AN EXCISION THEOREM FOR THE
K-THEORY OF C*-ALGEBRAS

Ian F. Putnam

DMS-726-IR

February 1996
[Revised October 1996]
AN EXCISION THEOREM FOR THE K-THEORY OF C*-ALGEBRAS

Ian F. Putnam

Department of Mathematics and Statistics
University of Victoria
Victoria, B.C. V8W 3P4
Canada

Abstract. We consider a pair of C*-algebras $A' \subseteq A$. The K-theory of the mapping cone for this inclusion can be regarded as a relative K-group. We describe a situation where two such pairs have isomorphic relative groups.

§1. Introduction

This paper is concerned with a certain excision result for K-theory of C*-algebras.

Let us begin by setting out some notation. Let $A$ be any C*-algebra. We let $A^\sim$ be the C*-algebra obtained by adjoining a unit to $A$ (even if $A$ is already unital). Let $M_n(A)$ denote the C*-algebra of $n \times n$ matrices with entries from $A$. For any $a$ in $A^\sim$ (respectively, $M_n(A^\sim)$), let $\bar{a}$ denote its image in $\mathbb{C}$, the complex numbers, (respectively, $M_n(\mathbb{C})$), under the map moding out by $A$. We also regard $\mathbb{C}$ and $M_n(\mathbb{C})$ implicitly as subalgebras of $A^\sim$ and $M_n(A^\sim)$, respectively.

Suppose $A'$ is a C*-subalgebra of $A$. We regard $A'^\sim \subseteq A^\sim$ as the natural unital inclusion. Recall [Sch, W-O, B1] that the mapping cone for the inclusion $A' \subseteq A$ is

$$C(A'; A) = \left\{ f : [0,1] \rightarrow A \mid f \text{ is continuous,} \right\}$$

$$f(0) = 0, \quad f(1) \in A' \}. \right.$$ 

It is a C*-algebra with pointwise operations and

$$\|f\| = \sup \{\|f(t)\| \mid 0 \leq t \leq 1\}$$

for $f$ in $C(A'; A)$. There is a natural short exact sequence

$$0 \rightarrow C_0(0,1) \otimes A \rightarrow C(A'; A) \rightarrow A' \rightarrow 0$$

1 Supported in part by an NSERC Operating Grant.
where
\[ ev(f) = f(1), \quad f \in C(A'; A) \]
\[ i(g \otimes a)(t) = g(t)a, \quad g \in C_0(0, 1), \quad a \in A, \quad 0 \leq t \leq 1. \]

Let \( b : K_i(A) \to K_{i+1}(C_0(0, 1) \otimes A) \) denote the usual isomorphism [B1]. After using \( b \) to replace the terms involving \( K_\ast(C_0(0, 1) \otimes A) \), the six-term exact sequence for \( K \)-groups associated with the sequence above becomes

\[
\begin{array}{cccccc}
K_1(A) & \xrightarrow{i_* b} & K_0(C(A'; A)) & \xrightarrow{ev_\ast} & K_0(A') \\
\uparrow j_* & & & & \downarrow j_* \\
K_1(A') & \xleftarrow{ev_*} & K_1(C(A'; A)) & \xleftarrow{i_* b} & K_0(A) \\
\end{array}
\]

where \( j : A' \to A \) denotes the inclusion map. We regard \( K_\ast(C(A'; A)) \) as a "relative group" for the \( C^* \)-algebra inclusion \( A' \subseteq A \). Indeed, if \( A' \) is actually an ideal in \( A \), then there is a natural isomorphism

\[ K_\ast(C(A'; A)) \cong K_\ast(A/A'). \]

To see this, let
\[ J = \{ f \in C(A'; A) \mid f(t) \in A' \text{ for all } 0 \leq t \leq 1 \}, \]
which is an ideal in \( C(A'; A) \). Moreover, \( J \cong C_0(0, 1) \otimes A' \) and so \( K_\ast(J) = 0 \), since \( C_0(0, 1) \) is contractible [W-O, B1]. We also have a short exact sequence

\[ 0 \to J \to C(A, A') \to C_0(0, 1) \otimes (A/A') \to 0. \]

Taking the six-term exact sequence for \( K \)-groups and noting \( K_\ast(J) = 0 \) yields the result. Thus, if \( A' \) is an ideal, \( K_\ast(C(A'; A)) \) depends only on \( A/A' \).

Our goal is to describe two pairs of inclusions \( A' \subseteq A \) and \( B' \subseteq B \) which are related in a specific way that we may conclude that there is an isomorphism

\[ K_\ast(C(A'; A)) \cong K_\ast(C(B'; B)), \]

which is natural in some sense. The rôles of \( A \) and \( B \) here will not be symmetric. In some sense, the inclusion \( A' \subseteq A \) will be the more tractible. We suppose that \( A \) and \( B \) are both
$C^*$-algebras of operators acting on the Hilbert space $\mathcal{H}$. We suppose that $z$ is a self adjoint unitary on $\mathcal{H}$ and that the following conditions are satisfied. First, $B$ should lie in the multiplier algebra of $A$. We should have $zAz = A$ and, for all $b$ in $B$, $zbz - b$ lies in $A$. One interesting case where this occurs is when $(\mathcal{H}, z)$ is a Fredholm module for $B$ [B1]. Let $A$ be the $C^*$-algebra of compact operators on $\mathcal{H}$. These conditions are satisfied. Returning to the general situation, we let $A'$ and $B'$ be those operators in $A$ and $B$, respectively, which commute with $z$. We require three more technical assumptions on $A$, $B$ and $z$ (given as 4, 5, 6 in section 3). Under these hypotheses, we construct a homomorphism

$$\alpha : K_*(C(B'; B)) \rightarrow K_*(C(A'; A))$$

and prove that it is an isomorphism.

The main applications of this result are in various situations arising from dynamical systems where $B$, $B'$, $A$ and $A'$ can all be described as groupoid $C^*$-algebras. For example, $B = C(X) \times_\phi \mathbb{Z}$ and $B' = A_Y$ of [Put1], where $\phi$ is a minimal homeomorphism of a Cantor set $X$, can be described in this way. Here, $A$ is the compact operator on $\ell^2(\mathbb{Z})$ and $A'$ is the direct sum of compact operators on two orthogonal subspaces. More applications can be found in [Put2]. (Also, see [GPS].)

In Section 2, we provide a description of $K_0(C(A'; A))$ which will be useful. In Section 3, we state and prove the main results (3.1 and 3.7).

§2. $K$-theory of Mapping Cones

Our aim in this section is to provide a natural description of $K_0(C(A', A))$.

We begin, as in Section 1, with $C^*$-algebras $A' \subseteq A$. For each $n = 1, 2, 3, \cdots$, we let $V_n(A'; A)$, or simply $V_n$, denote the set of elements $v$ in $M_n(A^\sim)$ such that

(i) $v$ is a partial isometry.
(ii) $v^*v$ is in $M_n(\mathbb{C})$.
(iii) $vv^*$ is in $M_n(A'^\sim)$.

