CERTAIN SUBCLASSES OF FUNCTIONS OF POSITIVE REAL PART WITH NEGATIVE COEFFICIENTS

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CERTAIN SUBCLASSES OF FUNCTIONS OF POSITIVE REAL
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ABSTRACT

The object of the present paper is to derive several useful properties of the class $\mathcal{P}(\alpha)$ which is related to the class $\mathcal{Q}(\alpha)$ defined earlier by H. Silverman and M. Ziegler [Houston J. Math. 4(1978), 269-275]. Relationships between $\mathcal{P}(\alpha)$ and various other classes including $\mathcal{Q}(\alpha)$, and some results for a modified convolution product of functions belonging to the class $\mathcal{P}(\alpha)$ are presented. Finally, a certain functional $\mathcal{J}(q)$ of functions $q(z)$ in $\mathcal{P}(\alpha)$ is considered.

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1. INTRODUCTION

Let \( Q(\alpha) \) denote the class of functions of the form

\[
q(z) = 1 - \sum_{n=1}^{\infty} b_n z^n \quad (b_n \geq 0)
\]

which are analytic in the unit disk \( \mathbb{U} = \{z : |z| < 1 \} \) satisfying the condition

\[
|q(z) - 1| \leq 1 - \alpha \quad (z \in \mathbb{U})
\]

for some \( \alpha \ (0 \leq \alpha < 1) \). The class \( Q(\alpha) \) was introduced by Silverman and Ziegler [9]. A function \( q(z) \) of the form (1.1) is said to be in the class \( \mathcal{R}(\alpha) \) if and only if

\[
zq'(z) + 1 \in Q(\alpha).
\]

In the present paper, we prove several interesting results for functions belonging to the class \( \mathcal{R}(\alpha) \).

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the unit disk \( \mathbb{U} \). Also let \( \mathcal{S} \) denote the subclass of \( \mathcal{A} \) consisting of analytic and univalent functions in the unit disk \( \mathbb{U} \). Then a function \( f(z) \) in \( \mathcal{S} \) is said to be starlike of order \( \alpha \ (0 \leq \alpha < 1) \) if and only if

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})
\]
for some $\alpha$ ($0 \leq \alpha < 1$). We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order $\alpha$ in the unit disk $\mathbb{U}$.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be convex of order $\alpha$ ($0 \leq \alpha < 1$) if and only if

$$\text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha$ ($0 \leq \alpha < 1$). We denote by $\mathcal{K}(\alpha)$ the class of all convex functions of order $\alpha$ in the unit disk $\mathbb{U}$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were first introduced by Robertson [4], and were studied subsequently by Schich [5], MacGregor [2], Pinchuk [3], Jack [1], and others.

Let $\mathcal{I}$ denote the subclass of $\mathcal{S}$ consisting of functions whose nonzero coefficients, from the second one on, are negative. Thus an analytic and univalent function $f(z)$ is in the class $\mathcal{I}$ if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

We denote by $\mathcal{I}^*(\alpha)$ and $\mathcal{G}(\alpha)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ with $\mathcal{I}$; that is,

$$\mathcal{I}^*(\alpha) = \mathcal{I} \cap \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{G}(\alpha) = \mathcal{I} \cap \mathcal{K}(\alpha).$$

The classes $\mathcal{I}^*(\alpha)$ and $\mathcal{G}(\alpha)$ were studied by Silverman [8]. Schich [6] considered a subclass of $\mathcal{I}$ consisting of polynomials having $|z| = 1$ as the radius of univalence, Silverman [8] proved coefficient inequalities, distortion theorems, and covering theorems for $\mathcal{I}^*(\alpha)$ and $\mathcal{G}(\alpha)$, and Schich and Silverman [7] gave some interesting results for the convolution product of functions in
the classes \( \mathcal{J}^*(\alpha) \) and \( \mathcal{E}(\alpha) \).

We require the following lemmas due to Silverman [8] in our investigation.

