QUALITATIVE STABILITY AND SOLVABILITY OF DIFFERENCE EQUATIONS

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ABSTRACT

We develop sufficient conditions for qualitative stability and solvability of the real discrete time system $x_{t+1} = Ax_t + b$. These conditions are a combination of qualitative and quantitative criteria and depend on signed digraphs.

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1. INTRODUCTION

Let x_t denote the state of a real n–dimensional $(n \geq 2)$ system at time t. We suppose that time is measured by the natural numbers and that the system evolves iteratively according to a finite difference equation

$$x_{t+1} = Ax_t + b \tag{1.1}$$

where A is a real $n \times n$ matrix and b a real n–vector.

In the context of many mathematical models, (for example, ecosystem, chemical network and economic models), the state variables in (1.1) have meaning only with nonnegative components. In particular, special significance can often be associated with versions of (1.1) which admit as a constant attractor trajectory a point $\hat{x}$ in the positive orthant $\mathbb{R}^n_+$ in n–space. The purpose of this article is to present sufficient conditions on the signs and magnitudes of entries in A and b which guarantee that all trajectories of (1.1) asymptotically approach some $\hat{x} \in \mathbb{R}^n_+$. Thus the system (1.1) is required to have both solvability and stability properties.

In qualitative matrix analysis equivalence classes of matrices are generally defined in terms of a partition of the real numbers. That is, suppose $C_1, C_2, \ldots, C_P$ are P nonempty, disjoint subsets of $\mathbb{R}^1$ such that $\bigcup_{p=1}^{P} C_p = \mathbb{R}^1$. With respect to this *partition* $\{C_p\}$, matrix A is *equivalent* to matrix B if every entry a_{ij} in A is in the same equivalence class C_p as the corresponding entry b_{ij} in B.

For example, if the partition is $\{\{0\}, \{\text{negative real numbers}\}, \{\text{positive real numbers}\}\}$, then the associated equivalence relation is called *sign equivalence*, see e.g. [JKV]. By contrast, the *modulus equivalence* (*m-equivalence*) of [KB] uses the partition $C_1 = \{x: 0 < |x| < 1\}$, $C_2 = \{x: |x| > 1\}$, $C_3 = \{\pm 1\}$, and $C_4 = \{0\}$.

Many types of mathematical models are linear, either differential equation systems $\dot{x} = Ax + b$, or difference equation systems (1.1). Often the specification of the matrix A and the forcing vector b is uncertain; only signs and rough magnitudes of entries might be known. Hence it is important to determine patterns of organization of linear models, that is, equivalence classes, with definite stability and solvability features.

For a discrete time system (1.1), asymptotic stability is equivalent to each eigenvalue of the matrix A having modulus less than one. Discrete qualitative stability is characterized by [KB] in terms of the above four modulus equivalence classes (see section 3). Thus discrete qualitative stability is more restrictive than the analogous sign stability for the continuous case (each eigenvalue having negative real part) which is characterized in [JKV]. Another related result [DPT, Appendix D] for (1.1) in the case with $b > 0$ and A nonnegative is that the existence of a positive constant trajectory $\hat{x} \in R^n_+$ automatically implies the asymptotic stability of this equilibrium and conversely. Qualitative stability and solvability of the continuous time model, with the system evolving according to the differential equation $\dot{x} = Ax + b$, have been studied using sign equivalence in [BJK]. A characterization in terms of digraphs is given for the existence of an asymptotically stable constant trajectory $\hat{x} \in \mathbb{R}^n_+$.

For our purpose with the discrete time system (1.1), we need to use both sign equivalence and modulus equivalence classes, and a combination of qualitative and quantitative criteria. We first present relevant digraph ideas in section 2; section 3 deals with stability, section 4 with solvability, and we conclude with our main result in section 5.

