GLOBAL EXISTENCE AND VALIDITY FOR
THE LIMITING ENSKOG HIERARCHY

by

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§1 INTRODUCTION

This paper arose in an attempt to generalize the celebrated result of Oscar Lanford [19] [20] (see also [25]). In those papers he presented, at least for short times and in full space, a derivation of the Boltzmann equation from the molecular dynamics of large numbers of particles. Since that time his result has been repeatedly used in investigations into nonequilibrium statistical mechanics, but the restriction to short times was a problem. Our goal was to obtain a result which was global in time and would also allow bounded spacial domains. However, we were only able to rigorously show that for all time, the rescaled correlation functions associated with the motion of finitely many hard spheres approach weak solutions of a limiting hierarchy of Enskog type in the Boltzmann -Grad limit. The results will be described in more detail shortly.

But a careful reading of Lanford’s papers revealed a gap which has apparently not been noticed until now. To indicate the difficulty, we now review Lanford’s argument. Lanford’s proof begins with a strong form of the BBGKY hierarchy for finitely many hard spheres. Lanford then considers an infinite hierarchy called the Boltzmann hierarchy, which is similar to the
BBGKY hierarchy except that the number $n$ of particles is infinite and the
diameter $\delta$ of the particles is zero. He then constructs solutions of
integrated versions of the strong forms of both the BBGKY and the Boltzmann
hierarchies by means of time-dependent perturbation series. Finally, he shows
that for short times and under suitable assumptions, these series converge
uniformly, and the terms of the first approach the terms of the second as $n \to \infty$.

The problem is that so far we have only been able to rigorously establish
a weak version of the BBGKY hierarchy. That there is a problem with the formal
derivation of the strong hierarchy used by Lanford (see Cercignani [9]) was
first pointed out by H. Spohn, who discussed a more complicated version in the
unpublished paper [27]. Thus the first thing we do in this paper is derive as
a lemma a version of the usual finite BBGKY hierarchy, but the crucial fact is
that it is only a weak version. After the research on the lemma reported here
was completed [13], the author learned of an independent and different
derivation of the hierarchy by Illner and Pulvirenti [15] [16], but it is
again only a weak version.

Thus what is not clear is whether the physically relevant correlation
functions, which are obtained from the $n$-particle functions by integrating
down, and as we have seen above are solutions of the weak BBGKY hierarchy, are
identical with the strong ones obtained from Lanford's series solution. This
would be established if we knew that the series solution satisfies the weak
BBGKY hierarchy, and if we could also could establish a uniqueness theorem for
weak solutions of the BBGKY hierarchy. This latter uniqueness result could be
difficult due to the fact that the collision integral is only defined almost
everywhere. R. Illner is working on the problem, but at present this is the
gap that still remains in Lanford's proof.
There is reason to believe that Lanford's result will not hold in certain
discrete velocity models in which angular dependence is crucial, but the
result of this paper should go through.

The method used here avoids the uniqueness question in Lanford's
development. It yields a general, global (i.e., holding for all time)
existence and validity result for weak solutions of a limiting Enskog
hierarchy in three dimensions in a bounded domain with sufficiently smooth
(e.g., $C^2$) boundary. In more detail, we show that, under very general
assumptions on the initial conditions for finite systems, the set of all weak
limits of the sequences of rescaled correlation evolutions in the
Boltzmann-Grad limit is nonempty, and that each such weak limit satisfies an
integrated weak version of a hierarchy which we call the Enskog hierarchy. It
is the natural limit as $n \to \infty$ of the BBGKY hierarchy. Furthermore, the
solutions have bounded energy and $H$-function, and so lie in the most general
physically reasonable class. In addition, if the initial conditions have, in a
natural sense, bounded moments, then the solutions also have bounded moments.
We also show in §5 that, under mild assumptions, the results are applicable in
the case of factored initial data. Our results are closely related to recent
work on the BBGKY hierarchy [14], especially in that nonstandard analysis is
used in the proof, but, as there, the results are standard. The reader should
also be aware of related applications of nonstandard analysis to the Boltzmann
equation by Arkeryd, e.g., [5] [6] [7].

The papers by Illner and Pulvirenti [15] and Pulvirenti [23] are also
devoted to the validity question, globally valid in time, but again they
follow the Lanford argument and so are subject to the same problems noted
above. They are based on the good contracting properties of free flow in full
space on functions rapidly decreasing at infinity. This idea was previously
used in the existence theory by Illner and Shinbrot [17] and Shinbrot [26].
Also, the Illner-Pulvirenti and Pulvirenti results are restricted to the
Cauchy problem in the case of a perturbation from vacuum for large mean free
paths, and in particular, do not apply in the more relevant context of bounded
domains.

This work is dedicated to the memory of Marvin Shinbrot who suggested a
nonstandard attack on the problem many years ago. His support in the
intervening years was unwavering, and his inspiration will be much missed. I
am also indebted to Reinhard Illner and Mario Pulvirenti for many helpful
discussions during the preparation of this paper. I especially want to thank
Leif Arkeryd for suggestions which resulted in essential improvements on an
earlier version of this paper.

§2. THE RESULTS

Throughout the paper, Λ denotes a domain in \( \mathbb{R}^2 \) with a \( C^2 \) boundary \( \partial \Lambda \)
(this condition could be weakened to the piecewise smooth conditions
introduced in [1]), and volume \( |\Lambda| \). \( N \) denotes the set of natural numbers. The
one particle phase space is the set \( \Lambda \times S \), where \( S = \mathbb{R}^2 \) equipped with the usual
Borel structure. Phase points in the one particle phase space are denoted by
\((q, p)\), where \( q \) is the position in \( \Lambda \) and \( p \) the momentum of a particle.
Throughout, we assume for convenience that each particle is of unit mass. We
let \( \pi_n = (\Lambda \times S)^n \) (n-fold cartesian product), where we define \( \pi_0 \) to be a single
point. On \( \pi_n \) we can define an equivalence relation \( \sim \) by putting \((x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)\) if there exists a permutation \( \xi \) of \( \{1, 2, \ldots, n\} \) with \( x_i = y_{\xi(i)} \) for
\( 1 \leq i \leq n \). The quotient measurable space of \( \pi_n \) under \( \sim \) is denoted by \( \pi_n / \sim \), with
a resulting measurable map \( \xi_n : \pi_n / \sim \to \pi_n \). The grand canonical phase space is
then, as usual, defined to be the measurable space \( \pi_\infty = \bigcup \pi_n \) (\( n \in \mathbb{N} \)), and is
thus the set of all collections of finitely many particles (with repetitions) in \( \Lambda \). If we define \( \pi = \bigcup \pi_n (n \geq 0) \), then there is a map \( \xi : \pi \rightarrow \pi_o \) which reduces to \( \xi_n \) on each summand. Points in \( \pi_o \) and \( \pi \) are denoted by \( x = (x_1, x_2, \ldots, x_n) = (x^n) \) and \( x = (x_1, \ldots, x_n) = (x^n) \), where \( x_i = (q_i, p_i) \), \( q_i \in \Lambda, p_i \in \mathbb{R} \) and we use the notation \( x^n = (x_1, \ldots, x_n) \).

We consider particles which are hard spheres, so it is necessary to discuss subsets of \( \pi_o \) in which phase points cannot come too close together, or too near the boundary \( \partial \pi \) of \( \Lambda \). Thus for \( \delta \geq 0 \) in \( \mathbb{R} \), we define \( \pi^\delta \) to be the set of points \( x \in \pi_o \) such that if \( x = (x_1, x_2, \ldots, x_n) \), and \( x_i = (q_i, p_i) \), then \( \inf \{ |q_i - q_j| : 1 \leq i < j \leq n \} \geq \delta \), and \( \inf \{ |q_i - q| : 1 \leq i \leq n, q \in \partial \pi \} \geq \delta/2 \), where \( | \cdot | \) denotes the norm in \( \mathbb{R}^3 \). The set \( \pi^\delta \) inherits its measurable structure from \( \pi_o \). In the same way, we define \( \pi^\delta_n = \{ x \in \pi_n : \inf \{ |q_i - q_j| : 1 \leq i < j \leq n \} \geq \delta \} \), \( \inf \{ |q_i - q| : 1 \leq i \leq n, q \in \partial \pi \} \geq \delta/2 \} \), and \( \pi^\delta = \bigcup \pi^\delta_n (n \geq 0) \). Clearly, \( \xi \) maps \( \pi^\delta \) onto \( \pi^\delta \).

