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STIELTJES TRANSFORM AND ITS
APPLICATION TO A CLASS OF SINGULAR
INTEGRAL EQUATIONS

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Dedicated to the memory of Professor David Vernon Widder (1898-1990)

A new convolution theorem is proved for the Stieltjes transform and is
then applied in solving a certain class of singular integral equations which
are related rather closely to the Riemann-Hilbert boundary value problem.
Some further extensions and consequences of the convolution theorem are
also considered.

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theorem, Stieltjes convolution, transcendental equation, classical inversion theorem.
1. INTRODUCTION

The convolution theorems for the Fourier, Laplace, and Mellin transforms are well-known (see, e.g., [2] and [8]). Each of these results has a great potential for applications in solving convolution integral equations (cf. [7] and [8]), in the investigation of convolution transforms (cf. [4], [9], and [10]), and in the evaluation of definite integrals (cf. [5]). The object of the present paper is first to prove a new convolution theorem for the Stieltjes transform:

\[ S\{f(t) : s\} = \int_0^\infty \frac{f(t)}{s + t} \, dt \quad (s \in D), \]

which arises naturally from the iteration of the classical Laplace transform, \( D \) being an arbitrary region of the complex \( s \)-plane cut along the nonpositive real axis. We then consider an interesting deduction from this convolution theorem leading to a Titchmarsh type theorem. We also show how our convolution theorem can be applied in solving a certain class of singular integral equations which are usually investigated by reducing the problem to an equivalent Riemann-Hilbert boundary value problem.

2. THE CONVOLUTION THEOREM

Let the functions \( f \) and \( g \) be defined on the interval

\[ \mathbb{R}_+ = (0, \infty) := \{ x : 0 < x < \infty \}. \]

We define a function \( h \) on \( \mathbb{R}_+ \) by

\[ h(t) = (f \otimes g)(t) := f(t) \int_0^\infty \frac{g(u)}{u - t} \, du + g(t) \int_0^\infty \frac{f(u)}{u - t} \, du, \]

where the integrals, when they exist, are understood as their Cauchy principal values.

Our main result is contained in the following

**Theorem.** Let

\[ f \in L_p(\mathbb{R}_+) \quad \text{and} \quad g \in L_q(\mathbb{R}_+) \]

\((1 < p < \infty; \ 1 < q < \infty; \ r^{-1} := p^{-1} + q^{-1} < 1). \)
Then the function $h$, defined by (2), belongs to $L_r(\mathbb{R}^+)$ and its Stieltjes transform is given by

$$S\{h(t) : s\} := S\{(f \otimes g)(t) : s\} = S\{f(t) : s\} S\{g(t) : s\}.$$  \hfill (3)

**Proof.** We begin by defining the functions $f_1$ and $f^*$ by

$$
\begin{aligned}
  f_1(t) &:= \int_0^\infty \frac{f(u)}{u-t} \, dt \quad (f \in L_p(\mathbb{R}^+); \ t \in \mathbb{R}^+) \\
  f^*(t) &:= \begin{cases} 
  f(t) & (t > 0) \\
  0 & (t \leq 0).
\end{cases}
\end{aligned}
$$  \hfill (4)

Clearly, for $f \in L_p(\mathbb{R}^+)$, we have

$$f^* \in L_p(\mathbb{R}) \quad (\mathbb{R} := \mathbb{R}^+ \cup \{0\}).$$

Furthermore, since the Hilbert transform

$$\tilde{f}(t) := \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{t-\varepsilon} + \int_{t+\varepsilon}^{\infty} \right) \frac{f^*(u)}{t-u} \, du \quad (t \in \mathbb{R})$$  \hfill (5)

is a bounded operator in $L_p(\mathbb{R})$ (cf., e.g., [1, p. 315, Theorem 8.1.12]), we conclude that

$$\tilde{f}(t) \in L_p(\mathbb{R}).$$

Combining this observation with the fact that [cf. Equations (4) and (5)]

$$f_1(t) = -\tilde{f}(t) \quad (t \in \mathbb{R}^+),$$

we have

$$f_1(t) \in L_p(\mathbb{R}^+),$$

which, in view of the Hölder inequality [6, p. 15], yields

$$g(t) f_1(t) \in L_r(\mathbb{R}^+)$$

under the hypothesis of the theorem.

