Kerr and Kerr-AdS Black Shells and Black Hole Entropy

by

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B.Sc., Nankai University, Tianjin, P. R. China, 2003

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Abstract

As an operational approach to the Bekenstein-Hawking formula $S_{\rm BH} = A/4l_{\rm Pl}^2$ for the black hole entropy, we consider the reversible contraction of a spinning thin shell to its event horizon and find that its thermodynamic entropy approaches $S_{\rm BH}$. In this sense the shell, called a "black shell", imitates and is externally indistinguishable from a black hole. Our work is a generalization of the previous result [10] for the spherical case. We assume the exterior space-time of the shell is given by the Kerr metric and match it to two different interior metrics, a vacuum one and a non-vacuum one. We find the vacuum interior embedding breaks down for fast spinning shells. The mechanism is not clear and worth further exploring. We also examine the case of a Kerr-AdS exterior, without trying to find a detailed interior solution. We expect the same behavior of the shell when the horizon limit is approached.

Contents

Chapter 1

Introduction

One of the most intriguing problems raised by Hawking's discovery of black hole evaporation is the nature of the enigmatic Bekenstein-Hawking relation $S_{\mathrm{BH}} = A/4l_{\mathrm{Pl}}^2$ between black hole area and entropy. Why does it possess such a universal form? How does the black hole forget all its past? Is the information inside recoverable when the black hole finally evaporates? This relation has been claimed to be the most proved and least understood formula in theoretical physics. The puzzle can only be unlocked at the birth of a complete theory of quantum gravity, as is hinted in the following remarks.

In classical general relativity, a *black hole* is a region of strong gravity where even light cannot escape. Its boundary is the *event horizon*, a one-way membrane for causal effects. Black holes are surprisingly simple objects. Their exterior geometries are characterized by only three parameters: mass, charge and angular momentum, even though their interiors contain all the complexities of their stellar progenitor. This feature makes the black hole resemble a thermodynamic system.

Consider, for example, a Reissner-Nordström black hole of mass m and charge e. The radius of the horizon is given by (we adopt $G = c = \hbar = 1$)

$$r_0 = 2\left(m - \frac{1}{2}\frac{e^2}{r_0}\right). \tag{1.1}$$

Now add an infinitesimal charge de to the hole. The mass increment can be written as

$$\mathrm{d}m = \frac{e}{r_0}\mathrm{d}e + \mathrm{d}E_{\mathrm{diss}}.$$

The first term is the work done in pushing the charge down to the horizon. The second term is nonnegative, representing the rest-mass and kinetic energy of the charge and any gravitational or electromagnetic waves that eventually fall across the horizon. It is easy to show that $\mathrm{d}E_{\mathrm{diss}}$ is proportional to an exact differential: by differentiating (1.1), we have

$$\frac{\kappa}{8\pi}\mathrm{d}A = \mathrm{d}m - \frac{e}{r_0}\mathrm{d}e = \mathrm{d}E_{\mathrm{diss}} \geq 0, \tag{1.2}$$

where $A = 4\pi r_0^2$ is the area of the horizon and

$$\kappa = \frac{m - e^2/r_0}{r_0^2}$$

the surface gravity. (1.2) states the "area law" that *the area of the horizon of a black hole can never decrease* (since $\kappa > 0$[1]), an analogue to the second law of thermodynamics. Observing this, Bekenstein (1973) postulated that A is actually a measure of (proportional to) the entropy of the black hole. It has also been proved that κ is constant over the horizon (even for spinning,

[1] The extremal case where $\kappa = 0$ will be looked at later.

non-spherical black holes), as is the temperature in a system in thermal equilibrium, the content of the zeroth law of thermodynamics. Thus an analogy can be drawn between (1.2) and the ordinary laws of thermodynamics

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V \geq 0, \tag{1.3}$$

if we make identifications between m and the energy E, which is so natural, and between the "work terms". This incorporates the first and second laws. A black hole version of the third law in its weaker (Nernst) form also exists [2]: *it is impossible by any process, no matter how idealized, to reduce* κ *to zero in a finite sequence of operations.*

It seems counterintuitive to assign a finite temperature to a black hole which, classically speaking, can emit nothing, but the pioneer work of Hawking (1975) shows that a black hole does have a physical temperature. His approach was semi-classical, involving calculating particle creation in the presence of an event horizon. It turns out that the black hole radiates to infinity with a black body spectrum at the Hawking temperature

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}.$$

This fixes the constant of proportionality between the entropy and the area:

$$S_{\mathrm{BH}} = \frac{1}{4}A, \tag{1.4}$$

where S_{BH} is called the Bekenstein-Hawking entropy associated with the black hole.

What is the nature of this $S_{\rm BH}$? (Note "BH" stands for Bekenstein-Hawking and not necessarily for Black Hole.) Does it represent the entropy of the matter that has fallen into the hole? We first notice that $S_{\rm BH}$ is proportional to the area, unlike the entropy of a non-gravitating system where it is proportional to the volume of the system. This seems peculiar to the gravitational theory, since in deriving the "first law of black hole mechanics" (the first equality in (1.2)) we only used the feature of the Reissner-Nordström solution, namely (1.1). In fact, it can even be shown that the validity of this law (in its generalized form for rotating black holes) depends only on very general properties of the Einstein's equation, without using the detailed form of it [3]. On the other hand, Hawking radiation is a process assumed to be happening in the classical space-time background—only the matter field is quantized. The radiation is interpreted as originating from particle pair creations near the horizon, which not surprisingly gives rise an entropy of the dimension of area through the identification of T with $\kappa/2\pi$. Also, a rough estimation can be made for $S_{\rm BH}$ and the entropy of a star of the same mass [2]. For the former, we have $S_{\rm BH} \sim 10^{78}$ for a $5M_{\odot}$ black hole. For the latter, its entropy is of order 10^{58} (roughly the number of particles in the star). The discrepancy is enormous and it is clear that $S_{\rm BH}$, as a universal property dependent only on a few macroscopic parameters, has no direct link to the normal matter entropy, except in some sense as a upper bound of the latter. The idea that $S_{\rm BH}$ keeps track of matter entropy obviously meets difficulties if we consider the collapse of a cold, pressureless and viscous-free dust shell into the hole, causing an increase in its mass, and thus $S_{\rm BH}$, but leaving the total matter entropy unchanged.

Since $S_{\rm BH}$ contains no information about the microscopic degrees of freedom of the initial matter, it can be understood as measuring our ignorance of the internal state of the black hole which is hidden beneath the horizon, in the sense of cosmic censorship. The presence of the horizon seems crucial. It appears to a distant observer as a hot surface—matters keep falling in only to feed this surface and lose their initial properties. From the observer's view, his inability to access the disappeared information is measured exactly by the "cross section" of the horizon, if we rewrite the Bekenstein-Hawking formula (1.4) a bit: $S_{\rm BH} = A/4 = \sigma$, where $\sigma = \pi r_0^2$ is the cross section. Ted Jacobson has also argued [4] that $S_{\rm BH}$ measures only those states that can influence the outside of the black hole and these states must be associated with the presence of the horizon, otherwise they would simply be counted as ordinary states of the exterior itself.

This interpretation coincides in spirit with the "entanglement entropy" which arises when tracing over (either) one of the two sets of quantum field modes in correlation across a geometric boundary, resulting in a density matrix describing a mixed state with entropy proportional to the area of the boundary. This is no surprise since their common boundary is the only thing that determines how the two subsystems are divided and correlated. The correlations would be strongest near the boundary. Now the black hole horizon naturally acts as such a dividing boundary. In this case we trace over the hidden modes under the horizon and remarkably the reduced density matrix describes a thermal state for outside modes. However, this entanglement entropy, while proportional to the area, diverges as α^{-2}, where α is a cutoff length above the horizon. This arises from the existence of modes of arbitrarily high angular momentum close to the horizon. We have to manually adjust α in order to reproduce the right coefficient of proportionality 1/4 between $S_{\rm BH}$ and A. α turns out to be of the order of the Planck length $l_{\rm Pl}$, which can be explained by the quantum fluctuations near the horizon which will prevent events closer to the horizon than α from being seen on the outside. This again implies the special role played by the horizon in accounting for the black hole entropy.

The brick wall model proposed by 't Hooft in 1985 uses the idea of the "horizon origination" of $S_{\rm BH}$ as discussed above. He considered a thermal atmosphere of quantum fields propagating in the black hole space-time background but outside a perfect reflecting surface ("brick wall") a proper distance α above the horizon. Like the entanglement entropy, the ordinary thermodynamic entropy of the quantum fields is also proportional to the area (of the wall) but again diverges as α^{-2}. By adjusting α to the Planck scale, one recovers the coefficient 1/4. It is notable that α turns out to be a universal constant: $\alpha = l_{\rm Pl}\sqrt{\mathcal{N}/90\pi}$, depending only on the number of physical fields $\mathcal{N}$ in nature.

A clearer account of the statistical origin of $S_{\rm BH}$ without cutoffs or ad hoc adjustment may require a full quantum theory of gravity. As the most promising candidate, string theory has succeeded in calculating $S_{\rm BH}$ for certain classes of extremal and nearly extremal black holes [5]. These black holes can be given as solutions of string theory at the low energy limit where it reduces to a 10-dimensional supergravity theory. Here it is necessary to introduce into the theory charges carried by D-branes in order to obtain black holes with nonzero area. Now, as one goes to the weak coupling limit of string theory (black hole corresponds to strong coupling), the black hole description is replaced by certain states comprised of D-branes with the same charges in a flat space-time background. Then the entropy of the system can be computed as ordinary statistical entropy by counting states of open strings on D-branes. The results turns out to be exactly the same as the Bekenstein-Hawking entropy for the corresponding black hole in the strong coupling limit.

It should be mentioned that there has not been an unambiguous interpretation of the entropy of extremal black holes. Usually people think it is still given by $S_{\rm BH} = A/4$ though $T_{\rm H} = 0$ ($\kappa = 0$), denying a black hole version of the strongest (Planck) form of the third law of thermodynamics, as is supported by the string-based state counting technique. However, there were arguments [6] that one should take the entropy of extremal black holes to be zero and abandon the Bekenstein-Hawking relation.

In this thesis, not to trouble ourselves with the statistical origin of $S_{\rm BH}$ and to further support the idea that $S_{\rm BH}$ is a purely surface property associated with the horizon, we present an operational approach based on thermodynamic arguments which reproduces the correct relation $S_{\rm BH} = A/4$ without ad hoc adjustment of parameters.

In thermodynamics, the entropy of any state can be found by devising a reversible process which arrives at the desired state from a state of known entropy and then using the first law of thermodynamics to compute the change in entropy during the process. The process considered here is the reversible quasi-static contraction of a massive thin shell towards its gravitational radius. The state of the shell is described by its temperature T, pressure P, proper surface density σ (or mass M) and radius R (or area A). T is determined from the requirement of reversibility that the shell be in thermal equilibrium with the acceleration radiation seen by observers on the shell. The ground state for quantum fields outside the shell is the Boulware state whose stress-energy will diverge to negative infinity when the shell's gravitational radius is approached. To control the resulting strong gravitational back reaction from this large negative mass, we draw on an energy source at infinity to form a thermal "topped-up Boulware (TUB) state" whose temperature at the shell's surface is raised to the local acceleration temperature so as to maintain reversibility. To find P and σ, we need the formalism describing the dynamics of general relativistic thin shells, which was developed by Werner Israel in the 1960s [7]. (Since then it has been extensively used as a framework for various problems in astrophysics and cosmology (see [11,12] for reviews), for example, relevant to the present paper, the study of gravitational collapse and its final states—black holes.) Geometrically, the history of the shell is a hypersurface separating two regions. The exterior is assumed to be a certain black hole solution. The interior solution is chosen to be nearly flat. They satisfy Israel's junction conditions on the shell, which contain relations between the surface stress-energy tensor (with P and σ as its components) and the exterior and interior extrinsic curvatures (describing how the shell is embedded in the bulk geometries of both sides respectively) of the shell. So once the geometries of the two sides are specified, P and σ will be determined uniquely.

The operational approach to black hole entropy has been investigated for the case of a spherical shell [9]. We are interested in generalizing to the rotating case. Though no exact interior solutions which match the exterior Kerr metric have been found yet, de la Cruz and Israel [10] were able to match a nearly flat interior to the Kerr exterior in the slowly rotating limit (without restrictions on the radius of the shell). In the present thesis we consider a spinning thin shell and are mainly interested in performing the match in the horizon limit since we are to examine its entropy in the black hole limit.