We can now use the constructions of the previous paragraph to define measures on \( \pi_o \). The natural measure on \( \pi \) is \( d\sigma = \sum d\sigma_n (x^n)/n! (n \geq 0) \), where \( d\sigma_n \) is an \( n \)-fold normalized Maxwellian product measure on \( \pi_n \) (\( d\sigma^0 \) being point mass on the single point \( \pi^0 \) \), defined by

\[
(2.1) \quad d\sigma_n (x^n) = \prod_{i=1}^{n} d\sigma_i ,
\]

where

\[
(2.2) \quad d\sigma_i = \frac{\beta}{2} \frac{p_i^2}{d\chi_i} dx_i, \quad 1 \leq i \leq n.
\]
Here \( \beta \) is a positive constant which plays no particular role in the discussion, \( c = |A|^{-\frac{3}{2}} (\beta / 2\pi) \) is a normalizing constant, and \( dx_i = dq_i dp_i \).

Since \( \xi \) is measurable, we may put \( \bar{\sigma} = \sigma \xi^{-\frac{1}{n}} \), the measure induced on \( X_n \) by \( \sigma \).

Also, \( \bar{\sigma}_n \) is \( \bar{\sigma} \) restricted to \( X_n \). This use of Maxweliens is for convenience only, and alters slightly the form of some standard expressions, in particular, the BBGKY hierarchy.

Our states (probability measures) \( \mu \) on \( X_n \) are obtained from measurable densities \( \bar{f} \) on \( X_n \), so that, for measurable \( \bar{F} \),

\[
(2.3) \quad \mu(\bar{F}) = \int_{\bar{F}} \bar{f} \ d\bar{\sigma}.
\]

Such states can also be given by

\[
(2.4) \quad \mu(\bar{F}) = \int_{\bar{F}} \bar{f} \ d\bar{\sigma} = \sum_n \frac{1}{n!} \int_{\bar{F}_n} \bar{f} \ d\bar{\sigma}^{(n)} \ (n \geq 0)
\]

where \( \bar{f} = \bar{f} \circ \xi, \bar{F} = \xi^{-1}(\bar{F}), \bar{f}_n = \bar{f}|_{\bar{F}_n}(A), \bar{F}_n = \bar{F}|_{\bar{F}_n} \).

Given a state \( \mu \) on \( X_n \), the \( k \)th correlation measure of \( \mu \) is a measure on \( \eta_k \) given \([21]\) by

\[
(2.5) \quad \rho(A) = \int N_A(x) \ d\mu(x),
\]

where, if \( x = \{x_1, x_2, \ldots, x_n\} \) enumerated in some order, then \( N_A(x) \) is the number of sequences \( (i_1, i_2, \ldots, i_k) \) with the \( i_s \) distinct, such that
\((x_1, x_2, \ldots, x_k) \in \mathcal{A}\). Using the representation (2.4), the \(k\)th correlation measure \(\rho_k\) of \(\mu\) has a density (correlation function) which takes the form

\[
(2.6) \quad \rho_k(x^k) = \sum_{n-k}^{1} \int \frac{1}{(n-k)!} \int f_n(x^n) \, d\sigma_{k+1} \ldots d\sigma_n \quad (n \geq k),
\]

where the integration is over \(x_{n-k}\).

Suppose now that \(\{T_t^x\}\) is a family (usually a group) of measurable transformations \(T_t^x : X_0 \to X_0\). Given an initial state \(\mu\), we define its time evolution \(\mu(t)\) by \(\mu(t)(F) = \mu(T_t^{-1}(F))\). Clearly, \(\mu(t)(F) = \int f(t) d\sigma\), and the correlation function of \(\mu(t)\) is given by

\[
(2.7) \quad \rho_k(t)(x^k) = \sum_{n-k}^{1} \int \frac{1}{(n-k)!} \int f_n(t)(x^n) \, d\sigma_{k+1} \ldots d\sigma_n \quad (n \geq k).
\]

Here and throughout, we use the notation \(f(t) = f \circ T_t^{-1}\) and \(f(t)(x^n) = f(t, x^n)\) for any function \(f\).

We now consider a family of structures on \(X_0\) indexed by \(m \in \mathbb{N}\). These structures include a family \(\{T_t^x\}_m\) of hard sphere flows on \(X_0\), a family of Borel probability measures \(\{\mu_k^m\}\) with an associated family of correlation functions \(\{\rho_k^m\}\), and an associated family \(\{\Psi_k^m\}\) of rescaled correlation functions to be defined shortly. Throughout the discussion we assume given a positive function \(\delta = \delta(m)\) which satisfies the Boltzmann – Grad condition

\[
(2.8) \quad \lim_{m \to \infty} m[\delta(m)]^2 = \kappa.
\]
First we define the dynamical flows. For each \( m \in \mathbb{N} \), we imagine each point in \( X^\delta(\Lambda) \) surrounded by a particle of diameter \( \delta = \delta(m) \). The particles move according to the hard sphere dynamics; that is, each particle moves in a straight line until it encounters another particle or the boundary. In a collision with the boundary, the particles reflect elastically. In a collision between two particles having positions and momenta \((q_i, p_i)\) and \((q_j, p_j)\), the ingoing momenta \( p_i \) and \( p_j \) are abruptly changed to the outgoing momenta \( p_i' \) and \( p_j' \), where

\[
p_i' = p_i - k_i(p_j, \omega_{ij}) \omega_{ij},
\]

\[
p_j' = p_j + k_i(p_j, \omega_{ij}) \omega_{ij},
\]

(2.9)

\( \omega_{ij} = (q_j - q_i) / |q_j - q_i| \) is the unit vector giving the direction between the centers, and \( k_i(p, \omega) = \omega \cdot (p_i - p) \). After this interchange of momenta, the particles continue in rectilinear motion.

There are several problems with this definition of the dynamics. Firstly, it is not clear that, beginning with given initial conditions, we can find the positions and momenta of the particles at any later time, and indeed, for some initial conditions, multiple and grazing collisions can occur and the trajectory cannot be uniquely continued for all time. But Alexander [11] has shown that these cases occur only on a set of measure zero in \( X^\delta \). Thus there exists a measurable subset \( \hat{X}^\delta \) of \( X^\delta \) of full measure, and a group \( \{ T^t_m \} : \hat{X}^\delta \to \hat{X}^\delta \) \( : t \in \mathbb{R} \) of measurable transformations, where \( T^t_m(x) \) is obtained by evolving the initial conditions \( x \in \hat{X}^\delta \) over a time \( t \) using the hard sphere dynamics associated with particles of diameter \( \delta = \delta(m) \). Similarly, there is a group,
which we also denote by \( \{ T_m^t \} \), of maps \( T_m^t : \hat{\Omega}^\delta \rightarrow \hat{\Omega}^\delta \), \( \hat{\Omega}^\delta = \xi^{-1}(\hat{\Omega}^\delta) \), obtained by labelling the particles and using the hard sphere dynamics; the flows \( T_m^t \) commute with the map \( \xi \). Note that the flows are so far defined only on \( \hat{\Omega}^\delta \) and \( \hat{\Omega}^\delta \), which may even be empty if \( \delta(m) \) is too large; we define the \( T_m^t \) everywhere by putting \( T_m^t(x) = x \) for \( x \in X^\delta - \hat{\Omega}^\delta \) or \( x \in \Omega - \hat{\Omega} \).

The second and intrinsic problem is that the maps \( T_m^t \) are not even continuous, let alone differentiable. Due to the abrupt changes in momentum, the trajectories are only piecewise continuous. To be definite we impose the condition that the trajectories are continuous from the left, that is,

\[
\lim_{t \to -0} T_m^t(x) = T_m^r(x),
\]

Of most concern to us is the fact that the measures \( d\sigma \) and \( \bar{d}\sigma \) are invariant under \( T_m^t \).

We want to consider a special class \( \{ \mu^m \} \) of states used for initial conditions. These have \( \{ \text{symmetric} \} \) densities \( f^m \) on \( \Omega \) of the form

\[
f^m_n = \begin{cases} m^! g^m, & n = m \\ \emptyset, & n \neq m \end{cases}
\]

where \( g^m \) satisfies the following conditions:

2.1  
(i) \( g^m \) is concentrated on \( \Omega_m \) and is zero on \( \Omega_m - \hat{\Omega}^\delta_m \);  
(ii) \( \int g^m d\sigma^m = 1; \)
(iii) There is a \( K \) so that 
\[
\sup_{m} \frac{1}{m} \int g^{m} \log g^{m} \, d\sigma^{m} < K.
\]
(iv) There is an \( L \) so that 
\[
\sup_{m} \frac{1}{m} \int g^{m} \left( \sum_{1 \leq i \leq m} p_{i} \right)^{2} \, d\sigma^{m} < L,
\]
where \( p_{i} = 1 \) if \( i \).

We will see in §5 that conditions 2.1 allow us to deal in the limit with the special case that the rescaled correlation functions initially factor. This is the important case of initial molecular chaos.

