THE FEKETE-SZEGÖ PROBLEM FOR
A SUBCLASS OF CLOSE-TO-CONVEX
FUNCTIONS

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Abstract

Let $C_1(\beta)$ be the class of normalized functions $f$, which are analytic in the open unit disk $\mathcal{U}$, given by the power series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

and satisfy the inequality:

$$\text{Re}\left\{ \frac{zf'(z)}{\phi(z)} e^{i\beta} \right\} > 0 \quad \left( z \in \mathcal{U}; \ -\frac{\pi}{2} < \beta < \frac{\pi}{2} \right)$$

for some normalized univalent and convex function $\phi$. In this paper we solve the Fekete-Szegö problem for the family:

$$C_1 := \bigcup_{\beta} C_1(\beta) \left( \ -\frac{\pi}{2} < \beta < \frac{\pi}{2} \right)$$

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by proving that

$$\max_{f \in \mathcal{C}_1} |a_3 - \lambda a_2^2| = \begin{cases} 
\frac{5}{3} - \frac{9\lambda}{4} & \left( 0 \leq \lambda \leq \frac{2}{9} \right) \\
\frac{2}{3} + \frac{1}{9\lambda} & \left( \frac{2}{9} \leq \lambda \leq \frac{2}{3} \right) \\
\frac{5}{6} & \left( \frac{2}{3} \leq \lambda \leq 1 \right) \end{cases}.$$  

1. Introduction

Let $\mathcal{S}$ be the class of (normalized) analytic and univalent functions in the open unit disk

$$\mathcal{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$$

that are given by the Taylor series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \quad (1.1)$$

Let $\mathcal{S}^*$ and $\mathcal{K}$ denote, respectively, the subsets of $\mathcal{S}$ consisting of starlike and convex functions in $\mathcal{U}$. A function $f$, analytic in $\mathcal{U}$ and given by the series (1.1), is said to be close-to-convex in $\mathcal{U}$ if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} e^{i\beta} \right\} > 0 \quad (z \in \mathcal{U}; \phi \in \mathcal{S}^*; -\frac{\pi}{2} < \beta < \frac{\pi}{2}). \quad (1.2)$$

We denote the family of close-to-convex functions in $\mathcal{U}$ by $\mathcal{C}$. This class was introduced and studied by Kaplan [10]. The number $e^{i\beta}$ is necessary in (1.2) for the definition of close-to-convex functions.

It is well known, for $f \in \mathcal{S}$ and given by (1.1), that

$$|a_3 - a_2^2| \leq 1, \quad (1.3)$$

where the equality holds true for the Koebe function:

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n.$$  

Since $k \in \mathcal{S}^* (\supset \mathcal{C})$, the estimate (1.3) is true for $\mathcal{S}^*$ and $\mathcal{C}$ without any further refinement. However, Trimble [18] has shown that, for $f \in \mathcal{K}$ and given by (1.1),

$$|a_3 - a_2^2| \leq \frac{1}{3}. \quad (1.4)$$

Earlier in 1933, Fekete and Szegö [4] made use of Löwner's parametric method in order to prove that, if $f \in \mathcal{S}$ and is given by (1.1),

$$|a_3 - \lambda a_2^2| \leq 1 + 2e - \frac{2\lambda}{1-\lambda} \quad (0 \leq \lambda \leq 1). \quad (1.5)$$

Equality in (1.5) holds true for the Koebe function $k(z)$ only for $\lambda = 0$ and $\lambda = 1$. The case $0 < \lambda < 1$ provides an example of an extremal problem over $\mathcal{S}$ in which the Koebe function $k(z)$ fails to be extremal. The determination of sharp upper bound for the nonlinear functional $|a_3 - \lambda a_2^2|$...
for any given family $\mathcal{F}$ of normalized analytic functions is popularly known as the Fekete-Szegő problem for $\mathcal{F}$. The result (1.5) was also proved later by Goluzin [7], Jenkins [9], Pfluger [14], Shaffer and Spencer [15], and others.

The Fekete-Szegő problem for the families $\mathcal{K}, \mathcal{S}^*$, and $\mathcal{C}$ has been completely solved in the literature. Thus we have (cf. [10] and [11])

$$\max_{f \in \mathcal{K}} |a_3 - \lambda a_2^2| = \max \left\{ \frac{1}{3}, |\lambda - 1| \right\},$$

$$\max_{f \in \mathcal{S}^*} |a_3 - \lambda a_2^2| = \begin{cases} |3 - 4\lambda| & \left( \lambda \leq \frac{1}{2} \text{ and } \lambda \leq 1 \right) \\ 1 & \left( \frac{1}{2} \leq \lambda \leq 1 \right) \end{cases},$$

and

$$\max_{f \in \mathcal{C}} |a_3 - \lambda a_2^2| = \begin{cases} 3 - 4\lambda & \left( 0 \leq \lambda \leq \frac{1}{3} \right) \\ \frac{1}{3} + \frac{4}{9\lambda} & \left( \frac{1}{3} \leq \lambda \leq \frac{2}{3} \right) \\ 1 & \left( \frac{2}{3} \leq \lambda \leq 1 \right) \end{cases}.$$

Many other recent works on the Fekete-Szegő problem include, for example, [1], [3], [6], [13], [14], and [17].