The thesis is organized as follows. First, to fix the shape of the shell, we introduce the concept of ZAM (zero angular mometum) equipotential hypersurface and investigate its properties in the general stationary axisymmetric space-time. Then we specialize the results to the Kerr exterior and match different interior solutions to the exterior across the shell. Finally we study the dynamics and thermodynamics of the shell. We also look at the Kerr-AdS exterior under the motivation of AdS/CFT correspondence.

Chapter 2

Kinematics of zero angular momentum (ZAM) observers

We consider an arbitrary 4-D stationary axisymmetric space-time with a metric of the general form

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2 + g_{\varphi\varphi}\mathrm{d}\varphi^2 + 2g_{\varphi t}\mathrm{d}\varphi \mathrm{d}t + g_{tt}\mathrm{d}t^2 \quad (2.1)$$

expressed in the coordinates $x^\alpha = (x^1, x^2, \varphi, t)$, where φ is the azimuthal angle about the axis of symmetry and t the time. Stationarity and axial symmetry imply that there exist two one-parameter groups of isometries generated respectively by the Killing vector fields $\xi^\alpha_{(t)}$ and $\xi^\alpha_{(\varphi)}$. Then the metric coefficients are functions of the other two spatial coordinates x^1 and x^2 only. The (normalized) 4-velocity of an observer orbiting in the φ direction can be written in terms of the two Killing vectors as

$$u^\alpha = U^{-1}U^\alpha = U^{-1}\Big[\xi^\alpha_{(t)} + \Omega\xi^\alpha_{(\varphi)}\Big] \tag{2.2a}$$

$$U^2 \equiv -U_\alpha U^\alpha = \frac{g_{\varphi t}^2 - g_{\varphi\varphi}g_{tt}}{g_{\varphi\varphi}} - g_{\varphi\varphi}(\Omega - \omega_{\rm B})^2, \tag{2.2b}$$

where $\Omega = {\rm d}\varphi/{\rm d}t$ is the angular velocity as measured by a stationary observer at infinity and $\omega_{\rm B} \equiv -g_{\varphi t}/g_{\varphi\varphi}$ is the Bardeen angular velocity.

A special class of these observers is that of ZAM (zero angular momentum) observers whose angular velocity is the Bardeen angular velocity, i.e., $\Omega = \omega_{\rm B}$. We write the ZAM 4-velocity as

$$v^\alpha = V^{-1}V^\alpha = V^{-1}\Big[\xi^\alpha_{(t)} + \omega_{\rm B}\xi^\alpha_{(\varphi)}\Big] \tag{2.3a}$$

$$V^2 = \frac{g_{\varphi t}^2 - g_{\varphi\varphi}g_{tt}}{g_{\varphi\varphi}} = \big(-g^{tt}\big)^{-1}. \tag{2.3b}$$

We see that the angular momentum of a ZAM observer does indeed vanish:

$$l \equiv v_\alpha \xi^\alpha_\varphi = v_\varphi = g_{\varphi\varphi}v^\varphi + g_{\varphi t}v^t = V^{-1}(g_{\varphi\varphi}\omega_{\rm B} + g_{\varphi t}) = 0.$$

The acceleration of a ZAM observer is

$$a_\alpha = v_{\alpha|\beta}v^\beta = -V^{-2}V_{\beta|\alpha}V^\beta = V^{-1}V_{,\alpha}, \tag{2.4}$$

where we have used Killing's equation

$$\xi_{(\alpha|\beta)} = 0$$

and the condition for axial symmetry and stationarity

$$V_{,\alpha}v^{\alpha} = (\omega_{\rm B})_{,\beta}v^{\beta} = 0.$$

Since $V_{\varphi} = v_{\varphi} = 0$ and $V^t = 1$,

$$V^2 = -V_{\alpha}V^{\alpha} = -V_t = -g^{tt}V_t^2 = \big(-g^{tt}\big)^{-1}.$$

This agrees with (2.3b) and shows that $V = \sqrt{-1/g^{tt}}$ is a natural generalization to stationary axisymmetric space-time of the potential $\sqrt{-g_{tt}}$ for static space-time and reduces to it when $g_{\varphi t} = 0$. Similarly, V^{α} is a generalization of the static timelike Killing vector $\xi^{\alpha}_{(t)}$. In fact we can write

$$V^{\alpha} = \Delta^{\alpha}{}_{\beta}\xi^{\beta}_{(t)}, \tag{2.5}$$

where $\Delta^{\alpha}{}_{\beta} \equiv \delta^{\alpha}{}_{\beta} - \xi^{\alpha}_{(\varphi)}\xi_{(\varphi)\beta}/\xi^2_{(\varphi)}$ projects onto tangent planes perpendicular to $\xi^{\alpha}_{(\varphi)}$. These planes ("blades"), in which all ZAM orbits lie, do not form a 3-space, i.e., $\xi^{\alpha}_{(\varphi)}$ is not proportional to a gradient in general. However, introducing an anholonomic coordinate $\bar{\Phi}$ defined by

$$d\!\!\!/\bar{\Phi} \equiv \Phi_{\alpha}{\rm d}x^{\alpha} = {\rm d}\varphi - \omega_{\rm B}{\rm d}t$$

where

$$\Phi^{\alpha} \equiv \xi^{\alpha}_{(\varphi)}/g_{\varphi\varphi}, \quad \Phi_{\alpha} = g_{\varphi\alpha}/g_{\varphi\varphi},$$

we can formally diagonalize (2.1) as

$$\left(\mathrm{d}s^2\right)^+ = g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2 + g_{\varphi\varphi}\not{\mathrm{d}}\bar{\Phi}^2 - V^2\mathrm{d}t^2. \tag{2.6}$$

Then for ZAM orbits,

$$\not{\mathrm{d}}\bar{\Phi} = 0 \quad \text{and} \quad \Phi_\alpha V^\alpha = \not{\partial}_\alpha \bar{\Phi} V^\alpha = \frac{\not{\mathrm{d}}\bar{\Phi}}{\mathrm{d}t} = 0.$$

Thus ZAM observers, who travel along constant $\bar{\Phi}$ world lines, play the role of static observers in static space-time, with correspondences $V^\alpha \longleftrightarrow \xi^\alpha_{(t)}$ and $-V^2$(= "g_{tt}" in (2.6)) $\longleftrightarrow \xi_{(t)\alpha}\xi^\alpha_{(t)}$(= g_{tt} in static space-time). As in the static case, we can interpret the potential V as the redshift factor. The infinite redshift surface $V = 0$ then represents an event horizon, which can also be understood as the limit where world lines of ZAM obserbers (as "generalized static observers") become lightlike and coincide with the null geodesic generators of the horizon.

Chapter 3

Extrinsic curvature of ZAM-equipotential surfaces $V = \text{const.}$

We have concluded that the event horizon is given by $V = 0$. This suggests that, in the near horizon thin shell model, we shall take the shell to lie on a ZAM-equipotential hypersurface, say Σ, consisting of ZAM orbits with the same value of V (which is infinitesimally small). We introduce the "redshifted surface gravity" of Σ given by

$$\kappa \equiv aV = \left|v_{\alpha|\beta}v^{\beta}\right|V = V^{-1}|\nabla V|V = |\nabla V|. \tag{3.1}$$

In the horizon limit, κ is just the surface gravity of the black hole. Note, however, points of the shell do not necessarily follow ZAM orbits. The shell's angular velocity, generally latitude dependent, will eventually be determined from its physical properties which are closely related to the geometry of Σ.

The geometry of a hypersurface in a given 4-geometry is characterized by two fundamental forms h_{ab} and K_{ab}, namely the induced metric and extrinsic curvature of the hypersurface. Latin indices denote intrinsic coordinates ξ^a of the hypersurface and run from 1 to 3. Naturally we choose $\xi^a = (\theta, \varphi, t)$, θ being the polar angle. The hypersurface Σ, as a 3-submanifold, can be given by a set of parametric equations $x^\alpha = x^\alpha(\xi^a)$:

$$\begin{aligned} &x^1 = x^1(\theta), \quad x^2 = x^2(\theta) \\ &\varphi = \varphi, \quad t = t, \end{aligned}$$

or equivalently by the equipotential condition $V(x^\alpha) = \text{const.}$. We start from the latter, which implies

$$\mathrm{d}V|_\Sigma = 0 \quad \Rightarrow \quad \mathrm{d}x^1_\Sigma = -\frac{V_{,2}}{V_{,1}}\mathrm{d}x^2_\Sigma.$$

Then the basis vectors $e^\alpha_{(a)} \equiv \partial x^\alpha / \partial \xi^a$ on Σ are

$$\begin{aligned} e^\alpha_{(\theta)} &= \left(-\frac{V_{,2}}{V_{,1}}, 1, 0, 0\right)\frac{\partial x^2_\Sigma}{\partial \theta} \\ e^\alpha_{(\varphi)} &= (0, 0, 1, 0) \\ e^\alpha_{(t)} &= (0, 0, 0, 1). \end{aligned}$$

Thus the induced intrinsic metric $h_{ab} \equiv g_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)}$ of Σ reads

$$\left(\mathrm{d}s^2\right)_\Sigma = h_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b = h_{\theta\theta}\mathrm{d}\theta^2 + g_{\varphi\varphi}\mathrm{d}\bar{\Phi}^2 - V^2\mathrm{d}t^2 \tag{3.2}$$

where

$$h_{\theta\theta}\left(\frac{\partial x^2_\Sigma}{\partial\theta}\right)^{-2} = g_{11}\left(\frac{V_{,2}}{V_{,1}}\right)^2 + g_{22} = \left(\frac{g^{22}V_{,2}^2 + g^{11}V_{,1}^2}{g^{11}V_{,1}^2}\right)g_{22}$$
$$= \frac{(\nabla V)^2}{(\nabla V)^2 - g^{22}V_{,2}^2}g_{22} = \frac{g_{22}}{1 - \kappa^{-2}g^{22}V_{,2}^2}. \tag{3.3}$$

The unit normal to Σ is given by

$$n_\alpha \equiv \frac{V_{,\alpha}}{|\nabla V|} = \kappa^{-1}V_{,\alpha}. \tag{3.4}$$

Then (2.4), (3.4) and (3.1) give

$$a_\alpha = V^{-1}\kappa n_\alpha = a n_\alpha, \tag{3.5}$$

so the acceleration of a ZAM observer is parallel to the normal vector of the ZAM-equipotential hypersurface to which he belongs.

Using these, we can calculate the extrinsic curvature from the defining relation

$$K_{ab} \equiv n_{\alpha|\beta}e^\alpha_{(a)}e^\beta_{(b)},$$

as follows:

$$
\begin{aligned}
& V_{,\alpha} e^{\alpha}_{(\theta)} = 0 \text{ and } (3.4) \\
\Rightarrow\quad & K_{\theta\theta} = \kappa^{-1} V_{|\alpha\beta} e^{\alpha}_{(\theta)} e^{\beta}_{(\theta)}
\end{aligned} \tag{3.6a}
$$

$$
\begin{aligned}
& n_{\alpha,A} = 0 \\
\Rightarrow\quad & K_{AB} = -n^{\alpha}\Gamma_{\alpha,AB} = \frac{1}{2} n^{\alpha} g_{AB,\alpha} = \frac{1}{2}\frac{\partial\big({}^{4}g_{AB}\big)}{\partial n}
\end{aligned} \tag{3.6b}
$$

$$
\begin{aligned}
& n_A = 0, \quad g_{\alpha A} = 0 \text{ (for } \alpha \neq \varphi, t) \quad \text{and} \quad g_{\alpha\beta,A} = 0 \\
\Rightarrow\quad & K_{\theta A} = -n^{\alpha}\Gamma_{\alpha,\beta A} e^{\beta}_{(\theta)} = 0,
\end{aligned} \tag{3.6c}
$$

with $A, B, \ldots = \varphi, t$. Furthermore we have, recalling (2.3a),

$$
\begin{aligned}
K_{\varphi A} v^{A} &= \frac{1}{2}\frac{\partial g_{\varphi A}}{\partial n} v^{A} = -\frac{1}{2} g_{\varphi A}\frac{\partial v^{A}}{\partial n} = -\frac{1}{2} g_{\varphi\varphi}\left[\frac{\partial(\omega_{\mathrm{B}} v^{t})}{\partial n} - \omega_{\mathrm{B}}\frac{\partial(v^{t})}{\partial n}\right] \\
&= -\frac{1}{2} V^{-1} g_{\varphi\varphi}\frac{\partial \omega_{\mathrm{B}}}{\partial n} \equiv \eta
\end{aligned} \tag{3.7a}
$$

$$
\begin{aligned}
K_{tA} v^{A} &= \frac{1}{2}\frac{\partial g_{tA}}{\partial n} v^{A} = \frac{1}{2} V^{-1}\left(\frac{\partial g_{tt}}{\partial n} + \frac{\partial g_{t\varphi}}{\partial n}\omega_{\mathrm{B}}\right) \\
&= \frac{1}{2} V^{-1}\left[\frac{\partial}{\partial n}\left(g_{tt} - \frac{g_{\varphi t}^{2}}{g_{\varphi\varphi}}\right) - g_{t\varphi}\frac{\partial\omega_{\mathrm{B}}}{\partial n}\right] \\
&= \frac{1}{2} V^{-1}\frac{\partial\big(-V^{2}\big)}{\partial n} - \frac{1}{2} V^{-1} g_{t\varphi}\frac{\partial\omega_{\mathrm{B}}}{\partial n} \\
&= -(\kappa + \eta\omega_{\mathrm{B}}) = V^{-1}(\kappa + \eta\omega_{\mathrm{B}}) v_{t}
\end{aligned} \tag{3.7b}
$$

$$
\begin{aligned}
K_{AB} v^{A} v^{B} &= K_{ab} v^{a} v^{b} = n_{\alpha|\beta} v^{\alpha} v^{\beta} = -n^{\alpha} v_{\alpha|\beta} v^{\beta} = -n^{\alpha} a_{\alpha} \\
&= -V^{-1}\kappa = -a \qquad\qquad (\text{since } n_{\alpha} v^{\alpha} = 0),
\end{aligned} \tag{3.7c}
$$

where v^a and its (φ, t) part v^A are given by

$$v^\alpha = v^a e^\alpha_{(a)}.$$

Chapter 4

Special case I: Kerr exterior

The hypersurface Σ partitions the 4-D space-time into two regions V^+ and V^-. We now allow the metrics of the two regions to be different but require that they induce the same intrinsic metric on Σ. In order to see more detailed properties of Σ we specialize the exterior metric to the Kerr metric, which is also of more physical interest since the Kerr metric is the only physically reasonable black hole solution for an isolated rotating source in vacuum.