The associated time-evolving correlation functions of the states \( \mu^{m}(t) \) are

\[
(2.12) \quad \rho_{k}^{m}(t,x^{k}) = \frac{m^{\dagger}}{(m-k)!} \int g^{m}(t,x^{m}) \, d\sigma_{k+1} \ldots d\sigma_{m} \quad \text{for} \quad k < m.
\]
\[
= m^{\dagger} g^{m}(t,x^{m}), \quad k = m
\]
\[
= 0, \quad k > m.
\]

Since the integrals of these correlation functions diverge as \( m \to \infty \), one defines the rescaled correlation functions \( \psi_{k}^{m} \) by

\[
(2.13) \quad \psi_{k}^{m}(t,x^{k}) = m^{-k} \rho_{k}^{m}(t,x^{k}).
\]

The first lemma we prove (in §3) is an integrated form of the BBGKY hierarchy for finitely many hard spheres. It states that for a certain class of test functions \( \varphi \), the function \( \int \varphi(t,x^{k}) \psi_{k}^{m}(t,x^{k}) d\sigma_{1} \ldots d\sigma_{k} \) is absolutely continuous and hence differentiable almost everywhere as a function of \( t \), and
satisfies an integrated form of the BBGKY hierarchy. The functions \( \Phi \) which are allowed are specified in the following definition.

2.2 Definition

Let \( \Phi_k^\delta \) consist of those \( \Phi \in C^1_o(\mathbb{R} \times \pi_k) \) (i.e., continuously differentiable functions of compact support) which also vanish if \( x_i^k \) is in a neighborhood of \( \partial \pi_k \cup (\pi_k(\Lambda) - \pi_k^\delta) \).

The set \( \Phi_k \) is similarly defined except that the functions vanish in a neighborhood of \( \partial \pi_k \cup \{ x_i \in \pi_k(\Lambda) : q_i = q_j \}, \) where \( 1 \leq i < j \leq k \).

Note that \( \Phi_k^\delta \) is dense in \( L^1(\mathbb{R} \times \pi_k) \).

In the following, we use the notation \( \langle \psi_k^m(t), \Phi(t) \rangle = \int \psi(t, x^k) \psi_k^m(t, x^k) d\sigma_1 \ldots d\sigma_k \). The BBGKY hierarchy for finitely many hard spheres takes the form described in

2.3 Lemma

Let \( t \in \mathbb{R}^+ \). For any \( \Phi \in \Phi_k^\delta \) and \( 1 \leq k < m \), we have

\[
(2.14) \quad \langle \psi_k^m(t), \Phi(t) \rangle = \langle \psi_k^m(\emptyset), \Phi(\emptyset) \rangle + \int_0^t \langle \psi_k^m(s), \Phi(s) \rangle + \int_0^t \langle \psi_k^m(s), H\Phi(s) \rangle + \int_0^t C_{k+1}^{m} \psi_{k+1}^m(s) \Phi(s) \rangle \, ds,
\]

where

\[
(2.15) \quad H\Phi = \frac{\partial \Phi}{\partial t} + \sum_{i=1}^k p_i \frac{\partial \Phi}{\partial q_i},
\]
\begin{equation}
C_{k+1}^m \psi_{k+1}(\varphi(t)) = \sum_{i=1}^{k} \int_A \varphi(t, x) \psi_{k+1}(t, \ldots, q_i, p', \ldots, q_i - \delta \omega, p') - \psi_{k+1}(t, x, q_i + \delta \omega, p') \, k_i(p, \omega) \, d\sigma - \frac{\beta^2}{2} \, d\omega dp,
\end{equation}

\( \delta = \delta(m), \) and \( A = \{ (x^k, \omega, p): |\omega| = 1, \omega \cdot (p_i - p) \geq 0 \}. \) The operator \( C_{k+1}^m \) is called the collision integral.

It is convenient for us to regard the sequence \( \{ \psi_{k+1}^m \}, k \in \mathbb{N} \) as an element of a space of sequences of functions, and hence we write \( \Psi^m = \{ \psi_{k+1}^m \}; k \in \mathbb{N} \}. \) The object of this paper is to show that the set of cluster points of the net \( \{ \Psi^m \} \) of time evolving rescaled sequences of correlation functions \( \Psi^m \) in a natural weak topology is non-empty, and that each such cluster point \( \psi \) satisfies an integrated version of the Boltzmann hierarchy, which is a hierarchy of the same form as \( (2.14) \) except that \( C_{k+1}^m \) is replaced by a similar operator with \( \delta = \emptyset \) (see \( (2.19) \) below).

We now specify the topologies in which the cluster points are to be taken. Recall \( [8] \) that the vague topology on the measures on a topological space \( E \) is the coarsest topology for which the maps \( \mu \rightarrow \int \varphi d\mu \) are continuous for all functions \( \varphi \in C_c(E) \). By restriction, the vague topology applies to \( (\text{measures with}) \) measurable densities. The weak topology of the following definition uses the vague topology.

2.4 Definition

Let \( B \) denote the set of sequences \( \Psi = \{ \psi_{k+1}(t, x^k) \}; k \in \mathbb{N}, t \in \mathbb{R}^+ \} \) such that, for each \( k \in \mathbb{N} \), \( \psi_{k+1}(t) \in L^1(\eta_k) \) for each \( t \in \mathbb{R}^+ \) and \( \psi_k \in L^1([a, b] \times \eta_k) \) for
each interval \([a,b] \subset \mathbb{R}^+\). A neighborhood subbase of \(\hat{\psi} \in \mathcal{B}\) in the weak topology is given by sets of the form

\[
U_{\alpha}^{\hat{\psi}} = \{ \psi : \int_{\mathbb{R}} \hat{\psi}(t)[\hat{\psi}_k(t) - \psi_k(t)] \, d\sigma^k \, dt < \varepsilon, \\
\int_{\mathbb{R}} \hat{\phi}(s)[\hat{\psi}_k(s) - \psi_k(s)] \, d\phi^k \, ds < \delta, \}
\]

where \(\alpha = (t, \varepsilon, \delta, \psi, k)\) and \(\psi \in C_0^1(\mathbb{R} \times \pi_k)\).

We can now state our main theorem.

2.5 Theorem

Suppose the initial states \(\mu_m\) satisfy condition 2.1. Then the set \(\Omega_\infty\) of all cluster points in the weak topology of the family \(\Omega = \{ \psi_m : m \in \mathbb{N} \}\) of rescaled correlation evolutions in \(\mathcal{B}\) is nonempty. Any \(\hat{\psi} = \{ \hat{\psi}_k : k \in \mathbb{N} \} \in \Omega_\infty\) has a representative which satisfies the integrated Boltzmann hierarchy

\[
\hat{\psi}_k(t, \psi(t)) = \langle \hat{\psi}_k(\emptyset), \psi(\emptyset) \rangle + \int_0^t \langle \hat{\psi}_k(s), \hat{\psi}(s) \rangle \, ds, \quad k \in \mathbb{N},
\]

for all \(\psi \in C_0^1(\mathbb{R} \times \pi_k)\). Here, with \(A = \{ (k^i, \omega, p) \in \pi_k \times \mathbb{R} \land : k_i(p, \omega) \geq \emptyset \}\), and \(S_2 = \{ \omega \in \mathbb{R}^3 : |\omega| = 1 \},

\[
C_{k+1} \hat{\psi}_{k+1}[\psi(t)] = \sum_{i=1}^{k} \int_{A} \hat{\psi}(t, x^k) \left\{ \hat{\psi}_{k+1}(\omega, t, \ldots, q_i, p', \ldots, q_i, p') - \hat{\psi}_{k+1}(\omega, t, \ldots, q_i, p, \omega) \right\} \, d\sigma^k \, ce - \frac{\beta^2}{2} \, d\omega \, dp.
\]
Furthermore, the functions \( \bar{\psi}_{k+1} (\omega, t, \ldots, q_i, p', \ldots, q_i, p') \) and \( \tilde{\psi}_{k+1} (\omega, t, \kappa, q_i, p) \) are vague cluster points of the functions \( \psi_{k+1}^m (t, \ldots, q_i, p', \ldots, q_i - \delta \omega, p') \) and \( \psi_{k+1}^m (t, \kappa, q_i + \delta \omega, p) \) in \((\omega, t, \kappa, p'_{-1}, \ldots, p'_{-k})\)-space.

Finally, for each \( k \) there are uniform energy and H-function bounds of the form

\[
\sup_{t} \int_{\mathbb{R}}^{k} \rho_i \psi_k(t) \left( \sum_{i=1}^{k} p_i^2 \right) \, d\sigma_k \leq B
\]

\[
\sup_{t} \int_{\mathbb{R}}^{k} \psi_k(t) \log \psi_k(t) \, d\sigma_k \leq B
\]

for some constant \( B > 0 \).

The important difference between the collision integral (2.19) and the usual collision integral is the dependence of the \( \tilde{\psi}_{k+1} \) on \( \omega \).