(In some ways, it would be more natural to required $v^*v$ to be in $M_n(A'^\sim)$; our definition will be more convenient, however.) We regard $V_n \subseteq V_{n+1}$ by identifying $v$ and $v \oplus 0$, for all $v$ in $V_n$. We let

$$V(A'; A) = \bigcup_n V_n(A'; A).$$
We will make use of the following two facts:

1. If \( h \) is a self-adjoint element of a \( C^\ast \)-algebra and \( \| h - h^2 \| < \delta < \frac{1}{2} \), then the spectrum of \( h \) is contained in \( (-2\delta, 2\delta) \cup (1 - 2\delta, 1 + 2\delta) \). The proof is an easy application of the spectral theorem.

2. If \( p_1 \) and \( p_2 \) are projections in a \( C^\ast \)-algebra with \( \| p_1 - p_2 \| < \delta < \frac{1}{2} \), then there is a unitary \( u \) in the \( C^\ast \)-algebra such that \( up_1u^* = p_2 \) and \( \| u - 1 \| < \pi\delta \). For a proof, see 4.3.2, 4.6.5 of [B1].

**Lemma 2.1.** Suppose \( 0 < \varepsilon < 100^{-1} \) and \( v \) in \( M_n(A^\sim) \) satisfies (i) and (ii) above and there exists \( q \) in \( M_n(A^\sim) \) such that \( \| vv^* - q \| < \varepsilon \). Then there exists a unitary \( u \) in \( M_n(A^\sim) \) such that \( \| u - 1 \| < 30\varepsilon \) and \( uv \) is in \( V_n(A'; A) \).

**Proof.** First replace \( q \) by \( (q + q^*)/2 \) so we may assume it is self-adjoint. Since \( v \) is a partial isometry, \( vv^* \) is a projection and so

\[
\| q^2 - q \| < 4\varepsilon.
\]

Then, using the first fact above, \( q_1 = \chi_{(\frac{1}{2}, \infty)}(q) \) is a projection and \( \| q_1 - q \| < 8\varepsilon \) hence

\[
\| q_1 - vv^* \| < 9\varepsilon.
\]

The second fact above then gives the desired \( u \). \hfill \blacksquare

We define a map

\[
\kappa : V(A'; A) \longrightarrow K_0 (C(A'; A)).
\]

Begin with \( v \) in \( V_n(A'; A) \). Consider

\[
v_1 = \begin{bmatrix}
1 - v^*v & v^* \\
v & 1 - vv^*
\end{bmatrix}
\]

in \( M_{2n}(A^\sim) \). It is easily verified that \( v_1 \) is a self-adjoint unitary. We define a path of self-adjoint unitaries in \( M_{2n}(A^\sim) \) by

\[
v_2(t) = \left[ \hat{v}_1 + 1 + e^{i\pi t}(1 - \hat{v}_1) \right]^{-1} \left[ v_1 + 1 + e^{i\pi t}(1 - v_1) \right],
\]

for \( 0 \leq t \leq 1 \). Notice that \( v_2 \) satisfies

(i) \( v_2(t) \) is unitary for all \( t \),

(ii) \( v_2 \) is in \( C[0, 1] \otimes M_{2n}(A^\sim) \),

for 0 ≤ t ≤ 1.
(iii) \( \dot{v}_2(t) = 1 \), for all \( t \),
(iv) \( v_2(0) = 1 \),
(v) \( v_2(1) = \dot{v}_1^{-1} v_1 \).

Together, (ii), (iii) and (iv) imply that \( v_2 \) may be regarded as an element of

\[ [C_0(0, 1] \otimes M_{2n}]^\sim. \]

Finally, we define

\[ p_v(t) = v_2(t) e_{11} v_2(t)^*, \]

for \( 0 \leq t \leq 1 \), where \( e_{11} \) denotes \( 1_n \otimes 0 \) in \( M_{2n}(A^\sim) \). It is easy to verify that

(i) \( p_v(0) = e_{11} \)
(ii) \( p_v(1) = (1_n - v^* v) \oplus vv^* \in M_{2n}(A'^\sim) \)
(iii) \( \dot{p}_v(t) = e_{11} \), for all \( 0 \leq t \leq 1 \).

Thus, \( p_v \) is in \( M_{2n}(C(A'; A)^\sim) \) and \([p_v] - [e_{11}]\) is in \( K_0(C(A'; A)) \). We denote this element by \( \kappa(v) \). We summarize the properties of \( \kappa \).

**Lemma 2.2.**

(i) For \( v, w \) in \( V(A'; A) \),

\[ \kappa(v \oplus w) = \kappa(v) + \kappa(w). \]

(ii) If \( v, w \) are in \( V_n(A'; A) \) and \( \|v - w\| < 200^{-1} \), then \( \kappa(v) = \kappa(w) \).

(iii) For \( v \) in \( V_n(A'; A) \), \( w_1 \) in \( U_n(A'^\sim) \) and \( w_2 \) in \( U_n(C) \), then \( w_1 v w_2 \) is in \( V_n(A'; A) \) and

\[ \kappa(w_1) = \kappa(w_2) = 0 \]

\[ \kappa(w_1 v w_2) = \kappa(v). \]

(iv) For any projection \( p \) in \( M_n(C) \), \( \kappa(p) = 0 \).

(v) If \( v \) is a partial isometry in \( M_n(A'^\sim) \), then \( \kappa(v) = 0 \).

**Proof.** Parts (i) and (iv) are verified by direct computations, which we omit.

In proving (ii), one notes that the construction of \( p_v \) depends continuously on \( v \). In fact, \( \|v - w\| < 200^{-1} \) implies \( \|p_v - p_w\| < \frac{1}{2} \) (we omit the details), which implies \([p_v] = [p_w]\) and the conclusion. As a consequence of (ii), if \( v \) and \( w \) are homotopic in \( V_n(A'; A) \) then \( \kappa(v) = \kappa(w) \).

In part (iii), we begin by considering \( v \oplus 0 \), \( w_1 \oplus w_1^* \) and \( w_2 \oplus w_2^* \). By standard methods (see 4.2.9 of \([W-O]\)), \( w_1 \oplus w_1^* \) and \( w_2 \oplus w_2^* \) are both homotopic to the identity in
$U_{2n}(A')$ and $U_{2n}(\mathbb{C})$ respectively. Thus, $w_1 w_2 \oplus 0$ is homotopic to $v \oplus 0$ in $V_{2n}(A'; A)$, so $\kappa(v) = \kappa(w_1 w_2)$ by (ii) and (i). Finally, $\kappa(w_1) = \kappa(w_2) = 0$ both following as special cases ($v = w_2 = 1, w_1 = v = 1$) of (iii) and (iv). As for (v), writing

$$v \oplus 0 = \begin{bmatrix} v & 1 - vv^* \\ 1 - v^* v & vv^* \end{bmatrix} \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix}$$

the conclusion follows from (iii) and (iv).

We now want to see how this map $\kappa$ relates to the six-term exact sequence (1.2).