**Lemma 1.** Let the function \( f(z) \) be defined by (1.6). Then \( f(z) \) is in the class \( \mathcal{J}^*(\alpha) \) if and only if

\[
(1.7) \quad \sum_{n=2}^{\infty} (n-\alpha) a_n \leq 1 - \alpha.
\]

**Lemma 2.** Let the function \( f(z) \) be defined by (1.6). Then \( f(z) \) is in the class \( \mathcal{E}(\alpha) \) if and only if

\[
(1.8) \quad \sum_{n=2}^{\infty} n(n-\alpha) a_n \leq 1 - \alpha.
\]

2. **Properties of the Class \( \mathcal{R}(\alpha) \)**

We begin by recalling here the following lemma due to Silverman and Ziegler [9].

**Lemma 3.** Let the function \( q(z) \) be defined by (1.1). Then \( q(z) \) is in the class \( \mathcal{R}(\alpha) \) if and only if

\[
(2.1) \quad \sum_{n=1}^{\infty} b_n \leq 1 - \alpha.
\]

Making use of Lemma 3, we shall prove...
THEOREM 1. Let the function \( q(z) \) be defined by (1.1). Then \( q(z) \) is in the class \( \mathcal{K}(\alpha) \) if and only if

\[
(2.2) \quad \sum_{n=1}^{\infty} n b_n n \leq 1 - \alpha.
\]

The result (2.2) is sharp.

PROOF. Since

\[
(2.3) \quad zq'(z) + 1 = 1 - \sum_{n=1}^{\infty} n b_n z^n,
\]

we prove the assertion (2.2) by substituting \( n b_n \) for \( b_n \) in Lemma 3. Further, for the function defined by

\[
(2.4) \quad q(z) = 1 - \left( \frac{1 - \alpha}{n} \right) z^n \quad (n \geq 1),
\]

we can easily see that the result (2.2) is sharp.

COROLLARY 1. Let the function \( q(z) \) defined by (1.1) be in the class \( \mathcal{K}(\alpha) \). Then

\[
(2.5) \quad b_n \leq \frac{1 - \alpha}{n} \quad (n \geq 1).
\]

Equality holds true for the function \( q(z) \) given by (2.4).

COROLLARY 2. Let \( 0 \leq \alpha < 1 \). Then

\[
(2.6) \quad \mathcal{K}(\alpha) \subset \mathcal{Q}(\alpha).
\]
Next, by using Theorem 1, we shall prove

**THEOREM 2.** The class \( \mathcal{K}(\alpha) \) is convex.

**PROOF.** Let the function \( q(z) \) defined by (1.1) and the function \( g(z) \) defined by

\[
g(z) = 1 - \sum_{n=1}^{\infty} c_n z^n \quad (c_n \geq 0)
\]

be in the class \( \mathcal{K}(\alpha) \). Then it suffices to prove that the function

\[
h(z) = \lambda q(z) + (1-\lambda)g(z) \quad (0 \leq \lambda \leq 1)
\]

is also in the class \( \mathcal{K}(\alpha) \). We note that

\[
h(z) = 1 - \sum_{n=1}^{\infty} \left\{ \lambda b_n + (1-\lambda)c_n \right\} z^n
\]

and

\[
\sum_{n=1}^{\infty} n \left\{ \lambda b_n + (1-\lambda)c_n \right\} = \lambda \sum_{n=1}^{\infty} n b_n + (1-\lambda) \sum_{n=1}^{\infty} n c_n \\
\leq 1 - \alpha,
\]

which evidently completes the proof of Theorem 2.

We know from Theorem 2 that there are some extreme points of \( \mathcal{K}(\alpha) \).

**THEOREM 3.** Let

\[
q_0(z) = 1
\]

and
(2.11) \[ q_n(z) = 1 - \left( \frac{1 - \alpha}{n} \right) z^n \quad (n \geq 1). \]

Then the function \( q(z) \) is in the class \( \mathcal{K}(\alpha) \) if and only if it can be expressed in the form

(2.12) \[ q(z) = \sum_{n=0}^{\infty} \lambda_n q_n(z), \]

where \( \lambda_n \geq 0 \quad (n \geq 0) \) and

(2.13) \[ \sum_{n=0}^{\infty} \lambda_n = 1. \]

PROOF. We assume that

(2.14) \[ q(z) = \sum_{n=0}^{\infty} \lambda_n q_n(z) \]

\[ = 1 - \sum_{n=1}^{\infty} \frac{(1-\alpha)\lambda_n n}{n} z^n. \]

Then, by appealing to Theorem 1, we have

(2.15) \[ \sum_{n=1}^{\infty} n \cdot \frac{(1-\alpha)\lambda_n n}{n} = (1-\alpha)(1-\lambda_0) \leq 1 - \alpha, \]

which implies that the function \( q(z) \) belongs to the class \( \mathcal{K}(\alpha) \).

