A UNIFIED PRESENTATION OF SOME CLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract
The authors introduce and investigate various properties of a general class
\[ U_k \{p, \alpha, \beta, A, B\} \]
\[(p, k \in \mathbb{N} := \{1, 2, 3, \ldots\}; 0 \leq \alpha < p; \beta \geq 0; \]
\[-1 \leq A < B \leq 1; 0 < B \leq 1), \]
which unifies and extends several (known or new) subclasses of meromorphically multivalent functions. The properties and characteristics of this general class, which are presented here, include growth and distortion theorems; they also involve Hadamard products (or convolution) of functions belonging to the class \( U_k[p, \alpha, \beta, A, B] \).

1. Introduction, Definitions, and Preliminaries
Let \( \sum_{p,k} \) denote the class of functions of the form:
\[ f(z) = \frac{1}{z^p} + \sum_{n=k}^{\infty} a_{n+p-1} z^{n+p-1} \quad (p, k \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1} \]
which are analytic and \( p \)-valent in the punctured unit disk
\[ U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}. \]

Many interesting families of analytic and multivalent functions were considered by earlier authors in Geometric Function Theory (cf., e.g., [4], [8], and [11]). For a function \( f(z) \) in \( \sum_{p,k} \), and for fixed parameters \( A \) and \( B \), with
\[-1 \leq A < B \leq 1, \quad A + B \geq 0, \text{ and } 0 < B \leq 1, \]
we say that \( f(z) \) is a member of the class \( Q_k[p, \alpha, A, B] \) if and only if it satisfies the inequality:
\[ \left| \frac{zf'(z)}{f(z)} + p \right| < 1 \quad (z \in U^*; 0 \leq \alpha < p). \tag{1.2} \]

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A function \( f(z) \in \sum_{p,k} \) is said to belong to the class \( \mathcal{R}_k[p, \alpha, A, B] \) if and only if
\[
-\frac{zf'(z)}{p} \in \mathcal{Q}_k[p, \alpha, A, B].
\] (1.3)

The classes \( \mathcal{Q}_1[p, \alpha, A, B] \) and \( \mathcal{Q}_1[p, 0, A, B] \) were introduced by Aouf [1] and Mogra [6], respectively. Some subclasses of \( \sum_{p,k} \) when \( k = p = 1 \) were considered by (for example) Miller [5], Pommerenke [9], Clunie [3], and Royster [10]. Furthermore, several subclasses of \( \sum_{p,k} \) when \( k = 1 \) were studied by (amongst others) Mogra ([6], [7]), Aouf ([1], [2]), and Uralegaddi and Ganigi [12].

Motivated essentially by many of these earlier works, we aim at investigating here various properties and characteristics of the above-defined general class
\[
\mathcal{U}_k[p, \alpha, \beta, A, B]
\]

\((p, k \in \mathbb{N}; 0 \leq \alpha < p; \beta \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1)\)
of meromorphically \( p \)-valent functions in
\[
\mathcal{U} := \mathcal{U}^* \cup \{0\} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.
\]
The following result can be proven fairly easily by appealing to the definition of the class \( \mathcal{Q}_k[p, \alpha, A, B] \).

**Lemma 1.** Let a function \( f(z) \) defined by (1.1) be in the class \( \sum_{p,k} \). If
\[
\sum_{n=k}^{\infty} C(p, \alpha, A, B; n) |a_{n+p-1}| \leq D(p, \alpha, A, B)
\]
(1.4)

\((0 \leq \alpha < p; -1 \leq A < B \leq 1; 0 < B \leq 1)\),

where, for convenience,
\[
C(p, \alpha, A, B; n) = (1 + B)(n - 1) + [2p + 2\alpha B + (B + A)(p - \alpha)] (n \geq k)
\]
(1.5)

and
\[
D(p, \alpha, A, B) = (B - A)(p - \alpha),
\]
(1.6)

then \( f(z) \in \mathcal{Q}_k[p, \alpha, A, B] \).