The standard Boyer-Lindquist form of the Kerr metric is[1] (i.e., we take $x^1 = r$ and $x^2 = \theta$ in (2.1))

$$\left(\mathrm{d}s^2\right)^+ = \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) + R^2 \sin^2\theta \mathrm{d}\varphi^2 - \frac{4mar}{\Sigma}\sin^2\theta \mathrm{d}\varphi \mathrm{d}t - \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2, \tag{4.1}$$

[1]The "exterior/interior" quantities are denoted with an upper $+/-$ sign which we sometimes omit for convenience or where it is obvious, while a lower $+/-$ sign means values taken at the outer/inner horizon of the Kerr solution.

where

$$\Sigma = r^2 + a^2\cos^2\theta$$
$$\Delta = r^2 - 2mr + a^2$$
$$R^2 = r^2 + a^2 + \frac{2mr}{\Sigma}a^2\sin^2\theta.$$

The Bardeen angular velocity for this metric reads

$$\omega_{\mathrm{B}} = -\frac{g_{\varphi t}}{g_{\varphi\varphi}} = \frac{2mar}{\Sigma R^2} \equiv \frac{a}{\Gamma},$$

where

$$\Gamma = \frac{\Sigma R^2}{2mr} = \frac{\Sigma}{2mr}\big(r^2+a^2\big)+a^2\sin^2\theta = \frac{\Sigma}{2mr}(\Delta+2mr)+a^2\sin^2\theta = \frac{\Sigma\Delta}{2mr}+r^2+a^2. \tag{4.2}$$

The ZAM potential is

$$V^2 = \frac{g_{\varphi t}^2 - g_{\varphi\varphi}g_{tt}}{g_{\varphi\varphi}} = \frac{\Delta\sin^2\theta}{g_{\varphi\varphi}} = \frac{\Sigma\Delta}{2mr\Gamma}, \tag{4.3}$$

showing that V vanishes on the (outer) horizon $r = r_+$, the larger root of $\Delta = 0$. The horizon itself is then a (null) ZAM-equipotential hypersurface. This explicit form of V allows us to find the near-horizon behavior of ZAM-equipotential hypersurfaces. In other words, we treat V as a small correction to the vanishing horizon case and express the extrinsic curvature K^+_{ab} (as in (3.6)) of Σ in terms of powers of V.

First, (4.2) and (4.3) combine to give

$$\Gamma = \frac{r^2 + a^2}{1 - V^2}. \tag{4.4}$$

Then

$$\omega_{\mathrm{B}} = \frac{a}{\Gamma} = \frac{a}{r^2 + a^2}\bigl(1 - V^2\bigr).$$

Its horizon limit is the "angular velocity of the black hole":

$$\omega_{\mathrm{H}} \equiv \omega_{\mathrm{B}}\Big|_{r\,=\,r_+\ (\text{or } V\,=\,0)} = \frac{a}{r_+^2 + a^2} = \frac{a}{2mr_+}.$$

It is easy to check that, noticing $r - r_+ = \Delta/(r - r_-) \sim \Delta \sim \mathcal{O}(V^2)$,

$$\Delta\omega \equiv \omega_{\mathrm{B}} - \omega_{\mathrm{H}} \sim \mathcal{O}\bigl(V^2\bigr). \tag{4.5}$$

Next, canceling Γ in (4.3) and (4.4), we have

$$\frac{V^2}{1 - V^2} = \Sigma \frac{\Delta}{2mr(r^2 + a^2)},$$

which neatly separates the θ and r dependence of V. We can then calculate derivatives of V, which are needed to construct K^+_{ab}. The first partial derivatives are

$$V_{,\theta} = \frac{1}{2} V \frac{\Sigma_{,\theta}}{\Sigma}\bigl(1 - V^2\bigr) \sim \mathcal{O}(V) \tag{4.6a}$$

$$V_{,r} = \frac{1}{2} V \bigl(1 - V^2\bigr)\left(\frac{\Delta_{,r}}{\Delta} + \sigma_{,r}\right) = \frac{1}{2} V \frac{\Delta_{,r}}{\Delta}\bigl[1 + \mathcal{O}\bigl(V^2\bigr)\bigr] \sim \mathcal{O}\bigl(V^{-1}\bigr), \tag{4.6b}$$

where

$$\sigma \equiv \ln\left[\frac{\Sigma}{2mr(r^2+a^2)}\right].$$

The second partial derivatives are

$$\begin{aligned}
V_{,\theta\theta} &= \frac{1}{2}V_{,\theta}\frac{\Sigma_{,\theta}}{\Sigma} + \frac{1}{2}V\left(\frac{\Sigma_{,\theta}}{\Sigma}\right)_{,\theta} + \mathcal{O}\big(V^3\big) \\
&= V_{,\theta}^2/V + \frac{1}{2}V\left(\frac{\Sigma_{,\theta}}{\Sigma}\right)_{,\theta} + \mathcal{O}\big(V^3\big) \\
&\sim \mathcal{O}(V) \qquad (4.7a)\\
V_{,r\theta} &= \frac{1}{2}V_{,\theta}\frac{\Delta_{,r}}{\Delta} + \mathcal{O}(V) \\
&= V_{,\theta}V_{,r}/V + \mathcal{O}(V) \\
&\sim \mathcal{O}\big(V^{-1}\big) \qquad (4.7b)\\
V_{,rr} &= \frac{1}{2}V_{,r}\frac{\Delta_{,r}}{\Delta} + \frac{1}{2}V\left(\frac{\Delta_{,r}}{\Delta}\right)_{,r} + \mathcal{O}\big(V^{-1}\big) \\
&= \frac{1}{2}V_{,r}\frac{\Delta_{,r}}{\Delta} - \frac{1}{2}V\left(\frac{\Delta_{,r}}{\Delta}\right)^2 + \mathcal{O}\big(V^{-1}\big) \\
&= -V_{,r}^2/V + \mathcal{O}\big(V^{-1}\big) \\
&\sim \mathcal{O}\big(V^{-3}\big). \qquad (4.7c)
\end{aligned}$$

Other quantities needed to be evaluated are

$$
\begin{aligned}
e^r_{(\theta)} &= -\frac{V_{,\theta}}{V_{,r}} \sim \mathcal{O}\big(V^2\big) \\
\kappa = |\nabla V| &= \sqrt{g^{rr}}V_{,r}\sqrt{1+\frac{g^{\theta\theta}V_{,\theta}^2}{g^{rr}V_{,r}^2}} = \sqrt{g^{rr}}V_{,r}\big[1+\mathcal{O}\big(V^2\big)\big] \sim \mathcal{O}(1) \\
n^r &= \kappa^{-1}g^{rr}V_{,r} \sim \mathcal{O}(V) \\
n^\theta &= \kappa^{-1}g^{\theta\theta}V_{,\theta} \sim \mathcal{O}(V) \\
\kappa_{,\theta} &\sim \mathcal{O}\big(V^2\big) \\
\kappa_{,r} &\sim \mathcal{O}(1),
\end{aligned}
$$

where (4.6) and (4.7) are used. The calculation of $\kappa_{,\theta}$ and $\kappa_{,r}$ is trickier, so we provide the details below. Starting from $\kappa^2 = |\nabla V|^2 = g^{rr}V_{,r}^2 + g^{\theta\theta}V_{,\theta}^2$, we have

$$
2\kappa\kappa_{,\theta} = \Delta\left(\frac{1}{\Sigma}\right)_{,\theta}V_{,r}^2 + 2\frac{\Delta}{\Sigma}V_{,r}V_{,r\theta} + \left(\frac{1}{\Sigma}\right)_{,\theta}V_{,\theta}^2 + 2\frac{1}{\Sigma}V_{,\theta}V_{,\theta\theta} \tag{4.8a}
$$

$$
2\kappa\kappa_{,r} = \left(\frac{\Delta}{\Sigma}\right)_{,r}V_{,r}^2 + 2\frac{\Delta}{\Sigma}V_{,r}V_{,rr} + \left(\frac{1}{\Sigma}\right)_{,r}V_{,\theta}^2 + 2\frac{1}{\Sigma}V_{,\theta}V_{,r\theta}. \tag{4.8b}
$$

Substitute in (4.6) and (4.7), in which we will only write explicitly the leading terms, and (4.8) now becomes

$$
\begin{aligned}
\kappa\kappa_{,\theta} &= \frac{\Delta}{\Sigma}V_{,r}\left(-\frac{1}{2}\frac{\Sigma_{,\theta}}{\Sigma}V_{,r}+V_{,r\theta}\right)+\frac{1}{\Sigma}V_{,\theta}\left(-\frac{1}{2}\frac{\Sigma_{,\theta}}{\Sigma}V_{,\theta}+V_{,\theta\theta}\right) \\
&= \frac{\Delta}{\Sigma}V_{,r}[-\cancel{V_{,\theta}V_{,r}/V}+\cancel{V_{,\theta}V_{,r}/V}+\mathcal{O}(V)] \qquad (4.9a) \\
&\quad+\frac{1}{\Sigma}V_{,\theta}\left[-\cancel{V_{,\theta}^2/V}+\cancel{V_{,\theta}^2/V}+\frac{1}{2}V\left(\frac{\Sigma_{,\theta}}{\Sigma}\right)_{,\theta}+\mathcal{O}\big(V^3\big)\right] \\
&\sim \mathcal{O}\big(V^2\big) \qquad (4.9b) \\
\kappa\kappa_{,r} &= \frac{\Delta}{\Sigma}V_{,r}\left[\frac{1}{2}\left(\frac{\Delta_{,r}}{\Delta}-\frac{\Sigma_{,r}}{\Sigma}\right)V_{,r}+V_{,rr}\right]+\frac{1}{\Sigma}V_{,\theta}\left(-\frac{1}{2}\frac{\Sigma_{,r}}{\Sigma}V_{,\theta}+V_{,r\theta}\right) \\
&= \frac{\Delta}{\Sigma}V_{,r}\left[\cancel{V_{,r}^2/V}-\frac{1}{2}\frac{\Sigma_{,r}}{\Sigma}V_{,r}-\cancel{V_{,r}^2/V}+\mathcal{O}\big(V^{-1}\big)\right] \qquad (4.9c) \\
&\quad+\frac{1}{\Sigma}V_{,\theta}\left[-\frac{1}{2}\frac{\Sigma_{,r}}{\Sigma}V_{,\theta}+V_{,\theta}V_{,r}/V+\mathcal{O}(V)\right] \\
&\sim \mathcal{O}(1), \qquad (4.9d)
\end{aligned}
$$

as stated above. The second covariant derivatives of V are

$$\begin{aligned}
V_{|rr} &= V_{,rr} - \frac{1}{2}V_{,r}g^{rr}g_{rr,r} + \frac{1}{2}V_{,\theta}g^{\theta\theta}g_{rr,\theta} \\
&= \sqrt{g_{rr}}\left(\frac{V_{,r}}{\sqrt{g_{rr}}}\right)_{,r} + \frac{1}{2}V_{,\theta}g^{\theta\theta}g_{rr,\theta} \\
&= \sqrt{g_{rr}}\kappa_{,r}\big[\mathcal{O}(1) + \mathcal{O}\big(V^2\big)\big] + \mathcal{O}\big(V^{-1}\big) \\
&\sim \mathcal{O}\big(V^{-1}\big) \\
V_{|r\theta} &= V_{,r\theta} - \frac{1}{2}V_{,r}g^{rr}g_{rr,\theta} - \frac{1}{2}V_{,\theta}g^{\theta\theta}g_{\theta\theta,r} \\
&= \sqrt{g_{rr}}\left(\frac{V_{,r}}{\sqrt{g_{rr}}}\right)_{,\theta} - \frac{1}{2}V_{,\theta}g^{\theta\theta}g_{\theta\theta,r} \\
&= \sqrt{g_{rr}}\kappa_{,\theta}\big[\mathcal{O}(1) + \mathcal{O}\big(V^2\big)\big] + \mathcal{O}(V) \\
&\sim \mathcal{O}(V) \\
V_{|\theta\theta} &= V_{,\theta\theta} + \frac{1}{2}V_{,r}g^{rr}g_{\theta\theta,r} - \frac{1}{2}V_{,\theta}g^{\theta\theta}g_{\theta\theta,\theta} \\
&\sim \mathcal{O}(V).
\end{aligned}$$