§3. PROOF OF THE LEMMA

A few remarks are perhaps in order concerning the proof of this lemma. In the usual derivation of the BBGKY hierarchy for hard spheres [10], the collision integral arises from an integration by parts, and it is not at all clear how the (infinite) interparticle forces enter into it. In [11, I.21], Grad makes some relevant comments in this regard. The proof presented here is of an elementary character, and it is made clear that the collision integral arises
from discontinuities in the momenta in collisions.

As in [14] the proofs use nonstandard analysis, for which the reader is referred to [12] [24] [28]. The analysis is carried out in a sufficiently saturated enlargement \( V(\mathbb{R}) \) of the superstructure \( V(\mathbb{R}) \) over \( \mathbb{R} \). We write \( r \approx s \) if \( r \) and \( s \) in \( \mathbb{R} \) are infinitesimally close. The standard part of \( r \in \mathbb{R} \) is denoted by \( ^o r \) or \( \text{st}(r) \), with the convention that \( ^o r = +\infty \) or \( -\infty \) if \( r \) is positive or negative infinite.

In the ensuing discussion we consider the set of points in \( \mathbb{R}_m^\delta \) which involve collisions between particles. To this end we define the set

\[
(3.1) \quad K_{ij} = \{ x \in \mathbb{R}_m^\delta : q_j = q_i + \delta \omega_{ij}, k_i(p_j, \omega_{ij}) \geq 0, \mid \omega \mid = 1 \}
\]

consisting of those phase points representing an ingoing collision between the \( i \)th and the \( j \)th particles. \( K_{ij} \) is coordinatized by replacing \( x_j = (q_j, p_j) \) in \( x = (x_1, \ldots, x_m) \) by \( (\omega_{ij}, p_j) \). Next we put

\[
(3.2) \quad K = \bigcup K_{ij} \ (1 \leq i < j \leq n).
\]

\( K \) inherits its topology from \( \mathbb{R}_m^\delta \), and on \( K \) there is defined the natural measure \( \nu \) given by \( d\nu = \sum d\nu_{ij} \ (1 \leq i < j \leq n) \), where \( d\nu_{ij} \) is the measure on \( K_{ij} \) given by

\[
(3.3) \quad d\nu_{ij} = \delta^2 \, d\sigma_1 \cdots d\sigma_{j-1} \, d\sigma_{j+1} \cdots d\sigma_n \, \frac{1}{\sqrt{p_j}} \, k_i(p_j, \omega) \, d\omega \, dp_j.
\]

Here \( d\omega \) represents the measure on the unit sphere with center \( q_i \) in \( q_j \) space.
Note that \( R^i_j \cap R^i_j = \emptyset \) since there are no multiple collisions. In the same way we define the set \( R^+ \), representing outgoing collisions, by replacing the condition \( k^i_j(p^i_j, \omega) \geq 0 \) by \( k^i_j(p^i_j, \omega) < 0 \) in (3.1). There is a measure preserving map \( \zeta : R^+ \to R \), where \( \zeta(y) \) is obtained from \( y \in R^+ \) by replacing outgoing by ingoing momenta using (2.8).

We now let \( \Sigma = \{ x \in \hat{\Sigma}_m^\delta : T^t_m(x) \in R \text{ for some } t > 0 \} \) denote the set of points which undergo a collision at some time \( t > 0 \). The set \( \Sigma \) has a representation as a special flow with base \( R \) [2] [3] [22]. To represent \( \Sigma \), we define a function \( \tau \) on \( R \) by

\[
(3.4) \quad \tau(y) = \min \{ t \in R^+ : T^{-t}_m(y) \in R^+ \}.
\]

The function \( \tau \) is never zero and may take the value \( \infty \), but it is lower semi-continuous on \( R \) and hence measurable since we have ruled out multiple and grazing collisions for \( x \in \hat{\Sigma}_m^\delta \) [1].

For any \( x \in \Sigma \), the set \( \{ t \in [0, \infty) : T^t_m(x) \in R \} \) has no finite point of accumulation since we have eliminated multiple collisions. We now let

\[
(3.5) \quad \tilde{\Sigma} = \{ (s, y) : 0 \leq s < \tau(y), y \in R \}.
\]

\( \tilde{\Sigma} \) inherits a topology and measure from \( R \times [0, \infty) \). Next we define a 1-1, bimeasurable, measure-preserving map \( \tilde{\phi}_m : \tilde{\Sigma} \to \tilde{\Sigma} \) by letting \( \tilde{\phi}_m(x) = (s, y) \), where \( s = s(x) = \inf \{ t : T^t_m(x) = y \in R \} \). Also the map \( \tilde{\phi}_m \) is continuous on \( \hat{\Sigma}_m^\delta \). That \( \tilde{\phi}_m \) is measure preserving is established by a straightforward calculation of the Jacobian of the map. We can use the rules (2.9) to define
a measurable map $\Theta : H \to H$ by the prescription $\Theta_m(y) = (T_m^{-\tau(y)}) \quad \text{if} \quad \tau(y) < \infty$

and $\Theta_m(y) = y$ if $\tau(y) = \infty$. Then corresponding to the semigroup $(T_m^{-t}; \quad t \geq 0)$ of maps on $\tilde{\Sigma}$ there is the semigroup $(S_m^{-t}; \quad t \geq 0)$ of maps on $\tilde{\Sigma}$ defined by

$$S_m^{-t}(s, y) = (t + s, y) \quad \text{for} \quad 0 \leq t < \tau(y) - s, \quad (3.6)$$

and $S_m^{-t}(s, y) = (t + s - \tau(y) - \ldots - \tau(\Theta_m^{n-1}y), \Theta_m^n y)$

for $\tau(y) + \ldots + \tau(\Theta_m^{n-1}y) - s \leq t < \tau(y) + \ldots + \tau(\Theta_m^n y) - s.$

$T_m^{-t}$ and $S_m^{-t}$ are related by the equation $\Theta_m \circ T_m^{-t} = S_m^{-t} \circ \Theta_m$. We can replace integration of a function $g$ on $\Sigma$ by integration of $\tilde{g} = g \phi_m^{-1}$ on $\tilde{\Sigma}$. The collision integral involves integration on $\tilde{\Sigma}$.

Throughout the following discussion we suppress the index $m$ on functions and in the dynamics.

Any function $\varphi \in \Phi_\delta_k$ can be regarded as a function on $\tilde{\Sigma}_m^\delta$ which is independent of the variables $x_n$, $k + 1 \leq n \leq m$. In the proof of the lemma we use the fact that for $\varphi \in \Phi_\delta_k$, and $g = g^m$ as in 2.1,

$$\int \varphi(t, x^k) g(t, x^m) d\sigma_1 \ldots d\sigma_m = \int \varphi(t, T_m^+(x^m)) g(x^m) d\sigma_1 \ldots d\sigma_m \quad (3.7)$$

(the integration can be assumed to be over $\tilde{\Sigma}_m^\delta$). Note that on the right-hand side we regard $x^k$ as being a point in $\tilde{\Sigma}_m^\delta$ by the obvious embedding, and the map $T_m^+$, though it acts only on $x^k$, involves the effects of all $m$ particles. In the following, we denote $T_m^+(x)$ by $x(t) = (x_1(t), \ldots, x_m(t))$, where $x_i(t) = (q_i(t), p_i(t)).$ Also we denote $\varphi(t, T_m^+(x))$ by $\varphi(t, x_k(t))$, remembering that $x_k(t)$ depends on $x_1, \ldots, x_m$, as well as $t.$
With the preliminaries over, we are now ready to begin the proof of the lemma. We first assume that the function \( g \) is continuous and supported in a compact set of the form \( \Sigma = A^m \times V \), where \( V = \{ p = (p_1, \ldots, p_m) : \sum p_i^2 (1 \leq i \leq m) \leq v, v \text{ a positive real number} \} \); the proof is completed by a limiting argument. Note that \( \Sigma \cap \hat{n}_m^\delta (\Lambda) \) is invariant under \( T^t \) since \( \sum p_i^2 \) is conserved under collisions.

The proof uses Kiesler's Infinite Sum Theorem \[12\] \[18\], which in our case can be stated as follows: In order to show that \((2.14)\) is true, we need only show that, if \( \psi_k^m = \psi_k \) and \( d\psi_k(t) \{ \psi(t) \} = \psi_k(t+dt) \{ \psi(t+dt) \} - \psi_k(t) \{ \psi(t) \}, \) then

\[
(3.8) \quad d\psi_k(t) \{ \psi(t) \} \bigg/ dt \cong \{\psi_k(t) [H\psi(t)] + m_0^{\frac{2}{k}} C_{k+1} \psi_{k+1}(t) [\psi(t)] \}
\]

for \( t \geq 0 \) in \( \mathbb{H} \mathbb{R} \), where \( dt \) is a positive infinitesimal increment in time and \( \cong \) denotes the relation of being infinitesimally close. Here and in the following we make no notational distinction between standard functions and their \( \mathbb{H} \)-transforms in the nonstandard model; the context makes clear what is meant.