Next, by observing that
\[
f(z) \in \mathcal{R}_k[p, \alpha, A, B] \iff -\frac{zf'(z)}{p} \in \mathcal{Q}_k[p, \alpha, A, B],
\]
(1.7)

we arrive at

**Lemma 2.** Let a function \( f(z) \) defined by (1.1) be in the class \( \sum_{p,k} \). If
\[
\sum_{n=k}^{\infty} \left( \frac{n+p-1}{p} \right) C(p, \alpha, A, B; n) |a_{n+p-1}| \leq D(p, \alpha, A, B)
\]
(1.8)

\((0 \leq \alpha < p; -1 \leq A < B \leq 1; 0 < B \leq 1)\),

where \( C(p, \alpha, A, B; n) \) and \( D(p, \alpha, A, B) \) are given by (1.5) and (1.6), respectively, then
\( f(z) \in \mathcal{R}_k[p, \alpha, A, B] \).

In view of Lemma 1 and Lemma 2, we define the subclasses \( \mathcal{Q}^*_k[p, \alpha, A, B] \) of \( \mathcal{Q}_k[p, \alpha, A, B] \) and \( \mathcal{R}^*_k[p, \alpha, A, B] \) of \( \mathcal{R}_k[p, \alpha, A, B] \) consisting of functions which, respectively, satisfy (1.5) and (1.8).
Furthermore, we introduce and investigate the various properties and characteristics of the following general class \( \mathcal{U}_k[p, \alpha, \beta, A, B] \) of functions \( f(z) \in \sum_{p,k} \) which also satisfy the inequality:

\[
\sum_{n=k}^{\infty} C(p, \alpha, A, B; n) \left[ 1 - \beta + \beta \left( \frac{n + p - 1}{p} \right) \right] |a_{n+p-1}| \leq D(p, \alpha, A, B)
\] (1.9)

\[0 \leq \alpha < p; \beta \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1\]

where \( C(p, \alpha, A, B; n) \) and \( D(p, \alpha, A, B) \) are given by (1.5) and (1.6), respectively. Clearly, we have

\[
\mathcal{U}_k[p, \alpha, \beta, A, B] = (1 - \beta) \mathcal{Q}_k^*[p, \alpha, A, B] + \beta \mathcal{R}_k^*[p, \alpha, A, B],
\]

so that

\[
\mathcal{U}_k[p, \alpha, 0, A, B] = \mathcal{Q}_k^*[p, \alpha, A, B]
\]

and

\[
\mathcal{U}_k[p, \alpha, 1, A, B] = \mathcal{R}_k^*[p, \alpha, A, B].
\]

2. Growth and Distortion Theorems

**Theorem 1.** If a function \( f(z) \) defined by (1.1) is in the class \( \mathcal{U}_k[p, \alpha, \beta, A, B] \), then

\[
\frac{1}{|z|^p} - \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right]} |z|^{k+p-1} \leq |f(z)|
\]

\[
\leq \frac{1}{|z|^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right]} |z|^{k+p-1} \] (2.1)

\[(\beta \geq 0; z \in \mathcal{U}^*)
\]

and

\[
\frac{p}{|z|^{p+1}} - \frac{(k + p - 1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right]} |z|^{k+p-2} \leq |f'(z)|
\]

\[
\leq \frac{p}{|z|^{p+1}} + \frac{(k + p - 1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right]} |z|^{k+p-2} \] (2.2)

\[(\beta \geq 0; z \in \mathcal{U}^*)
\]

The bounds in (2.1) and (2.2) are attained for the function \( f(z) \) given by

\[
f(z) = \frac{1}{z^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right]} z^{k+p-1}.
\]

**Proof.** Noting that

\[
\sum_{n=k}^{\infty} |a_{n+p-1}| \leq \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right]}
\]