We now introduce the class $\mathcal{C}_1(\beta)$ of (normalized) analytic functions $f$ in $\mathcal{U}$, which are given by (1.1) and satisfy the inequality (1.2) with $\phi \in \mathcal{K}$ (instead of $\phi \in \mathcal{S}^*$), and let

$$\mathcal{C}_1 := \bigcup_{\beta} \mathcal{C}_1(\beta) \quad \left( -\frac{\pi}{2} < \beta < \frac{\pi}{2} \right).$$

Since $\mathcal{K} \subset \mathcal{S}^*$, it follows that $\mathcal{C}_1 \subset \mathcal{C}$. Also, by taking $f = \phi$ in (1.2), we have $\mathcal{K} \subset \mathcal{C}_1$. Furthermore, the choice $\phi(z) = z$ in (1.2) exhibits the fact that the class of (normalized) analytic functions satisfying the inequality:

$$\text{Re} \left\{ e^{i\beta} f'(z) \right\} > 0 \quad \left( z \in \mathcal{U}; -\frac{\pi}{2} < \beta < \frac{\pi}{2} \right)$$

is contained in the class $\mathcal{C}_1$.

Problems involving growth and distortion inequalities, coefficient estimates, convex hull, extreme points, and so on, for the family $\mathcal{C}_1$ were investigated by Silverman and Telage [16]. In this paper, we completely solve the Fekete-Szegő problem for the family $\mathcal{C}_1$. In particular, one of our results (Theorem 4 below) gives a refinement of (1.3) for the smaller set $\mathcal{C}_1$, and it also includes some recent results of Abdel-Gawad and Thomas [2].

**Theorem 1.** Let $f \in \mathcal{C}_1$ and be given by (1.1). Then

$$\left| a_3 - \frac{2}{9} a_2^2 \right| \leq \frac{7}{6}.$$
The result is sharp.

**Theorem 2.** Let \( f \in C_1 \) and be given by (1.1). Then
\[
|a_3 - \lambda a_2^2| \leq \frac{5}{3} - \frac{9\lambda}{4} \quad \left( \lambda \leq \frac{2}{9} \right).
\]
The result is sharp.

**Theorem 3.** Let \( f \in C_1 \) and be given by (1.1). Then
\[
|a_3 - \lambda a_2^2| \leq \frac{2}{3} + \frac{1}{9\lambda} \quad \left( \frac{2}{9} < \lambda < \frac{2}{3} \right).
\]
The result is sharp.

**Theorem 4.** Let \( f \in C_1 \) and be given by (1.1). Then
\[
|a_3 - \frac{2}{3} a_2^2| \leq \frac{5}{6}.
\]
The result is sharp.

**Theorem 5.** Let \( f \in C_1 \) and be given by (1.1). Then
\[
|a_3 - a_2^2| \leq \frac{5}{6}.
\]
The result is sharp.

**Theorem 6.** Let \( f \in C_1 \) and be given by (1.1). Then
\[
|a_3 - \lambda a_2^2| \leq \frac{5}{6} \quad \left( \frac{2}{3} \leq \lambda \leq 1 \right).
\]
The result is sharp.

It follows from the definition that, if \( f \in C_1 \), then \( f' \) can be written as
\[
f'(z) = \frac{\phi(z)}{z} h(z) e^{-i\phi} \quad \left( z \in \mathcal{U}; \phi \in \mathcal{K}; -\frac{\pi}{2} < \beta < \frac{\pi}{2} \right),
\]
where
\[
h(z) = \left( \frac{1 + w(z)}{1 - w(z)} \right) \cos \beta + i \sin \beta
\]
for some Schwarz function \( w \) analytic in \( \mathcal{U} \) such that
\[
w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathcal{U}).
\]
Thus, if \( f \) is given by (1.1),
\[
\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathcal{U})
\]
and
\[
w(z) = \sum_{n=1}^{\infty} \alpha_n z^n \quad (z \in \mathcal{U})
\]
then

\[ 2a_2 = \left(2e^{-i\beta} \cos \beta\right) a_1 + b_2 \]  

(1.10)

and

\[ 3a_3 = 2e^{-i\beta} \left(\alpha_2 + \alpha_1^2 + \alpha_1 b_2\right) \cos \beta + b_3. \]  

(1.11)

Equations (1.9) and (1.10), together, yield

\[ a_3 - \lambda a_2^2 = \frac{1}{3} \left(b_3 - \frac{3}{4} \lambda b_2^2\right) + \frac{2}{3} e^{-i\beta} \left[ a_2 + \left(1 - \frac{3}{2} \lambda e^{-i\beta} \cos \beta\right) \alpha_1^2 \right] \cos \beta \]

\[ + \frac{2}{3} e^{-i\beta} \left(1 - \frac{3}{2} \lambda\right) \alpha_1 b_2 \cos \beta. \]  

(1.12)

The expression (1.12) shall be used throughout this paper. We shall also need the following results for the proof of Theorems 1 to 6.