Finally from (3.6) and (3.7),

$$K^{+}_{\theta\theta} = \kappa^{-1}\big[V_{|\theta\theta} + V_{|rr}\mathcal{O}\big(V^4\big) + V_{|r\theta}\mathcal{O}\big(V^2\big)\big] \sim \mathcal{O}(V) \tag{4.10a}$$

$$K^{+}_{AB} = \frac{1}{2}n^{\alpha}g_{AB,\alpha} \sim \mathcal{O}(V) \tag{4.10b}$$

and

$$K_{\varphi A}v^A = -\frac{1}{2}V^{-1}g_{\varphi\varphi}\frac{\partial\omega_{\mathrm{B}}}{\partial n} \equiv \eta \sim \mathcal{O}(1) \tag{4.11a}$$

$$K_{tA}v^A = -(\kappa + \eta\omega_{\mathrm{B}}) \sim \mathcal{O}(1) \tag{4.11b}$$

$$K^{+}_{ab}v^a v^b = -V^{-1}\kappa, \tag{4.11c}$$

where in (4.11a) we have used

$$\frac{\partial\omega_{\mathrm{B}}}{\partial n} \simeq \frac{\Delta\omega}{\mathrm{d}n} \simeq \frac{\Delta\omega}{V/|\nabla V|} = \kappa\frac{\Delta\omega}{V} \sim \mathcal{O}(V). \tag{4.12}$$

Chapter 5

Shell dynamics: review of basic formulae

We now put the matter in. We assume it is compressed into a thin shell. The term "thin shell" refers to a *singular hypersurface of order one*, or (the history of) a *surface layer* [7], where matter is concentrated. In the present case we assume the source of the exterior stationary axisymmetric (Kerr) metric to be a spinning shell located at Σ (we then also call the shell Σ). Clearly the interior metric $g^{-}_{\alpha\beta}$ will not be the Kerr metric and not necessarily be written in coordinates that match continuously with the Boyer-Lindquist coordinates, i.e., $[x^{\alpha}] \neq 0$ and $[g_{\alpha\beta}] \neq 0$, where $[A] \equiv A^{+}\big|_{\Sigma} - A^{-}\big|_{\Sigma}$ denotes the jump of any tensorial quantity A across Σ. Even so, we have Israel's junction conditions (or "jump condition" for the second) that

$$[h_{ab}] = 0$$

and

$$-8\pi S_{ab} = [K_{ab} - h_{ab}K], \tag{5.1}$$

where $K \equiv h^{ab}K_{ab}$. The first, already stated in Chapter 4, says that the induced 3-metric must be continuous across Σ. The second relates the jump in the extrinsic curvature K_{ab} of Σ to its intrinsic surface stress-energy tensor S_{ab} due to the presence of matter. Both are expressed independently of the 4-D coordinates.

The angular velocity $\Omega(\theta)$ and proper surface density σ of Σ will be determined from the eigenvalue equation

$$S^a{}_b u^b = -\sigma u^a, \tag{5.2}$$

with (c.f. (2.2))

$$u^a = g^{ab} e^{\alpha}_{(b)} u_\alpha = U^{-1} g^{ab} e^{\alpha}_{(b)} \left[\xi_{(t)\alpha} + \Omega(\theta)\xi_{(\varphi)\alpha}\right] = U^{-1}\left[\delta^a_t + \Omega(\theta)\delta^a_\varphi\right]. \tag{5.3}$$

The surface pressure of Σ is given by

$$P_\theta = S_\theta{}^\theta$$
$$P_\phi = S_\phi{}^\phi.$$

Chapter 6

ZAM-equipotential shell in Kerr

The usual Boyer-Lindquist form of the Kerr metric (4.1) is expressed in a coordinate frame that is non-rotating with respect to the inertial frame at infinity (frame of the "fixed stars"), where the metric becomes Minkowskian. However, according to the effect of *dragging of inertial frames*, local inertial frames near the shell will partially co-rotate with it, so for our purpose it would be better to choose a "co-rotating" azimuthal coordinate Φ for the Kerr metric. Since the co-rotating angular velocity becomes the constant ω_{H} at the horizon and we are working with the near-horizon approximation, it is natural to define Φ as follows:

$$\Phi \equiv \varphi - \omega_{\mathrm{H}} t, \quad \mathrm{d}\Phi = \mathrm{d}\varphi - \omega_{\mathrm{H}} \mathrm{d}t = \not{d}\bar{\Phi} + \Delta\omega \mathrm{d}t.$$

Then the Kerr metric is transformed into

$$\left(\mathrm{d}s^2\right)^+ = g_{rr}\mathrm{d}r^2+g_{\theta\theta}\mathrm{d}\theta^2+g_{\varphi\varphi}\mathrm{d}\Phi^2-2g_{\varphi\varphi}\Delta\omega\mathrm{d}\Phi\mathrm{d}t-\left[V^2-g_{\varphi\varphi}(\Delta\omega)^2\right]\mathrm{d}t^2 \quad (6.1)$$

in the new coordinates (r,θ,Φ,t) and we see that

$$\begin{aligned} g_{\Phi\Phi} &= g_{\varphi\varphi} \\ g_{\Phi t} &= -g_{\varphi\varphi}\Delta\omega \\ \tilde{g}_{tt} &= -V^2 + g_{\varphi\varphi}(\Delta\omega)^2, \end{aligned}$$

where $\tilde{g}_{tt}$ refers to the metric coefficient in (6.1) as distinguished from the Boyer-Lindquist one g_{tt}. For the new metric, the Bardeen angular velocity is $\tilde{\omega}_{\mathrm{B}} \equiv -g_{\Phi t}/g_{\Phi\Phi} = \Delta\omega$, while the ZAM potential $\tilde{V}$ is the same as the old one:

$$\tilde{V}^2 = \frac{g_{\Phi t}^2 - g_{\Phi\Phi}\tilde{g}_{tt}}{g_{\Phi\Phi}} = \frac{g_{\varphi t}^2 - g_{\varphi\varphi}g_{tt}}{g_{\varphi\varphi}} = V^2. \quad (6.2)$$

So for the same ZAM-equipotential shell, its extrinsic curvature in the new coordinates would take the same form as (3.6). This is checked by performing a direct coordinate transformation, e.g., for K^+_{AB} $(A,B,\ldots = \Phi, t)$:

$$K^+_{tt} = \frac{1}{2}\left[\frac{\partial g_{\varphi\varphi}}{\partial n}(\omega_{\mathrm{H}})^2 + 2\frac{\partial g_{\varphi t}}{\partial n}\omega_{\mathrm{H}} + \frac{\partial g_{tt}}{\partial n}\right] = \frac{1}{2}\frac{\partial \tilde{g}_{tt}}{\partial n} \quad (6.3a)$$

$$K^+_{\Phi t} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial n}\omega_{\mathrm{H}} + \frac{\partial g_{\varphi t}}{\partial n}\right) = \frac{1}{2}\frac{\partial g_{\Phi t}}{\partial n} \quad (6.3b)$$

$$K^+_{\Phi\Phi} = \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n} = \frac{1}{2}\frac{\partial g_{\Phi\Phi}}{\partial n}. \quad (6.3c)$$

To estimate the magnitude of K^+_{ab}, we notice from (3.1) and (4.12) that

$$\frac{\partial V}{\partial n} = |\nabla V| = \kappa \sim \mathcal{O}(1)$$
$$\frac{\partial \Delta\omega}{\partial n} = \frac{\partial \omega_{\mathrm{B}}}{\partial n} \simeq \kappa \frac{\Delta\omega}{V} \sim \mathcal{O}(V). \tag{6.4}$$

Then

$$\begin{aligned}
K^+_{tt} &= \frac{1}{2}\frac{\partial \tilde{g}_{tt}}{\partial n} \\
&= -V\frac{\partial V}{\partial n} + g_{\varphi\varphi}\Delta\omega\frac{\partial \Delta\omega}{\partial n} + \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}(\Delta\omega)^2 \\
&= -V\kappa + g_{\varphi\varphi}\frac{(\Delta\omega)^2}{V}\kappa + \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}(\Delta\omega)^2 \\
&= -V\kappa(1 + 2\Delta\omega\eta/\kappa) + \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}(\Delta\omega)^2 \sim \mathcal{O}(V) && (6.5a)\\
K^+_{\Phi t} &= \frac{1}{2}\frac{\partial g_{\Phi t}}{\partial n} \\
&= -\frac{1}{2}g_{\varphi\varphi}\frac{\partial \Delta\omega}{\partial n} - \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\Delta\omega \\
&= -\frac{1}{2}g_{\varphi\varphi}\frac{\Delta\omega}{V}\kappa - \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\Delta\omega \\
&= V\eta - \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\Delta\omega \sim \mathcal{O}(V) && (6.5b)\\
K^+_{\Phi\Phi} &= \frac{1}{2}\frac{\partial g_{\Phi\Phi}}{\partial n} \\
&= \frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n} \sim \mathcal{O}(V) && (6.5c)\\
K^+_{\theta\theta} &= \kappa^{-1}V_{|\alpha\beta}e^{\alpha}_{(\theta)}e^{\beta}_{(\theta)} \sim \mathcal{O}(V), && (6.5d)
\end{aligned}$$

where

$$\eta = -\frac{1}{2}V^{-1}g_{\varphi\varphi}\frac{\partial \omega_{\mathrm{B}}}{\partial n} = -\frac{1}{2}g_{\varphi\varphi}\frac{\Delta\omega}{V^2}\kappa. \tag{6.6}$$

The contracted forms (3.7) or (4.11) become (note that v^a is not the 4-velocity of the shell)

$$K_{\Phi A} v^A\big|^+ = \eta \sim \mathcal{O}(1) \tag{6.7a}$$

$$K_{tA} v^A\big|^+ = -(\kappa + \eta\Delta\omega) = V^{-1}(\kappa + \eta\Delta\omega)v_t \sim \mathcal{O}(1) \tag{6.7b}$$

$$K_{ab} v^a v^b\big|^+ = -V^{-1}\kappa, \tag{6.7c}$$

with ω_{B} replaced by $\Delta\omega$.

