A NOTE ON A CERTAIN INVERSE PAIR
OF MULTIPLE SERIES IDENTITIES

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DMS-676-IR
August 1994
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Abstract

We develop a multiple-series generalization of certain series identities considered by T.J. Osler [2] and H.M. Srivastava [4] and derive its inverse. We also apply our pair of multiple series identities to the generalized Kampé de Fériet hypergeometric function in several variables.

1. Introduction

For a suitably bounded multiple sequence \( \{C(n_1, \cdots, n_r)\} \), let a general function of \( r \) variables be defined by

\[
f(z_1, \cdots, z_r) = \sum_{n_1, \cdots, n_r = 0}^{\infty} C(n_1, \cdots, n_r) z_1^{n_1} \cdots z_r^{n_r} \quad (1.1)
\]

\(|z_j| < R_j; \quad R_j > 0; \quad j \in \{1, \cdots, r\} \),

provided that the multiple series converges absolutely.

Srivastava [4, p. 197] gave the multiple series identity (see also Srivastava and Manocha [6, p. 217, Problem 12]):

\[
\sum_{n_1, \cdots, n_r = 0}^{\infty} C(n_1, \cdots, n_r) = \sum_{M_1 = 0}^{N_1 - 1} \cdots \sum_{M_r = 0}^{N_r - 1} \left( \sum_{n_1, \cdots, n_r = 0}^{\infty} C(n_1N_1 + M_1, \cdots, n_rN_r + M_r) \right) \quad (1.2)
\]

1991 Mathematics Subject Classification. Primary 33C65, 33C70; Secondary 33B15.

Key words and phrases. Multiple sequence, multiple series identities, Kampé de Fériet function, inverse series relation, Srivastava’s identity, Gauss’s multiplication formula.
where \( N_j (j = 1, \ldots, r) \) are arbitrary positive integers.

The single-series analogue of (1.2), when \( r = 1 \), stated in [4, p. 193, Equation (8)] is, in fact, an inverse series relation of a result given earlier by Osler [2, p. 889, Equation (2)]. The general series identity in both of these papers generates precisely the same hypergeometric series identity. The particular case of (1.2), when \( r = 2 \), was also considered by Sharma [3]. It would thus seem worthwhile to develop a multiple-series generalization of the aforementioned series identity and also derive its inverse. The applications of our pair of series identities would be manifold. We, however, illustrate the use by deducing a series identity involving the generalized Kampé de Fériet function.

2. A Theorem on Multiple Series Identities

The inverse pair of multiple series identities which we propose to derive is contained in the following:

**Theorem.** For each positive integer \( N_j \), let \( M_j \in \{0, 1, 2, \ldots, N_j - 1\} (j \in \{1, \ldots, r\}) \), and suppose that there exists a number \( w_j = \exp(2\pi i/N_j) \) \((j \in \{1, \ldots, r\})\). Then, corresponding to the multivariable function \( f(z_1, \ldots, z_r) \) defined by (1.1), there exist the following pair of multiple series identities:

\[
\sum_{m_1, \ldots, m_r = 0}^{\infty} C(m_1 N_1 + M_1, \ldots, m_r N_r + M_r) z_1^{m_1 N_1 + M_1} \cdots z_r^{m_r N_r + M_r} = \left( \prod_{s=1}^{r} N_s \right)^{-1} \sum_{k_1 = 0}^{N_1 - 1} \cdots \sum_{k_r = 0}^{N_r - 1} f(z_1 w_1^{k_1}, \ldots, z_r w_r^{k_r}) w_1^{-M_1 k_1} \cdots w_r^{-M_r k_r} \tag{2.1}
\]

and

\[
f(z_1, \ldots, z_r) = \sum_{M_1 = 0}^{N_1 - 1} \cdots \sum_{M_r = 0}^{N_r - 1} \sum_{m_1, \ldots, m_r = 0}^{\infty} \cdot C(m_1 N_1 + M_1, \ldots, m_r N_r + M_r) z_1^{m_1 N_1 + M_1} \cdots z_r^{m_r N_r + M_r}, \tag{2.2}
\]

provided that each side of (2.1) and (2.2) has a meaning.
Proof. Denoting the right-hand side of (2.1) by $I$, we have

