AN IDENTITY FOR THE DETERMINANT

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Abstract

When the directed graph of an $n$-by-$n$ matrix $A$ does not contain a Hamilton cycle, we exhibit a formula for $\det A$ in terms of proper principal minors of $A$. The set of minors involved depends upon the zero/nonzero pattern of $A$. 
1 The Determinantal Identity

For a positive integer \( n \), let \( \langle n \rangle \equiv \{1, 2, \ldots, n\} \). A partition, \( p \), of \( \langle n \rangle \) is a collection of pairwise disjoint subsets of \( \langle n \rangle \), called the components of \( p \), whose union is \( \langle n \rangle \). The number of components of \( p \) is denoted by \( |p| \). We denote the set of all partitions of \( \langle n \rangle \) by \( P_{\langle n \rangle} \). We define on \( P_{\langle n \rangle} \) a partial order, \( \leq \), as follows: for \( p_1, p_2 \in P_{\langle n \rangle} \), \( p_1 \preceq p_2 \) if every component of \( p_1 \) is a subset of a component of \( p_2 \) (i.e., \( p_1 \) is at least as fine as \( p_2 \)). We also define the intersection of \( p_1 \) and \( p_2 \), \( p_1 \cap p_2 \), as a partition of \( \langle n \rangle \) whose components are the intersection of the components of \( p_1 \) and \( p_2 \). Notice that \( p_1 \cap p_2 \preceq p_i \), \( i = 1, 2 \).

Consider now an \( n \)-by-\( n \) matrix \( A = (a_{ij}) \) and its directed graph \( D(A) \). Let \( P_{\langle n \rangle}(A) \) denote the set of partitions of \( \langle n \rangle \) each component of which is ordered corresponding to a simple cycle in \( D(A) \). We associate to any \( p \in P_{\langle n \rangle}(A) \) with components
\[
p : \{i_1, \ldots, i_u\}, \{j_1, \ldots, j_v\}, \ldots, \{l_1, \ldots, l_w\}
\]
the cyclic product
\[
C(p) = (a_{i_1i_2}a_{i_2i_3} \cdots a_{i_{u-1}i_u}a_{i_u1}) (a_{j_1j_2}a_{j_2j_3} \cdots a_{j_{v-1}j_v}a_{j_vj_1}) \cdots
\]
\[
\cdots (a_{l_1l_2}a_{l_2l_3} \cdots a_{l_{w-1}l_w}a_{lwl_1}).
\]

Let \( \alpha = \{k_1, k_2, \ldots, k_r\} \subseteq \langle n \rangle \). By \( A[\alpha] = A[k_1, k_2, \ldots, k_r] \), we denote the principal submatrix of \( A \) with rows and columns indexed by \( \alpha \). The product of the principal minors corresponding to the partition in (1.1) is denoted by
\[
A(p) = \text{det}A[i_1, i_2, \ldots, i_u] \text{det}A[j_1, j_2, \ldots, j_v] \cdots \text{det}A[l_1, l_2, \ldots, l_w].
\]

It is well known (see e.g., [MOVW]) that the determinant of \( A \) can be expressed in terms of the cyclic products as follows:
\[
\text{det}A = \sum_{p \in P_{\langle n \rangle}(A)} (-1)^{|p|+n} C(p).
\]

We refer to (1.2) as the cyclic expansion of \( \text{det}A \). Our aim is to derive an expansion of the determinant of \( A \) entirely in terms of principal minors. To describe this result we need to introduce the following graph-theoretic notion.

Definition 1.1 The set \( \{p_1, p_2, \ldots, p_m\} \subseteq P_{\langle n \rangle}(A) \) is called the maximal cycle cover of \( A \) if \( p_j \npreceq p_k \) for all \( j \neq k \), and if \( q \in P_{\langle n \rangle}(A) \) implies that there exists an \( i \in \{1, 2, \ldots, m\} \) such that \( q \preceq p_i \).
Observe that if $D(A)$ contains a Hamilton cycle, then the maximal cycle cover of $A$ has cardinality one and it consists of the trivial partition $p = (n)$. When $D(A)$ does not contain a Hamilton cycle the cardinality of the maximal cycle cover is at least one.

We can now state and prove the following identity.