Conversely, let us assume that the function \( q(z) \) defined by (1.1) is in the class \( \mathcal{K}(\alpha) \). Then, since \[ b_n \leq \frac{1 - \alpha}{n} \quad \text{for} \quad n \geq 1, \]
we can set

\[(2.16) \quad \lambda_n = \frac{n b_n}{1 - \alpha} \quad (n \geq 1)\]

and

\[(2.17) \quad \lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n.\]

Consequently, we have the representation (2.12), and the proof of Theorem 3 is completed.

**COROLLARY 3.** The extreme points of \(\mathcal{R}(\alpha)\) are \(q_n(z)\) \((n \geq 0)\) given by (2.10) and (2.11).

3. SOME INTERESTING RELATIONSHIPS

Silverman and Ziegler [9] gave a relationship between \(Q(\alpha)\) and \(\mathcal{F}^*(\alpha)\). We derive several interesting relationships between \(Q(\alpha)\) and \(\mathcal{F}^*(\alpha)\), and between \(\mathcal{R}(\alpha)\) and \(\mathcal{C}(\alpha)\).

**THEOREM 4.** Let the function \(q(z)\) defined by (1.1) be in the class \(Q(\alpha)\). Then

\[\int_0^z q(z)dz\]

is in the class \(\mathcal{F}^*(\alpha)\).

**PROOF.** We note that
\[ (3.1) \quad \int_0^z q(z)dz = z - \sum_{n=1}^{\infty} \left( \frac{b_n}{n+1} \right) z^{n+1}. \]

Hence, by Lemma 3,

\[ (3.2) \quad \sum_{n=1}^{\infty} (n+1-\alpha) \left( \frac{b_n}{n+1} \right) \leq \sum_{n=1}^{\infty} b_n \leq 1 - \alpha, \]

which, in view of Lemma 1, implies that

\[ \int_0^z q(z)dz \in \mathcal{F}^*(\alpha). \]

COROLLARY 4. Let the function \( q(z) \) defined by (1.1) be in the class \( \mathcal{R}(\alpha) \). Then

\[ \int_0^z \{ zq'(z) + 1 \}dz \]

is in the class \( \mathcal{F}^*(\alpha) \).

PROOF. Since \( q(z) \in \mathcal{R}(\alpha) \) if and only if

\[ zq'(z) + 1 \in Q(\alpha), \]

the proof of Corollary 4 is straightforward.

THEOREM 5. Let the function \( f(z) \) defined by (1.6) be in the class \( \mathcal{F}^*(\alpha) \). Then \( f'(z) \) is in the class \( Q \left[ \frac{\alpha}{z - \alpha} \right] \).

PROOF. Note that Lemma 1 gives
(3.3) \[ \sum_{n=1}^{\infty} a_n \leq \frac{1 - \alpha}{2 - \alpha}. \]

Therefore, we have

(3.4) \[ \sum_{n=1}^{\infty} (n+1)a_{n+1} \leq 1 - \alpha + \alpha \sum_{n=1}^{\infty} a_{n+1} \]

\[ \leq 1 - \frac{\alpha}{2 - \alpha}, \]

which implies that

\[ f'(z) \in \mathcal{Q}\left(\frac{\alpha}{2 - \alpha}\right). \]

COROLLARY 5. Let the function \( f(z) \) be defined by (1.6). Then \( f(z) \) is in the class \( \mathcal{F}^*(0) \) if and only if \( f'(z) \) is in the class \( \mathcal{Q}(0) \).

PROOF. Corollary 5 follows easily upon setting \( \alpha = 0 \) in Theorem 5.