**Lemma 2.3.**

(i) For $v$ in $V_n(A'; A)$,

$$ev_*(\kappa(v)) = [vv^*] - [v^* v].$$

(ii) For $v$ in $U_n(A^\sim)$

$$i_* b[v] = \kappa(v).$$

**Proof.**

(i) We compute

$$ev_*(\kappa(v)) = [p_v(1)] - [e_{11}]$$

$$= [(1_n - v^* v) \oplus vv^*] - [e_{11}]$$

$$= [vv^*] - [v^* v].$$

(ii) In the construction of $\kappa(v)$, $v_2$ is a path of unitaries in $M_{2n}(A^\sim)$ from 1 to $\dot{v}_{-1} v_1$. Let $v_3(t)$ be any path of unitaries in $M_{2n}(\mathbb{C})$ from 1 to $\dot{v} \oplus \dot{v}^*$. Then $v_3(t) v_2(t)$ is a path from 1 to $v \oplus v^*$. By the definition of $b$

$$b[v] = [v_3 v_2 e_{11} v_2 v_3^*] - [e_{11}]$$

$$= [v_3 p_v v_3^*] - [e_{11}]$$

$$= [p_v] - [e_{11}]$$

$$= \kappa(v),$$

since $v_3(t)$ is in $M_{2n}(\mathbb{C})$.

**Lemma 2.4.** $\kappa : V(A'; A) \to K_0(C(A'; A))$ is onto.
Proof. Let $p, q$ be projections in $M_m(C(A'; A)\cdot)$ with $[p] = [q]$ in $K_0(C)$; i.e. $[p] - [q]$ is in $K_0(C(A'; A))$. By exactness of (1.2), $j_*ev_*([p] - [q]) = 0$ in $K_0(A)$. This means $[p(1)] = [q(1)]$ in $K_0(A)$. So there exists positive integers $k, n = 2m + k$ and a partial isometry $v$ in $M_n(A\cdot)$ such that

$$v^*v = 1_m \oplus 0_m \oplus 1_k$$

$$vv^* = p(1) \oplus (1_m - q(1)) \oplus 1_k.$$ 

Then $v$ is in $V_n(A'; A)$ and by (i) of 2.3, we have

$$ev_*([p] - [q]) = ev_*(\kappa(v)).$$

Hence, $\kappa(v) - [p] + [q]$ is in the kernel of $ev_*$ which is the image of $i_*$. For some unitary $w$ in $M_*(A'\cdot)$, $i_*(w) = \kappa(v) - [p] + [q]$. Using (ii) of 2.3, we have

$$\kappa(v \oplus w^*) = \kappa(v) + \kappa(w^*)$$

$$= \kappa(v) - i_*(w)$$

$$= [p] - [q].$$

Lemma 2.5. Let $\sim$ denote the equivalence relation on $V(A'; A)$ generated by

(i) $v \sim v \oplus p, v \in V(A'; A), p$ a projection in $M_n(C)$.

(ii) If $\nu(t)$ is a continuous path in $V_n(A'; A)$, then $\nu(0) \sim v(1)$.

Then $\kappa : V(A'; A)/\sim \rightarrow K_0(C(A'; A))$ is a well-defined bijection.

Proof. It follows from 2.2 (i), (ii) and (iv) that $\kappa$ is well-defined. From 2.4, we see that $\kappa$ is onto. It remains to show that if $v_1, v_2$ are in $V_n(A'; A)$ and $\kappa(v_1) = \kappa(v_2)$, then $v_1 \sim v_2$.

First, note that if $v, w_1$ and $w_2$ are as in 2.2(iii), then

$$w_1vw_2 = w_1vw_2 \oplus 0$$

$$= (w_1 \oplus w_1^*)(v \oplus 0)(w_2 \oplus w_2^*).$$

By homotoping the first and third terms of the right hand side, we see that $w_1vw_2 \sim v$.

Returning to $v_1$ and $v_2$ with $\kappa(v_1) = \kappa(v_2)$, we may first assume that by taking direct sums with (different) scalar projections that the ranks of $v_1^*v_2$ and $v_2^*v_2$ are equal. We can
then right multiply $v_1$ by a scalar unitary — without changing its $\approx$-equivalence class — to obtain $v_1^*v_1 = v_2^*v_2$.

From $\kappa(v_1) = \kappa(v_2)$, we apply $ev_*$ to both sides, use 2.3(i) and $v_1^*v_1 = v_2^*v_2$ to conclude that $[v_1v_1^*] = [v_2v_2^*]$ in $K_0(A'^\sim)$. Again we may take direct sum with a scalar projection and reduce to the case $v_1v_1^*$ and $v_2v_2^*$ are unitarily equivalent. By left multiplying $v_1$ be a unitary in $M_n(A'^\sim)$, we obtain $v_1v_1^* = v_2v_2^*$, $v_1^*v_1 = v_2^*v_2$, without changing the $\approx$-equivalence class of $v_1$ or $v_2$.

Let
\[
R_n(t) = \begin{bmatrix}
t & -\sqrt{1-t^2} \\
\sqrt{1-t^2} & t
\end{bmatrix}, \quad 0 \leq t \leq 1
\]
be in $M_{2n}(\mathbb{C})$ and define the path in $M_{2n}(A'^\sim)$
\[
v(t) = R_n(t) [v_1 \oplus v_1^*v_1] R_n(t)^{-1} \left( [(v_1^*v_2 + 1 - v_1^*v_1) \oplus 1] \right)
\]
for $0 \leq t \leq 1$. Observe that for all $t$, $v(t)$ is in $V_{2n}(A'; A)$, $v(0) = v_1^*v_2 \oplus v_1$ and $v(1) = v_2 \oplus v_1^*v_1$. We have $v_1^*v_2$ is in $V_n(A'; A)$ and
\[
\kappa(v_1^*v_2) = \kappa(v(0)) - \kappa(v_1)
\]
\[
= \kappa(v(1)) - \kappa(v_1)
\]
\[
= \kappa(v_2) - \kappa(v_1)
\]
\[
= 0.
\]
Now, consider the unitary $v = v_1^*v_2 + (1 - v_1^*v_1)$ in $M_n(A'^\sim)$. We have
\[
i_* b[v] = \kappa(v) = \kappa(v_1^*v_2) = 0,
\]
which implies $[v]$ is in the image of $j_*$. That is, $v$ is homotopic (after direct summing with the identity) to a unitary in $M_n(A'^\sim)$. Let $v'(t)$ be any path of unitaries in $M_n(A'^\sim)$ with $v'(0) = v$ and $v'(1) \in M_n(A'^\sim)$.