**Theorem A** (Keogh and Merkes [11]). Let \( \phi \) given by (1.8) be a univalent convex function in \( U \) (that is, \( \phi \in \mathcal{K} \)). Then, for any complex number \( s \),

\[ |b_3 - sb_2^2| \leq \max \left\{ \frac{1}{3}, |s - 1| \right\}. \]  

(1.13)

**Theorem B** (Keogh and Merkes [11]). Let the Schwarz function \( w \) be given by (1.7) and the power series (1.9). Then, for any complex number \( s \),

\[ |\alpha_2 - sa_1^2| \leq 1 + (|s| - 1)|\alpha_1|^2. \]  

(1.14)

**Theorem C** (Silverman and Telage [16]). Let \( f \in C_1 \) and be given by (1.1). Then

\[ |a_n| \leq 2 - \frac{1}{n} \quad (n \in \mathbb{N} \setminus \{1\}). \]  

(1.15)

Equality in (1.15) holds true for the function:

\[ h(z, \zeta) = \bar{\zeta} \log(1 - \zeta z) + \frac{2z}{1 - \zeta z} \quad (|\zeta| = 1). \]

2. Proofs of Theorems 1 to 6

**Proof of Theorem 1.** Putting \( \lambda = \frac{2}{9} \) in (1.12), we get

\[ \left| a_3 - \frac{2}{9} a_2^2 \right| \leq \frac{1}{3} \left| b_3 - \frac{1}{6} b_2^2 \right| + \left\{ \frac{2}{3} \left| a_2 - \left(\frac{1}{3} e^{-i\beta} \cos \beta - 1\right) \alpha_1^2 \right| + \frac{4}{9} |\alpha_1 b_2| \right\} \cos \beta \]

Using Theorem A, Theorem B, and the inequality \( |b_2| \leq 1 \), we get

\[ \left| a_3 - \frac{2}{9} a_2^2 \right| \leq \frac{5}{18} + \left\{ \frac{2}{3} \left[ 1 + \left(\frac{1}{3} (\cos \beta - i \sin \beta) \cos \beta - 1\right) |\alpha_1|^2 \right] \right. 

\[ \quad + \frac{4}{9} |\alpha_1 b_2| \right\} \cos \beta \]

\[ \leq \frac{17}{18} + \left\{ \frac{2}{3} \left(\sqrt{1 - \frac{5}{9} \cos^2 \beta - 1}\right) |\alpha_1|^2 + \frac{4}{9} |\alpha_1 b_2| \right\} \cos \beta. \]  

(2.1)
Since
\[ \sqrt{1 - \frac{5}{9} \cos^2 \beta - 1 \leq \frac{2}{3} - 1 = -\frac{1}{3},} \]
we find from (2.1) that
\[
\left| a_3 - \frac{2}{9} a_2 \right| \leq \frac{17}{18} + \left\{ \frac{2}{3} \left( -\frac{1}{3} \right) |\alpha_1|^2 + \frac{4}{9} |\alpha_1 b_2| \right\} \cos \beta
\]
\[
= \frac{17}{18} + \frac{2}{9} |b_2|^2 \cos \beta - \left\{ \frac{2}{9} |b_2|^2 + \frac{2}{9} |\alpha_1|^2 - \frac{4}{9} |\alpha_1 b_2| \right\} \cos \beta
\]
\[
\leq \frac{7}{6} - \frac{2}{9} (|b_2| - |\alpha_1|)^2 \cos \beta
\]
\[
\leq \frac{7}{6},
\]
which completes the proof of Theorem 1.

**Proof of Theorem 2.** We begin by considering
\[
|a_3 - \lambda a_2^2| = |a_3 - \frac{2}{9} a_2 + \frac{2}{9} a_2^2 - \lambda a_2^2|
\]
\[
\leq |a_3 - \frac{2}{9} a_2^2| + \left( \frac{2}{9} - \lambda \right) |a_2|^2,
\]
which, in view of Theorem 1, yields
\[
|a_3 - \frac{2}{9} a_2^2| \leq \frac{7}{6}.
\]
Thus, using Theorem C, we have
\[
|a_3 - \lambda a_2^2| \leq \frac{7}{6} + \left( \frac{2}{9} - \lambda \right) \left( \frac{3}{2} \right)^2
\]
\[
= \frac{5}{3} - \frac{9 \lambda}{4},
\]
which completes the proof of Theorem 2.

Equality can be shown to hold true by putting \( b_2 = b_3 = 1, \alpha_1 = 1, \alpha_2 = 0, \) and \( \beta = 0 \) in (1.12).