In an analogous manner, we can show that

$$g_1(t) := \int_0^\infty \frac{g(u)}{u-t} \, du \in L_q(\mathbb{R}^+).$$
and
\[ f(t)g_1(t) \in L_r(\mathbb{R}_+). \]

Therefore \( h \in L_r(\mathbb{R}_+) \), which is precisely the first assertion of the theorem.

Next, since the Stieltjes transform (1) is a bounded operator in \( L_r(\mathbb{R}_+) \) \((1 < r < \infty)\) (cf. [6, p. 225]), we can take the Stieltjes transform of the function \( h \) defined by (2), and we have
\[
S\{h(t) : s\} = \int_0^\infty \frac{f(t)}{s + t} \left( \int_0^\infty \frac{g(u)}{u - s} \, du \right) \, dt
+ \int_0^\infty \frac{g(t)}{s + t} \left( \int_0^\infty \frac{f(u)}{u - s} \, du \right) \, dt.
\]

Putting
\[
f_\varepsilon(t) := \left( \int_0^{t-\varepsilon} + \int_{t+\varepsilon}^\infty \right) \frac{f(u)}{u - t} \, dt \quad (\varepsilon > 0),
\]
and applying the Riesz inequality [1, p. 315, Theorem 8.1.12], we obtain
\[
\|f_\varepsilon\|_{L_p(\mathbb{R}_+)} \leq C\|f^*\|_{L_p(\mathbb{R})} = C\|f\|_{L_p(\mathbb{R}_+)},
\]
where \( f^* \) is defined by (4). We note also that
\[
(s + t)^{-1} \in L_p(\mathbb{R}_+) \quad (r^{-1} + \rho^{-1} = 1).
\]
Hence, using the Hölder inequality for three functions (cf. [6, p. 154] and [8, p. 97]), we see that
\[
\varphi_\varepsilon(t) := (s + t)^{-1} g(t) f_\varepsilon(t) \in L_1(\mathbb{R}_+)
\]
and, moreover, that
\[
\|(s + t)^{-1} g f_\varepsilon\|_1 \leq \|(s + t)^{-1}\|_p \|f\|_p \|g\|_q.
\]

Thus one can apply the Lebesgue theorem to obtain
\[
\int_{\mathbb{R}_+} \left\{ \lim_{\varepsilon \to 0^+} \varphi_\varepsilon(t) \right\} \, dt = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}_+} \varphi_\varepsilon(t) \, dt
\]
or, equivalently,
\[
\int_0^\infty \frac{g(t)}{s + t} \left( \int_0^\infty \frac{f(u)}{u - t} \, du \right) \, dt
\]
\[= \int_0^\infty f(u) \left( \int_0^\infty \frac{g(t)}{(s + t)(u - t)} \, dt \right) \, du,
\]
and similarly for the first term on the right-hand side of (6). Consequently, the formula (6) becomes
\[
\mathcal{S}\{h(t) : s\} = \int_0^\infty f(t) \left( \int_0^\infty \frac{g(u)}{(s + t)(u - t)} \, du \right) \, dt
\]
\[+ \int_0^\infty f(t) \left( \int_0^\infty \frac{g(u)}{(s + u)(t - u)} \, du \right) \, dt
\]
\[= \int_0^\infty f(t) \left( \int_0^\infty \frac{g(u)}{u - t} \left\{ \frac{1}{s + t} - \frac{1}{s + u} \right\} \, du \right) \, dt.
\]
Simplifying this last double integral, we finally have
\[
\mathcal{S}\{h(t) : s\} = \int_0^\infty \frac{f(t)}{s + t} \, dt \int_0^\infty \frac{g(u)}{s + u} \, du
\]
\[= \mathcal{S}\{f(t) : s\} \mathcal{S}\{g(t) : s\}.
\]

Since the Stieltjes transform, when it exists, is an analytic function in the complex plane cut along the nonpositive real axis \((\text{cf. } [9, \text{ p. 328}])\), the formula (12) holds true everywhere in the complex \(s\)-plane except along the nonpositive real axis. This evidently completes the proof of the theorem.

**REMARK.** In view of the property (3), the function \(h\) defined by (2) may be called the *Stieltjes convolution* of the functions \(f\) and \(g\).