THEOREM 6. Let the function \( f(z) \) defined by (1.6) be in the class \( \mathcal{C}(\alpha) \). Then \( f'(z) \) is in the class \( \mathcal{K}(\alpha) \cap \mathcal{Q}\left(\frac{1}{2 - \alpha}\right). \)

PROOF. To prove that \( f'(z) \in \mathcal{K}(\alpha) \), we need only show that

(3.5) \[ \sum_{n=1}^{\infty} n(n+1)a_{n+1} \leq 1 - \alpha. \]

In fact, from Lemma 2, we obtain
\begin{align*}
(3.6) \quad \sum_{n=1}^{\infty} n(n+1)a_{n+1} & \leq \sum_{n=1}^{\infty} (n+1)(n+1-a)a_{n+1} \leq 1 - \alpha,
\end{align*}

and

\begin{align*}
(3.7) \quad \sum_{n=1}^{\infty} (n+1)a_{n+1} & \leq 1 - \frac{1}{2 - \alpha},
\end{align*}

showing that

\[ f'(z) \in \mathcal{Q}\left(\frac{1}{2 - \alpha}\right). \]

Thus we have Theorem 6.

**THEOREM 7.** Let the function \( f(z) \) defined by (1.6) be in the class \( \mathcal{F}^*(\alpha) \). Then

\begin{align*}
(3.8) \quad \frac{f(z)}{z} & \in \mathcal{R}(\alpha) \cap \mathcal{Q}\left(\frac{1}{2 - \alpha}\right).
\end{align*}

**PROOF.** Since

\begin{align*}
(3.9) \quad \frac{f(z)}{z} = 1 - \sum_{n=1}^{\infty} a_{n+1}z^n,
\end{align*}

we obtain

\begin{align*}
(3.10) \quad \sum_{n=1}^{\infty} n a_{n+1} = \sum_{n=2}^{\infty} (n-1)a_n \leq \sum_{n=2}^{\infty} (n-\alpha)a_n \leq 1 - \alpha
\end{align*}

and

\begin{align*}
(3.11) \quad \sum_{n=1}^{\infty} a_{n+1} = \sum_{n=2}^{\infty} a_n \leq 1 - \frac{1}{2 - \alpha}.
\end{align*}

Now the assertion (3.8) of Theorem 7 follows at once from (3.10) and (3.11).
Similarly, we have

**THEOREM 8.** Let the function $f(z)$ defined by (1.6) be in the class $C(\alpha)$. Then

\begin{equation}
\frac{f(z)}{z} \in B\left(\frac{1 + \alpha}{2}\right) \cap J\left(\frac{3 - \alpha}{2(2-\alpha)}\right).
\end{equation}

4. A MODIFIED CONVOLUTION PRODUCT

Let $q_j(z)$ $(j = 1, 2)$ be defined by

\begin{equation}
q_j(z) = 1 - \sum_{n=1}^{\infty} b_{n,j} z^n \quad (b_{n,j} \geq 0).
\end{equation}

We denote by $q_1 \ast q_2(z)$ a modified convolution product of two functions $q_1(z)$ and $q_2(z)$, defined by

\begin{equation}
q_1 \ast q_2(z) = 1 - \sum_{n=1}^{\infty} b_{n,1} b_{n,2} z^n.
\end{equation}

We now consider the modified convolution products of functions in the classes $Q(\alpha)$ and $R(\alpha)$.

**THEOREM 9.** Let the functions $q_i(z)$ $(i = 1, 2)$ be defined by (4.1). Also let $q_1(z) \in R(\alpha)$ and $q_2(z) \in R(\beta)$. Then the modified convolution product $q_1 \ast q_2(z)$ defined by (4.2) is in the class $R(\alpha + \beta - \alpha\beta)$.

**PROOF.** We have to find the largest $\gamma = \gamma(\alpha, \beta)$ such that
\[(4.3) \quad \sum_{n=1}^{\infty} n b_{n,1} b_{n,2} \leq 1 - \gamma,\]

or equivalently,

\[(4.4) \quad \sum_{n=1}^{\infty} \left(\frac{n}{1 - \gamma}\right) b_{n,1} b_{n,2} \leq 1.\]

We note from Theorem 1 that

\[(4.5) \quad \sum_{n=1}^{\infty} \left(\frac{n}{1 - \alpha}\right) b_{n,1} \leq 1\]

and

\[(4.6) \quad \sum_{n=1}^{\infty} \left(\frac{n}{1 - \beta}\right) b_{n,2} \leq 1.\]