**Proof of Theorem 3.** By (1.12), we have
\[
|a_3 - \lambda a_2^2| \leq \frac{1}{3} \left| b_3 - \frac{3}{4} \lambda b_2^2 \right| + \frac{2}{3} \left| \alpha_2 - \left( \frac{3}{2} \lambda e^{-i\theta} \cos \beta - 1 \right) \alpha_1^2 \right| \cos \beta
\]
\[
+ \frac{2}{3} \left( 1 - \frac{3}{2} \lambda \right) |\alpha_1| |b_2| \cos \beta.
\]
(2.2)
Now, using Theorem A, Theorem B, and the fact that $|b_2| \leq 1$, we get
\[
|a_3 - \lambda a_2^2| \leq \frac{4 - 3\lambda}{12} + \frac{2}{3} \left\{ 1 + \left[ \frac{3}{2} \lambda e^{-i\beta} \cos \beta - 1 \right] |\alpha_1|^2 \right\} \cos \beta + \frac{2 - 3\lambda}{3} |\alpha_1| \cos \beta \\
= \frac{4 - 3\lambda}{12} + \left\{ \frac{2}{3} + \frac{2}{3} |\alpha_1|^2 \left( \sqrt{1 - \left( \frac{3\lambda - \frac{9}{4}\lambda^2}{\cos^2 \beta - 1} \right)} \right) \cos \beta \right\} \\
+ \frac{2 - 3\lambda}{3} |\alpha_1| \\
(2.3)
\]
Putting $\cos \beta = y$ and $|\alpha_1| = \alpha$ in (2.3), we get
\[
|a_3 - \lambda a_2^2| \leq \frac{4 - 3\lambda}{12} + y \left\{ \frac{2}{3} + \frac{2}{3} \alpha^2 \left( \sqrt{1 - \left( \frac{3\lambda - \frac{9}{4}\lambda^2}{y^2 - 1} \right)} \right) \right\} + \frac{2 - 3\lambda}{3} \alpha \\
= F_\lambda(\alpha, y) \quad \text{(say)}.
(2.4)
\]
In the rest of the proof, we shall show that $F_\lambda(\alpha, y)$ attains its maximum value for $(\alpha, y) \in [0, 1] \times [0, 1]$ at the point $\left( \frac{2 - 3\lambda}{6\lambda}, 1 \right)$. Note that
\[
F_\lambda \left( \frac{2 - 3\lambda}{6\lambda}, 1 \right) = \frac{2}{3} + \frac{1}{9\lambda}.
(2.5)
\]
We shall first show that $F_\lambda$ does not have a local maximum at any interior point of the open rectangle $(0, 1) \times (0, 1)$. For, if $F_\lambda$ had a local maximum at some point $(\alpha_0, y_0) \in (0, 1) \times (0, 1)$, then the partial derivatives $\frac{\partial F_\lambda}{\partial \alpha}$ and $\frac{\partial F_\lambda}{\partial y}$ must vanish at $(\alpha_0, y_0)$. Now
\[
\frac{\partial F_\lambda}{\partial \alpha} = y \left[ \frac{4\alpha}{3} \left( \sqrt{1 - y^2 \left( \frac{3\lambda - \frac{9\lambda^2}{4}}{y^2 - 1} \right)} - 1 \right) + \frac{2 - 3\lambda}{3} \right].
\]
Thus the first requirement:
\[
\frac{\partial F_\lambda}{\partial \alpha} \bigg|_{(\alpha_0, y_0)} = 0
\]
implies that
\[
4\alpha_0 \left( \sqrt{1 - y_0^2 \left( \frac{3\lambda - \frac{9\lambda^2}{4}}{y_0^2} \right)} - 1 \right) = -\frac{4}{3} \gamma \left( \gamma = \frac{3}{4} (2 - 3\lambda) \right).
\]
Note that $0 < \gamma < 1$. Therefore
\[
y_0^2 \left( \frac{3\lambda - \frac{9\lambda^2}{4}}{y_0^2} \right) = \frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{9\alpha_0^2}.
(2.6)
\]
Similarly, we have
\[
\frac{\partial F_\lambda}{\partial y} = \left[ \frac{2}{3} + \frac{2\alpha_0^2}{3} \left( \sqrt{1 - y^2 \left( 3\lambda - \frac{9\lambda^2}{4} \right)} - 1 \right) + \left( \frac{2 - 3\lambda}{3} \right) \alpha \right] + y \left[ \frac{2\alpha_0^2}{3} \left( \frac{-y}{\sqrt{1 - y^2 \left( 3\lambda - \frac{9\lambda^2}{4} \right)}} \right) \right].
\]