Chapter 7

Interior embedding — vacuum interior

By virtue of the junction condition, the intrinsic metrics of the shell induced by the exterior and interior metrics must agree. In this chapter we first assume a stationary axisymmetric vacuum interior and write its metric in the form given by Lewis [10]:

$$\left(\mathrm{d}s^2\right)^- = e^{2(\nu-\lambda)}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \rho^2 e^{-2\lambda}\mathrm{d}\Phi'^2 - e^{2\lambda}(\mathrm{d}t' - \psi\mathrm{d}\Phi')^2, \qquad (7.1)$$

for which the vacuum field equations reduce to (subscripts indicate partial differentiation)

$$\nu_\rho = \rho\big(\lambda_\rho^2 - \lambda_z^2\big) - \frac{1}{4}\rho^{-1}e^{4\lambda}\big(\psi_\rho^2 - \psi_z^2\big) \tag{7.2a}$$

$$\nu_z = 2\rho\lambda_\rho\lambda_z - \frac{1}{2}\rho^{-1}e^{4\lambda}\psi_\rho\psi_z \tag{7.2b}$$

$$\lambda_{\rho\rho} + \rho^{-1}\lambda_\rho + \lambda_{zz} = -\frac{1}{2}\rho^{-2}e^{4\lambda}\big(\psi_\rho^2 + \psi_z^2\big) \tag{7.2c}$$

$$\psi_{\rho\rho} - \rho^{-1}\psi_\rho + \psi_{zz} = -4(\lambda_\rho\psi_\rho + \lambda_z\psi_z). \tag{7.2d}$$

To match (7.1) to the exterior Kerr metric we notice that (6.1) induces the following intrinsic metric on the shell

$$\big(\mathrm{d}s^2\big)_\Sigma = h_{\theta\theta}\mathrm{d}\theta^2 + g_{\varphi\varphi}\mathrm{d}\Phi^2 - 2g_{\varphi\varphi}\Delta\omega\mathrm{d}\Phi\mathrm{d}t - \big[V^2 - g_{\varphi\varphi}(\Delta\omega)^2\big]\mathrm{d}t^2, \tag{7.3}$$

where from (3.3)

$$h_{\theta\theta} = \frac{\Sigma}{1-\kappa^{-2}V_{,\theta}^2/\Sigma}.$$

Assuming Φ' and t' are proportional to Φ and t by a constant respectively, we can drop the prime signs by absorbing them into the metric coefficients, leaving the field equations (7.2) unaffected. Then a first comparison of the coefficients of $\mathrm{d}t^2$ in (7.1) and (7.3) suggests that we may have, remembering (4.5),

$$e^{2\lambda} = V^2$$

on the shell, so that

$$e^{4\lambda} = V^4$$

which makes the RHS of (7.2c) negligible under the near-horizon approximation[1] and by the uniqueness of the solution

$$\lambda = \text{const.} = \ln V$$

all over the interior as well as on the shell. Then (7.2a) and (7.2b) (whose RHS terms now all vanish) give

$\nu = \text{const.}(= 0$ by "elementary flatness", which requires $\nu = 0$ on the axis [10])

and (7.2d) becomes[2]

$$\psi_{\rho\rho} - \rho^{-1}\psi_\rho + \psi_{zz} = \nabla^2\psi - 2\rho^{-1}\psi_\rho = 0. \tag{7.4}$$

Rescale $(\rho, z) \to V(\rho, z)$, and finally we reach the following expression for the interior metric

$$\left(\mathrm{d}s^2\right)^- = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\mathrm{d}\Phi^2 - V^2(\mathrm{d}t - \psi\mathrm{d}\Phi)^2. \tag{7.5}$$

Note here V means the constant value of the ZAM potential on the shell. The metric induced by (7.5) on the shell at $\rho = \rho_\Sigma(\theta)$ and $z = z_\Sigma(\theta)$ should

[1]We assume $e^{4\lambda}$ is of the same order in the interior.

[2]Refer to the Appendix for a detailed discussion of ψ.

match (7.3), giving

$$\mathrm{d}\rho_\Sigma{}^2 + \mathrm{d}z_\Sigma{}^2 = h_{\theta\theta}\mathrm{d}\theta^2 \tag{7.6a}$$

$$\rho_\Sigma^2 - V^2\psi_\Sigma^2 = g_{\varphi\varphi} \tag{7.6b}$$

$$V^2\psi_\Sigma = -g_{\varphi\varphi}\Delta\omega \tag{7.6c}$$

$$V^2 = V^2 - g_{\varphi\varphi}(\Delta\omega)^2. \tag{7.6d}$$

Note that (6.6) and (7.6c) give

$$\psi_\Sigma = 2\eta/\kappa. \tag{7.7}$$

Writing (7.6) out explicitly for the Kerr metric and neglecting terms of $\mathcal{O}(V^2)$ and higher, i.e., letting $r = r_+$, we have from (7.6b) and (7.6a) [(7.6c) and (7.6d) are automatically satisfied to this order]

$$\rho_\Sigma = R_+ \sin\theta = \frac{r_+^2 + a^2}{\Sigma_+^{\frac{1}{2}}} \sin\theta$$

$$\mathrm{d}z_\Sigma{}^2 = \Sigma_+ \mathrm{d}\theta^2 - \mathrm{d}\rho_\Sigma{}^2.$$

So

$$\mathrm{d}\rho_\Sigma = \frac{\left(r_+^2 + a^2\right)^2}{\Sigma_+^{\frac{3}{2}}} \cos\theta \mathrm{d}\theta = \Sigma_+^{\frac{1}{2}} F \mathrm{d}\theta \tag{7.8}$$

$$\mathrm{d}z_\Sigma = -\sqrt{\frac{\Sigma_+^4 - \left(r_+^2 + a^2\right)^4 \cos^2\theta}{\Sigma_+^3}} \mathrm{d}\theta = -\Sigma_+^{\frac{1}{2}} G \mathrm{d}\theta \tag{7.9}$$

$$z_\Sigma = -\int_{\frac{\pi}{2}}^{\theta} \Sigma_+^{\frac{1}{2}} G \mathrm{d}\theta, \tag{7.10}$$

where

$$F(\theta) \equiv \left(\frac{r_+^2 + a^2}{\Sigma_+}\right)^2 \cos\theta$$

$$G(\theta) \equiv \sqrt{1 - \left(\frac{r_+^2 + a^2}{\Sigma_+}\right)^4 \cos^2\theta} = \sqrt{1 - F^2}.$$

The minus signs in front of (7.9) and (7.10) are a result of the fact that z decreases in the direction of increasing positive θ.

For the embedding to hold, it must be true that

$$\mathrm{d}{z_\Sigma}^2 \geq 0, \tag{7.11}$$

which means (for $0 \leq \cos\theta < 1$)

$$
\begin{aligned}
& \Sigma_+^4 - \left(r_+^2 + a^2\right)^4 \cos^2\theta \geq 0 \\
\Rightarrow \quad & r_+^2 + a^2\cos^2\theta \geq \left(r_+^2 + a^2\right)\cos^{\frac{1}{2}}\theta \\
\Rightarrow \quad & r_+^2 \geq \frac{\cos^{\frac{1}{2}}\theta - \cos^2\theta}{1 - \cos^{\frac{1}{2}}\theta} a^2 \\
& \quad = \frac{\cos^{\frac{1}{2}}\theta - \cos\theta + \cos\theta - \cos^{\frac{3}{2}}\theta + \cos^{\frac{3}{2}}\theta - \cos^2\theta}{1 - \cos^{\frac{1}{2}}\theta} a^2 \\
& \quad = \left(\cos^{\frac{1}{2}}\theta + \cos\theta + \cos^{\frac{3}{2}}\theta\right) a^2 \\
\Rightarrow \quad & \sqrt{m^2 - a^2} \geq \sqrt{C}a - m,
\end{aligned}
\tag{7.12}
$$

where $C \equiv \cos^{\frac{1}{2}}\theta + \cos\theta + \cos^{\frac{3}{2}}\theta$. Solving for a, we have

(1) $C \leq 1$

$$a \leq m;$$

(2) $1 < C < 3$

$$a \leq \frac{2m\sqrt{C}}{1 + C} \equiv a_{\max}. \tag{7.13}$$

Similar results hold for $-1 < \cos\theta \leq 0$ by symmetry. (7.13) marks a failure of the interior embedding in the regions around the poles for fast rotating shells. The breakdown of the embedding starts from the poles and proceeds to the latitude with $C = 1$ as a increases. Note that the poles themselves are exceptional points since the equal sign in (7.11) holds identically. However this is not going to change the fate of the shell and so is of little physical interest.

Chapter 8

Interior embedding — non-vacuum interior

In an attempt to avoid these embedding difficulties for rapidly spinning shells, let us look at a slightly different, non-vacuum interior geometry, with the metric

$$\left(\mathrm{d}s^2\right)^- = \mathrm{d}\rho^2 + \mathrm{d}z^2 + N^2(\rho)\mathrm{d}\Phi^2 - \mathrm{d}t'^2. \tag{8.1}$$

Using the oblate spheroidal coordinates defined by

$$\rho = \sqrt{r^2 + a^2}\sin\theta, \quad z = r\cos\theta$$

we can transform the above metric into

$$\left(\mathrm{d}s^2\right)^- = \Sigma\left(\frac{\mathrm{d}r^2}{r^2 + a^2} + \mathrm{d}\theta^2\right) + N^2(\rho)\mathrm{d}\Phi^2 - \mathrm{d}t'^2. \tag{8.2}$$

The intrinsic metrics induced on both sides at $r = r_+$ agree if we choose

$$N(\rho) = \rho\gamma^{-\frac{1}{2}}, \quad \gamma \equiv 1 - \frac{a^2\rho^2}{\left(r_+^2 + a^2\right)^2}.$$

This can be done for all $a \leq m$, which means there is no breakdown of the embedding.

Chapter 9

Inner extrinsic curvature

To compute the inner extrinsic curvature K_{ab}^{-} we work with the vacuum interior case and use $\zeta^a = \zeta^a(\theta, \Phi, t)$ as intrinsic coordinates of the shell. Then from (7.8) and (7.9) the basis vectors ${e^{\alpha}_{(a)}}^{-} = \partial(x^\alpha)^-/\partial\zeta^a$ are

$$\begin{aligned}
{e^{\alpha}_{(\theta)}}^{-} &= \Big(\Sigma_{+}^{\frac{1}{2}}F, -\Sigma_{+}^{\frac{1}{2}}G, 0, 0\Big) \\
{e^{\alpha}_{(\Phi)}}^{-} &= (0, 0, 1, 0) \\
{e^{\alpha}_{(t)}}^{-} &= (0, 0, 0, 1) \\
n^{\alpha -} = n_{\alpha}^{-} &= \big(G, F, 0, 0\big).
\end{aligned}$$

Thus (note that V is constant in the interior)

$$K^-_{tt} = -n^\alpha \Gamma_{\alpha,tt}\Big|^- = \frac{1}{2}\frac{\partial g_{tt}}{\partial n}\Big|^- = 0 \tag{9.1a}$$

$$K^-_{\Phi t} = K^-_{t\Phi} = -n^\alpha \Gamma_{\alpha,\Phi t}\Big|^- = \frac{1}{2}V^2 \frac{\partial \psi}{\partial n}\Big|^- \sim \mathcal{O}\big(V^2\big) \tag{9.1b}$$

$$\begin{aligned} K^-_{\Phi\Phi} &= -n^\alpha \Gamma_{\alpha,\Phi\Phi}\Big|^- = \frac{1}{2} G g_{\Phi\Phi,\rho} + \frac{1}{2} F g_{\Phi\Phi,z} \\ &= G\rho_\Sigma = G\frac{r_+^2 + a^2}{\Sigma_+^{\frac{1}{2}}} \sin\theta \sim \mathcal{O}(1) \end{aligned} \tag{9.1c}$$

$$\begin{aligned} K^-_{\theta\theta} &= \Big(n_{\rho,\theta} e^\rho_{(\theta)} + n_{z,\theta} e^z_{(\theta)}\Big)\Big|^- = \Sigma_+^{\frac{1}{2}}(G'F - F'G) = \Sigma_+^{\frac{1}{2}} G'/F \\ &= \frac{\big(r_+^2 + a^2\big)^2}{\Sigma_+^{\frac{1}{2}}} \frac{\big(r_+^2 - 3a^2\cos^2\theta\big)\sin\theta}{\sqrt{\Sigma_+^4 - \big(r_+^2 + a^2\big)^4 \cos^2\theta}} \sim \mathcal{O}(1) \end{aligned} \tag{9.1d}$$

We see that the numerator of (9.1d) is always positive, for by (7.12)

$$\big(r_+^2 - 3a^2\cos^2\theta\big) \geq \big(\cos^{\frac{1}{2}}\theta + \cos\theta + \cos^{\frac{3}{2}}\theta - 3\cos^2\theta\big)a^2 > 0$$

for $\theta \neq 0$. The equal sign corresponds to the limiting case where the embedding starts to break down, with $K^-_{\theta\theta} \to +\infty$, but otherwise all K^-_{ab}'s are regular even in the horizon limit.