Suppose now that the functions \(f\) and \(g\) satisfy the hypothesis of the convolution theorem (3) and that

\[
f \otimes g = 0 \quad \text{almost everywhere on } \mathbb{R}_+.
\]

Then
\[
\mathcal{S}\{f(t) : s\} \mathcal{S}\{g(t) : s\} = \mathcal{S}\{(f \otimes g)(t) : s\} = 0
\]
everywhere in the complex $s$-plane cut along the nonpositive real axis. Since
\[ S\{f(t) : s\} \quad \text{and} \quad S\{g(t) : s\} \]
are analytic functions in this cut $s$-plane, it follows that either
\[ S\{f(t) : s\} = 0 \]
or
\[ S\{g(t) : s\} = 0. \]
Making use of the uniqueness of the Stieltjes transform (cf. [9, p. 336]), we conclude that either $f = 0$ or $g = 0$ almost everywhere on $\mathbb{R}_+$. Thus we have proved the following result:

*If the functions $f$ and $g$ satisfy the hypothesis of the theorem, and if $f \otimes g = 0$ almost everywhere on $\mathbb{R}_+$, then either $f = 0$ or $g = 0$ almost everywhere on $\mathbb{R}_+$.*

This is a Titchmarsh type theorem.

3. AN APPLICATION OF THE CONVOLUTION THEOREM

We consider the following interesting class of *singular integral equations:*
\[ f(t) + \lambda \int_0^\infty \frac{f(u)}{u - t} \, du = g(t) \quad (\lambda \neq 0), \tag{13} \]
where $g(t)$ is prescribed and $f(t)$ is an unknown function to be determined. The solution of the integral equation (13) was investigated earlier by reducing the problem to an equivalent Riemann-Hilbert boundary value problem (see [3, Chapter 3, Section 21] for details). In this section we shall show how the convolution theorem (3) can be applied to solve the integral equation (13).

We begin by assuming $\alpha_0$ to be a (unique) root of the transcendental equation:
\[ \tan \pi \alpha = -\pi \lambda \quad (0 < \Re(\alpha) < 1; \quad \lambda \neq 0). \tag{14} \]
Then, in view of the well-known integral (cf., e.g., [2, Vol. II, p. 249, Entry 15.2(28); p. 216, Entry 14.2(5))):
\[ \int_0^\infty \frac{u^{\alpha-1}}{u - \zeta} \, du = \begin{cases} -\pi \zeta^{\alpha-1} \cot \pi \alpha & (\Re(\zeta) > 0; \quad 0 < \Re(\alpha) < 1) \\ \pi(-\zeta)^{\alpha-1} \csc \pi \alpha & (\Re(\zeta) < 0; \quad 0 < \Re(\alpha) < 1), \end{cases} \tag{15} \]
the integral equation (13) can be written in the form:

\[ f(t) \int_0^\infty \frac{u^{\alpha_0 - 1}}{u - t} \, du + t^{\alpha_0 - 1} \int_0^\infty \frac{f(u)}{u - t} \, du = -\pi t^{\alpha_0 - 1} g(t) \cot \pi \alpha_0 \]

or, equivalently,

\[ f(t) \otimes t^{\alpha_0 - 1} = -\pi t^{\alpha_0 - 1} g(t) \cot \pi \alpha_0, \tag{16} \]

where we have made use of the definition (2).

Applying the convolution theorem (3), this last relationship (16) yields

\[ \mathcal{S}\{f(t) : s\} \int_0^\infty \frac{u^{\alpha_0 - 1}}{u + s} \, du = -\pi \mathcal{S}\{t^{\alpha_0 - 1} g(t) : s\} \cot \pi \alpha_0, \]

which, in view of the integral (15) again, becomes

\[ \mathcal{S}\{f(t) : s\} = -s^{1-\alpha_0} \mathcal{S}\{t^{\alpha_0 - 1} g(t) : s\} \cos \pi \alpha_0. \tag{17} \]

Finally, by appealing to the classical inversion theorem for the Stieltjes transform (see, e.g., [10, p. 126, Theorem 14.1]), we obtain

\[ f(t) = \lim_{\varepsilon \to 0^+} \frac{\cos \pi \alpha_0}{2\pi i} \left[ (-t + i\varepsilon)^{1-\alpha_0} \mathcal{S}\{u^{\alpha_0 - 1} g(u) : -t + i\varepsilon\} \right. \]

\[ \left. - (-t - i\varepsilon)^{1-\alpha_0} \mathcal{S}\{u^{\alpha_0 - 1} g(u) : -t - i\varepsilon\} \right], \tag{18} \]

which provides the solution of the singular integral equation (13), \( \alpha_0 \) being a (unique) root of the transcendental equation (14).
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