Thus the second requirement:
\[
\frac{\partial F_\lambda}{\partial y} \bigg|_{\alpha_0, y_0} = 0
\]
implies that
\[
\frac{2}{3} + \frac{2\alpha_0^2}{3} \left( \sqrt{1 - y_0^2 \left( 3\lambda - \frac{9\lambda^2}{4} \right)} - 1 \right) + \left( \frac{2 - 3\lambda}{3} \right) \alpha_0 = \frac{2\alpha_0^2}{3} \frac{y_0^2 \left( 3\lambda - \frac{9\lambda^2}{4} \right)}{\sqrt{1 - y_0^2 \left( 3\lambda - \frac{9\lambda^2}{4} \right)}}. \tag{2.7}
\]

Substituting the value of \( y_0^2 \left( 3\lambda - \frac{9\lambda^2}{4} \right) \) from (2.6) into (2.7), we find that
\[
\frac{2}{3} + \frac{2\alpha_0^2}{3} \left( -\frac{\gamma}{3\alpha_0} \right) + \frac{4\gamma\alpha_0}{9} - \frac{2}{3} \frac{\alpha_0^2}{\alpha_0} \left[ \frac{2\gamma}{3\alpha_0} - \frac{\gamma^2}{3\alpha_0} \right] = 0
\]
or, equivalently, that
\[
\alpha_0^2 \gamma - 3\alpha_0 + \gamma = 0. \tag{2.8}
\]

Solving the quadratic equation (2.8) for \( \alpha_0 \), we have
\[
\alpha_0 \gamma = \frac{3 - \sqrt{9 - 4\gamma^2}}{2}. \tag{2.9}
\]
The value of \( F_\lambda (\alpha_0, y_0) \) written in terms of \( \gamma \) becomes
\[
F_\lambda (\alpha_0, y_0) = \frac{6 + 4\gamma}{36} + y_0 \left( \frac{2}{3} + \frac{2}{9} \gamma \alpha_0 \right),
\]
which, upon substituting the value of $\alpha_0 \gamma$ from (2.9), yields

\[
F_\lambda (\alpha_0, y_0) = \frac{6 + 4\gamma}{36} + y_0 \left(1 - \frac{1}{3} \sqrt{1 - 4\frac{\gamma^2}{9}}\right)
\leq \frac{6 + 4\gamma}{36} + 1 - \frac{1}{3} \sqrt{1 - 4\frac{\gamma^2}{9}}
= \frac{21 + 2\gamma - 6\sqrt{1 - 4\frac{\gamma^2}{9}}}{18}.
\]

Since $y \in (0, 1)$, there exists $\eta > 0$ such that

\[
\gamma = \frac{3}{2} \cos \delta \quad \text{and} \quad \sqrt{1 - 4\frac{\gamma^2}{9}} = \sin \delta \quad \left(0 < \eta < \delta < \frac{\pi}{2}\right).
\]

Moreover, the inequality:

\[
1 < 2 \cos \delta + \sin \delta
\]

gives

\[
(1 - \cos \delta)(1 - \sin \delta) < \frac{\cos^2 \delta}{2},
\]

so that

\[
(1 - \cos \delta) \left(21 + 2\gamma - 6\sqrt{1 - 4\frac{\gamma^2}{9}}\right)
= (1 - \cos \delta)(15 + 3 \cos \delta) + 6(1 - \cos \delta)(1 - \sin \delta)
< 15 + 3 \cos \delta - 15 \cos \delta - 3 \cos^2 \delta + 3 \cos^2 \delta
= 15 - 12 \cos \delta = 3 + 12(1 - \cos \delta).
\]

Therefore, we have

\[
\frac{21 + 2\gamma - 6\sqrt{1 - 4\frac{\gamma^2}{9}}}{18} < \frac{1}{6(1 - \cos \delta)} + \frac{2}{3} = \frac{2}{3} + \frac{1}{9\lambda},
\]

which shows that

\[
F_\lambda (\alpha_0, y_0) < \frac{2}{3} + \frac{1}{9\lambda}.
\]

Hence $F_\lambda (\alpha, y)$ does not have a local maximum in $(0, 1) \times (0, 1)$, so that the maximum must be attained at a boundary point. Since

\[
F_\lambda (\alpha, 0) = \frac{4 - 3\lambda}{12} < \frac{2}{3} + \frac{1}{9\lambda},
\]

there is no maximum on the line $y = 0$. Similarly, we have

\[
F_\lambda (0, y) = \frac{4 - 3\lambda}{12} + \frac{2}{3}y \leq \frac{4 - 3\lambda}{12} + \frac{2}{3} \leq \frac{2}{3} + \frac{1}{9\lambda},
\]
so that there is no maximum on the line $\alpha = 0$ either. On the line $y = 1$, we get

$$F_\lambda(\alpha, 1) = 1 - \frac{\lambda}{4} - \lambda \alpha^2 + \left(\frac{2}{3} - \lambda\right) \alpha = G_\lambda(\alpha) \quad \text{(say)}.$$