(2.4)
for $f(z) \in \mathcal{U}_k [p, \alpha, \beta, A, B]$, we have

$$|f(z)| \geq \frac{1}{|z|^p} - |z|^{k+p-1} \sum_{n=k}^{\infty} |a_{n+p-1}|$$

$$\geq \frac{1}{|z|^p} - \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} \left[ 1 - \beta + \beta \left( \frac{k+p-1}{p} \right) \right] |z|^{k+p-1}$$

($\beta \geq 0; \ z \in \mathcal{U}^*$) \hspace{4cm} (2.5)

and

$$|f(z)| \leq \frac{1}{|z|^p} + |z|^{k+p-1} \sum_{n=k}^{\infty} |a_{n+p-1}|$$

$$\leq \frac{1}{|z|^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} \left[ 1 - \beta + \beta \left( \frac{k+p-1}{p} \right) \right] |z|^{k+p-1}$$

($\beta \geq 0; \ z \in \mathcal{U}^*$). \hspace{4cm} (2.6)

We also observe that

$$C(p, \alpha, A, B; k) \left[ 1 - \beta + \beta \left( \frac{k+p-1}{p} \right) \right] \sum_{n=k}^{\infty} (n+p-1) |a_{n+p-1}|$$

$$\leq \sum_{n=k}^{\infty} C(p, \alpha, A, B; n) \left[ 1 - \beta + \beta \left( \frac{n+p-1}{p} \right) \right] |a_{n+p-1}| \leq D(p, \alpha, A, B) \ (\beta \geq 0),$$

which readily yields the following distortion inequalities:

$$|f'(z)| \geq \frac{p}{|z|^{p+1}} - |z|^{k+p-2} \sum_{n=k}^{\infty} (n+p-1) |a_{n+p-1}|$$

$$\geq \frac{p}{|z|^{p+1}} - \frac{(k+p-1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-2}$$

($\beta \geq 0; \ z \in \mathcal{U}^*$) \hspace{4cm} (2.8)

and

$$|f'(z)| \leq \frac{p}{|z|^{p+1}} + |z|^{k+p-2} \sum_{n=k}^{\infty} (n+p-1) |a_{n+p-1}|$$

$$\leq \frac{p}{|z|^{p+1}} + \frac{(k+p-1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-2}$$

($\beta \geq 0; \ z \in \mathcal{U}^*$). \hspace{4cm} (2.9)

Now it is easy to see that the bounds in (2.1) and (2.2) are attained for the function $f(z)$ given by (2.3).

Taking $\beta = 0$ in Theorem 1, we have
Corollary 1. If a function $f(z)$ defined by (1.1) is in the class $Q_k^* [p, \alpha, A, B]$, then
\[
\frac{1}{|z|^p} - \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-1} \leq |f(z)| \leq \frac{1}{|z|^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-1} (z \in \mathcal{U}^*) \tag{2.10}
\]
and
\[
\frac{p}{|z|^{p+1}} - \frac{(k+p-1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-2} \leq |f'(z)| \leq \frac{p}{|z|^{p+1}} + \frac{(k+p-1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-2} (z \in \mathcal{U}^*) \tag{2.11}
\]
The bounds in (2.10) and (2.11) are attained for the function:
\[
f(z) = \frac{1}{z^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} z^{k+p-1}. \tag{2.12}
\]
Letting $\beta = 1$ in Theorem 1, we have

Corollary 2. If a function $f(z)$ defined by (1.1) is in the class $R_k^* [p, \alpha, A, B]$, then
\[
\frac{1}{|z|^p} - \frac{pD(p, \alpha, A, B)}{(k+p-1)C(p, \alpha, A, B; k)} |z|^{k+p-1} \leq |f(z)| \leq \frac{1}{|z|^p} + \frac{pD(p, \alpha, A, B)}{(k+p-1)C(p, \alpha, A, B; k)} |z|^{k+p-1} (z \in \mathcal{U}^*) \tag{2.13}
\]
and
\[
\frac{p}{|z|^{p+1}} - \frac{pD(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-2} \leq |f'(z)| \leq \frac{p}{|z|^{p+1}} + \frac{pD(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} |z|^{k+p-2} (z \in \mathcal{U}^*) \tag{2.14}
\]
The bounds in (2.13) and (2.14) are attained for the function:
\[
f(z) = \frac{1}{z^p} + \frac{pD(p, \alpha, A, B)}{(k+p-1)C(p, \alpha, A, B; k)} z^{k+p-1}. \tag{2.15}
\]