Now define a path in $M_{4n}(A'^\sim)$
\[
w(t) = \begin{bmatrix}
v'(t)v_1 & v'(t)(1 - v_1v_1^*) & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 - v_1^*v_1 & 0 & 0 & 0 \\
0 & v_1v_1^* & 0 & 0
\end{bmatrix}.
\]
It is straightforward to verify that, for all $0 \leq t \leq 1$,

$$w(t)w(t)^* = 1_n \oplus 1_n \oplus 0_n \oplus 0_n$$

and so $w(t)$ is a path in $V_{4n}(A'; A)$. Evaluating at $t = 0$, we see

$$w(0) = \begin{bmatrix}
v_2 & 1 - v_1^* v_1^* & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 - v_1^* v_1 & 0 & 0 & 0 \\
0 & v_1^* v_1 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
v_1 v_1^* & 1 - v_1 v_1^* & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 - v_1 v_1^* & v_1 v_1^* & 0 & 0
\end{bmatrix} \begin{bmatrix}
v_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 - v_2^* v_2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
v_2^* v_2 & 0 & 1 - v_2^* v_2 & 0 \\
0 & 1 & 0 & 0 \\
1 - v_2^* v_2 & 0 & v_2^* v_2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$

The first matrix in this product is a unitary in $M_{4n}(A'\sim)$, the last in $M_{4n}(\mathbb{C})$ and so

$$w(0) \approx v_2 \oplus 1 \oplus (1 - v_2^* v_2) \oplus 0 \approx v_2.$$

A similar calculation shows $w(1) \approx v_1$ and we are done.  

Regarding the relation $\approx$, it is clear that if $v_0$ and $v_1$ are homotopic, then for any scalar projection $p$, $v_0 \oplus p$ and $v_1 \oplus p$ are homotopic. Therefore, if $v_0 \approx v_1$ then there are scalar projections $p_0$ and $p_1$ such that $v_0 \oplus p_0$ and $v_1 \oplus p_1$ are homotopic.

A few other remarks are in order. Following exactly as in the beginning of the proof (before $\kappa(v_1) = \kappa(v_2)$ was used), given any $v_1$ and $v_2$ in $V(A'; A)$ we may direct sum scalar
projections and right multiply by one by a scalar unitary to get $v_1^*v_1 = v_2^*v_2$. Finally, if $v(0)$ is a path in $V_n(A'; A)$, we may right multiply by a path of scalar unitaries so that $v(0)^*v(0) = v(0)^*v(0)$, for all $r$.

For each $0 < \varepsilon < 400^{-1}$, we let $V_n^\varepsilon(A'; A)$ denote the set of $v$ in $M_n(A^\sim)$ such that

(i) $v$ is a partial isometry,
(ii) $v^*v$ is in $M_n(\mathbb{C})$,
(iii) $\|vv^* - q\| < \varepsilon$, for some $q$ in $M_n(A'^\sim)$.

We let $V^\varepsilon(A'; A)$ denote the union of the $V_n^\varepsilon(A'; A)$, with the usual inclusion of $V_n^\varepsilon$ in $V_{n+1}^\varepsilon$. For any $a$ in $V^\varepsilon(A'; A)$, let $v$ be as in 2.1. We define $\kappa(a) = \kappa(v)$. This is independent of the choice of $v$ by 2.2(ii). It is also easy to see that 2.2 is valid if we replace $V(A'; A)$ by $V^\varepsilon(A'; A)$. We extend the definition of $\approx$ to $V^\varepsilon(A', A)$ in the obvious way.

**Lemma 2.6.** Suppose $A$ has a countable approximate unit $\{e_n\}_{1}^{\infty}$ contained in $A'$. Then for every $v$ in $V_n(A'; A)$ and $0 < \varepsilon < 400^{-1}$, $v \approx w$, for some $w$ in $V_{2n}^\varepsilon(A'; A)$ such that

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^*w_0)^{1/2} & 0 \end{bmatrix},$$

where $w_0$ is in $M_n(A)$, $p$ is a projection in $M_n(\mathbb{C})$ and $0 \leq w_0^*w_0 \leq p$. Moreover if

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^*w_0)^{1/2} & 0 \end{bmatrix}, \quad w' = \begin{bmatrix} w_0' & 0 \\ (p - w_0'^*w_0')^{1/2} & 0 \end{bmatrix}$$

are homotopic in $V_{2n}^\varepsilon(A'; A)$ then there is a path

$$w(t) = \begin{bmatrix} w_0(t) & 0 \\ (p - w_0(t)^*w_0(t))^{1/2} & 0 \end{bmatrix}$$

joining them.

(The point here is that $w_0$ lies in $M_n(A)$ and not just $M_n(A^\sim)$.)

**Proof.** Notice that $v \approx \hat{v}^*v$ — see the proof of 2.5 — and $(\hat{v}^*v) = \hat{v}^*\hat{v} = p$ is a projection in $M_n(\mathbb{C})$. Thus, we may assume $\hat{v} = p$. Using $e_m$ to denote $1_n \otimes e_m$ in $M_n(A)$, notice that

$$e'_m = \begin{bmatrix} e_m & -(1 - e_m^2)^{1/2} \\ (1 - e_m^2)^{1/2} & e_m \end{bmatrix}$$
is a unitary in $M_{2n}(A'\sim)$ so

$$v \approx e'_m(v \oplus 0) = \begin{bmatrix} e_m v & 0 \\ (1 - e_m^2)^{1/2} v & 0 \end{bmatrix}.$$ 

We will let $w_0 = e_m v$, for some sufficiently large $m$, which is in $M_n(A)$. It is clear that $w_0^* w_0 \leq p$. Consider

$$\left\| (1 - e_m^2)^{1/2} v - (p - w_0^* w_0)^{1/2} \right\|$$

$$\leq \left\| (1 - e_m^2)^{1/2} (v - p) \right\|$$

$$+ \left\| (1 - e_m^2)^{1/2} p - (p - w_0^* w_0)^{1/2} \right\|.$$

The first term tends to zero since $v - p$ is in $M_n(A)$ and $e_m$ is an approximate unit. As for the second, since $(1 - e_m^2)$ and $p$ commute, their product is positive and

$$\left\| (1 - e_m^2)^{1/2} p - (p - w_0^* w_0)^{1/2} \right\|$$

$$\leq \left\| (1 - e_m^2) p - (p - w_0^* w_0) \right\|^{1/2}$$

$$= \left\| (p - v)^* (1 - e_m^2) (p - v) \right\|^{1/2}$$

which tends to zero as $m$ goes to infinity. Therefore, we may choose $m$ so that $e'_m(v \oplus 0)$ and

$$\begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{1/2} & 0 \end{bmatrix}$$

are sufficiently close so that the latter is in $V_{2n}^e(A'\sim; A)$ and is $\approx$-equivalent to the former.

For the final part, consider the $C^*$-algebra $C[0,1] \otimes A$. We omit the details. \(\blacksquare\)

§3. The Excision Theorem

Here, we state and prove our main results (Theorems 3.1-3.7). We describe the hypotheses. We suppose that $A$ and $B$ are $C^*$-algebras acting on the Hilbert space $\mathcal{H}$. We also suppose that $z$ is a self-adjoint unitary operator on $\mathcal{H}$. Note that we regard $M_n(A)$ and $M_n(B)$ as acting on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, the $n$-fold direct sum. We also let $z$ denote the operator $z \oplus \cdots \oplus z$ on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. We let $[a, b] = ab - ba$ for any operators $a, b$ on $\mathcal{H}$. 
We will assume conditions 1-6 hold.