By using the Cauchy-Schwarz inequality, we have

\[(4.7) \quad \sum_{n=1}^{\infty} \sqrt{\frac{n}{1 - \alpha}} \sqrt{\frac{n}{1 - \beta}} \sqrt{b_{n,1} b_{n,2}} \leq 1.\]

Hence, if

\[(4.8) \quad \left(\frac{n}{1 - \gamma}\right) \sqrt{b_{n,1} b_{n,2}} \leq \sqrt{\frac{n}{1 - \alpha}} \sqrt{\frac{n}{1 - \beta}}\]

for \(n \geq 1\), we have (4.4).

Further, it is sufficient to prove that

\[(4.9) \quad \frac{1}{1 - \gamma} \leq \frac{n}{(1 - \alpha)(1 - \beta)}\]

for \(n \geq 1\).
It follows from (4.9) that

\[
\gamma \leq 1 - \frac{(1-\alpha)(1-\beta)}{n}.
\]

Since

\[
\phi(n) = 1 - \frac{(1-\alpha)(1-\beta)}{n}
\]

is an increasing function of \( n \) \((n \geq 1)\), putting \( n = 1 \) in (4.11), we obtain

\[
\gamma \leq \phi(1) = \alpha + \beta - \alpha\beta < 1.
\]

Thus

\[
q_1 * q_2(z) \in \mathcal{P}(\alpha + \beta - \alpha\beta),
\]

which proves Theorem 7.

In a similar manner, we can prove

THEOREM 10. Let the functions \( q_i(z) \) \((i = 1, 2)\) be defined by (4.1). Also let \( q_1(z) \in \mathcal{Q}(\alpha) \) and \( q_2(z) \in \mathcal{Q}(\beta) \). Then the modified convolution product \( q_1 * q_2(z) \) defined by (4.2) is in the class \( \mathcal{Q}(\alpha + \beta - \alpha\beta) \).

THEOREM 11. Let the functions \( q_i(z) \) \((i = 1, 2)\) be defined by (4.1). Also let \( q_1(z) \in \mathcal{Q}(\alpha) \) and \( q_2(z) \in \mathcal{Q}(\beta) \). Then the modified convolution product \( q_1 * q_2(z) \) defined by (4.2) is in the class \( \mathcal{Q}(\alpha + \beta - \alpha\beta) \).
5. THE FUNCTIONAL $\mathcal{J}(q)$

We introduce the functional $\mathcal{J}(q)$ defined by

\begin{equation}
\mathcal{J}(q) = \frac{1}{z} \int_0^z q(z) \, dz,
\end{equation}

and prove

**THEOREM 12.** Let the function $q(z)$ defined by (1.1) be in the class $\mathcal{Q}(\alpha)$. Then $\mathcal{J}(q)$ is in the class $\mathcal{K}(\alpha) \cap \mathcal{Q}\left(\frac{1 + \alpha}{2}\right)$.

**PROOF.** Note that

\begin{equation}
\mathcal{J}(q) = 1 - \sum_{n=1}^{\infty} \left(\frac{b_n}{n + 1}\right) z^n.
\end{equation}

Hence

\begin{equation}
\sum_{n=1}^{\infty} n \left(\frac{b_n}{n + 1}\right) \leq \sum_{n=1}^{\infty} b_n \leq 1 - \alpha,
\end{equation}

which implies that $\mathcal{J}(q) \in \mathcal{K}(\alpha)$. Furthermore,

\begin{equation}
\sum_{n=1}^{\infty} \frac{b_n}{n + 1} \leq \frac{1}{2} \sum_{n=1}^{\infty} b_n \leq 1 - \left(\frac{1 + \alpha}{2}\right),
\end{equation}

which shows that

$\mathcal{J}(q) \in \mathcal{Q}\left(\frac{1 + \alpha}{2}\right)$. 

In a similar way, we prove

**THEOREM 13.** Let the function $q(z)$ defined by (1.1) be in the class $\mathcal{K}(\alpha)$. Then $f(q)$ is in the class $\mathcal{K}\left(\frac{1 + \alpha}{2}\right)$.

**REFERENCES**