3. Convolution Properties
For functions
\[
f_j(z) = \frac{1}{z^p} + \sum_{n=k}^{\infty} a_{n+p-1,j} z^{n+p-1} \quad (j = 1, 2) \tag{3.1}
\]
belonging to the class $\sum_{p,k}$, we denote by $(f_1 * f_2)(z)$ the convolution (or Hadamard product) of the functions $f_1(z)$ and $f_2(z)$, that is,
\[
(f_1 * f_2)(z) := \frac{1}{z^p} + \sum_{n=k}^{\infty} a_{n+p-1,1} a_{n+p-1,2} z^{n+p-1}. \tag{3.2}
\]
Theorem 2. Let the functions \( f_j(z) \) \((j = 1, 2)\) defined by (3.1) be in the class \( \mathcal{U}_k[p, \alpha, \beta, A, B] \). Then

\[ (f_1 * f_2)(z) \in \mathcal{U}_k[p, \gamma, \beta, A, B], \]

where

\[ \gamma = p - \frac{(B - A)(1 + B)(k + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; k)\}^2 \left[ 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right] + \{D(p, \alpha, A, B)\}^2}. \]  

(3.3)

The result is sharp for the functions:

\[ f_j(z) = \frac{1}{z^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} \left( 1 - \beta + \beta \left( \frac{k + p - 1}{p} \right) \right)^{k + p - 1} \quad (j = 1, 2). \]  

(3.4)

Proof. In order to prove Theorem 2, we must find the largest \( \gamma \) such that

\[ \sum_{n=k}^{\infty} \frac{C(p, \gamma, A, B; n)}{D(p, \gamma, A, B)} \left[ 1 - \beta + \beta \left( \frac{n + p - 1}{p} \right) \right] |a_{n+p-1,1}| |a_{n+p-1,2}| \leq 1 \]  

for \( f_j(z) \in \mathcal{U}_k[p, \gamma, \beta, A, B] \) \((j = 1, 2)\). Since \( f_j(z) \in \mathcal{U}_k[p, \alpha, \beta, A, B] \) \((j = 1, 2)\), we readily see that

\[ \sum_{n=k}^{\infty} \frac{C(p, \gamma, A, B; n)}{D(p, \gamma, A, B)} \left[ 1 - \beta + \beta \left( \frac{n + p - 1}{p} \right) \right] |a_{n+p-1,j}| \leq 1 \quad (j = 1, 2). \]  

(3.6)

Therefore, by the Cauchy-Schwarz inequality, we obtain

\[ \sum_{n=k}^{\infty} \frac{C(p, \gamma, A, B; n)}{D(p, \gamma, A, B)} \left[ 1 - \beta + \beta \left( \frac{n + p - 1}{p} \right) \right] \sqrt{|a_{n+p-1,1}| |a_{n+p-1,2}|} \leq 1. \]  

(3.7)

This implies that we need only show that

\[ \frac{C(p, \gamma, A, B; n)}{p - \gamma} |a_{n+p-1,1}| |a_{n+p-1,2}| \leq \frac{C(p, \alpha, A, B; n)}{p - \alpha} \sqrt{|a_{n+p-1,1}| |a_{n+p-1,2}|} \quad (n \geq k) \]  

or, equivalently, that

\[ \sqrt{|a_{n+p-1,1}| |a_{n+p-1,2}|} \leq \frac{(p - \gamma)C(p, \alpha, A, B; n)}{(p - \alpha)C(p, \gamma, A, B; n)} \quad (n \geq k). \]  

(3.8)