1. For all \( a \) in \( A \), \( b \) in \( B \), \( ab \) is in \( A \); i.e. \( B \) acts as multipliers of \( A \).

2. \( zA z = A \).

3. For all \( b \) in \( B \), \( zb z - b \) is in \( A \).

4. There is a continuous path \( \{ e_t \mid t \geq 0 \} \) in \( A \) such that
   
   (i) \( 0 \leq e_t \leq e_s \leq 1 \), for \( t \leq s \),
   
   (ii) \( e_s e_t = e_t \) for \( s \geq t + 2 \),
   
   (iii) for all \( a \) in \( A \),
   
   \[
   \lim_{t \to \infty} \|e_t a - a\| = 0 = \lim_{t \to \infty} \|ae_t - a\|.
   \]

   (iv) \( [e_t, z] = 0 \), for all \( t \).

   We define \( C^*-\)subalgebras

   \[
   A' = \{ a \in A \mid [a, z] = 0 \}
   \]

   \[
   B' = \{ b \in B \mid [b, z] = 0 \}.
   \]

5. For all \( b \) in \( B \), there exists \( b' \) in \( B' \) such that

   \[
   \|b - b'\| \leq 2\|[b, z]\|.
   \]

   (In the terminology of M.-D. Choi, almost commuting with \( z \) implies nearly commuting with \( z \).)

6. There is a dense \( * \)-subalgebra \( \mathcal{A} \subseteq A \) such that for \( a \) in \( A \), there is \( t_0 \geq 1 \) such that

   (i) \( ae_t = e_t a = a \), for all \( t \geq t_0 \),

   and, for any such \( t_0 \) as above, there is \( b \) in \( B \) such that

   (ii) \( be_t = e_t b = a \), \( t_0 \leq t \leq t_0 + 2 \).

   (iii) \( [b, z] = [a, z] \).

   (iv) \( \|b\| \leq \|a\| \).

   (The choice of \( b \) will depend on \( t_0 \) as well as \( a \).)

   Note that the condition on \( A \) analogous to 5 is valid; let \( a' = (a + za)/2 \).

   Many examples are found in the theory of \( C^* \)-algebras associated to dynamical systems via the crossed product or groupoid \( C^* \)-algebra constructions. Let us mention one explicit example.
Fix an irrational number \( \theta, 0 < \theta < 1 \). Let \( \mathcal{H} = \ell^2(\mathbb{Z}) \) and let \( u \) and \( v \) denote the unitary operators
\[
(u\xi)(n) = \xi(n - 1)
\]
\[
(v\xi)(n) = \exp(2\pi i \theta) \xi(n),
\]
for \( \xi \) in \( \ell^2(\mathbb{Z}) \), \( n \) in \( \mathbb{Z} \). Then \( u \) and \( v \) satisfy the relation \( uv = \exp(2\pi i \theta) vu \) and generate a \( \mathcal{C}^* \)-algebra, \( \mathcal{B} \), isomorphic to the irrational rotation \( \mathcal{C}^* \)-algebra, \( A_\theta \). We let \( A = K(\mathcal{H}) \), the compact operators, and
\[
(z\xi)(n) = \begin{cases} 
\xi(n) & n \geq 1 \\
-\xi(n) & n \leq 0.
\end{cases}
\]
It is easy to verify 1, 2 and 3. It is also easy to see that
\[
A' = K(\ell^2\{n \mid n \leq 0\}) \oplus K(\ell^2\{n \mid n \geq 1\}).
\]
The proofs that 4, 5 and 6 hold can be found in [Put2]. Also the techniques of [Put2] show that \( \mathcal{B}' \) is the \( \mathcal{C}^* \)-subalgebra of \( \mathcal{B} \) generated by \( v \) and \( u(v - 1) \). (See example 2.6 of [Put2].)

**Theorem 3.1.** Let \( A, \mathcal{B}, z \) satisfy 1-6 as above. Then there is an isomorphism
\[
\alpha : K_0(\mathcal{C}(\mathcal{B}'; \mathcal{B})) \rightarrow K_0(\mathcal{C}(A', A))
\]
which is natural in a sense to be described.

Let us take a moment to try to justify our description of 3.1 as an “excision” theorem. Section 2 describes the \( K \)-theory of the mapping cone \( \mathcal{C}(A'; A) \) as partial isometries in \( A \) with initial and final projection in \( A' \). The extent to which an element \( a \) lies in \( A' \) can be measured by \( zaz - a = z[a, z] \). A similar remark applies to \( \mathcal{B}' \) and \( \mathcal{B} \). Conditions 2, 3 and 6(iii) essentially mean that the sets
\[
\{zaz - a \mid a \in A\}
\]
\[
\{zbx - b \mid b \in B\}
\]
“agree”. The conclusion is then that the corresponding “relative \( K \)-groups” are isomorphic.

We begin by describing the map \( \alpha \). We use \( e_i \) to also denote the element \( 1_n \otimes e_i \) in \( M_n(A) \), for any \( n = 1, 2, 3, \ldots \). We will use the description of \( K_0(\mathcal{C}(\mathcal{B}'; \mathcal{B})) \) provided by
Lemma 2.5 and the discussion following it. Let \( v \) be in \( V_n^c(B'; B) \). For all \( t \geq 1 \), we define \( \alpha(v)_t \) by
\[
\alpha(v)_t = \begin{bmatrix}
ve_t & 0 \\
(v^*v - e_t v^* v e_t) & 0 \\
\end{bmatrix}^{\frac{1}{2}}
\]
Since \( B \) acts as multipliers of \( A \), \( ve_t \) is in \( M_n(A) \). Also, \( v^*v \) is a projection in \( M_n(\mathbb{C}) \) and it follows that \( \alpha(v)_t \) lies in \( M_{2n}(A^\sim) \). It is also worth noting that \( e_t \) and \( v^*v \) commute so that
\[
(v^*v - e_t v^* v e_t)^{\frac{1}{2}} = v^*v \left( 1 - e_t^2 \right)^{\frac{1}{2}}.
\]
It is easy to check that
\[
\alpha(v)_t^* \alpha(v)_t = v^*v \oplus 0,
\]
which is in \( M_{2n}(\mathbb{C}) \) and is a projection.

**Lemma 3.2.** For \( v \) in \( V_n^c(B'; B) \) and \( 0 < \epsilon < 400^{-1} \), there is \( t \geq 1 \) such that \( \alpha(v)_s \) is in \( V_{2n}^c(A'; A) \) for all \( s \geq t \).