Chapter 10

Surface stress-energy tensor and angular velocity of shell

With both K^+_{ab} and K^-_{ab} obtained, we are now ready to calculate the stress-energy tensor and angular velocity of the shell as given by (5.1) and (5.2). First, we find the contravariant components of the intrinsic metric (7.3) using

$$h^{ab} = \frac{\mathrm{cof} h_{ab}}{\det h_{ab}}.$$

Specifically, noting (7.6),

$$
\begin{aligned}
h^{tt} &= -V^{-2} \\
h^{\Phi\Phi} &= \frac{1}{g_{\varphi\varphi}} - \frac{(\Delta\omega)^2}{V^2} \\
&= \frac{1}{\rho_\Sigma^2}\left(1 + \frac{V^2\psi_\Sigma^2}{\rho_\Sigma^2}\right) - V^2\frac{\psi_\Sigma^2}{g_{\varphi\varphi}^2} \\
&= \frac{1}{\rho_\Sigma^2}\left(1 + \frac{V^2\psi_\Sigma^2}{\rho_\Sigma^2}\right) - \frac{V^2\psi_\Sigma^2}{\rho_\Sigma^4}\left(1 + 2\frac{V^2\psi_\Sigma^2}{\rho_\Sigma^2}\right) \\
&= \frac{1}{\rho_\Sigma^2}\left(1 - 2\frac{V^4\psi_\Sigma^4}{\rho_\Sigma^4}\right) \sim \mathcal{O}(1) \\
h^{\Phi t} &= \frac{\Delta\omega}{-V^2} = \frac{\psi_\Sigma}{g_{\varphi\varphi}} = \frac{\psi_\Sigma}{\rho_\Sigma^2}\left(1 + \frac{V^2\psi_\Sigma^2}{\rho_\Sigma^2}\right) \sim \mathcal{O}(1) \\
h^{\theta\theta} &= \frac{1}{h_{\theta\theta}} = \Sigma^{-1} + \mathcal{O}\big(V^2\big) \sim \mathcal{O}(1).
\end{aligned}
$$

Then from (6.5) and (9.1), we find for ${K_a}^b$

$$
\begin{aligned}
\left({K_t}^t\right)^+ &= h^{tt}K^+_{tt} + h^{t\Phi}K^+_{\Phi t} \\
&= \left[V^{-1}\kappa(1+2\Delta\omega\eta/\kappa) - \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{(\Delta\omega)^2}{V^2}}\right] - \left[\frac{\Delta\omega}{V}\eta - \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{(\Delta\omega)^2}{V^2}}\right] \\
&= V^{-1}\kappa\big(1+\Delta\omega\eta/\kappa\big) \qquad (10.1a)
\end{aligned}
$$

$$
\begin{aligned}
\left({K_\Phi}^\Phi\right)^+ &= h^{\Phi t}K^+_{t\Phi} + h^{\Phi\Phi}K^+_{\Phi\Phi} \\
&= \left[-\frac{\Delta\omega}{V}\eta + \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{(\Delta\omega)^2}{V^2}}\right] + \left[\frac{1}{2g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial n} - \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{(\Delta\omega)^2}{V^2}}\right] \\
&= -\frac{\Delta\omega}{V}\eta + \frac{1}{2g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial n} \sim \mathcal{O}(V) \qquad (10.1b)
\end{aligned}
$$

$$
\left({K_\theta}^\theta\right)^+ = h^{\theta\theta}K^+_{\theta\theta} \sim \mathcal{O}(V) \qquad (10.1c)
$$

$$
\begin{aligned}
\left({K_\Phi}^t\right)^+ &= h^{tt}K^+_{t\Phi} + h^{t\Phi}K^+_{\Phi\Phi} \\
&= \left[-V^{-1}\eta + \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{\Delta\omega}{V^2}}\right] - \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{\Delta\omega}{V^2}} \\
&= -V^{-1}\eta \sim \mathcal{O}\big(V^{-1}\big) \qquad (10.1d)
\end{aligned}
$$

$$
\begin{aligned}
\left({K_t}^\Phi\right)^+ &= h^{\Phi t}K^+_{tt} + h^{\Phi\Phi}K^+_{\Phi t} \\
&= \left[\frac{\Delta\omega}{V}\kappa + 2\frac{(\Delta\omega)^2}{V}\eta - \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{(\Delta\omega)^3}{V^2}}\right] \\
&\quad + \left[\frac{V\eta}{g_{\varphi\varphi}} - \frac{1}{2g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial n}\Delta\omega - \frac{(\Delta\omega)^2}{V}\eta + \cancel{\frac{1}{2}\frac{\partial g_{\varphi\varphi}}{\partial n}\frac{(\Delta\omega)^3}{V^2}}\right] \\
&= \frac{1}{2}\frac{\Delta\omega}{V}\kappa + \frac{(\Delta\omega)^2}{V}\eta - \frac{1}{2g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial n}\Delta\omega \sim \mathcal{O}(V) \qquad (10.1e)
\end{aligned}
$$

$$\left(K_t{}^t\right)^- = h^{tt}K^-_{tt} + h^{t\Phi}K^-_{\Phi t} \sim \mathcal{O}\left(V^2\right) \tag{10.2a}$$

$$\left(K_\Phi{}^\Phi\right)^- = h^{\Phi t}K^-_{t\Phi} + h^{\Phi\Phi}K^-_{\Phi\Phi} = \mathcal{O}\left(V^2\right) + G/\rho = \mathcal{O}\left(V^2\right) + \frac{\Sigma_+^{\frac{1}{2}}}{r_+^2 + a^2}\frac{G}{\sin\theta}$$
$$\sim \mathcal{O}(1) \text{ in both } \theta \text{ and } V \tag{10.2b}$$

$$\left(K_\theta{}^\theta\right)^- = h^{\theta\theta}K^-_{\theta\theta} = \frac{\left(r_+^2 + a^2\right)^2\left(r_+^2 - 3a^2\cos^2\theta\right)\sin\theta}{\Sigma_+^{\frac{7}{2}}G} + \mathcal{O}\left(V^2\right)$$
$$\sim \mathcal{O}(1) \text{ in both } \theta \text{ and } V \tag{10.2c}$$

$$\left(K_\Phi{}^t\right)^- = h^{tt}K^-_{t\Phi} + h^{t\Phi}K^-_{\Phi\Phi} \sim \mathcal{O}(1) \tag{10.2d}$$

$$\left(K_t{}^\Phi\right)^- = h^{\Phi t}K^-_{tt} + h^{\Phi\Phi}K^-_{\Phi t} \sim \mathcal{O}\left(V^2\right). \tag{10.2e}$$

By the jump conditions

$$-8\pi S_a{}^b = [K_a{}^b - \delta_a^b K]$$

we have (we neglect higher order terms in the calculations below)

$$
\begin{aligned}
8\pi S_t{}^t &= -\big(K_\theta{}^\theta\big)^- - \big(K_\Phi{}^\Phi\big)^- \\
&= -\frac{\big(r_+^2+a^2\big)^2\big(r_+^2-3a^2\cos^2\theta\big)\sin\theta}{\Sigma_+^{\frac{7}{2}}G} - \frac{\Sigma_+^{\frac{1}{2}}}{r_+^2+a^2}\frac{G}{\sin\theta} \sim \mathcal{O}(1)
\end{aligned} \tag{10.3a}
$$

$$
8\pi S_\theta{}^\theta = \big(K_t{}^t\big)^+ = V^{-1}\kappa \tag{10.3b}
$$

$$
8\pi S_\Phi{}^\Phi = \big(K_t{}^t\big)^+ - \big(K_\theta{}^\theta\big)^- = V^{-1}\kappa - \frac{\big(r_+^2+a^2\big)^2\big(r_+^2-3a^2\cos^2\theta\big)\sin\theta}{\Sigma_+^{\frac{7}{2}}G} \tag{10.3c}
$$

$$
8\pi S_\Phi{}^t = -\big(K_\Phi{}^t\big)^+ \sim \mathcal{O}\big(V^{-1}\big) \tag{10.3d}
$$

$$
8\pi S_t{}^\Phi = -\big(K_t{}^\Phi\big)^+ \sim \mathcal{O}(V). \tag{10.3e}
$$

In coordinates (r,θ,Φ,t) the shell's 4-velocity (5.3) takes the form

$$
u^a = U^{-1}\big[\delta_t^a + (\Omega(\theta)-\omega_{\rm H})\delta_\Phi^a\big]. \tag{10.4}
$$

Solving the eigenvalue equation

$$
S^a{}_b u^b = -\sigma u^a
$$

we find

$$
\begin{aligned}
\sigma &= -S_\tau{}^\tau = -S_t{}^t \\
\Omega(\theta)-\omega_{\rm H} &= \frac{S_t{}^\Phi}{S_t{}^t - S_\Phi{}^\Phi} \sim \mathcal{O}\big(V^2\big).
\end{aligned}
$$

Remembering (4.5)

$$\omega_{\mathrm{B}} - \omega_{\mathrm{H}} \sim \mathcal{O}\big(V^2\big),$$

we have[1]

$$\Omega(\theta) - \omega_{\mathrm{B}} \sim \mathcal{O}\big(V^2\big). \tag{10.5}$$

By (2.2b) and (2.3b) we have

$$U^2 - V^2 = -g_{\varphi\varphi}[\Omega(\theta) - \omega_{\mathrm{B}}]^2 \sim \mathcal{O}\big(V^4\big). \tag{10.6}$$

[1]If we approximate $\frac{\partial \Delta\omega}{\partial n}$ to $2\kappa\frac{\Delta\omega}{V}$ instead of $\kappa\frac{\Delta\omega}{V}$ in (6.4), observing $\Delta\omega$ is quadratic in V, the two leading terms in $\big({K_t}^{\Phi}\big)^+$ will cancel, giving $\big({K_t}^{\Phi}\big)^+ \sim \mathcal{O}\big(V^3\big)$. Consequently, ${S_t}^{\Phi} \sim \mathcal{O}\big(V^3\big)$ and $\Omega(\theta) - \omega_{\mathrm{H}} \sim \mathcal{O}\big(V^4\big)$. However, other ${S_a}^{b}$'s are unaffected and (10.5) still holds.

Chapter 11

Brief view of thermodynamics of ZAM-equipotential shell

From the results of the preceding chapters, we can study the thermodynamics of the shell and examine whether it resembles that of a black hole in the horizon limit.

As explained in the Introduction chapter, the shell's local temperature is equal to the acceleration temperature seen by observers sitting on the outer surface of the shell, whose acceleration would be [8]

$$a_\alpha^+ = u_{\alpha|\beta} u^\beta = U^{-1}(U_{,\alpha} + l\Omega(\theta)_{,\alpha}),$$

where l is the angular momentum of the observer. Using (10.6) and (10.5), we get

$$
\begin{aligned}
&U - V \sim \mathcal{O}\big(V^3\big)\\
&U_{,\alpha} - V_{,\alpha} \sim \mathcal{O}\big(V^2\big)\\
&U^{-1} - V^{-1} \sim \mathcal{O}(V)\\
&l = U^{-1} g_{\varphi\varphi}(\Omega(\theta) - \omega_{\mathrm{B}}) \sim \mathcal{O}(V).
\end{aligned}
$$

So to the zeroth order, we can approximate a_α^+ to the acceleration of a ZAM orbit, i.e., (2.4)

$$a_\alpha^+ = V^{-1} V_{,\alpha}.$$

Then by (10.1a) and (3.1)

$$\big(K_t{}^t\big)^+ \simeq V^{-1}\kappa = a^+.$$

On the other hand, the surface pressure and surface density of the shell are related to the extrinsic curvature through the jump conditions as follows, noticing $K_\tau{}^\tau = K_t{}^t$:

$$
\begin{aligned}
-8\pi P_\theta &= -8\pi S_\theta{}^\theta = \big[K_\theta{}^\theta - \delta_\theta{}^\theta K\big] = \big[-K_t{}^t - K_\Phi{}^\Phi\big]\\
-8\pi P_\Phi &= -8\pi S_\Phi{}^\Phi = \big[K_\Phi{}^\Phi - \delta_\Phi{}^\Phi K\big] = \big[-K_t{}^t - K_\theta{}^\theta\big]\\
8\pi\sigma &= -8\pi S_\tau{}^\tau = \big[K_\tau{}^\tau - \delta_\tau{}^\tau K\big] = \big[-K_\theta{}^\theta - K_\Phi{}^\Phi\big].
\end{aligned}
$$

Combining the above, we have

$$8\pi(\sigma + P_\theta + P_\Phi) = \big[2K_t{}^t\big] \simeq 2\big(K_t{}^t\big)^+ \simeq 2a^+. \tag{11.1}$$

Now the acceleration temperature is given by $T = a^+/2\pi$ $\big[$note T is nearly constant along the shell since $T_{,\alpha} e^{\alpha}_{(\theta)} = \kappa_{,\alpha} e^{\alpha}_{(\theta)}/(2\pi V) \sim \mathcal{O}(V)$; for the red-shifted temperature $T_\infty = TV$ by Tolman's law we would have $(T_\infty)_{,\alpha} e^{\alpha}_{(\theta)} \sim \mathcal{O}\big(V^2\big)\big]$, so we have

$$2(\sigma + P_\theta + P_\Phi) = T.$$

Substituting this into the Gibbs-Duhem relation

$$s = \frac{\sigma + P}{T}$$

where s is the entropy per unit area and we assume[1]

$$P = \frac{P_\theta + P_\Phi}{2},$$

we obtain

$$s = \frac{1}{2}\frac{\sigma + \frac{P_\theta + P_\Phi}{2}}{\sigma + P_\theta + P_\Phi} = \frac{1}{4}\frac{2\sigma + P_\theta + P_\Phi}{\sigma + P_\theta + P_\Phi}.$$

In the horizon limit $P \to +\infty$ and σ is bounded, so

$$s \to \frac{1}{4}.$$

We thus recover the Bekenstein-Hawking formula $S = A/4$.