**Theorem 1.2** Let $\{p_1, p_2, \ldots, p_m\}$ be the maximal cycle cover of $A$. Then,

$$
\text{det} A = \sum_{s=1}^{m} (-1)^{s-1} \sum_{1 \leq i_1 < \cdots < i_s \leq m} A(p_{i_1} \cap \ldots \cap p_{i_s}).
$$

(1.3)

**Proof.**

We prove the theorem by showing that the right hand side of (1.3) is equal to the right hand side of (1.2). We do this by considering the cyclic expansion of (each principal minor in) every summand $A(p_{i_1} \cap p_{i_2} \cap \ldots \cap p_{i_s})$ in (1.3).

First take a summand from (1.2)

$$
\alpha_p \equiv (-1)^{|p|+n}C(p), \quad p \in P_{(n)}(A).
$$

By definition of the maximal cycle cover, for any such $p \in P_{(n)}(A)$ there exists $i \in \{1, 2, \ldots, m\}$ such that $p \preceq p_i$. Let $p_i$ have components $\gamma_j$, $j = 1, 2, \ldots, t$, and consider the cyclic expansion of each $\text{det} A[\gamma_j]$. Then, on letting $k_j$ denote the number of components of $p$ that are contained in $\gamma_j$, we have that

$$
\sum_{j=1}^{t} |\gamma_j| = n \quad \text{and} \quad \sum_{j=1}^{t} k_j = |p|,
$$

and thus the cyclic expansion of $A(p_i) = \prod_{j=1}^{t} \text{det} A[\gamma_j]$ contains the term

$$
\left( \prod_{j=1}^{t} (-1)^{|\gamma_j|+k_j} \right) C(p) = (-1)^{|p|+n}C(p) = \alpha_p.
$$

This in particular shows that every summand $\alpha_p$ on the right hand side of (1.2) is a term in the cyclic expansion of a summand on the right hand side of (1.3) with $s = 1$.

Conversely, we claim that every term in the cyclic expansion of $A(p_{i_1} \cap \ldots \cap p_{i_s})$ is equal to a term $\alpha_p$, $p \in P_{(n)}(A)$. For that purpose, let $\delta_1, \delta_2, \ldots, \delta_r$ be the components of the partition $p_{i_1} \cap \ldots \cap p_{i_s}$, recall that

$$
A(p_{i_1} \cap \ldots \cap p_{i_s}) = \prod_{j=1}^{r} \text{det} A[\delta_j].
$$
and consider the cyclic expansion of every \( \text{det}A[\delta_j] \). The claim then follows from an argument similar to the first part of the theorem and the fact that every component \( \delta_j \) is contained in a component of a partition belonging to the maximal cycle cover of \( A \).

To complete the proof we must show that, after we expand and collect terms on the right hand side of (1.3), the coefficient of each \( \alpha_p, p \in P(n)(A) \), is equal to one. Since \( p_1, p_2, \ldots, p_m \) is a maximal cycle cover of \( A \), there exist \( p_{i_1}, p_{i_2}, \ldots, p_{i_k}, \ k \leq m \), such that

\[
p \begin{cases}
  \leq p_{i_j} & \text{if } j \in \{1, 2, \ldots, k\} \\
  \leq p_{i_l} & \text{otherwise.}
\end{cases}
\]

Observe then that \((-1)^{l-1} \alpha_p\) is a term in the cyclic expansion of \( \binom{k}{l} \) summands of (1.3), where \( l = 1, 2, \ldots, k \). Therefore, the coefficient of \( \alpha_p \) on the right hand side of (1.3) is given by

\[
\binom{k}{l} - \binom{k}{2} + \ldots (-1)^{k-1} \binom{k}{k} = 1,
\]

completing the proof of the theorem. \( \square \)

We remark that if \( D(A) \) contains a Hamilton cycle, (1.3) reduces to the trivial identity \( \text{det} A = \text{det} A[\langle n \rangle] \).

**Corollary 1.3** Let \( \{p_1, p_2, \ldots, p_m\} \) be the maximal cycle cover of \( A \). If \( \lambda \) is an eigenvalue of at least one principal submatrix in each of the summands of (1.3), then \( \lambda \) is an eigenvalue of \( A \).