**Proof.** We claim that
\[
\limsup_{t \to \infty} \| [\alpha(v)_t, \alpha(v)_t^*, z] \| \leq \epsilon.
\]
To see this,
\[
\alpha(v)_t \alpha(v)_t^* = \begin{bmatrix}
ve_t^2 v^* & ve_t \left( 1 - e_t^2 \right)^{\frac{1}{2}} \\
(1 - e_t^2)^{\frac{1}{2}} e_t v^* & v^*v \left( 1 - e_t^2 \right)^{\frac{1}{2}} \\
\end{bmatrix}
\]
and we will check the commutators of the four entries with \( z \) separately. The lower right entry actually commutes with \( z \) since \( e_t \) does and \( v^*v \) is in \( M_n(\mathbb{C}) \). As for the upper right (or lower left)
\[
\lim_{t \to \infty} \left[ ve_t \left( 1 - e_t^2 \right)^{\frac{1}{2}}, z \right] = \lim_{t \to \infty} [v, z] e_t \left( 1 - e_t^2 \right)^{\frac{1}{2}} = 0
\]
since \( z[v, z] \) is in \( M_n(A) \) and \( e_t \) is an approximate unit for \( A \). For the upper left entry, we have
\[
\limsup_{t \to \infty} \| [ve_t^2 v^*, z] \| = \limsup_{t \to \infty} \| [v, z] e_t^2 v^* + ve_t^2 [v^*, z] \|.
\]
Since \( z[v, z] \) and \( z[v^*, z] \) are both in \( A \), \( e_t \) will asymptotically commute both, so this equals
\[
\limsup_{t \to \infty} \| e_t^2 [v, z] v^* + v[v^*, z] e_t^2 \|.
\]
Applying the same argument and noting \([v, z] v^*\) is in \(M_n(A)\) since \(v^*\) is in the multiplier algebra of \(M_n(A)\), this equals

\[
\limsup_{t \to \infty} \|([v, z] v^* + v[v^*, z]) e_t^2\| = \limsup_{t \to \infty} \|v v^*, z\| e_t^2\|
\leq \epsilon
\]

since \(v v^*\) is within \(\epsilon\) of an element of \(M_n(A\sim)\). The claim is established.

To see the conclusion, let

\[
q = \frac{z \alpha(v) t \alpha(v)^* z + \alpha(v) t \alpha(v)^*}{2}.
\]

Now, (iii) follows from the claim and it is clear that \(q\) is in \(M_{2n}(A\sim)\).

Notice that

\[
\alpha(v \oplus w)_t = \alpha(v)_t \oplus \alpha(w)_t
\]

(at least after a change of basis which we will suppress). It follows from 3.2 that letting

\[
\alpha(\kappa(v)) = \kappa(\alpha(v)_s),
\]

for any sufficiently large \(s\) defines an element in \(K_0(C(A'; A))\). To see that \(\alpha\) is well-defined it suffices to apply Lemma 2.5 and observe the following. If \(p\) is a projection in \(M_n(C)\) then

\[
\alpha(p)_t = e^{'t}(p \oplus 0),
\]

where \(e'_t\) is as in 2.6. So then \(\kappa(\alpha(p)_t) = 0\) by 2.2(ii), (iii).

Also observe that if \(v(r), 0 \leq r \leq 1\) is a path in \(V_n'(B'; B)\) then the limit in 3.2 can be made uniform over \(r\), and, hence, for \(s\) large \(\alpha(v(r))_s\) will be a homotopy in \(V_{2n}^2(A'; A)\).

The proof of 3.1 will require several technical Lemmas.

**Lemma 3.3.** Let \(w_0 \in M_n(A)\) and \(p\) be a projection in \(M_n(C)\) such that \(p \geq w_0^* w_0\). Then there is \(t_0 \geq 1\) and \(v_0\) in \(M_n(B)\) with \(v_0^* v_0 \leq p\) such that

(i) \(w_0 e_s = e_s w_0 = w_0, \text{ for } s \geq t_0\)

(ii) \(v_0 e_s = e_s v_0 = w_0, \text{ for } t_0 + 2 \geq s \geq t_0\)

(iii) \([v_0, z] = [w_0, z]\)
(iv) \([v_0^*v_0, z] = [w_0^*w_0, z]\)

(v) \([v_0v_0^*, z] = [w_0w_0^*, z]\)

(vi) \([ (p - v_0^*v_0)^{\frac{1}{2}}, z] = [(p - w_0^*w_0)^{\frac{1}{2}}, z].\)

Proof. Choose any \(t_0\) and \(b\) as in hypothesis 6 for \(a = w_0\). Then let \(v_0 = bp\) so \(v_0^*v_0 = p b^*b p = p ||b||^2 p = p\).

Conditions (i), (ii) and (iii) follow at once from hypothesis 6.

We have

\[
[v_0^*v_0, z] = [v_0^*, z] v_0 + v_0^*[v_0, z]
\]

\[
= [w_0^*, z] v_0 + v_0^*[w_0, z]
\]

\[
= [w_0^*, z] v_0 + v_0^*[e_t w_0, z], \quad \text{for } t_0 \leq t \leq t_0 + 2
\]

\[
= [w_0^*, z] e_t v_0 + v_0^* e_t [w_0, z]
\]

\[
= [w_0^*, z] w_0 + w_0^* [w_0, z] \quad \text{by (ii)}
\]

\[
= [w_0^* w_0, z]
\]

and so (iv) holds. A similar argument establishes (v). As for (vi), it follows from (iv) that

\[
[f(p - v_0^*v_0), z] = [f(p - w_0^*w_0), z]
\]

for any polynomial \(f\). By standard approximation arguments, the same holds for \(f(t) = t^{\frac{1}{2}}\). \(\blacksquare\)

Lemma 3.4. Let \(w_0, p, t_0, v_0\) be as in 3.3. Define \(w\) in \(M_{2n}(A^\sim)\) and \(v\) in \(M_{2n}(B^\sim)\) by

\[
w = \begin{bmatrix}
w_0 & 0 \\
(p - w_0^*w_0)^{\frac{1}{2}} & 0
\end{bmatrix}
\]

\[
v = \begin{bmatrix}
v_0 & 0 \\
(p - v_0^*v_0)^{\frac{1}{2}} & 0
\end{bmatrix}
\]

Then

(i) \(w^*w = v^*v = p \oplus 0\),

(ii) \(e_s[v, z] = [v, z] e_s = [v, z] = [w, z]\) for \(s \geq t_0\),
(iii) \( [ww^*, z] = [vv^*, z] \).

The proof is an easy consequence of 3.3; we omit the details.

**Lemma 3.5.** Let \( w_0 \) be in \( M_n(A) \), \( p \) a projection in \( M_n(C) \) with \( p \geq w_0^* w_0 \). Let \( t_0, v_0 \) be as in 3.3, \( w, v \) as in 3.4 and assume \( w \) is in \( V_{2n}^\epsilon(A'; A) \) for some \( 0 < \epsilon < 400^{-1} \). Then

(i) \( v \) is in \( V_{2n}^{4\epsilon}(B'; B) \),

(ii) \( \alpha(v)_s \) is in \( V_{2n}^{4\epsilon}(A'; A) \), for all \( s \geq t_0 \),

(iii) \( \kappa(\alpha(v)_s) = \kappa(w) \), for \( t_0 \leq s \leq t_0 + 2 \).