[1] This is good as long as the embedding does not break down, as can be seen from (10.3).

Chapter 12

Summary remarks on Kerr black shell

In summary, we have considered a spinning thin shell as a source of the Kerr gravitational field for a mass m and angular momentum ma. We have studied how the physical properties of this shell evolve as it contracts slowly and reversibly toward the Kerr black hole horizon at $r_+ = m + \sqrt{m^2 - a^2}$. We found that the shell's angular velocity approaches the rigid horizon angular velocity $\omega_{\rm H} = a/(r_+^2 + a^2) = a/(2mr_+)$, and that its proper mass per unit area σ stays finite and is given by (10.3a). However, both the surface pressure and temperature of the shell diverge in the horizon limit. Specifically, $P \simeq V^{-1}\kappa/8\pi$ and $T = V^{-1}\kappa/2\pi$, where V, the gravitational potential at the shell's surface, tends to zero in the horizon limit and κ, the redshifted surface gravity of the shell, tends to the (constant) surface gravity of the horizon. The entropy s per unit area, generally given by $s = (\sigma + P)/T$, accordingly tends to 1/4, in agreement with the Bekenstein-Hawking entropy $S = A/4$.

Chapter 13

Special case II: Kerr-AdS exterior

In Chapters 4–11 we considered the case where the geometry outside the shell is the asymptotically flat Kerr geometry. We now turn to an alternative case of special interest, the rotating AdS metric.

Black hole solutions with asymptotically AdS behavior, though not likely representing our real universe, are of interest in the study of the AdS/CFT correspondence. Moreover, in the rotating case, the adding of a (negative) cosmological constant also makes it possible for a thermal bath to be in equilibrium and co-rotate with the black hole all the way to infinity, with the speed never becoming faster than light [14].

Rotating AdS black holes are given by the Kerr-AdS metric which reads in Boyer-Lindquist type coordinates [14]

$$\begin{aligned}\left(\mathrm{d}s^2\right)^+ =&\frac{\Sigma}{\Delta_r}\mathrm{d}r^2 + \frac{\Sigma}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\Delta_\theta\left(r^2+a^2\right)^2 - \Delta_r a^2\sin^2\theta}{\Xi^2\Sigma}\sin^2\theta\mathrm{d}\varphi^2\\ &- 2\frac{\Delta_\theta\left(r^2+a^2\right)-\Delta_r}{\Xi\Sigma}a\sin^2\theta\mathrm{d}\varphi\mathrm{d}t - \frac{\Delta_r - \Delta_\theta a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2, \quad (13.1)\end{aligned}$$

where

$$\begin{aligned}\Sigma &= r^2 + a^2\cos^2\theta\\ \Delta_r &= \left(r^2+a^2\right)\left(1+l^{-2}r^2\right) - 2mr\\ \Delta_\theta &= 1 - l^{-2}a^2\cos^2\theta\\ \Xi &= 1 - l^{-2}a^2.\end{aligned}$$

The AdS radius l is related to the cosmological constant Λ by $l^2 = -3/\Lambda$. These coordinates have the advantage that when $l \to \infty$ the metric (13.1) reduces to the normal Kerr metric written in the Boyer-Lindquist coordinates. This can be seen more easily from the following alternative expressions for the metric coefficients:

$$\begin{aligned}g_{\varphi\varphi} &= \left[\Xi\left(r^2+a^2\right) + \frac{2mr}{\Sigma}a^2\sin^2\theta\right]\frac{\sin^2\theta}{\Xi^2}\\ g_{\varphi t} &= \left[l^{-2}\left(r^2+a^2\right) - \frac{2mr}{\Sigma}\right]\frac{a\sin^2\theta}{\Xi}\\ g_{tt} &= -\left[l^{-2}\left(r^2+a^2\sin^2\theta\right) + 1 - \frac{2mr}{\Sigma}\right].\end{aligned}$$

One needs to be careful when defining “zero angular momentum” (ZAM), which was first introduced in the asymptotically flat geometry where energy and angular momentum are defined with the timelike and axial Killing vectors $\xi^\alpha_{(t)}$ and $\xi^\alpha_{(\varphi)}$ respectively. In the asymptotically AdS geometry, there is not one unique timelike Killing vector at infinity but a lot of them: $\xi^\alpha_{(t)} + \Omega\xi^\alpha_{(\varphi)}$, with Ω taking a range of values. However, luckily, there *is* a unique one which is perpendicular to $\xi^\alpha_{(\varphi)}$: $V^\alpha = \xi^\alpha_{(t)} + \omega_{\mathrm{B}}\xi^\alpha_{(\varphi)}$ (actually $V_\alpha\xi^\alpha_{(\varphi)} = 0$ everywhere outside the horizon, c.f. (2.5)). So if we still use $\xi^\alpha_{(\varphi)}$ to define angular momentum, V^α happens to be the (unnormalized) ZAM 4-velocity, and ω_{B} and V the ZAM angular velocity and ZAM potential. Now energy is defined with the asymptotic Killing vector $V^\alpha_\infty = \xi^\alpha_{(t)} + \omega^\infty_{\mathrm{B}}\xi^\alpha_{(\varphi)}$, which plays the role of $\xi^\alpha_{(t)}$ in the asymptotically flat case. It would be natural to use a frame which is co-rotating with this "ZAM Killing vector", i.e., in which $\omega^\infty_{\mathrm{B}}$ vanishes, and this frame is to be interpreted as the non-rotating inertial frame at infinity (frame of the "fixed stars"). ([15] discussed the advantages of using V^α_∞ and the non-rotating frames in the context of Kerr-AdS black hole thermodynamics.) Unfortunately, the Boyer-Lindquist type coordinates do not form such a frame, that is, it is rotating with angular velocity $(-\omega^\infty_{\mathrm{B}})_{\text{B-L}} \neq 0$ with respect to that (non-rotating) frame. To see this, we work out the explicit expressions of ω_{B} and V in metric (13.1):

$$\omega_{\mathrm{B}} = a\Xi\frac{\Delta_\theta\big(r^2+a^2\big)-\Delta_r}{\Delta_\theta(r^2+a^2)^2-\Delta_r a^2\sin^2\theta} = \frac{a\Xi}{\Gamma} \tag{13.2}$$

$$V^2 = \frac{\Sigma\Delta_r\Delta_\theta}{\Delta_\theta(r^2+a^2)^2-\Delta_r a^2\sin^2\theta} = \frac{\Sigma\Delta_r\Delta_\theta}{[\Delta_\theta(r^2+a^2)-\Delta_r]\Gamma}, \tag{13.3}$$

where

$$\Gamma \equiv r^2 + a^2 + \frac{\Sigma\Delta_r}{\Delta_\theta(r^2+a^2)-\Delta_r}.$$

Then the asymptotic form of the metric reads

$$\begin{aligned}\left(\mathrm{d}s^2\right)^+\Big|_{r\to\infty} &= g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\varphi\varphi}(\mathrm{d}\varphi - \omega_{\mathrm{B}}\mathrm{d}t)^2 - V^2\mathrm{d}t^2\Big|_{r\to\infty} \\ &= \frac{\mathrm{d}r^2}{1+l^{-2}r^2} + \frac{r^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{r^2\sin^2\theta}{\Xi}\left(\mathrm{d}\varphi + l^{-2}a\mathrm{d}t\right)^2 - \frac{\Delta_\theta\left(1+l^{-2}r^2\right)}{\Xi}\mathrm{d}t^2,\end{aligned} \tag{13.4}$$

where we have included the sub-leading term in the $r\to\infty$ limit $\Delta_r\big|_{r\to\infty} = r^2\left(1+l^{-2}r^2\right)$ instead of simply $\Delta_r\big|_{r\to\infty} = l^{-2}r^4$, which ensures that (13.4) reduces to the flat metric when $l^{-1}\to 0$. As shown in (13.4),

$$\omega_{\mathrm{B}}^{\infty} = -l^{-2}a \tag{13.5}$$

$$V_\infty^2 = \frac{\Delta_\theta\left(1+l^{-2}r^2\right)}{\Xi}, \tag{13.6}$$

which follows from (13.2) and (13.3). (13.6) shows that the boundary surface of constant r at infinity does not coincide with the ZAM-equipotential surface and is an inhomogeneously distorted 2-sphere.

To bring the metric (13.4) to the standard AdS form

$$\mathrm{d}s^2 = \frac{\mathrm{d}y^2}{1+l^{-2}y^2} + y^2\mathrm{d}\Theta^2 + y^2\sin^2\Theta\mathrm{d}\Phi^2 - \left(1+l^{-2}y^2\right)\mathrm{d}T^2 \tag{13.7}$$

we perform the following coordinate transformations, comparing (13.4) and

(13.7),

$$
\begin{aligned}
\frac{\mathrm{d}y^2}{1+l^{-2}y^2} + y^2\mathrm{d}\Theta^2 &= \frac{\mathrm{d}r^2}{1+l^{-2}r^2} + \frac{r^2}{\Delta_\theta}\mathrm{d}\theta^2 \\
\mathrm{d}\Phi &= \mathrm{d}\varphi + l^{-2}a\mathrm{d}t \\
\mathrm{d}T &= \mathrm{d}t \\
y^2\sin^2\Theta &= \frac{r^2\sin^2\theta}{\Xi} \\
1+l^{-2}y^2 &= \frac{\Delta_\theta\left(1+l^{-2}r^2\right)}{\Xi},
\end{aligned}
$$

which simplify to

$$
\begin{aligned}
T &= t \\
\Phi &= \varphi + l^{-2}at \\
y^2 &= \frac{\Delta_\theta r^2 + a^2\sin^2\theta}{\Xi} \\
\sin^2\Theta &= \frac{r^2\sin^2\theta}{\Delta_\theta r^2 + a^2\sin^2\theta}.
\end{aligned}
$$

Note this is the "AdS $\leftrightarrow$ Kerr-AdS$|_{r\to\infty}$" transformation. [14] gives an "AdS $\leftrightarrow$ Kerr-AdS$|_{m=0}$" transformation. The only difference is that in the latter case

$$
\sin^2\Theta = \frac{\left(r^2+a^2\right)\sin^2\theta}{\Delta_\theta r^2 + a^2\sin^2\theta},
$$

which will agree with the former case when $r \to \infty$. Now in terms of the new coordinates,

$$\omega_{\mathrm{B}}^{\infty} = -\frac{g_{\Phi t}}{g_{\Phi\Phi}} = 0 \tag{13.8}$$

$$V_{\infty}^{\alpha} = \xi_{(T)}^{\alpha} \ \big(\text{in old coordinates } V_{\infty}^{\alpha} = \xi_{(t)}^{\alpha} - l^{-2} a \xi_{(\varphi)}^{\alpha}\big) \tag{13.9}$$

$$V_{\infty}^{2} = -g_{TT} = 1 + l^{-2} y^{2} \left(= \frac{\Delta_\theta \big(1 + l^{-2} r^{2}\big)}{\Xi} \right). \tag{13.10}$$

So the frame is non-rotating, in which the ZAM Killing vector reduces to just the time translation Killing vector and the ZAM equipotential surface at infinity is given by constant radial coordinate y.