$$I = \left( \prod_{s=1}^{r} N_s \right)^{-1} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_r=0}^{N_r-1} \left( \sum_{n_1, \ldots, n_r=0}^{\infty} C(n_1, \ldots, n_r) \cdot z_1^{n_1} \cdots z_r^{n_r} \right) w_1^{(n_1-M_1)k_1} \cdots w_r^{(n_r-M_r)k_r} \quad (2.3)$$

$$= \left( \prod_{s=1}^{r} N_s \right)^{-1} \sum_{n_1, \ldots, n_r=0}^{\infty} \left( \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_r=0}^{N_r-1} \cdot w_1^{(n_1-M_1)k_1} \cdots w_r^{(n_r-M_r)k_r} \right) C(n_1, \ldots, n_r) z_1^{n_1} \cdots z_r^{n_r}.$$

Now noting that (see also Osler [2])

$$\sum_{k_j=0}^{N_j-1} w_j^{(n_j-M_j)k_j} = \sum_{k_j=0}^{N_j-1} \exp \left( 2\pi i (n_j - M_j) k_j / N_j \right)$$

$$= N_j, \quad \text{if } n_j - M_j = N_j m_j \quad (m_j = 0, 1, 2, \cdots)$$

$$= 0, \quad \text{otherwise}, \quad (2.4)$$

(2.3) leads to the desired left-hand side of (2.1), and this proves the multiple series identity (2.1).

To prove the inverse relation (2.2), let us effect the multiple sum over $M_j$ from 0 to $N_j - 1$, for all $j = 1, \ldots, r$, on both the sides of (2.1); then, keeping in mind the relation:

$$\sum_{M_j=0}^{N_j-1} w_j^{-M_j k_j} = \sum_{M_j=0}^{N_j-1} \exp \left( 2\pi i (-M_j k_j) / N_j \right)$$

$$= N_j, \quad \text{if } k_j = 0,$$

$$= 0, \quad \text{otherwise}, \quad (2.5)$$

the multiple series identity (2.2) follows.

Remark 1. The identity (2.2) would also follow at once from Srivastava’s identity [4, p. 197, Equation (24)] on replacing

$$C(n_1, \ldots, n_r) \quad \text{by} \quad C(n_1, \ldots, n_r) z_1^{n_1} \cdots z_r^{n_r}.$$
3. Applications

We choose the multiple sequence \( \{C(n_1, \ldots, n_r)\} \) as follows:

\[
C(n_1, \ldots, n_r) = \frac{\prod_{j=1}^{p} (a_j)_{n_1+\cdots+n_r} \prod_{j=1}^{p_1} (b_j')_{n_1} \cdots \prod_{j=1}^{p_r} (b_j^{(r)})_{n_r}}{\prod_{j=1}^{q} (\alpha_j)_{n_1+\cdots+n_r} \prod_{j=1}^{q_1} (\beta_j')_{n_1} \cdots \prod_{j=1}^{q_r} (\beta_j^{(r)})_{n_r}} \cdot (n_1! \cdots n_r!)^{-1} \tag{3.1}
\]

where, as usual, \((\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)\).

From Gauss’s multiplication formula and the simple relations [6, pp. 22-23], we have

\[
(\lambda)_{N_{m+M}} = (\lambda)_M N^{N_m} \prod_{j=0}^{N-1} \left( \frac{\lambda + M + j}{N} \right)_m. \tag{3.2}
\]

If we denote by \((a^{(i)}_{p_i})\) the array of \(p_i\) parameters

\[
a^{(i)}_{1}, \ldots, a^{(i)}_{p_i} \quad (p_i = 1, 2, 3, \ldots; \; i = 1, \ldots, r),
\]

and by \(([a^{(i)}_{p_i}] : N)\) the array of \(Np_i\) parameters

\[
a^{(i)}_{j}/N, \; (a^{(i)}_{j} + 1)/N, \ldots, (a^{(i)}_{j} + N - 1)/N \quad (j = 1, \ldots, p_i; \; i = 1, \ldots, r),
\]

then, upon substituting from (3.1) into (2.1), setting \(N_i = N \; (i = 1, \ldots, r)\), and suitably using the relation (3.2), we arrive at the following result (cf. [6, p. 65]; see also [5, p. 454]):