Let \( t \in \mathbb{H} \mathbb{R}^+ \) and \( dt \) be an infinitesimal, positive increment in time. Using \((3.7)\), we have, with \( \lambda = \frac{m^k (m-k)}{m!} \) and \( g = \mathbf{g}^m \),

\[
(3.9) \quad \lambda d\psi_k(t) \{ \psi(t) \} = \int [\varphi(t+dt, x^k(t+dt)) - \varphi(t, x^k(t))] g(x^m) d\sigma^m.
\]

We write the integral on the right as the sum of integrals, depending on the collision patterns between the particles in the time interval \([t, t+dt]\). More explicitly, we let \( A_o \) and \( \tilde{A}_{1j} \) denote the sets of points in \( \hat{n}_m^\delta \) which, under
the action of $T^t$, lead to no collisions, and precisely one collision which is between the $i^{th}$ and $j^{th}$ particles, respectively, in the time interval $[0, dt)$. We also define $\mathcal{A}_i$ to be the complement of $\bigcup (\mathcal{A}_i \cup \mathcal{A}_{ij})(1 \leq i < j \leq m)$.

Finally we put $\mathcal{A}_o(t) = T^{-t}(\mathcal{A}_o)$, etc. Thus $\mathcal{A}_o(t), \mathcal{A}_i(t)$ and $\mathcal{A}_{ij}(t)$ consist, respectively, of the sets of initial conditions which lead to no collisions, more than one collision, and precisely one collision which is between the $i^{th}$ and $j^{th}$ particle, respectively, in the interval $[t, t + dt)$.

Carrying on, with $f(t,x^k) = \varphi(t+dt, x^k(t+dt)) - \varphi(t, x^k(t))$, we now have

$$\int f(t,x^k)g(x^m)\,d\sigma^m = \int f(t)g(x^m)\,d\sigma^m + \int f(t)g(x^m)\,d\sigma^m + \int \mathcal{A}_o(t)g(x^m)\,d\sigma^m + \sum \int f(t)g(x^m)\,d\sigma^m (1 \leq i < j \leq m), \mathcal{A}_{ij}(t)$$

First we show that the second integral on the right of (3.10) is $O(dt^2)$ and hence can, as we shall see, be neglected in the following calculations. Since $f$ and $g$ are bounded, it suffices to show that $\int \mathcal{A}_1(t)g(x^m)\,d\sigma^m = O(dt^2)$. But, with $\eta = dt$,

$$\int \mathcal{A}_1(t)g(x^m)\,d\sigma^m \leq C \int \mathcal{A}_1(t)\,d\sigma^m \leq \frac{m}{2} [2\pi \eta^2/3]^2 [4\eta^2/\delta^2].$$

Here, $A = \{ x \in \pi_m : \| q_2 - q_1 \| \leq \delta + 2\eta, \| q_3 - q_4 \| \leq \delta + 2\eta \}$, and $C$ is a finite constant arising from integration in momentum space; the expressions on the right arise from integration over the shells (for similar calculations, see [1]). This yields the desired result.
On $A_0(t)$, $q_i(t+dt) - q_i(t) = p_i(t+dt) - p_i(t)$. Using the mean value theorem and the fact that $\varphi \in C^1$, we get

\begin{equation}
\frac{1}{dt} \int A_0(t) \left( \frac{\partial \varphi}{\partial t} + \sum_{i=1}^{k} p_i \frac{\partial \varphi}{\partial q_i} \right) (x^m) \, d\sigma^m \approx \int \frac{\partial \varphi}{\partial t} (x^m) \, d\sigma^m.
\end{equation}

Now note that we can replace integration over $A_0$ in the last integral in (3.12) by integration over $\tilde{A}_0$ since the subset of $\tilde{A}_0$ involving initial conditions leading to a collision in $[0, dt)$ has a measure which is $O(dt)$ (see (3.16) and the remarks which follow).

There results

\begin{equation}
(3.13) \quad \frac{1}{dt} \int \varphi \, d\sigma^m \approx \lambda \int \varphi_k \, d\sigma^k.
\end{equation}

Next we consider terms of the form $\int \varphi \, d\sigma^m$. If $1 \leq i < j \leq k$, We see from Definition 2.2 that $\varphi = 0$, and so the corresponding integral is zero if $1 \leq i < j \leq k$. Since $\varphi$, regarded as a function of all $m$ variables, is independent of the $x^m$, $k < m$, we reach the same conclusion if $k < i < j \leq m$. On the other hand, if $1 \leq i \leq k$ then
\[
(3.14) \quad \sum_{j=k+1}^{m} \int f g \, d\sigma^m = \sum_{j=k+1}^{m} \int [\varphi(\ldots, q_i(t+dt), p_i(t+dt), \ldots) - \varphi(\ldots, q_i(t), p_i(t+dt), \ldots)] g(x^m) \, d\sigma^m + \sum_{j=k+1}^{m} \int [\varphi(\ldots, q_i(t), p_i(t+dt), \ldots) - \varphi(\ldots, q_i(t), p_i(t), \ldots)] g(x^m) \, d\sigma^m.
\]

To deal with the first term on the right in (3.14), we note that since the dynamics is continuous in the coordinates \( q_i \), there is an infinitesimal \( \hat{\eta} \) so that \(|\varphi(\ldots, q_i(t+dt), p_i(t+dt), \ldots) - \varphi(\ldots, q_i(t), p_i(t+dt), \ldots)| < \hat{\eta}\) on \( \hat{\mathcal{N}}_{ij}(t) \). Thus we have

\[
(3.15) \quad \int [\varphi(\ldots, q_i(t+dt), p_i(t+dt), \ldots) - \varphi(\ldots, q_i(t), p_i(t+dt), \ldots)] g(x^m) \, d\sigma^m \leq \hat{\eta} \int g(x^m) \, d\sigma^m \int_{\hat{\mathcal{N}}_{ij}(t)} \leq \hat{\eta} \int g(t, x^m) \, d\sigma^m,
\]

Furthermore,

\[
(3.15) \quad \int g(t, x^m) \, dx^m \leq \int_{\hat{\mathcal{N}}_{ij}} \int g(S^{-t}(s, y)) \, dv_{ij}(y) \, ds = \int_{\hat{\mathcal{N}}_{ij}} \int g(t-s)(s, y) \, dv_{ij}(y) \, ds = \int_{\hat{\mathcal{N}}_{ij}} \int g(t-s)(\theta, y) \, dv_{ij}(y) \, ds.
\]
where here and later we use the notation \( \tilde{g}(t) = g(t) \alpha \hat{\Phi}_m^{-1} \). Using the continuity of \( g \) and the mean-value theorem of the integral calculus in (3.16), we see that the left-hand side of (3.15) is infinitesimally close to \( \tilde{r} B dt \) for some positive constant \( B \), and so the first sum on the right in (3.14) can be neglected in the calculation.

We now consider the second sum on the right in (3.14). In \( \tilde{\alpha}_{i,j}(t) \), the \( i \)th particle collides with the \( j \)th particle in the time interval \([t, t+dt)\). The (outgoing) velocity \( p_{i,j}(t+dt) \) of the \( i \)th particle after such a collision is given by

\[
(3.17) \quad p_{i,j}(t) = \left[ \omega_{i,j}(t+\alpha dt) \left[ p_{i,j}(t) - p_{j,i}(t) \right] \right] \omega_{i,j}(t+\alpha dt)
\]

\[
= p_{i,j}(t) - \left[ \omega_{i,j}(t), [p_{i,j}(t) - p_{j,i}(t)] \right] \omega_{i,j}(t),
\]

\[
= p'_{i,j}(t)
\]

where \( 0 < \alpha < 1 \) and \( \omega_{i,j}(t) = [q_{i,j}(t) - q_{j,i}(t)]/[q_{i,j}(t) - q_{j,i}(t)] \). Thus we have

\[
(3.18) \quad \sum_{j=k+1}^m \int \{ \tilde{\phi}(\ldots, q_{i,j}(t), p_{i,j}(t+dt), \ldots) - \\
\quad - \tilde{\phi}(\ldots, q_{i,j}(t), p_{i,j}(t+dt), \ldots) \} \ g(x^m) \, d\sigma^m
\]

\[
= \sum_{j=k+1}^m \int \{ \tilde{\phi}(\ldots, q_{i,j}(t), p'_{i,j}(t), \ldots) - \\
\quad - \tilde{\phi}(\ldots, x_i(t), \ldots) \} \ g(x^m) \, d\sigma^m
\]

\[
= \sum_{j=k+1}^m \int \{ \tilde{\phi}(\ldots, q_{i,j}, p'_{i,j}, \ldots) - \\
\quad - \tilde{\phi}(\ldots, x_i, \ldots) \} \ g(t, x^m) \, d\sigma^m
\]
\[ = (m-k) \int \{ \varphi(t, \ldots, q_i, p_i', \ldots) - \varphi(t, x_i^k) \} g(t, x_i^m) \, d\sigma^m, \]

the last by the symmetry of \( g \), where \( p_i' = p_i', k+1 \). The last expression in (3.18) can again be treated as an integral over \( \tilde{Z} \). Using the definition of the surface measure \( d\sigma_{ij} \) and the continuity as before, we find

\[ (3.19) \frac{1}{dt} \int \varphi(t, x^k)^m g(t, x^m) \, d\sigma = \int \frac{d}{dt} \int \tilde{\varphi}(t)(\tau, y)^m g(t)(\tau, y) \, d\nu_{i, k+1} \, d\tau \]

\[ = \delta^2 \int K_{i, k+1} \varphi(t, x^k) g(t, x^m) \, d\sigma^k + \ldots \, d\sigma_m \, ce^2 \int \frac{B}{p^2} \, d\omega dp. \]