2. SIGNED DIGRAPHS

We proceed by describing a class of patterns of organization in the system (1.1), specifically the signed digraph SD(A) of an $n \times n$ matrix A. This digraph consists of n labelled vertices and some signed, directed edges, one edge for each nonzero entry in $A = [a_{ij}]$. If the matrix entry $a_{ij} \neq 0$, then the digraph has an edge directed from vertex j to vertex i and signed as the sign of a_{ij}. Note that the direction of the edge associated with nonzero a_{ij} is from j to i, not the reverse. This convention is more suggestive of causal relations or flows in modeling, even though the reverse is usually used in the graph theory literature.

As an example with $n = 3$ we associate with matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & .9 & 0 \end{bmatrix}, \tag{2.1}$$

the signed digraph SD(A) in Figure 1.

Figure 1

Let $\{i_1, i_2, \ldots, i_k\}$, $k \geq 2$, be a subset of $\{1,2,\ldots,n\}$. If $a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_k i_{k-1}} \neq 0$, then we say a *k-path* exists in the digraph from vertex i_1 to vertex i_k with *path product* $a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_k i_{k-1}}$. Given such a path and $a_{i_1 i_k} \neq 0$, we say a *k-cycle*, $k \geq 2$, exists in the digraph with *cycle product* $a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_k i_{k-1}} a_{i_1 i_k}$. Thus paths and cycles are always simple and of finite length. We also associate with any $a_{ii} \neq 0$ a 1-cycle. We denote the identity matrix by I, so the signed digraph SD(A−I) for the matrix A in (2.1) is the signed digraph SD(A) in Figure 1 with the addition of a 1–cycle (with cycle product -1) at each of the three vertices.

A *strongly connected component* of a digraph is a certain maximal subset of vertices and edges, namely a maximal subgraph with the property that between any two distinct vertices i_1, i_k there is a path from i_1 to i_k; and matrix A is *irreducible* iff SD(A) is strongly connected. (We think of paths and cycles as either lists of matrix entries or the associated vertices with connecting edges.) A *strong unipathic digraph* is a strongly connected digraph in which each edge is in exactly one k cycle, $k \geq 2$. For example, SD(A) in Figure 1 is a strong unipathic digraph.

3. STABILITY

First we suppose a constant trajectory $\hat{x} \in \mathbb{R}^n_+$ exists for (1.1). Suppose x_t is some other trajectory and define $y_t = x_t - \hat{x}$. It follows that $y_{t+1} = x_{t+1} - \hat{x} = Ax_t + b - \hat{x} = Ax_t + b - A\hat{x} - b = A(x_t - \hat{x})$, that is,

$$y_{t+1} = Ay_t. \tag{3.1}$$

It is well known that all trajectories of (3.1) converge to the origin of $\mathbb{R}^n$ (so x_t converges to $\hat{x}$) iff all the eigenvalues of A have modulus less than 1. Thus the stability of (1.1) concerns only A, not b.

LEMMA 3.1. *Suppose* $A = [a_{ij}]$ *is a matrix with signed digraph* SD(A) *which has no 1–cycles and* $\sum_{j=1}^{n} |a_{ij}| < 1$ *for each* i. *Then each eigenvalue of* A *has modulus less than* 1.

Proof. By Geršgorin's Theorem [see, e.g. H.J., p. 344], all eigenvalues of A lie in the n discs centered at the origin of $\mathscr{C}$ (as $a_{ii} = 0$ for all i), and having radii $\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| < 1$. ∎

Note that in the inequality in lemma 3.1 the roles of all i and j can be interchanged, giving radii in terms of column sums rather than row sums as in the lemma. If A is irreducible then we can improve on lemma 3.1.

LEMMA 3.2. *Suppose* $A = [a_{ij}]$ *is an irreducible matrix with signed digraph* SD(A) *which has no 1-cycles and* $\sum_{j=1}^{n} |a_{ij}| \le 1$ *for each* i, *with strict inequality for at least one* i. *Then each eigenvalue of* A *has modulus less than* 1.