Hence, by the inequality (3.7), it is sufficient to prove that

\[ \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; n)} \left[ 1 - \beta + \beta \left( \frac{n + p - 1}{p} \right) \right] \leq \frac{(p - \gamma)C(p, \alpha, A, B; n)}{(p - \alpha)C(p, \gamma, A, B; n)} \quad (n \geq k). \]  

(3.10)

It follows from (3.10) that

\[ \gamma \leq p - \frac{(B - A)(1 + B)(n + p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; n)\}^2 \left[ 1 - \beta + \beta \left( \frac{n + p - 1}{p} \right) \right] + \{D(p, \alpha, A, B)\}^2} \quad (n \geq k). \]  

(3.11)
Now, defining the function $\varphi(n)$ by
\begin{equation}
\varphi(n) := p - \frac{(B - A) (1 + B) (n + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; n)\}^2 \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right] + \{D(p, \alpha, A, B)\}^2} \quad (n \geq k),
\end{equation}
we see that $\varphi(n)$ is an increasing function of $n$. Therefore, we conclude that
\begin{equation}
\gamma \leq \varphi(k) = p - \frac{(B - A) (1 + B) (k + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; k)\}^2 \left[1 - \beta + \beta \left(\frac{k+p-1}{p}\right)\right] + \{D(p, \alpha, A, B)\}^2},
\end{equation}
which evidently completes the proof of Theorem 2.

Letting $\beta = 0$ in Theorem 2, we arrive at

**Corollary 3.** Let the functions $f_j(z) \,(j = 1, 2)$ defined by (3.1) be in the class $Q_k^*\,[p, \gamma, A, B]$. Then
\begin{equation}
(f_1 \ast f_2) (z) \in Q_k^*\,[p, \gamma, A, B],
\end{equation}
where
\begin{equation}
\gamma = p - \frac{(B - A) (1 + B) (k + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; k)\}^2 + \{D(p, \alpha, A, B)\}^2}.
\end{equation}
The result is sharp for the functions:
\begin{equation}
f_j(z) = \frac{1}{z^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)} z^{k+p-1} \quad (j = 1, 2).
\end{equation}

Putting $\beta = 1$ in Theorem 2, we have

**Corollary 4.** Let the functions $f_j(z) \,(j = 1, 2)$ defined by (3.1) be in the class $R_k^*\,[p, \alpha, A, B]$. Then
\begin{equation}
(f_1 \ast f_2) (z) \in R_k^*\,[p, \gamma, A, B],
\end{equation}
where
\begin{equation}
\gamma = p - \frac{p (B - A) (1 + B) (k + 2p - 1)(p - \alpha)^2}{(k + p - 1) \{C(p, \alpha, A, B; k)\}^2 + p \{D(p, \alpha, A, B)\}^2}.
\end{equation}
The result is sharp for the functions:
\begin{equation}
f_j(z) = \frac{1}{z^p} + \frac{pD(p, \alpha, A, B)}{(k + p - 1)C(p, \alpha, A, B; k)} z^{k+p-1} \quad (j = 1, 2).
\end{equation}

Finally, we prove

**Theorem 3.** Let the functions $f_j(z) \,(j = 1, 2)$ defined by (3.1) be in the class $U_k\,[p, \alpha, \beta, A, B]$. Then the function $h(z)$ defined by
\begin{equation}
h(z) := \frac{1}{z^p} + \sum_{n=k}^{\infty} (a_{n+p-1,1} + a_{n+p-1,2}) z^{n+p-1}
\end{equation}
belongs to the class $\mathcal{U}_k[p, \gamma, \beta, A, B]$, where

$$\gamma = p - \frac{2(B - A) (1 + B) (k + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; k)\}^2 \left[1 - \beta + \beta \left(\frac{k+p-1}{p}\right)\right] + 2 \{D(p, \alpha, A, B)\}^2}. \quad (3.19)$$

The result is sharp for the functions $f_j(z)$ ($j = 1, 2$) given by (3.4).

