γ is real, and δ satisfies (2.9).

Next we prove

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and let

$$\text{(2.19)} \qquad \mathcal{L}(2,1)\mathcal{L}(2,1-\lambda) f(z) \neq 0 \qquad (z \in \mathcal{U} - \{0\})$$

for $\lambda \notin \mathcal{N}$. Then $\Lambda(\lambda)f(z)$ is also in the class $\mathcal{K}$.

PROOF. Note that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. By using Theorem 1, we know that

$$\begin{aligned} f(z) \in \mathcal{K} &\Leftrightarrow zf'(z) \in \mathcal{S}^* \\ &\Rightarrow \Lambda(\lambda)zf'(z) \in \mathcal{S}^* \\ &\Rightarrow z\left(\Lambda(\lambda)f\right)' \in \mathcal{S}^* \\ &\Leftrightarrow \Lambda(\lambda)f(z) \in \mathcal{K}, \end{aligned}$$

which completes the proof of Theorem 2.

3. FURTHER APPLICATIONS OF THE OPERATOR $\Lambda(\lambda)$

In order to derive some further results involving the operator $\Lambda(\lambda)$, we recall here the following lemmas.

LEMMA 4 (Ruscheweyh and Sheil-Small [7]; see also Duren [2, Theorem 8.6′]). If $f(z) \in \mathcal{S}^*$ and $g(z) \in \mathcal{K}$, then $f*g(z) \in \mathcal{S}^*$.

LEMMA 5 (Lewis [3, p. 435, Theorem 1]). Given μ, with $-\infty < \mu < \infty$, let

$$f_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\mu} z^{n+1} \tag{3.1}$$

for $z \in \mathcal{U}$. Then $f_\mu(z)$ is in the class $\mathcal{K}$ whenever $\mu \geqq 0$.

Now we state and prove

THEOREM 3. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and let

$$\mathcal{L}(2,1-\lambda)\left(f_\mu * f(z)\right) \neq 0 \qquad (z \in \mathcal{U} - \{0\}) \tag{3.2}$$

for $\mu \geqq 0$ *and* $\lambda \notin \mathcal{N}$, *with the function* $f_\mu(z)$ *being given by* (3.1). *Then* $\Lambda(\lambda)\{f_\mu * f(z)\}$ *is in the class* $\mathcal{S}^*$.

PROOF. Note that

$$(3.3) \qquad \mathrm{Re}\left\{\frac{z\left[\Lambda(\lambda)\{f_\mu * f(z)\}\right]'}{\Lambda(\lambda)\{f_\mu * f(z)\}}\right\} = \mathrm{Re}\left\{\frac{\mathcal{L}(2,1-\lambda)\{f * zf'_\mu(z)\}}{\mathcal{L}(2,1-\lambda)\{f * f_\mu(z)\}}\right\}$$

$$= \mathrm{Re}\left\{\frac{\mathcal{L}(2,1-\lambda)f(z) * zf'_\mu(z)}{\mathcal{L}(2,1-\lambda)f(z) * f_\mu(z)}\right\}$$

$$= \mathrm{Re}\left\{\frac{\left(\sum_{n=0}^{\infty} \frac{(2)_n}{(1-\lambda)_n} a_{n+1} z^{n+1}\right) * \{zf'_\mu(z)\}}{\left(\sum_{n=0}^{\infty} \frac{(2)_n}{(1-\lambda)_n} a_{n+1} z^{n+1}\right) * f_\mu(z)}\right\}$$

and that, by Lemma 5, $f_\mu(z) \in \mathcal{K} \subset \mathcal{S}^*$ for $\mu \geqq 0$.

Putting $\sigma = 1$, $\rho = -1$,

$$h(z) = \sum_{n=0}^{\infty} \frac{(2)_n}{(1-\lambda)_n} a_{n+1} z^{n+1},$$

$$F(z) = \frac{zf'_\mu(z)}{f_\mu(z)},$$

and $g(z) = f_\mu(z)$ in Lemma 1, we observe that $\Lambda(\lambda)\{f_\mu * f(z)\} \in \mathcal{S}^*$, which completes the proof of Theorem 3.

COROLLARY 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$ *and satisfy the condition* (3.2) *for* $\mu \geqq 0$, *and let* $f_\mu(z)$ *be given by* (3.1). *Then* $\Lambda(\lambda)\{f_\mu * f(z)\}$ *is also in the class* $\mathcal{S}^*$.

REMARK 1. Letting $f(z) \in \mathcal{S}^*$ and $\mu = 0$ in Theorem 3, we have Theorem 1.

Finally, we prove

THEOREM 4. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}$ and let

$$\mathcal{L}(2,1)\mathcal{L}(2,1-\lambda)\left(f_\mu * f(z)\right) \neq 0 \qquad (z \in \mathcal{U} - \{0\}) \tag{3.4}$$

for $\mu \geqq 0$ and $\lambda \notin \mathcal{N}$, with the function $f_\mu(z)$ being given by (3.1). Then $\Lambda(\lambda)\left(f_\mu * f(z)\right)$ is also in the class $\mathcal{K}$.

PROOF. Since $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$, it follows from Lemma 4 and Lemma 5 that

$$f_\mu * \left(zf'(z)\right) \in \mathcal{S}^*$$

for $\mu \geqq 0$. Hence, with the aid of Corollary 3, we obtain

$$\begin{aligned} f(z) \in \mathcal{K} \Leftrightarrow zf'(z) &\in \mathcal{S}^* \\ \Rightarrow f_\mu * \left(zf'(z)\right) &\in \mathcal{S}^* \\ \Rightarrow \Lambda(\lambda)\left(f_\mu * zf'(z)\right) &\in \mathcal{S}^* \\ \Rightarrow z\left\{\Lambda(\lambda)\left(f_\mu * f(z)\right)\right\}' &\in \mathcal{S}^* \\ \Leftrightarrow \Lambda(\lambda)\left(f_\mu * f(z)\right) &\in \mathcal{K}, \end{aligned}$$

thus completing the proof of Theorem 4.

REMARK 2. Letting $\mu = 0$ in Theorem 4, we have Theorem 2.

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