**Proof.**

(i) From 3.4(i), \( v^* v = p \oplus 0 \) and we must check only that \( vv^* \) is close to an element of \( M_{2n}(B'\sim) \). From 3.4(iii)

\[
[[vv^*, z]] = [[ww^*, z]] \leq 2\epsilon
\]

since \( w \) is in \( V_{2n}^\epsilon(A'; A) \). Apply hypothesis 5 to find \( q \) in \( M_{2n}(B'\sim) \) so that \( \|q - vv^*\| \leq 4\epsilon \), and (i) is complete.

(ii) As before, we must compute

\[
[[\alpha(v)_s, \alpha(v)_s^*], z]].
\]

Now, for \( s \geq t_0 \),

\[
\alpha(v)_s, \alpha(v)_s^* = \begin{bmatrix}
ve_2^2 v^* & ve_2 (1 - e_2^2)^{1/2} v^* \\
v^* v (1 - e_2^2)^{1/2} e_2^* v^* & v^* v (1 - e_2^2)
\end{bmatrix}
\]

and commutators with \( z \) for each of the entries is done separately. The off-diagonal entries commute with \( z \) because \( v^* v = p \) and by condition (ii) of 3.4, so \( (1 - e_2)[v, z] = 0 \).

The lower right entry also commutes with \( z \) while

\[
[ve_2^2 v^*, z] = [ww^*, z] \quad \text{for} \quad s \geq t_0.
\]

This completes the proof of (ii).
(iii) By direct computation

\[ a(v) = \begin{bmatrix}
  v_0 e_s & 0 & 0 & 0 \\
  (p - v_0^*v_0)^{\frac{1}{2}} e_s & 0 & 0 & 0 \\
  p(1 - e_s^2)^{\frac{1}{2}} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \]

\[ = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & e_s & -(1 - e_s^2)^{\frac{1}{2}} & 0 \\
  0 & (1 - e_s^2)^{\frac{1}{2}} & e_s & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  w_0 & 0 & 0 & 0 \\
  (p - w_0^*w_0)^{\frac{1}{2}} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \]

for \( t_0 \leq s \leq t_0 + 2 \), using Lemma 3.2. The first matrix above is in \( M_{4n} \) \((A'\sim)\) and so the result follows from 2.2(iii).  

**Lemma 3.6** Suppose \( v \) is in \( V_n(B'; B) \) and \( \| [v, z] \| \leq \epsilon \leq 10^{-6} \). Then \( \kappa(v) = 0 \).

**Proof.** By hypothesis 5, there is a \( v' \) in \( M_n(B'\sim) \) such that \( \| v' \| \leq 1 \) and \( \| v - v' \| \leq 2\epsilon \).

Let

\[ w = \begin{bmatrix}
  v'p & 0 \\
  (p - pv'^*v'p)^{\frac{1}{2}} & 0
\end{bmatrix}, \]

where \( p = v^*v \), so \( w \) is in \( V_{2n}(B'; B) \) and in \( M_{2n}(B'\sim) \) and

\[ \| v \oplus 0 - w \| \leq 4\epsilon^{\frac{1}{2}}. \]

Moreover, \( \kappa(w) = 0 \) by 2.2(v) and \( \kappa(v) = \kappa(w) \) by 2.2(ii). 

Let us describe the naturality of the isomorphism described in 3.1. Suppose \( (A_1, B_1, z_1, \{ e_t^{(1)} \}) \) and \( (A_2, B_2, z_2, \{ e_t^{(2)} \}) \) are two systems satisfying 1-6. Also suppose

\[ \sigma : A_1 \longrightarrow A_2 \]

\[ \pi : B_1 \longrightarrow B_2 \]
a $*$-homomorphisms such that

\[ \sigma(ab) = \sigma(a)\pi(b), \quad a \in A_1, \ b \in B_1 \]
\[ \sigma(z_1 az_1) = z_2 \sigma(a) z_2, \quad a \in A_1 \]
\[ \pi(z_1 bz_1) = z_2 \pi(b) z_2, \quad b \in B_1 \]
\[ \sigma(z_1 bz_1 - b) = z_2 \pi(b) z_2 - \pi(b), \quad b \in B_1 \]
\[ \sigma(e_t^{(1)}) = e_t^{(2)} , \quad \text{for all} \ t. \]

It is easy to see that $\sigma$ and $\pi$ induce $*$-homomorphisms

\[ \tilde{\sigma} : C(A_1'; A_1) \longrightarrow C(A_2'; A_2) \]
\[ \tilde{\pi} : C(B_1'; B_1) \longrightarrow C(B_2'; B_2). \]

The map $\alpha$ is natural in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
K_0(C(B_1'; B_1)) & \xrightarrow{\alpha} & K_0(C(A_1'; A_1)) \\
\downarrow \tilde{\pi}_* & & \downarrow \tilde{\sigma}_* \\
K_0(C(B_2'; B_2)) & \xrightarrow{\alpha} & K_0(C(A_2'; A_2))
\end{array}
\]

The proof of this is immediate. We omit the details.

As an application, suppose $(A, B, z, e_t)$ satisfies 1-6 and suppose $X$ is a compact second countable Hausdorff space. Fix some regular Borel measure $\mu$ on $X$ with full support. Then we can regard $A \otimes C(X), B \otimes C(X)$ and $z \otimes 1$ as operators on $\mathcal{H} \otimes L^2(X, \mu)$. Hypotheses 1-3 are easily checked and $e_t \otimes 1$ satisfies 4. We also have

\[ (A \otimes C(X))' = A' \otimes C(X) \]
\[ (B \otimes C(X))' = B' \otimes C(X) \]

and 5 follows. The algebraic tensor produce of $\mathcal{A}$ and $C(X)$ can be seen to satisfy 6.

**Proof of 3.1.** First of all, it is fairly clear that $\alpha$ is additive. The surjectivity of $\alpha$ follows at once from Lemmas 2.6 and 3.5.
Suppose \( v \) is in \( V_n(B'; B) \) and \( \alpha(\kappa(v)) = 0 \) in \( K_0(C(A'; A)) \). Let \( p = v^*v \) which is a projection in \( M_n(C) \). Fix \( \varepsilon = 10^{-7} \). Choose \( t_1 \geq 1 \) such that

\[
\|[v, z]e_t - [v, z]\| \leq \varepsilon
\]

\[
\|[v, z] - [v, z]e_t\| \leq \varepsilon, \quad t \geq t_1
\]

and such that

\[
\alpha(v)_t \in V_{2n}^\varepsilon(A'; A), \quad t \geq t_1.
\]