Putting $\alpha = 1$, we get

$$G_\lambda(1) = \frac{5}{3} - \frac{9\lambda}{4}.$$

Since $G_\lambda(1)$ is not maximal, the local maximum of $G_\lambda(\alpha)$ is attained at

$$\alpha_0 = \frac{2 - 3\lambda}{6\lambda} \quad \text{for which} \quad \frac{dG_\lambda(\alpha)}{d\alpha}\bigg|_{\alpha=\alpha_0} = 0.$$

This leads to the maximal value given by (2.4). The proof will be complete if we show that

$$F_\lambda(1, y) \leq \frac{2}{3} + \frac{1}{9\lambda} \quad (0 < y < 1). \quad (2.10)$$

Since

$$F_\lambda(1, y) = \frac{1}{3} \left[ \frac{3 + 2\gamma}{6} + 2y \left( \sqrt{1 - y^2 \left(1 - \frac{4}{9}\gamma^2\right)} + \frac{2\gamma}{3} \right) \right]$$

$$= \frac{1}{3} \left[ \frac{3 + 2\gamma}{6} + H_\gamma(y) \right],$$

where

$$H_\gamma(y) = 2y \left( \sqrt{1 - y^2 \left(1 - \frac{4}{9}\gamma^2\right)} + \frac{2\gamma}{3} \right),$$

the assertion (2.10) is equivalent to

$$H_\gamma(y) \leq \frac{2}{3} \frac{(3 - \gamma)^2}{(3 - 2\gamma)} \quad (0 < y < 1). \quad (2.11)$$

Therefore, it suffices to prove (2.11) at the points $y \in (0, 1)$ for which

$$\frac{dH_\gamma(y)}{dy} = 0,$$

which implies that

$$\frac{2\gamma}{3} \sqrt{1 - y^2 \left(1 - \frac{4}{9}\gamma^2\right)} = 2y^2 \left(1 - \frac{4}{9}\gamma^2\right) - 1. \quad (2.12)$$

Squaring both sides in (2.12), we get

$$\left\{2y^2 \left(1 - \frac{4}{9}\gamma^2\right)\right\}^2 - 2 \left\{2y^2 \left(1 - \frac{4}{9}\gamma^2\right)\right\} \left(1 - \frac{\gamma^2}{9}\right) + \left(1 - \frac{4}{9}\gamma^2\right) = 0.$$

Thus we have

$$2y^2 \left(1 - \frac{4}{9}\gamma^2\right) = \frac{9 - \gamma^2 - \gamma\sqrt{18 + \gamma^2}}{9}, \quad (2.13)$$

since $0 \leq y \leq 1$. 
We first square both sides of the inequality (2.11), substitute the value of \( y^2 \) from (2.13), and get the equivalent inequality:
\[
(3 - 2\gamma) \left( 9 - \gamma^2 - \gamma\sqrt{18 + \gamma^2} \right) \left( \sqrt{18 + \gamma^2} + 5\gamma \right)^2 \leq 8(3 + 2\gamma)(3 - \gamma)^4.
\]
For simplicity of our calculations, we put \( t = \sqrt{18 + \gamma^2} \); then (after a routine calculation) this becomes
\[
18t\gamma(3 - 2\gamma)(2 - \gamma^2) \leq 729 - 486\gamma - 270\gamma^2 + 324\gamma^3 - 30\gamma^4 - 28\gamma^5.
\]  \hspace{1cm} (2.14)