Here, in analogy with the notation above, \( \tilde{K}_{i, k+1} \) is the subset of \( K_{i, k+1} \) on which \( |q_i - q_n| > \delta \) unless \( m = i \) and \( n = k+1 \). The first equality in (3.19) results from the fact that \( \tau(y) \) is not infinitesimal at any standard point in \( K \), and hence the same is true at any near-standard point since \( \tau \) is lower semi-continuous; we need only consider near standard points since \( \varphi \) has compact support. In the last line of (3.19) we have been able to expand the integration to \( K_{i, k+1} \) since the set \( K_{i, k+1} - \tilde{K}_{i, k+1} \) involves collisions other than that between \( i \) and \( k+1 \), and so is of lower dimension in \( K_{i, k+1} \) and therefore of measure zero. Finally,

\[ (3.20) \sum_{j=k+1}^m \int \tilde{\varphi} g \, d\sigma = \int \frac{dt}{\delta^2} \lambda \int K_{i, k+1} \varphi(t, x^k) \varphi_{k+1}(t, x^k, q_i + \delta \omega, p) \, d\sigma^k \, ce^2 \int \frac{B}{p^2} \, d\omega dp, \]
where the integration is over the set $A = \{ (x^k, \omega, p); |\omega| = 1, k \in \mathbb{Z}, \omega \neq 0 \}$.

To deal with the integral $\int \varphi(t, \ldots, q^i, p^i, \ldots) g(t, x^m) \, d\sigma^m$ over $\tilde{A}_{j,k+1}$ we first use the fact that $\omega \cdot (p^i - p) = -\omega \cdot (p^i - p')$ and the above argument to obtain

\begin{equation}
(3.21) \quad \langle m-k \rangle \int \varphi(t, \ldots, q^i, p^i, \ldots) g(t, x^m) \, d\sigma^m \\
\approx -\frac{\partial}{\partial t} m_0^2 \lambda \int \tilde{A} \varphi(t, \ldots, q^i, p^i, \ldots) x \\
\times \psi_{k+1}(t, x^k, q^i + \delta \omega, p) \, d\sigma^k \cdot \omega \cdot \frac{\beta}{2p^2} \, dp \, dp.
\end{equation}

Since the transformation (2.9) is orthogonal, and hence $dp \, dp = dp^i \, dp'$, we may replace $\omega$ by $-\omega$, and we see that the last line in (3.21), when divided by $dt$, is infinitesimally close to

\begin{equation}
(3.22) \quad m_0^2 \lambda \int \tilde{A} \varphi(t, x^k) x \\
\times \psi_{k+1}(t, \ldots, q^i, p^i, \ldots, q^i + \delta \omega, p') \, d\sigma^k \cdot \omega \cdot \frac{\beta}{2p^2} \, dp \, dp.
\end{equation}

Lastly we note that it is not necessary to consider the discontinuities arising from collisions with the boundary in the time interval $[t, t+dt]$ since $\varphi(x^k(t))$ vanishes when any point in $x^k(t)$ is near the boundary $\partial \Omega_k$.

Putting all of the above together with Kiesler's Infinite Sum Theorem, we obtain the desired result for continuous functions of compact support.

To prove the result for general $g \in L^1(\eta^\delta_n)$ is now simply a matter of approximating $g$ in $L^1$ by continuous functions with compact support. The one
point to notice in this connection is that the collision integral, when
integrated over \( t \), corresponds to a full volume integral on \( \tilde{S} \). To be precise,
note that, as in (3.18) - (3.21), \( \int_{0}^{t} \int_{k+1}^{m} \psi_{k+1}^{m} [\varphi(s)] \, ds \) is a finite sum of
terms of the form

\[
(3.23) \quad m_{0}^{2} \int F(s, x^{k}) \, k_{i}(\omega, p) \, \psi_{k+1}^{m}(s, x^{k}, q_{i}, \delta \omega, p) \, c_{e} \frac{\beta}{2} \, d\sigma \, d\omega \, dp \, ds,
\]

where \( F(t, x^{k}) = \varphi(t, \ldots, q_{i}, p_{i}', \ldots) - \varphi(t, x^{k}) \), and the integration with respect
to \( s \) is from 0 to \( t \). Now

\[
(3.24) \quad m_{0}^{2} \int F \, k_{i}(\omega, p) \, \psi_{k+1}^{m}(s, x^{k}, q_{i}, \delta \omega, p) \, c_{e} \frac{\beta}{2} \, d\sigma \, d\omega \, dp \, ds
\]

\[
= \lambda^{-1} \int \int_{S} \tilde{g}(s, y) \, d\nu_{i, k+1} \, ds,
\]

\[
= \lambda^{-1} \int \int_{B} \tilde{g}(s, y) \, d\nu \, ds
\]
as in (3.18) and (3.19) (the set \( B \) is obtained by using the flow \( S^{-s} \)). The last
integral can be carried back to \( \pi \) using \( \Phi^{-1} \). This completes the proof.

§4. PROOF OF THE THEOREM

First a short sketch of the main ideas before we begin the proof. From
now on we let \( \psi_{k}^{m} \) denote the reduced correlation functions based on the \( g_{m}^{n} \).
Next we pass to the nonstandard model, and consider the internal functions \( \psi_{k}^{m} \)
with \( m \) an infinite integer. It follows from the assumptions 2.1 that these
functions are \( S \)-integrable. This allows us to define functions \( \hat{\psi}_{k} \) on \( R \times \pi_{k} \)
which are weak cluster points of the \( \psi_{k}^{m} \) by mapping the functions \( \psi_{k}^{m} \) down to \( R \).
x m_k using conditional expectation. The last step is to show that these functions satisfy the limiting Enskog hierarchy. This follows by taking standard parts on both sides of the (transferred) EBGHY hierarchy. The S-integrability estimates allow us straightaway to interchange the standard part operation and integration on all but the collision integral. That the same is true for this last integral makes essential use of the fact that it involves integration with respect to time as well as space. We now begin the proof.

The set \( \Omega_\infty \) of all cluster points is now obtained by considering the elements in \( \hat{\Omega} = \{ \psi^m(t) : m \in \hat{N}, t \in \hat{R}^+ \} \) for which \( m \in \hat{N}_\infty \), the set of infinite natural numbers. Throughout the following discussion, \( m \) denotes a fixed element in \( \hat{N}_\infty \).

First we show that the functions \( \psi_k^m(t, x^k) \) are S-integrable as functions of \( x^k \) for any \( t \in \hat{R} \), \( k \in N \). To do so, we need the following proposition, which is an elaboration of a standard argument in information theory (see also Cercignani [10]).

4.1 Proposition

Let \( k \in N \) be finite. There is a \( B > 0 \) so that

\[
(i) \int \psi_k^m(t) \ln \psi_k^m(t) \, d\sigma^k \leq B
\]

and

\[
(ii) \int \psi_k^m(t) \sum_{i} p_i^2 \, d\sigma^k \leq B
\]

for all \( m \in \hat{N} \) and all \( t \in \hat{R}^+ \).