Proof. Noting that

$$\sum_{n=k}^{\infty} \frac{\{C(p, \alpha, A, B; n)\}^2 \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right]^2}{\{D(p, \alpha, A, B)\}^2} |a_{n+p-1,j}|^2 \leq \left(\sum_{n=k}^{\infty} \frac{C(p, \alpha, A, B; n) \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right]}{D(p, \alpha, A, B)} |a_{n+p-1,j}|\right)^2 \leq 1 \quad (j = 1, 2) \quad (3.20)$$

for $f_j(z) \in \mathcal{U}_k[p, \alpha, \beta, A, B]$ ($j = 1, 2$), we have

$$\sum_{n=k}^{\infty} \frac{\{C(p, \alpha, A, B; n)\}^2 \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right]^2}{2 \{D(p, \alpha, A, B)\}^2} |a_{n+p-1,1}^2 + a_{n+p-1,2}^2| \leq 1. \quad (3.21)$$

Therefore, we have to find the largest $\gamma$ such that

$$\frac{C(p, \gamma, A, B; n)}{p - \gamma} \leq \frac{\{C(p, \alpha, A, B; n)\}^2 \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right]}{2(B - A) (p - \alpha)^2} \quad (n \geq k), \quad (3.22)$$

that is, that

$$\gamma \leq p - \frac{2(B - A) (1 + B) (n + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; n)\}^2 \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right] + 2 \{D(p, \alpha, A, B)\}^2} \quad (n \geq k). \quad (3.23)$$

Now, defining a function $\psi(n)$ by

$$\psi(n) := p - \frac{2(B - A) (1 + B) (n + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; n)\}^2 \left[1 - \beta + \beta \left(\frac{n+p-1}{p}\right)\right] + 2 \{D(p, \alpha, A, B)\}^2} \quad (n \geq k), \quad (3.24)$$

we observe that $\psi(n)$ is an increasing function of $n$. Thus we conclude that

$$\gamma \leq \psi(k) = p - \frac{2(B - A) (1 + B) (k + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; k)\}^2 \left[1 - \beta + \beta \left(\frac{k+p-1}{p}\right)\right] + 2 \{D(p, \alpha, A, B)\}^2}, \quad (3.25)$$

which completes the proof of Theorem 3.

By setting $\beta = 0$, Theorem 3 leads us to

**Corollary 5.** Let the functions $f_j(z)$ ($j = 1, 2$) defined by (3.1) be in the class $\mathcal{Q}_k^*[p, \alpha, A, B]$. Then the function $h(z)$ defined by (3.18) belongs to the class $\mathcal{Q}_k^*[p, \gamma, A, B]$, where

$$\gamma = p - \frac{2(B - A) (1 + B) (k + 2p - 1)(p - \alpha)^2}{\{C(p, \alpha, A, B; k)\}^2 + 2 \{D(p, \alpha, A, B)\}^2}. \quad (3.26)$$

The result is sharp for the functions $f_j(z)$ ($j = 1, 2$) given by (3.15).
Letting $\beta = 1$ in Theorem 3, we have

**Corollary 6.** Let the functions $f_j(z)$ ($j = 1, 2$) defined by (3.1) be in the class $\mathcal{R}_K^+[p, \alpha, A, B]$. Then the function $h(z)$ defined by (3.18) belongs to the class $\mathcal{R}_K^+[p, \gamma, A, B]$, where

$$
\gamma = p - \frac{2p(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{(k+p-1)(C(p, \alpha, A, B; k))^2 + 2p(D(p, \alpha, A, B))^2}.
$$

(3.27)

The result is sharp for the functions $f_j(z)$ ($j = 1, 2$) given by (3.17).

Many of our results in this paper (especially Corollaries 1 to 6) would simplify considerably when we set

$$A = -1 \quad \text{and} \quad B = 1.
$$

The details involved in the derivation of these and other special cases of our results may be left as an exercise for the interested reader.

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