The right-hand side of (2.14) turns out to be positive, since
\[
729 - 486\gamma - 270\gamma^2 + 324\gamma^3 - 30\gamma^4 - 28\gamma^5
= (1 - \gamma)(729 + 243\gamma - 27\gamma^2 + 297\gamma^3 + 267\gamma^4) + 239\gamma^5
> 0.
\]
Thus, squaring (2.14) once again, we get the equivalent inequality:
\[
324(18 + \gamma^2)\gamma^2 \left( 36 - 48\gamma - 20\gamma^2 + 48\gamma^3 - 7\gamma^4 - 12\gamma^5 + 4\gamma^6 \right)
\leq 729^2 - 2 \cdot 729 \cdot 486\gamma + (486^2 - 2 \cdot 729 \cdot 270)\gamma^2 + (2 \cdot 729 \cdot 342 + 2 \cdot 486 \cdot 270)\gamma^3
+ (270^2 - 2 \cdot 729 \cdot 30 - 2 \cdot 486 \cdot 324)\gamma^4 + (-2) \cdot 729 \cdot 28
+ 2 \cdot 486 \cdot 30 - 2 \cdot 270 \cdot 324)\gamma^5 + (324^2 + 2 \cdot 486 \cdot 28 + 2 \cdot 270 \cdot 30)\gamma^6
+ (2 \cdot 270 \cdot 28 - 2 \cdot 324 \cdot 30)\gamma^7 + (900 - 2 \cdot 324 \cdot 28)\gamma^8 + 60 \cdot 28\gamma^9 + 784\gamma^{10}.
\]
that is,
\[
324\gamma^2 \left( 648 - 864\gamma - 324\gamma^2 + 816\gamma^3 - 146\gamma^4 - 168\gamma^5 + 65\gamma^6 - 12\gamma^7 + 4\gamma^8 \right)
\leq 729^2 - 2 \cdot 729 \cdot 486\gamma + (486^2 - 2 \cdot 729 \cdot 270)\gamma^2 + (2 \cdot 729 \cdot 342 + 2 \cdot 486 \cdot 270)\gamma^3
+ (270^2 - 2 \cdot 729 \cdot 30 - 2 \cdot 486 \cdot 324)\gamma^4 + (-2) \cdot 729 \cdot 28
+ 2 \cdot 486 \cdot 30 - 2 \cdot 270 \cdot 324)\gamma^5 + (324^2 + 2 \cdot 486 \cdot 28 + 2 \cdot 270 \cdot 30)\gamma^6
+ (2 \cdot 270 \cdot 28 - 2 \cdot 324 \cdot 30)\gamma^7 + (900 - 2 \cdot 324.28)\gamma^8 + 60 \cdot 28\gamma^9 + 784\gamma^{10}.
\]
Thus the reformulated inequality is given by
\[
729^2 - 2 \cdot 729 \cdot 486\gamma + (486^2 - 2 \cdot 729 \cdot 270 - 324 \cdot 648)\gamma^2
+ (2 \cdot 729 \cdot 324 + 2 \cdot 486 \cdot 270 + 324 \cdot 864)\gamma^3 + (270^2 - 2 \cdot 729 \cdot 30
- 2 \cdot 486 \cdot 324 + 324^2)\gamma^4 + (-2) \cdot 729 \cdot 28 + 2 \cdot 486 \cdot 30 - 2 \cdot 270 \cdot 324
- 324 \cdot 816\gamma^5 + (324^2 + 2 \cdot 486 \cdot 28 + 2 \cdot 270 \cdot 30 + 324 \cdot 146)\gamma^6
+ (2 \cdot 270 \cdot 28 - 2 \cdot 324 \cdot 30 + 324 \cdot 168)\gamma^7(900 - 2 \cdot 324 \cdot 28 - 324 \cdot 65)\gamma^8
+ (1680 + 324 \cdot 12)\gamma^9 + (784 - 324 \cdot 4)\gamma^{10} \geq 0.
\]  \hspace{1cm} (2.15)
Now the left-hand side of (2.14) is
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\[ 531441 - 708588\gamma - 367416\gamma^2 + 1014768\gamma^3 - 180792\gamma^4 - 451008\gamma^5 \\
+ 195696\gamma^6 + 50112\gamma^7 - 38304\gamma^8 + 5568\gamma^9 - 512\gamma^{10} \\
= 531441(1 - \gamma) - 171147\gamma(1 - \gamma) - 544563\gamma^2(1 - \gamma) + 470205\gamma^3(1 - \gamma) \\
+ 289413\gamma^4(1 - \gamma) - 161595\gamma^5(1 - \gamma) + 34101\gamma^6(1 - \gamma) + 84213\gamma^7(1 - \gamma) \\
+ 45909\gamma^8(1 - \gamma) + 51477\gamma^9(1 - \gamma) + 50965\gamma^{10} \\
= (1 - \gamma)(531441 - 177147\gamma - 544563\gamma^2 + 470205\gamma^3 + 289413\gamma^4 - 16595\gamma^5 \\
+ 34101\gamma^6 + 84213\gamma^7 + 45909\gamma^8 + 51477\gamma^9 + 50965\gamma^{10}) \\
= (1 - \gamma)[(1 - \gamma)(531441 + 354294\gamma - 190269\gamma^2 + 161595\gamma^4) + 279936\gamma^3 \\
+ 127818\gamma^4 + 34101\gamma^6 + 84213\gamma^7 + 45909\gamma^8 + 51477\gamma^9] + 50965\gamma^{10} \\
= (1 - \gamma)((1 - \gamma)(531441 + 164025\gamma + 190260\gamma(1 - \gamma) + 161595\gamma^4) \\
+ 279936\gamma^3 + 127818\gamma^4 + 34101\gamma^6 + 84213\gamma^7 + 45909\gamma^9 + 51477\gamma^9] + 59065\gamma^{10} \\
> 0.

Hence (2.15) is true. This completes the proof of the main assertion of Theorem 3. The result can be shown to be sharp by setting \( b_2 = b_3 = 1, \beta = 0, \alpha_1 = \frac{2 - 3\lambda}{6\lambda}, \) and \( \alpha_2 = 1 - \alpha_1^2 \) in (1.12).