Proof: Let \( m \in N \) be finite, \( f(x^m) \) be a function of \( m \) variables with \( \int f \, d\sigma^m = 1 \). If \( y \) is any subset of the \( m \) variables, we denote by \( f(y) \) the function obtained by integrating out the variables in \( x^m - y \). Now let \( y_o =
\( (x_1, \ldots, x_k), y_1 = (x_{k+1}, \ldots, x_{2k}), \ldots, y_\mu = (x_{\mu k + 1}, \ldots, x_{\mu k + k}) \), where \( \mu = [m/k] - 1 \) and the brackets denote "greatest integer in". Using the inequality \( \ln x \leq x - 1 \), we find

\[
(4.1) \quad \sum_{i=0}^{\mu} \int f(y_i) \ln f(y_i) \, d\sigma^k(y_i) - \int f \ln f \, d\sigma^m = \int f \ln \frac{g}{\epsilon} \, d\sigma^m \leq \int (g - \epsilon) \, d\sigma^m = 0,
\]

where \( g(x^m) = \prod_{i=1}^{m} f(y_i) \) \( (0 \leq i \leq \mu) \). If \( f \) is symmetric, we conclude that

\[
(4.2) \quad [m/k] \int f(y_\sigma) \ln f(y_\sigma) \, d\sigma^k \leq \int f \ln f \, d\sigma^m.
\]

The inequality (i) now follows by transfer on replacing \( f \) by \( g \) and using 2.i(iii), after noting that the integral \( \int g \ln g \, d\sigma^m \) is invariant under \( T_m^+ \). Inequality (ii) is almost immediate from 2.i(iv). This completes the proof of the lemma.

We now infer from Proposition 4.1 and results of Arkeryd [5] that the functions \( \psi^m_k(t) \) are \( S \)-integrable with respect to \( \sigma^k \) as functions of \( x^k \) for any \( t \in \mathbb{R}^+ \), \( k \in \mathbb{N} \). It also follows from the fact that \( \int \psi^m_k(t) \, d\sigma^k \) is bounded (by one) that it is \( S \)-integrable as a function of \( t \in \mathbb{R}^+ \) on any finite interval \([a, b]\) \( \subseteq \mathbb{R}^+ \), the near-standard points in \( \mathbb{R}^+ \).

From here on, we use the standard part notation to denote Loeb measures; for example, \( \sigma^k \) denotes the Loeb measure on \( \mathbb{P}^k \) obtained from \( \sigma^k \). Also, \( \varphi \) generically denotes a function in \( \Phi^k \); the extension to more general test functions is by approximation. For any \( k \in \mathbb{N} \), the measures \( \sigma^k \) are
near-standardly concentrated. It follows from this and the fact that the \( \psi^m_k \) are \( S \)-integrable that the functions \( \psi^m_k \) are integrable with respect to \( \sigma^k \).

Thus, using the notation \( \mathring{\psi} = \mathring{\psi}^{(\ast)} \), \( \mathring{\pi}_k = \mathring{\pi}_k^{(\ast)} \), and \( \mathring{\mathbb{R}} = \mathring{\mathbb{R}}^{(\ast)} \), we have

\[
(4.3) \quad \mathring{\int}_{\mathring{\pi}_k} \mathring{\psi}(t, x^k) \psi^m_k(t, x^k) \, d\sigma_k = \mathring{\int}_{\mathring{\pi}_k} \mathring{\psi}(t, x^k) \psi^m_k(t, x^k) \, d\sigma_k,
\]

and

\[
(4.4) \quad \mathring{\int}_\mathring{\mathbb{R}} \mathring{\int}_{\mathring{\pi}_k} \mathring{\psi}(t, x^k) \psi^m_k(t, x^k) \, d\sigma_k = \mathring{\int}_\mathring{\mathbb{R}} \mathring{\int}_{\mathring{\pi}_k} \mathring{\psi}(t, x^k) \psi^m_k(t, x^k) \, d\sigma_k,
\]

for any \( \mathring{\psi} \in C^1(\mathring{\mathbb{R}} \times \mathring{\pi}_k) \).

We now proceed to characterize the set \( \Omega_\infty \). By §10.2 in [8] there are, for any \( m \in \mathring{\mathbb{N}}_\infty \), \( t \in \mathring{\mathbb{R}} \), and \( k \in \mathbb{N} \), functions \( \mathring{\psi}^m_k(t, \sigma) \) and \( \mathring{\psi}^m_k(t, \sigma) \), defined on \( \mathring{\pi}_k \) and \( \pi_k \) respectively, so that \( E^{\mathring{\pi}_k}(\mathring{\psi}^m_k) = \mathring{\psi}^m_k = \mathring{\psi}^m_k \circ \sigma \), where \( E^{\mathring{\pi}_k} \) denotes conditional expectation with respect to the \( \sigma \)-algebra generated by \( \sigma \): \( \mathring{\pi}_k \rightarrow \pi_k \). If \( \mathring{\psi} \in \mathring{\psi}^m_k \), \( k \in \mathbb{N} \), we have, using (4.3), that

\[
(4.5) \quad \mathring{\int}_{\mathring{\pi}_k} \mathring{\psi}(t) \psi^m_k(t) \, d\sigma_k \approx \mathring{\int} \mathring{\psi}(t) \psi^m_k(t) \, d\sigma_k,
\]

Since, as we shall show, the function \( \mathring{\psi}(t) \psi^m_k(t) \, d\sigma_k \) satisfies (2.18) and so is (standard) continuous in \( t \), we also have
\[ (4.5) \int_{a}^{b} ds \int_{a}^{b} \varphi(s) \psi^{m}_{k}(s) d\sigma^{k} \approx \int_{a}^{b} ds \int_{a}^{b} \varphi(t) \psi^{m}_{k}(s) d\sigma^{k}. \]

By Theorem 4.2.6 in [23], the set \( \{ \hat{\varphi}^{m}; m \in \mathbb{N}_{\infty} \} \) is precisely the set \( \Omega_{\infty} \) of all cluster points in the weak topology of the evolutions in \( \Omega \).

Next we show that any \( \hat{\varphi}^{m} = \{ \hat{\varphi}^{m}_{k}(t); k \in \mathbb{N}, m \in \mathbb{N}_{\infty} \} \in \Omega_{\infty} \) has a representative which satisfies the Enskog hierarchy. Since \( m \in \mathbb{N}_{\infty} \), we have \( \delta \approx 0 \) and \( m_{\delta}^{2} \approx k \). Since \( \varphi \in \Phi_{k}^{\delta} \) and \( \delta \) is infinitesimal, we see that \( \varphi \in \Phi_{k}^{d} \). By transfer from (2.14) we have

\[ (4.7) \int_{0}^{t} \{ \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) d\sigma^{k} = \int_{0}^{t} \{ \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) d\sigma^{k} = \int_{0}^{t} \{ \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) d\sigma^{k}. \]

Taking standard parts on both sides of (4.7) and using (4.5), there results

\[ (4.8) \psi^{m}_{k}(t)[\varphi(t)] - \psi^{m}_{k}(0)[\varphi(0)] = \int_{0}^{t} \{ \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) d\sigma^{k}. \]

Consider now the first term on the right. Note that \( \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) \) is finitely bounded since \( \varphi \in C_{\infty} \), and so the integrand is \( S \)-integrable and also bounded. As in (4.6) we obtain

\[ (4.9) \int_{0}^{t} \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) d\sigma^{k} = \int_{0}^{t} \psi^{m}_{k}(s)\}^{\kappa} \varphi(s) d\sigma^{k}. \]

There remains to consider the collision term. Notice again as in (3.18)
through (3.23) that \( \int_{0}^{\infty} \chi_{k+1}^{m} \int_{k+1}^{m} \chi_{k+1}^{m} \left[ \Psi(s) \right] * ds \) is a finite sum of terms of the form

\[
(4.10) \quad \int \Psi(s, x_{k}, q_{i}, p_{i}, \omega, p) \chi_{k+1}^{m} (s, x_{k}, q_{i} + \delta\omega, p) \left( \Psi(s, x_{k}, q_{i}, p_{i}, \omega, p) \right) e^{-\frac{B_{p}^{2}}{2} \Psi} \, d\sigma k \, d\omega \, dp \, ds,
\]

where \( \Psi(t, x_{k}) = \Psi(t, \ldots, q_{i}, p_{i}, \ldots) - \Psi(t, x_{k}) \), and the integration with respect to \( s \) is from 0 to \( t \). We first show that the integrand in (4.10) is \( S \)-integrable. To do so, it suffices to show that \( \int \Psi_{k+1}^{m} \ln \Psi_{k+1}^{m} \, d\sigma_{k+1} \, d\alpha \) is finite, where \( d\alpha = k_{k+1} \, d\alpha_{k} \), \( d\alpha_{k} \), \( d\omega \), \( dp \), and \( ds \), and \( k \) is a finite constant chosen so that \( \int d\alpha = 1 \). Here \( \Psi_{k+1}^{m} \) is evaluated at \( (s, x_{k}, q_{i} + \delta\omega, p) \). Note that there is a finite (by the remarks following (3.24)) and non-infinitesimal (since \( \Psi \) is measure preserving) constant \( B \) so that

\[
(4.11) \quad \int \psi(s, x_{1}, \ldots, x_{k}, q_{i} + \delta\omega, p, x_{i+2}, \ldots, x_{m}) \times d\sigma^{m-2} \, d\alpha = B^{-1},
\]