Proof. As A is irreducible, if λ is an eigenvalue of A on the boundary of the union of the n discs, then *every* Geršgorin circle passes through λ (see, e.g. [HJ; p. 356]). So if there is such a λ with $|\lambda| = 1$, the every row sum must equal 1. The conditions of the lemma exclude this, thus every eigenvalue lies strictly inside the unit circle. ■

In [KB], two matrices are called *m-equivalent* if each corresponding entry belongs to the same m-equivalence class C_p, $p = 1, \ldots, 4$, given in the introduction and a matrix is called *m-stable* if all matrices that are m-equivalent to it are asymptotically stable. The main result in [KB] is that an irreducible matrix of size ≥ 2 is m-stable iff each entry in the matrix has modulus less than or equal to 1 and there is only one cycle in the digraph of the matrix, this cycle having length n and cycle product less than 1. If SD(A) consists of only one cycle of length n then the inequality in lemma 3.2 becomes $|a_{ij}| \le 1$ for all i, j with strict inequality for at least one i. Such a matrix is m-stable by the theorem of [KB]; for example, the matrix given in (2.1) is m-stable.

4. SOLVABILITY

If (1.1) has a constant trajectory $\hat{x}$, then it satisfies the equation

$$(A-I)x + b = 0. \tag{4.1}$$

Thus solvability of the discrete system concerns the matrix $(A-I)$ and vector b.

We assume in all that follows that $a_{ii} = 0$ for all i, so $(A-I)$ has a negative 1-cycle with product -1 at each vertex. Following previous work on sign solvability, for example, [BMQ, BJK, Man, May], we also assume that A is irreducible and only one component of b is nonzero. Without loss of generality we take $b_1 \neq 0$.

We define (4.1) to be *positively sign solvable* if given A, b with $a_{ii} = 0$ for all i, $b_1 > 0$ and $b_i = 0$ for all $i \neq 1$, then equation (4.1) admits a strictly positive solution $\hat{x}$ (that is, each $\hat{x}_i > 0$) for every pair in the same sign equivalence class as A, b. Note that our edge directions and assumptions differ from the standard forms used in the references above.

Given A, we thus know SD(A–I), and if there is a path in this digraph from i to j, we say that $j > i$. This defines a poset relation. We define a *poset digraph* (partially ordered set digraph) as a connected digraph having a "–" 1-cycle at each vertex, no other cycles, no other "–" edge, a root vertex 1, and "+" edges so that to any vertex $j \neq 1$ is at least one path from 1.

Note that we have labelled the root vertex as 1 (we retain this subsequently), and that every path product from this root is positive. We call a path from one vertex to another in a poset digraph (consisting entirely of "+" edges) a *poset path*. Consider the set of poset paths from the root vertex 1 to a fixed vertex j. We define the *level* of vertex j from 1 to be the maximum of the number of edges in this set of paths.

LEMMA 4.1. *If* SD(A–I) *is a poset digraph, if* $b_1 > 0$, *and if all other* $b_i = 0$, *then* $(A-I)x + b = 0$ *has a unique solution* $\hat{x}$ *with each* $\hat{x}_i > 0$.

Proof. Given a vertex j in SD(A–I), let us consider all the paths in SD(A–I) from 1 to j. Suppose j has level 0, that is, suppose j is 1. The corresponding row equation (4.1) is $-\hat{x}_1 + b_1 = 0$, so $\hat{x}_1$ is positive. Due to the absence of k–cycles, $k \geq 2$, and the assumption $1 < n < \infty$, there must exist at least one vertex j of level 1. The corresponding row equation in (4.1) is $-\hat{x}_j + a_{j1}\hat{x}_1 = 0$, so $\hat{x}_j$ has the same sign as $\hat{x}_1$, that is, $\hat{x}_j > 0$. This procedure can be iterated up the levels until all entries in $\hat{x}$ are expressed as positive multiples of $\hat{x}_1$, that is, all entries in $\hat{x}$ are positive and unique. ■

To every vertex in a poset digraph there exists at least 1 positive and at least 1 negative edge (the negative edge is a 1–cycle, and b_1 is thought of as the positive edge to the root vertex).

Starting with a poset digraph, we can add "–" edges to form a *same sign branch* according to the following rule:

(ssb) if $j \neq 1$ is a vertex, consider all the poset paths from 1 to j and then the set of all vertices common to all such paths. A *same sign branch* can include a "–" edge from j to any such common vertex.