**Proof.**

Apply (1.3) to \( \text{det}(A - \lambda I) \). \( \square \)

Note that this eigenvalue result is true independently of the values of the nonzero entries of \( A \) not contained in the particular set of principal submatrices with common eigenvalue \( \lambda \). This result generalizes the statement that if \( A \) is reducible and in Frobenius normal form, and if \( \lambda \) is an eigenvalue of an irreducible diagonal block, then \( \lambda \) is an eigenvalue of \( A \).

### 2 Examples

In the following examples we apply the determinantal identity of Theorem 1.2 to matrices with a specified zero/nonzero pattern.

**Example 2.1** Let
\[ A = \begin{bmatrix}
\times & \times & \times & \times & 0 & 0 \\
\times & \times & \times & \times & 0 & 0 \\
\times & 0 & \times & 0 & 0 & \times \\
0 & 0 & 0 & \times & \times & 0 \\
\times & 0 & 0 & \times & \times & \times \\
\times & 0 & 0 & 0 & 0 & \times 
\end{bmatrix} \]

The maximal cycle cover of \( A \) consists of the partitions

\[ p_1 : \{1, 2, 4, 5, 6\}, \{3\} \text{ and } p_2 : \{1, 2, 3, 6\}, \{4, 5\}. \]

According to Theorem 1.2 we compute the intersection \( p_1 \cap p_2 \),

\[ p_1 \cap p_2 : \{1, 2, 6\}, \{4, 5\}, \{3\} \]

and the determinant of \( A \) can be expressed in terms of its principal minors as follows:

\[

As in the previous example, whenever the cardinality of the maximal cycle cover is 2, the identity (1.3) expresses \( det A \) as a sum of three products of principal minors of \( A \). The next example illustrates that (1.3) may give a very complicated expression for \( det A \) when the cardinality of the maximal cycle cover is large.

**Example 2.2** Let

\[ A = \begin{bmatrix}
\times & \times & \times & 0 & 0 \\
0 & \times & \times & 0 & \times \\
\times & 0 & \times & \times & 0 \\
\times & \times & 0 & \times & \times \\
0 & \times & 0 & \times & \times 
\end{bmatrix} \]

The maximal cycle cover of \( A \) has cardinality 6 and is given by

\[ p_1 : \{1, 2, 3, 4\}, \{5\}, \ p_2 : \{1\}, \{2, 3, 4, 5\}, \]

\[ p_3 : \{1, 2, 4, 5\}, \{3\}, \ p_4 : \{1, 2, 3\}, \{4, 5\}, \]

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\[ p_5 : \{2, 4, 5\}, \{1, 3\} \text{ and } p_6 : \{1, 3, 4\}, \{2, 5\}. \]

Applying Theorem 1.2 there are \(2^6 - 1\) possible summands on the right hand side of (1.3). Many of the intersections of these partitions are repeated (e.g., \(p_2 \cap p_3 = p_2 \cap p_5 = p_3 \cap p_5 = p_2 \cap p_4 \cap p_5 : \{1\}, \{3\}, \{2, 4, 5\}\)). As a consequence, (1.3) reduces to 27 distinct summands. Not all of these summands occur with a coefficient of \(\pm 1\). For example, the reduced identity contains a summand of the form


\[ \square \]

**Example 2.3** Consider a matrix \(A\) with the following zero/nonzero pattern:

\[
A = \begin{bmatrix}
  x & x & x & x & 0 \\
  x & x & x & 0 & 0 \\
  x & x & x & 0 & 0 \\
  0 & 0 & 0 & x & x \\
  x & 0 & 0 & 0 & x
\end{bmatrix}.
\]

Here the maximal cycle cover consists of

\[ p_1 : \{1, 2, 3\}, \{4\}, \{5\} \text{ and } p_2 : \{1, 4, 5\}, \{2, 3\}. \]

From Theorem 1.2 we obtain

\[
\]


(2.1)

Applying Corollary 1.3, if \(\lambda = A[4] \) (or \(A[5]\)) is an eigenvalue of \(A[2, 3]\), then \(\lambda\) is also an eigenvalue of \(A\), independently of the values of the other nonzero entries in \(A\). (Note that \(A\) is irreducible.)

We remark that the directed graph of \(A\) contains a cut-point (vertex 1). In this case, a formula for \(\det A\) in terms of principal minors is known (see [MOVW]), which yields

\[
\det A = \det A[1, 2, 3] \det A[4, 5] + \det A[1, 4, 5] \det A[2, 3]
\]


\[ \square \]
References