where \( d\sigma^{m-2} = d\sigma_{1} \cdots d\sigma_{j-1} d\sigma_{j+1} \cdots d\sigma_{k} \cdots d\sigma_{m} \), and \( \psi = \lambda^{-1} \psi \). As in the proof of Proposition 4.1, we obtain

\[
(4.12) \quad [m-2/k-1] \int \nu_{k+1} \ln (\nu_{k+1} B) \, d\sigma_{k-1} \, d\alpha \leq \int \nu \ln (\nu) \, d\sigma^{m-2} \, d\alpha,
\]

where \( \nu_{k+1} \) is obtained from \( \nu \) (with the variables shown in (4.11)) by integrating out all except \( k-1 \) of the variables \( x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k}, x_{k+2}, \ldots, x_{m} \). From (4.12) we get

\[
(4.13) \quad [m-2/k-1] \int \nu_{k+1} \ln \nu_{k+1} \, d\sigma_{k-1} \, d\alpha \leq \int \nu \ln \nu \, d\sigma^{m-2} \, d\alpha +
\]
\[ + \int v \ln B \quad ^{\wedge} d\alpha^{m-2} \quad ^{\wedge} d\alpha - [m-2/k-1] \int v_{k+1} \ln B \quad ^{\wedge} d\alpha^{k-1} \quad ^{\wedge} d\alpha \]

Now \( \int v \ln B \quad ^{\wedge} d\alpha^{m-2} \quad ^{\wedge} d\alpha \leq \int v B \quad ^{\wedge} d\alpha^{m-2} \quad ^{\wedge} d\alpha = 1 \), and \( \int v_{k+1} \ln u \quad ^{\wedge} d\alpha^{k-1} = \ln B \), so there results

\[ (4.14) \quad [m-2/k-1] \int \psi_i^{m} \ln \psi_j^{m} \quad ^{\wedge} d\alpha^{k-1} \quad ^{\wedge} d\alpha \leq \int v \ln v \quad ^{\wedge} d\alpha^{m-2} \quad ^{\wedge} d\alpha + 1 + [m-2/k-1] \ln B \]

The desired bound follows since the integral on the right is bounded by \( m^k \) as in (3.24) and the following remarks, with \( g \) replaced by \( g \ln g \).

Since the integrand in the collision integral is \( S \) - integrable, we conclude that

\[ (4.15) \quad \o_{m} \o_{l} \o_{i} \psi_i^{m} \ln \psi_j^{m} \quad ^{\wedge} d\alpha^{k-1} \quad ^{\wedge} d\alpha = \kappa \o_{l} \o_{i} \psi_j^{m} \quad ^{\wedge} d\alpha^{k-1} \quad ^{\wedge} d\alpha. \]

Notice that the equations in Loeb space resulting at this stage have the same \( \o_{i} \psi_i^{k} \) to the right at level \( k \) and to the left at level \( k+1 \).

Now we project the collision term down onto the standard space using conditional expectation as before. To be precise, let \( E \) be the conditional expectation generated by the standard part map, again denoted by \( st \), on the near-standard points in \( (\omega,s,x^k,p) \) space. Define \( \tilde{\psi}_{i,k+1} \) by \( E(f_i) = \tilde{\psi}_{i,k+1} \circ st \), where \( f_i(\omega,s,x^k,p) = \psi_i^{m}(s,x^k,q_i + \delta \omega,p) \). Then we have
\[
(4.16) \quad \int_{t=0}^{t_{k+1}} \int_{\omega} \int_{p} \int_{s} \psi_{k+1}^{m}(s, \omega, p) \psi_{k+1}^{m}(\omega, s, P, p) \frac{\beta^2}{2} d\alpha d\omega dp ds,
\]

where \( \psi_{k+1}^{m}(s, \omega, p) = \psi(s, \omega, p) - \psi(s, x^k) \). The final form of the collision term in (2.19) now follows as in (3.18) through (3.23). The facts about the vague topology follow as above.

Finally we establish the bounds (2.20) and (2.21). The bound (2.20) follows from Proposition 4.1(ii) and the fact that \( \int \alpha^f \geq \int f \) for any non-negative \( \int \alpha^f \) measurable function \( f \). To prove (2.21), notice first that by convexity, \( x \log x = \sup_{L} f_L(x) \) where \( L \) is a countable family of affine functions. If \( H(x^k) = \psi_{k}^{m}(x^k) \log \psi_{k}^{m}(x^k) \), we have \( H^m(x^k) \geq (\log \psi_{k}^{m})(x^k) \). From inequality (10.1.11) in [8] there results \( \hat{H}(x^k) \geq (\log \psi_{k}^{m})(x^k) \) a.e. and hence \( \hat{H}(x^k) \geq \psi_{k}^{m}(x^k) \log \psi_{k}^{m}(x^k) \), where \( \hat{H} \) is defined by conditional expectation as before. The result now follows from the fact that \( \int H(x^k) d\alpha_k \) is bounded above. We follow Arkeryd (see e.g. [7]) in establishing this last fact. We have, with \( f = \psi_{k}^{m}(t) \), that

\[
(4.17) \quad \int_{n_k} \int_{n_k} \int_{n_k} \int_{n_k} f \log^{+} f d\alpha_k \leq \int_{n_k} \int_{n_k} f \log^{+} f d\alpha_k \\
\leq \int_{n_k} f \log^{+} f d\alpha_k + \int_{n_k} f \log^{+} f d\alpha_k \\
\leq \int_{n_k} f \log^{+} f d\alpha_k + \int_{n_k} f \log^{+} f d\alpha_k \\
\leq \int_{n_k} f \log^{+} f d\alpha_k + \int_{n_k} f \log^{+} f d\alpha_k + \int_{n_k} f \log^{+} f d\alpha_k +
\]

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\[ + \prod_{k=1}^{n} \exp \left( - \sum_{1}^{k} p_{1}^{2} \right) \delta \sigma_{k} \]

The last inequality follows from the inequality \( f \log f \geq -g + f \log g \) with \( g = \exp(-\sum_{1}^{k} p_{1}^{2}) \). The desired bound now follows from Proposition 4.1. This completes the proof.

§5. REMARKS ON MOLECULAR CHAOS

In order to establish the global validity of the Boltzmann equation from the results presented here, two additional steps are necessary:

(a) We must be assured that, in the limit, the results apply to factored initial data, i.e., to the case of initial chaos.

(b) We must show that if the initial conditions factor, then the solutions of the Boltzmann hierarchy factor for all time. This is the "subsequent chaos" problem. It could presumably be solved by establishing a uniqueness theorem in our context. This is because solutions of the Boltzmann equation can be used to construct solutions of the Boltzmann hierarchy by taking products, but we must be assured that these solutions (for given initial conditions) coincide with our solutions.

We have nothing further to say about (b). To see that our results apply to the case of factored initial data, suppose that \( g: \mathbb{R}_{+}^{1} \rightarrow \mathbb{R}_{+} \) satisfies the conditions that \( g(1 + p^{2}) \) and \( (g \log g) \) are both in \( L^{1}(\mathbb{R}_{+}^{1}) \), together with the
following additional condition:

\[ 5.1 \quad \lim_{m \to \infty} m k_m = 0, \]

where

\[ k_m = \sup_{A(q_0, m)} \int_{A(q_0, m)} g \, d\sigma, \]

and

\[ A(q_0, q, m) = \{ x \in \Pi_1 : |q_0 - q| < \delta(m) \}. \]

This condition is satisfied, for example, if \( g \in L^p(\Pi_1) \) for some \( p > 3 \), by virtue of Holder's inequality and (2.8).

We now define \( g^m(x^m) = \alpha_m \chi_{\hat{A}} \prod_{i=1}^{m} g(x_i) \) (1 \( \leq i \leq m \)), where \( \hat{A} = \Pi_1^m \), \( \chi_{\hat{A}} \) is the characteristic function of \( \hat{A} \), and \( \alpha_m \) is a normalizing constant. To ensure that conditions 2.1(i)-(iv) are satisfied, we need only show that \( \alpha_m \) is bounded in \( m \). But

\[ (5.1) \quad \alpha_m^{-1} = 1 - \beta_m, \]

where, using inclusion-exclusion, we find

\[ (5.2) \quad \beta_m \leq \sum (-1)^{n+1} {m-1 \choose n} k_m^n (1 \leq n \leq m-1), \]

and the last sum approaches zero by 5.1. Similarly we see that the rescaled correlation functions based on the \( g^m \) approach those based on the factored initial data as \( m \to \infty \). Condition 2.1(v)(b) will be satisfied if \( \int h \ln h \, d\sigma \)
is bounded, where $h$ is $g$ mollified by $\hat{k}$.

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