However, the horizon itself is not given by constant y, so it seems more convenient to treat the near horizon shell in the old Boyer-Lindquist type coordinates. This is justified by observing that the defining function (2.3b) for V, $V^2 = (g_{\varphi t}^2 - g_{\varphi\varphi} g_{tt})/g_{\varphi\varphi} = (-g^{tt})^{-1}$, which only involves the contravariant time component, is form invariant under the coordinate transformation (c.f. (6.2)), that is, observers in the two frames will agree on whether a surface is ZAM-equipotential or not as well as on the value of V. So we can follow a similar procedure to calculate the extrinsic curvature of ZAM-equipotential hypersurfaces as in the Kerr case in the near-horizon approximation. As in Chapter 3, we write in terms of V

$$\Gamma = \frac{r^2 + a^2}{1 - V^2/\Delta_\theta}$$

$$\omega_{\mathrm{B}} = \frac{a\Xi}{r^2 + a^2} \big(1 - V^2/\Delta_\theta\big),$$

and

$$\omega_{\rm H} = \frac{a\Xi}{r_+^2 + a^2}.$$

Again,

$$\Delta\omega \sim \mathcal{O}\big(V^2\big)$$

From (13.3) we have

$$\frac{1}{V^2} = \frac{\big(r^2+a^2\big)^2}{\Sigma\Delta_r} - \frac{\big(r^2+a^2\big)}{\Sigma\Delta_\theta} + \frac{1}{\Delta_\theta}. \tag{13.11}$$

Take derivatives of both sides with respect to θ and r, yielding

$$2\frac{V_{,\theta}}{V^3} = \frac{\big(r^2+a^2\big)^2}{\Sigma\Delta_r}\frac{\Sigma_{,\theta}}{\Sigma} + \big(r^2+a^2\big)\left(\frac{1}{\Sigma\Delta_\theta}\right)_{,\theta} - \left(\frac{1}{\Delta_\theta}\right)_{,\theta} \tag{13.12a}$$

$$2\frac{V_{,r}}{V^3} = \frac{\big(r^2+a^2\big)^2}{\Sigma\Delta_r}\frac{\Delta_{r,r}}{\Delta_r} - \frac{1}{\Delta_r}\left[\frac{\big(r^2+a^2\big)^2}{\Sigma}\right]_{,r} + \left[\frac{\big(r^2+a^2\big)}{\Sigma\Delta_\theta}\right]_{,r}. \tag{13.12b}$$

Noting $\Delta_r \sim \mathcal{O}\big(V^2\big)$ and $\Delta_{r,r} \sim \mathcal{O}(1)$, we have from (13.11) and (13.12)

$$V_{,\theta} = \frac{1}{2}V\frac{\Sigma_{,\theta}}{\Sigma} + \mathcal{O}\big(V^3\big) \sim \mathcal{O}(V)$$
$$V_{,r} = \frac{1}{2}V\frac{\Delta_{r,r}}{\Delta_r} + \mathcal{O}(V) \sim \mathcal{O}\big(V^{-1}\big),$$

which is quite the same as the result (4.6) for the Kerr case. Then we can go through the same arguments again and get

$$K^+_{\theta\theta} \sim K^+_{AB} \sim \mathcal{O}(V)$$
$$K^+_{ab}v^a v^b \sim \mathcal{O}\big(V^{-1}\big).$$

Though here we will not try to match the Kerr-AdS metric to an interior one, we expect that $(K_a{}^b)^-$'s are bounded and negligible compared to $(K_t{}^t)^+$. Actually the divergence of $(K_t{}^t)^+$ is true for general stationary axisymmetric space-times, since

$$(K_t{}^t)^+ = (K_\tau{}^\tau)^+ = h^{\tau\tau} K^+_{\tau\tau} = -n_{\alpha|\beta} v^\alpha v^\beta = V^{-1}\kappa,$$

where in the last step we have used (3.7c) which holds for the general case. The smallness of $(K_t{}^t)^-$ can be expected if

$$K^-_{tt} = \frac{1}{2}\frac{\partial g^-_{tt}}{\partial n}$$

vanishes or at least is of higher order than K^+_{tt}, i.e., the interior gravitational potential g^-_{tt} is constant or nearly constant. So the shell's surface pressure P, which has $(K_t{}^t)^+$ as the main contribution, will dominate over its surface density σ and is related to the outer acceleration temperature $T = a^+/2\pi = V^{-1}\kappa/2\pi$ through $8\pi P \simeq (K_t{}^t)^+ = 2\pi T$. Then the thermodynamic relation

$$\frac{S}{A} = \frac{\sigma + P}{T}$$

produces the universal constant $1/4$ of proportionality between the entropy and area, as stated by the Bekenstein-Hawking relation.

Chapter 14

Conclusion

In this thesis, we have studied properties of a near-horizon spinning thin shell ("black shell") and found that its thermodynamic entropy approaches the Bekenstein-Hawking entropy $S_{\mathrm{BH}} = A/4$ for the black hole that it is about to form, providing an operational definition of the latter.

We first introduced the notion of ZAM (zero angular momentum) observer and ZAM-equipotential hypersurface in the general stationary axisymmetric space-time. The ZAM potential is a generalization of the gravitational potential for the static space-time. We chose the shell to lie on a ZAM-equipotential hypersurface. The shell's physical properties (angular velocity, surface density and pressure) are determined from the way it is embedded between the exterior and interior geometries, which is described by Israel's junction conditions. We examined two different exterior geometries: Kerr and Kerr-AdS solutions. We worked out the detailed results for the Kerr case, matched to a vacuum, nearly flat interior. We found the surface pressure of the shell diverges in the horizon limit while the surface density stays finite. We also found the 4-velocity of the shell (at each point) can be approximated to that of a ZAM observer in the limit, whose acceleration is parallel to the normal vector of the shell. Since the extrinsic curvature of the shell, which describes the embedding geometry of the shell, is essentially the derivative of the normal vector along the shell's surface, the acceleration is related through the junction conditions to the surface pressure and density of the shell. Since this acceleration also accounts for the acceleration radiation which shares the temperature of the shell, we thus established a relation between the shell's pressure, density and temperature. Then with the help of the thermodynamic relation, we were able to recover the Bekenstein-Hawking entropy $S = A/4$.

Appendix: Vacuum interior solution

We will find out the function ψ in the metric (7.5) as a solution of equation (7.4), which we write again for convenience:

$$\left(\mathrm{d}s^2\right)^- = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\mathrm{d}\Phi^2 - V^2(\mathrm{d}t - \psi\mathrm{d}\Phi)^2 \tag{7.5}$$

$$\psi_{\rho\rho} - \rho^{-1}\psi_\rho + \psi_{zz} = \nabla^2\psi - 2\rho^{-1}\psi_\rho = 0. \tag{7.4}$$

In oblate spheroidal coordinates defined by

$$\rho = \sqrt{r^2 + a^2}\sin\theta, \quad z = r\cos\theta$$

they take the form

$$\left(\mathrm{d}s^2\right)^- = \Sigma\left(\frac{\mathrm{d}r^2}{r^2 + a^2} + \mathrm{d}\theta^2\right) + \left(r^2 + a^2\right)\sin^2\theta\mathrm{d}\Phi^2 - V^2(\mathrm{d}t - \psi\mathrm{d}\Phi)^2 \tag{A.1}$$

$$\left(r^2 + a^2\right)\psi_{rr} + \psi_{\theta\theta} - \cot\theta\psi_\theta = 0, \tag{A.2}$$

subject to the boundary condition given by (7.6c):

$$\begin{aligned} V^2\psi_\Sigma &= -g_{\varphi\varphi}\Delta\omega \\ &\simeq \frac{V^2}{\kappa}\frac{2a}{\left(r_+^2+a^2\right)}\left[\frac{r_+\left(r_+^2+a^2\right)}{\Sigma_+}+(r_+-m)\right]\sin^2\theta. \end{aligned} \tag{7.6c}$$

By separation of variables

$$\psi(r,\theta)=\phi(r)\varphi(\theta)$$

(A.2) is transformed into the following set of equations

$$\varphi_{\theta\theta}-\cot\theta\varphi_\theta=-\lambda\varphi \tag{A.3a}$$

$$\left(r^2+a^2\right)\phi_{rr}=\lambda\phi. \tag{A.3b}$$

Let $\mu=\cos\theta$. Then (A.3a) becomes

$$\left(1-\mu^2\right)\varphi_{\mu\mu}+\lambda\varphi=0. \tag{A.4}$$

Let $\varphi=f(\mu)y(\mu)$ and we get

$$\left(1-\mu^2\right)\left[y''+\frac{2f'}{f}y'+\left(\frac{f''}{f}+\frac{\lambda}{1-\mu^2}\right)y\right]f=0. \tag{A.5}$$

This agrees with the associated Legendre equation

$$\left(1-\mu^2\right)y''-2\mu y'+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]y=0 \tag{A.6}$$

with solutions $y = P_n^m(\mu),\ Q_n^m(\mu)$ provided

$$\frac{2f'}{f} = -\frac{2\mu}{1-\mu^2},$$

i.e., $f = \sqrt{1-\mu^2}$. Substitute f back into (A.5) and we get

$$\left(1-\mu^2\right)y'' - 2\mu y' + \left[\lambda - \frac{1}{1-\mu^2}\right]y = 0. \tag{A.7}$$

This agrees with (A.6) with $m = 1$ and $n(n+1) = \lambda$. Then (A.3b) becomes

$$\left(r^2 + a^2\right)\phi_{rr} = n(n+1)\phi$$

which we write for short as

$$L[\phi] = 0.$$

Its solutions can be given as contour integrals (p, $g(s)$ to be determined)

$$\phi(r) = \int_C (r-s)^{p+1} g(s)\mathrm{d}s.$$

Then

$$L[\phi] = \int_C \mathrm{d}s(r-s)^{p-1} g(s)\left[p(p+1)\left(r^2+a^2\right) - n(n+1)(r-s)^2\right] \tag{A.8}$$

Choose p so that the coefficient of r^2 in (A.8) vanishes, i.e., $p = n$(or $-(n+1)$):

$$L[\phi] = \int_C \mathrm{d}s(r-s)^{n-1} g(s)\left(a^2 + 2rs - s^2\right)n(n+1). \tag{A.9}$$

Choose $g(s)$ so that

$$\text{“integrand”} = n(n+1)\frac{\mathrm{d}}{\mathrm{d}s}\big[(r-s)^n F(s)\big],$$

$F(s)$ being determined by equating coefficients of $r^1 \times (r-s)^{n-1}$ and $r^0 \times (r-s)^{n-1}$:

$$\begin{aligned} &\big(a^2 - s^2\big)g(s) + 2sg(s)r \equiv (r-s)F'(s) - nF(s), \quad \forall r \\ \Rightarrow \quad &F(s) = \big(a^2 + s^2\big)^{-n}, \quad g(s) = -n\big(a^2 + s^2\big)^{-n-1}. \end{aligned}$$

So finally

$$\psi = n \sin\theta \big[c_n P_n^1(\cos\theta) + d_n Q_n^1(\cos\theta)\big] \int_C (r-s)^{n+1}\big(a^2 + s^2\big)^{-n-1} \mathrm{d}s. \quad \text{(A.10)}$$

Bibliography

[1] W. Israel, *Third Law of Black-Hole Dynamics: A Formulation and Proof*, Phys. Rev. Lett. **57**, 397 (1986).

[2] W. Israel, *Black Hole Thermodynamics*, Lect. Notes Phys. **617** (Berlin: Springer), 15 (2003).

[3] R. M. Wald, *The Thermodynamics of Black Holes*, Living Rev. Rel. **4**, 6 (2001), gr-qc/9912119.

[4] T. Jacobson, *On the Nature of Black Hole Entropy*, gr-qc/9908031.

[5] G. T. Horowitz, *Quantum States of Black Holes*, gr-qc/9704072.

[6] S. W. Hawking, G. T. Horowitz and S. F. Ross, *Entropy, area, and black hole pairs*, Phys. Rev. D **51**, 4302 (1995), gr-qc/9409013.

[7] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, Nuovo Cimento **44B** (1966).

[8] C. Barrabès, B. Boisseau and W. Israel, *Orbits, forces and accretion dynamics near spinning black holes*, Mon. Not. R. Astron. Soc. **276**, 432 (1995), gr-qc/9505025.

[9] F. Pretorius, D. Vollick and W. Israel, *An operational approach to black hole entropy*, Phys. Rev. D **57**, 6311 (1998).

[10] V. de la Cruz and W. Israel, *Spinning shell as a source of the Kerr metric*, Phys. Rev. **170**, 1187 (1968).

[11] S. Ansoldi, *WKB metastable quantum states of a de Sitter-Reißner-Nordström dust shell*, Class. Quantum Grav. **19**, 6321 (2002), gr-qc/0310004.

[12] J. Kijowski, G. Magli and D. Malafarina, *Relativistic dynamics of spherical timelike shells*, Gen. Relativ. Gravit. **38**, 1697 (2006).

[13] C. Lanczos, Physik. Z. **23**, 529 (1922); Ann. Physik **74**, 518 (1924).

[14] S. W. Hawking, C. J. Hunter and M. M. Taylor-Robinson, *Rotation and the AdS-CFT correspondence*, Phys. Rev. D **59**, 064005 (1999), hep-th/9811056.

[15] G. W. Gibbons, M. J. Perry and C. N. Pope, *Bulk-boundary thermodynamic equivalence, and the Bekenstein and cosmic-censorship bounds for rotating charged AdS black holes*, Phys. Rev. D **72**, 084028 (2005).