Since \( \kappa(\alpha(v)) = 0 \), we may direct sum \( \alpha(v)_{t_1} \) with a scalar projection \( q \) so that the result is homotopic to a scalar projection in \( V^\varepsilon(A'; A) \). By replacing \( v \) by \( v \oplus q \), we may assume simply that \( \alpha(v)_{t_1} \) is homotopic to \( \begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix} \), which is homotopic to \( p \oplus 0 \). We apply Lemma 2.6 to obtain a path as described there. We may then approximate the "\( w_0 \)" part of this path by a path in \( M_n(A) \). We right multiply this path by \( p \) and we obtain a path \( a(s), 0 \leq s \leq 1 \), such that \( a \) is in the algebraic tensor product of \( C[0, 1] \) and \( M_n(A) \),

\[
w(s) = \begin{bmatrix} a(s) \\ (p - a(s)^*a(s))^\frac{1}{2} \end{bmatrix}, \quad 0 \leq s \leq 1,
\]

\[
in V_{2n}^{2\varepsilon}(A'; A)
\]

\[
a(1) = 0
\]

\[
\|[w(0) - \alpha(v)_{t_1}]\| \leq 2\varepsilon,
\]

hence,

\[
\|[a(0) - ve_{t_1}]\| \leq 2\varepsilon,
\]

\[
\left\|\left( p - a(0)^*a(0) \right)^{\frac{1}{2}} - p \left(1 - e_{t_1}^2 \right)^{\frac{1}{2}} \right\| \leq 2\varepsilon.
\]

We may apply the sequence of Lemmas 3.3, 3.4 and 3.5 to the element \( a \) in \( M_n(C[0, 1] \otimes A) \) (algebraic tensor product) and \( p \) in \( M_n(C) \) to obtain a path \( b(s), 0 \leq s \leq 1 \)

\[
v_1(s) = \begin{bmatrix} b(s) \\ (p - b(s)^*b(s))^\frac{1}{2} \end{bmatrix}
\]
0 ≤ s ≤ 1 and \( t_2 ≥ t_1 + 2 \) such that

\[(6) \quad [b(s), z] = [a(s), z],\]

\[(7) \quad b(s) e_t = e_t b(s), \quad t_2 ≤ t ≤ t_2 + 2,\]

\[(8) \quad a(s) e_t = e_t a(s) = a(s), \quad t ≥ t_2,\]

\[(9) \quad [b(s)^* b(s), z] = [a(s)^* a(s), z]\]

\[(10) \quad [b(s) b(s)^*, z] = [b(s) b(s)^*, z]\]

\[(11) \quad \left[ (p - b(s)^* b(s))^{\frac{1}{2}}, z \right] = \left[ (p - a(s)^* a(s))^{\frac{1}{2}}, z \right],
\]

\[v_1(s) \text{ is in } V_{2n}^{4\epsilon}(B'; B)\]

\[\alpha(v_1(s))_t \text{ is in } V_{4n}^{4\epsilon}(A'; A), \quad t ≥ t_2.\]

Let us evaluate \( v_1 \) at \( s = 1 \). Making use of (4), (6) and (9), we see that

\[(12) \quad [v_1(1), z] = 0\]

and so \( v_1(1) \) is in \( M_n(B'^\sim) \). Next, we claim that

\[(13) \quad \|[v b(0)^*, z]\| ≤ 3\epsilon,\]

\[(14) \quad \|[v (p - b(0)^* b(0))^{\frac{1}{2}}, z]\| ≤ 3\epsilon.\]

To see the first, we have

\[\|[v b(0)^*, z]\| = \|[v, z] b(0)^* + v [b(0)^*, z]\|
\]

\[≤ \|[v, z] e_{t_1} b(0)^* + v [a(0)^*, z]\| + \epsilon\]

by (1) and (6),

\[≤ \|[v, z] e_{t_1} e_{t_2} b(0)^* + v [e_{t_1} v^*, z]\| + \epsilon\]
by hypothesis 4(ii) and (5),

\[ = \| [v, z] e_{\eta_1} a(0)^* + v e_{\eta_1} [v^*, z] \| + \epsilon \]

by (7)

\[ \leq \| [v, z] e_{\eta_1}^2 v^* + v e_{\eta_1}^2 [v^*, z] \| + 2\epsilon \]

by (5) and (1)

\[ = \| [v e_{\eta_1}^2 v^*, z] \| + 2\epsilon \]

\[ \leq 3\epsilon \]

because of (2). To see the second, there is a similar computation which we omit.

Now consider

\[ v_2(s) = (v \oplus 0) v_1(s)^*, \quad 0 \leq s \leq 1. \]

This is a path of partial isometries in \( M_{2n}(B^\sim) \). For each \( s \), its range projection is the range projection of \( v \) which is in \( M_{2n}(B'^\sim) \). Its initial projection is the range projection of \( v_1(s) \) which is in \( M_{2n}(B'^\sim) \), for all \( s \). As noted in (12), when \( s = 1 \), this projection is actually Murray-von Neumann equivalent to \( p \oplus 0 \) in \( M_{2n}(B'^\sim) \). So we may find a path of unitaries \( u(s), 0 \leq s \leq 1 \) in \( M_{2n}(B'^\sim) \) (actually, it may be necessary to pass to \( M_{4n}(B'^\sim) \)) such that

\[ v_1(1)^* u(1) = p \oplus 0 \]

\[ v_1(s)^* u(s) \text{ has initial projection } p \oplus 0, \]

\[ 0 \leq s \leq 1. \]

Now, consider the path

\[ v_3(s) = (v \oplus 0) v_1(s)^* u(s), \quad 0 \leq s \leq 1. \]

It is a path in \( V_{2n}(B'; B) \). Moreover, for \( s = 1 \),

\[ v_3(1) = v \oplus 0 \]

while for \( s = 0 \),

\[ v_3(0) = \begin{bmatrix} v b(0)^* & v (p - b(0)^* b(0))^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} u(0) \]
which commutes with $z$, to within $3\epsilon$, by (13) and (14). By Lemma 2.2(v) and the homotopy invariance of $\kappa$,

$$\kappa(v) = \kappa(v_3(1)) = \kappa(v_3(0)) = 0.$$  

This proves that $\alpha$ is injective and we are done. 

**Theorem 3.7.** Let $A, B, z$ satisfy 1-6 as before. Then there are isomorphisms

$$\alpha : K_i(C(B';B)) \rightarrow K_i(C(A';A)),$$

which are natural, for $i = 0, 1$.

**Proof.** The case $i = 0$ is done. For the other case, let $B_1 = C(S^1) \otimes B$, $A_1 = C(S^1) \otimes A$, $z_1 = 1 \otimes z$ and $\sigma : A_1 \rightarrow A$, $\pi : B_1 \rightarrow B$ be given by evaluation at some fixed point of the circle, $S^1$. There is a split exact sequence

$$0 \rightarrow C_0(0,1) \otimes C(B';B) \rightarrow C(B_1';B_1) \rightarrow C(B';B) \rightarrow 0$$

and a corresponding one for $A$ and $A_1$. Using the naturality of $\alpha$ on $K_0$ and the usual isomorphism

$$K_1(C(B';B)) \cong K_0(C_0(0,1) \otimes C(B';B))$$

and the usual techniques, one obtains the result for $K_1$ groups as well. 

**References**


