SPREAD OF A SYMMETRIC RANDOM WALK

(Amer. Math. Monthly Advanced Problem 6665)

by

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6665. Proposed by José Luis Palacios, New Jersey Institute of Technology, Newark, NJ, and Dennis P. Sandell, Swedish University of Agricultural Sciences, Varberg, Sweden.

Let \( S_n \) \((n \geq 0)\) be a simple symmetric random walk, i.e., \( S_0 = 0 \) and \( S_n = X_1 + X_2 + \cdots + X_n \) for \( n > 0 \), where the \( X_i \) are independent identically distributed random variables with \( P(X_i = 1) = P(X_i = -1) = \frac{1}{2} \).

Let \( N \) be an arbitrary positive integer and let \( T \) be the first time that the difference between the maximum and minimum of the random walk is \( N \), i.e., let

\[
T = \min \left\{ n : \max_{0 \leq k \leq n} S_k - \min_{0 \leq k \leq n} S_k = N \right\}.
\]

Find the expected value of \( T \).

Solution by Bruce R. Johnson, University of Victoria, Victoria, B.C., Canada. We will show that \( E(T) = N(N+1)/2 \).

Let \( \max_{0 \leq k \leq n} S_k - \min_{0 \leq k \leq n} S_k \) be called the spread of the random walk after \( n \) steps. For \( i \in \{1,2,\ldots,N\} \) define

\[
Y_i = \text{the number of steps after the spread first reaches } i-1 \text{ until the spread first reaches } i.
\]

When the spread first reaches \( i-1 \), the random walk is at either a current minimum or maximum and is \( i-1 \) units away from the other extreme. Thus, \( Y_i \) has the same probability distribution as the duration of play in the classical gambler's ruin problem where

(a) the gambler is just as likely to win one unit as lose one unit on each play,
(b) the gambler has initial capital 1 unit, and
(c) the game continues until the gambler's capital is either reduced to zero or increased to \(i+1\).


\[ E(Y_1) = \text{expected duration of play} = (1)((i+1)-1) = i. \]

Hence,

\[
E(T) = E(Y_1 + Y_2 + \cdots + Y_N) = E(Y_1) + E(Y_2) + \cdots + E(Y_N)
\]
\[= 1 + 2 + \cdots + N = N(N+1)/2. \]

\(^1\)Letting \(e_k\) denote the expected duration of play for a gambler with initial capital \(k\) units, we condition on the outcome of the first play to obtain the difference equations

\[ e_k = (e_{k-1} + 1)^{1/2} + (e_{k+1} + 1)^{1/2} \quad \text{for} \quad k = 1, 2, \ldots, i. \]

Expressing these equations in the form

\[ e_k - e_{k-1} = e_{k+1} - e_k + 2 \]

and equating the telescoping sum of left-hand-sides to the telescoping sum of right-hand-sides, we obtain

\[ e_i - e_0 = e_{i+1} - e_1 + 2i. \]

The boundary conditions \(e_0 = 0 = e_{i+1}\) and the symmetry condition \(e_1 = e_i\) now yield the desired result \(e_1 = i\).