**Proof of Theorem 4.** Putting \( \lambda = \frac{2}{3} \) in (1.12), we get

\[
\alpha_3 - \frac{2}{3}\alpha_2^2 = \frac{1}{3} \left( b_3 - \frac{1}{2} b_2^2 \right) + \frac{2}{3} e^{-i\beta} \left[ \alpha_2 + \left( 1 - e^{-i\beta} \cos \beta \right) \alpha_1^2 \right] \cos \beta.
\]

Hence

\[
\left| \alpha_3 - \frac{2}{3}\alpha_2^2 \right| \leq \frac{1}{3} \left| b_3 - \frac{1}{2} b_2^2 \right| + \frac{2}{3} \left| \alpha_2 - \left( e^{-i\beta} \cos \beta - 1 \right) \alpha_1^2 \right| \cos \beta.
\]

Using Theorem A and Theorem B, we get

\[
\left| \alpha_3 - \frac{2}{3}\alpha_2^2 \right| \leq \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \left[ 1 + \left( \left| e^{-i\beta} \cos \beta - 1 \right| - 1 \right) |\alpha_1|^2 \right] \cos \beta \\
= \frac{1}{6} + \frac{2}{3} \left[ 1 + \left( \sqrt{1 - \cos^2 \beta} - 1 \right) |\alpha_1|^2 \right] \cos \beta \\
= \frac{1}{6} + \frac{2}{3} \cos \beta + \frac{2}{3} (|\sin \beta| - 1) |\alpha_1|^2 \cos \beta \\
\leq \frac{5}{6} - \frac{2}{3} (1 - |\sin \beta|) |\alpha_1|^2 \cos \beta \\
\leq \frac{5}{6},
\]

which completes the proof of Theorem 4.
Proof of Theorem 5. Putting $\lambda = 1$ in (1.12), we get

$$a_3 - a_2^2 = \frac{1}{3} \left( b_3 - \frac{3}{4} b_2^2 \right) + \frac{2}{3} e^{-i\beta} \left[ \alpha_2 + \left( 1 - \frac{3}{2} e^{-i\beta} \cos \beta \right) \alpha_1^2 \right] \cos \beta - \frac{1}{3} \alpha_1 b_2 e^{-i\beta} \cos \beta.$$ 

Hence

$$|a_3 - a_2^2| \leq \frac{1}{3} |b_3 - b_2^2| + \frac{|b_2|^2}{12} + \frac{2}{3} \left| \alpha_2 - \left( \frac{3}{2} e^{-i\beta} \cos \beta - 1 \right) \alpha_1^2 \right| + \frac{\alpha_1 b_2}{3} \cos \beta.$$ 

(2.16)

Using Theorem B and a result of Trimble [18], we find from (2.16) that

$$|a_3 - a_2^2| \leq \frac{1}{3} \left( \frac{1 - |b_2|^2}{3} \right) + \frac{|b_2|^2}{12} + \frac{2}{3} \left( \frac{3}{2} \cos^2 \beta - 1 \right) \left| \alpha_1 \right|^2 + \frac{|\alpha_1 b_2|}{3}$$

$$= \frac{7}{9} + \frac{|b_2|^2}{18} - \frac{|b_2|^2}{12} + \frac{2}{3} \left( \sqrt{1 - \frac{3}{4} \cos^2 \beta - 1} \right) |\alpha_1|^2 + \frac{|\alpha_1 b_2|}{3}$$

$$= \frac{7}{9} + \frac{|b_2|^2}{18} - \frac{1}{3} \left[ \frac{|b_2|^2}{4} - 2 \left( \sqrt{1 - \frac{3}{4} \cos^2 \beta - 1} \right) |\alpha_1|^2 - |\alpha_1 b_2| \right].$$ 

(2.17)

The elementary inequality:

$$1 - \frac{3}{4} \cos^2 \beta \geq \frac{1}{4}$$

immediately yields

$$\sqrt{1 - \frac{3}{4} \cos^2 \beta} \geq \frac{1}{2}.$$ 

Thus we find from (2.17) that

$$|a_3 - a_2^2| \leq \frac{7}{9} + \frac{1}{18} - \frac{1}{3} \left( \frac{|b_2|^4}{4} + |\alpha_1|^2 - |\alpha_1 b_2| \right)$$

$$= \frac{5}{6} - \frac{1}{3} \left( \frac{|b_2|^2}{2} - |\alpha_1| \right)^2$$

$$\leq \frac{5}{6}.$$ 

This completes the proof of Theorem 5.

Proof of Theorem 6. Observe that

$$a_3 - \lambda a_2^2 = (3\lambda - 2) (a_3 - a_2^2) + 3(1 - \lambda) \left( a_3 - \frac{2}{3} a_2^2 \right).$$

The main assertion of Theorem 6 follows from Theorem 4 and Theorem 5.
Equality can be shown to hold true by setting $b_2 = b_3 = 1$, $\alpha_2 = 1 - \alpha_1^2$, $\beta = 0$, and

$$\alpha_1 = \frac{(2 - 3\lambda) \pm i\sqrt{6\lambda - 4}}{6\lambda}$$

in (1.12).

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