Note that the vertex 1 is common to all such poset paths. The addition of some "−" edges to a poset digraph is illustrated below in Figure 2. Each vertex also has a negative 1–cycle.

Figure 2

As a special case we could start with a (directed) tree as poset digraph. We could add "−" edges until each "+" edge is in a "+, −" 2–cycle. This special case (with different assumptions on the 1–cycles) was called a same sign branch in [BJK].

We now develop some properties of our same sign branches.

LEMMA 4.2. *Every cycle in a same sign branch has exactly one negative edge.*

Proof. Of course, all 1–cycle products are negative. By the construction of same sign branches from "+" edge posets, no cycle could consist entirely of "+" edges. Suppose a k–cycle, $k \geq 2$, exists in a same sign branch with at least two "−" edges. Chose a "−" edge so that no other "−" edge involves a vertex in some poset path with higher level. Denote the vertices of this edge by i and j, so it is directed from i to j. The remainder of the k–cycle cannot consist of a poset path from j to i because all edges in such a path would be "+". Since the k–vertices in the k–cycle must be unique, the next "−" edge must be directed to a vertex below j (in level). Any other "−" edges must be directed to vertices even lower. To complete the cycle, we must use edges back up to i. But (ssb) implies any such path would necessarily include vertex j. Thus no such (simple) k–cycle exists. ∎

This lemma shows that a same sign branch has the property that every cycle product is negative. Certain families of digraphs with this property have been identified in the literature, see [HLM, Ch]. In particular, [HLM] show that every (properly signed) strong unipathic digraph has this property.

LEMMA 4.3. *If* SD(A–I) *is a same sign branch, then the sign of* det(A–I) *is* $(-1)^n$ *and consequently* A–I *is nonsingular.*

Proof. By the previous lemma 4.2 every cycle product is negative. A well known formula for the determinant (see, e.g. [C, May et al]) uses a sum of products of cycle products taken over all decompositions of the digraph of the matrix into complementary cycles. Namely,

$$\det(A-I) = (-1)^n \sum (-1)^P \mathscr{C}_1 \mathscr{C}_2 \cdots \mathscr{C}_P$$

where $\left\{\mathscr{C}_i\right\}$ are cycle products of P complementary cycles. If SD(A–I) is a same sign branch, then each $\mathscr{C}_i$ is negative. Furthermore there is at least one summand in this expansion, namely that corresponding to the decomposition of SD(A–I) into n negative 1–cycles. Thus the sign of det(A–I) is that of $(-1)^n$. ∎

This lemma shows that if SD(A–I) is a same sign branch, then every matrix which is sign equivalent to (A–I) is nonsingular. Thus (A–I) is sign nonsingular (see, e.g. [KLM, Man]).

According to [D], every term in the expansion of a cofactor of a matrix is a path product multiplied by certain cycle products (see also [May, May et al]). We use this expansion to prove our result on sign solvability.

LEMMA 4.4. *If* SD(A–I) *is a same sign branch, if* $b_1 > 0$, *and if all other* $b_i = 0$, *then* $(A-I)x + b = 0$ *is positively sign solvable.*

Proof. By lemmas 4.2, 4.3, the sign of $\det(A-I)$ is $(-1)^n$ and every cycle product is negative. Thus in $(A-I)x + b = 0$, for $i \neq 1$,

$$x_i = \frac{1}{\det(A-I)} \sum_{j=1}^{n} \operatorname{cof}(A-I)^T_{ij}(-b_j), \tag{4.2}$$

so x_i has the sign of $(-1)^{n+1} \operatorname{cof}(A-I)_{1i}\, b_1$,

where $\operatorname{cof}(A-I)_{1i}$ is $(-1)^{1+i}$ times the determinant of the matrix obtained from A–I by deleting row 1, col i.