\[
F_{p^pN; q^qN_1, N_2; \ldots; q^qN; N}^{p^pN_1, N_2 + 1; \ldots; p^pN_1, N_2 + 1} \left( \begin{array}{c}
([a_p] + M_1 + \cdots + M_r; \; N] : ([b_{p1}'] + M_1; \; N], 1; \cdots;
([\alpha_q] + M_1 + \cdots + M_r; \; N] : ([\beta_{q1}'] + M_1; \; N]; \cdots;
([b_{q1}'] + M_r; \; N]_1;
([\beta_{qr}'] + M_r; \; N];
\end{array} \right)
\right)
\]

\[
= N^{-r} \Delta(M_1, \ldots, M_r) \prod_{i=1}^{r} \left\{ \sum_{k_i=0}^{N-1} z_i^{-M_i} w^{-M_i k_i} \right\} \cdot F_{q^qN_1, \ldots, q^qN_1}^{p^pN_1, \ldots; p^pN_1} \left( z_1 w^{k_1}, \ldots, z_r w^{k_r} \right),
\]

where
(i) $\beta'_{q_1+1} = \cdots = \beta^{(r)}_{q_r+1} = 1$,

(ii) $1 + q_j + p - p_j \geq 0 \ (j \in \{1, \cdots, r\})$; the equality holds when $|z_i|$ are suitably restricted;

$$\lambda_i = p + p_i - q - q_i - 1 \quad (i \in \{1, \cdots, r\}), \quad (3.4)$$

and

$$\Delta(M_1, \cdots, M_r) = \frac{\prod_{i=1}^{q} (\alpha_i)_{M_1 + \cdots + M_r} \prod_{i=1}^{q_1+1} (\beta'_i)_{M_1} \cdots \prod_{i=1}^{q_r+1} (\beta^{(r)}_i)_{M_r}}{\prod_{i=1}^{p} (a_i)_{M_1 + \cdots + M_r} \prod_{i=1}^{p_1+1} (b'_i)_{M_1} \cdots \prod_{i=1}^{p_r+1} (b^{(r)}_i)_{M_r}}. \quad (3.5)$$

The inverse of the identity (3.3) can be deduced in a similar manner from (2.2). The result thus obtained is

$$F_{p_1; \cdots; p_r}^{p; p_1; \cdots; p_r} (z_1, \cdots, z_r) = \sum_{M_1=0}^{N-1} \cdots \sum_{M_r=0}^{N-1} [\Delta(M_1, \cdots, M_r)]^{-1} \cdot z_1^{M_1} \cdots z_r^{M_r} \cdot F_N^{p.N; p_1 N + 1; \cdots; p_r N + 1} \left( [(\alpha_p) + M_1 + \cdots + M_r; N] : [(\alpha_q) + M_1 + \cdots + M_r; N] : \right.$$  

$$\left. [(b'_p) + M_1; N], \cdots; [(b^{(r)}_p) + M_r; N], \right.$$  

$$\left. [(\beta'_{q_1+1}) + M_1; N], \cdots; [(\beta^{(r)}_{q_1+1}) + M_r; N] ; \right.$$  

$$\left. (z_1 N^{\lambda_1})^N, \cdots, (z_r N^{\lambda_r})^N \right), \quad (3.6)$$

provided that Conditions (i) and (ii) stated with (3.3) are satisfied, $\Delta(M_1, \cdots, M_r)$ and $\lambda_i$ ($i = 1, \cdots, r$) being defined by (3.4) and (3.5) above.

**Remark 2.** It should be noted that any one of the numerator parameters 1 can get cancelled by one of the denominator parameters

$$(\beta'_{q_1+1} + M_1)/N, \ (\beta'_{q_1+1} + M_1 + 1)/N, \cdots, (\beta'_{q_1+1} + M_1 + N - 1)/N,$$

and so on, which is due to the fact that

$$\beta'_{q_1+1} = \cdots = \beta^{(r)}_{q_r+1} = 1.$$

**Remark 3.** For $r = 1$, the pair of identities (3.3) and (3.6) would correspond essentially to the identities given in [2]. Furthermore, the pair of identities stated in [3] would
correspond to (3.3) and (3.6), when \( r = 2 \), and when \( z_i \) is replaced by \( z_i N^{-\lambda_i} \) \((i = 1, \ldots, r)\), \( \lambda_i \) being defined by (3.4).

Acknowledgements

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353. The first-named author wishes to thank the University Grants Commission of India for providing financial support for this project.
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