From [May, May et al], this cofactor for $i \neq 1$ is given by

$$\operatorname{cof}(A-I)_{1i} = \sum (-1)^{\ell_k} \mathcal{P} \det_{\mathcal{P}}(A-I)$$

where the sum is over all paths in SD(A–I) from 1 to i, ℓ_k is the length of such a path, $\mathcal{P}$ is the (positive) path product and $\det_{\mathcal{P}}(A-I)$ is the principal minor of A–I on vertices *not* on the path. So each cofactor has sign $(-1)^{\ell_k}(-1)^{n-\ell_k-1}$, that is $(-1)^{n-1}$. For $i = 1$, $\operatorname{cof}(A-I)_{11}$ is the determinant of the matrix obtained from (A–I) by deleting row 1 and column 1, which also has sign $(-1)^{n-1}$. Thus (4.2) implies x_i is positive for all i. ∎

We now connect our same sign branch constructions with the characterization of positively sign solvable systems given by [Man] for one strong component. (We remark again that our edge directions and assumptions differ from the standard form in [Man].) Specifically we show that if even one path product from vertex 1 is negative, then sign solvability is lost.

LEMMA 4.5. *Suppose* A *is an* n×n *matrix,* $a_{ii} = 0$ *for all* i, $b_1 > 0$, *and* $b_i = 0$ *for all* $i \neq 1$. *Suppose for any vertex* $j \neq 1$ *there exists a path in* SD(A–I) *from* 1 *to* j, *and suppose one such path contains a* "–" *edge. Then* $(A-I)x + b = 0$ *is not positively sign solvable.*

Proof. Select $b_1 = 1$ and $\tilde{A}$ as follows. First select a path of the above type of minimal length (having exactly one "–" edge). Let 1, $j_2, j_3, \ldots, j_p$ be the vertices in the path. Let $\tilde{a}_{j_2 1} = \tilde{a}_{j_3 j_2} = \ldots = \tilde{a}_{j_{p-1} j_{p-2}} = 1$ and $\tilde{a}_{j_p j_{p-1}} = -1$. Let all other entries in $\tilde{A}$ be zero (so $\tilde{A}$ and A are not necessarily sign equivalent). Then $(\tilde{A}-I)x + b = 0$ gives

$$-x_1 + 1 = 0, \quad -x_{j_2} + x_1 = 0, \quad -x_{j_3} + x_{j_2} = 0, \quad \ldots,$$
$$-x_{j_{p-2}} + x_{j_{p-1}} = 0, \quad -x_{j_p} - x_{j_{p-1}} = 0$$

and $-x_k = 0$ for other components. These are solved by $x_1 = \ldots = x_{j_{p-1}} = 1$, $x_{j_p} = -1$, other $x_k = 0$. Clearly $\tilde{A}-I$ is invertible. Thus allowing $\bar{A}$ with other small magnitude entries (near $\tilde{A}$ in $\mathbb{R}^{n^2}$ but sign equivalent to A) leads to a solution of $(\bar{A}-I)x + b = 0$ with x_{j_p} negative. Thus $(A-I)x + b$ is not positively sign solvable. ■

If we start with a same sign branch (even a poset digraph) and add one or more "–" edges which do *not* fulfill (ssb), then we create a path with a negative edge and, by the above lemma, lose positive sign solvability.

5. MAIN THEOREM

We now have a specification of some sufficient conditions for qualitative stability and solvability of (1.1).

THEOREM 5.1. *Let* A *be an* $n \times n$ *real matrix and* b *a real vector. Then the equation* $x_{t+1} = Ax_t + b$ *always has a constant trajectory* $\hat{x}$ *in the positive orthant and every trajectory asymptotically approaches* $\hat{x}$ *if* A *and* b *fulfill the following conditions*:

(a) $a_{ii} = 0$ *and* $\sum_{j=1}^{n} |a_{ij}| \leq 1$ *for all* $i = 1, \ldots, n$, *with strict inequality for at least one* i.

(b) $b_1 > 0$, $b_i = 0$ *for all* $i = 2, \ldots, n$,

and

(c) SD(A–I) *is a same sign branch.*

Proof. The positive sign solvability follows from lemma 4.4, and the stability from lemma 3.2. ∎

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